CONGRUENCES CONCERNING LEGENDRE POLYNOMIALS

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ABSTRACT. Let p be an odd prime. In this paper, by using the properties of Legendre polynomials we prove some congruences for $\sum_{k=0}^{\frac{p-1}{2}} \binom{2k}{k}^2 m^{-k} \pmod{p^2}$. In particular, we confirm several conjectures of Z.W. Sun. We also pose 13 conjectures on supercongruences.

1. INTRODUCTION

Let p be an odd prime. In 2003, Rodriguez-Villegas [11] conjectured the following congruence:

$$(1.1) \qquad \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{16^k} \equiv (-1)^{\frac{p-1}{2}} \pmod{p^2}.$$

This was later confirmed by Mortenson [7] via the Gross-Koblitz formula. See also [9] and [10, p. 204]. Recently the author's twin brother, Zhi-Wei Sun [13], obtained the congruences for $\sum_{k=0}^{p-1} \binom{2k}{k}^2 m^{-k} \pmod{p}$ in the cases $m = 8, -16, 32$ and made several conjectures for $\sum_{k=0}^{p-1} \binom{2k}{k}^2 m^{-k} \pmod{p^2}$. For example, he conjectured
(1.2)
$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{32^k} \equiv \begin{cases} 0 \pmod{p^2} & \text{if } 4 \mid p - 3, \\ 2a - \frac{p}{2a} \pmod{p^2} & \text{if } 4 \mid p - 1 \text{ and } p = a^2 + b^2 \text{ with } 4 \mid a - 1. \end{cases}$$

Let $\{P_n(x)\}$ be the Legendre polynomials given by

$$\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n \quad (|t| < 1).$$

It is well known that (see [6, pp. 228-232], [4, (3.132)-(3.133)])

$$(1.3) \qquad P_n(x) = \frac{1}{2^n} \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n - 2k)!}{k!(n-k)!(n-2k)!} x^{n-2k} = \frac{1}{2^n \cdot n!} \cdot \frac{d^n}{dx^n}(x^2 - 1)^n$$

and $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$, where $[x]$ is the greatest integer not exceeding x.

In the paper, by using the expansions of Legendre polynomials we obtain some congruences for $P_{\frac{p-1}{2}}(x)$ modulo p^2, where p is an odd prime and x is a rational p-integer. For example, we have

$$(1.4) \qquad \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{16^k}\left(x^k - (-1)^{\frac{p-1}{2}}(1-x)^k\right) \equiv 0 \pmod{p^2}$$

and

$$(1.5) \qquad \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{16^k}\left(x^k - \left(\frac{x}{p}\right)x^{-k}\right) \equiv 0 \pmod{p} \quad \text{for} \quad x \not\equiv 0 \pmod{p},$$

where $\left(\frac{x}{p}\right)$ is the Jacobi symbol. Taking $x = 1$ in (1.4) we obtain (1.1) immediately, and taking $x = \frac{1}{2}$ in (1.4) we deduce (1.2) for $p \equiv 3 \pmod 4$. We also determine $\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{32^k}k(k-1)\cdots(k-r+1) \pmod{p^2}$ for $r \in \{1, 2, \ldots, \frac{p-1}{2}\}$, $\sum_{k=0}^{[p/3]} \frac{(3k)!}{54^k \cdot k!^3} \pmod{p}$ and pose some conjectures on supercongruences concerning binary quadratic forms.

Throughout this paper we use $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{Z}_p$ to denote the sets of integers, positive integers and rational p-integers for a prime p, respectively.

2. MAIN RESULTS

Lemma 2.1. *For $n \in \mathbb{N}$ we have*

$$P_n(x) = \sum_{k=0}^{n} \binom{n+k}{2k}\binom{2k}{k}\left(\frac{x-1}{2}\right)^k.$$

Proof. From [4, (3.135)] we have the following result due to Murphy:

$$(2.1) \qquad P_n(x) = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}\left(\frac{x-1}{2}\right)^k.$$

As $\binom{n+k}{2k}\binom{2k}{k} = \binom{n+k}{k}\binom{n}{k}$, we obtain the result. $\qquad\qquad\square$

Lemma 2.2. *Let p be an odd prime and $k \in \{1, 2, \ldots, (p-1)/2\}$. Then*

$$\binom{\frac{p-1}{2}+k}{2k} \equiv \frac{\binom{2k}{k}}{(-16)^k}\left(1 - p^2\sum_{i=1}^{k}\frac{1}{(2i-1)^2}\right) \pmod{p^4}.$$

Proof. Clearly

$$\begin{aligned}
\binom{\frac{p-1}{2}+k}{2k} &= \frac{(\frac{p-1}{2}+k)(\frac{p-1}{2}+k-1)\cdots(\frac{p-1}{2}-k+1)}{(2k)!} \\
&= \frac{(p+2k-1)(p+2k-3)\cdots(p-(2k-3))(p-(2k-1))}{2^{2k}\cdot(2k)!} \\
&= \frac{(p^2-1^2)(p^2-3^2)\cdots(p^2-(2k-1)^2)}{2^{2k}\cdot(2k)!} \\
&\equiv \frac{(-1)^k\cdot 1^2\cdot 3^2\cdots(2k-1)^2}{2^{2k}\cdot(2k)!}\left(1 - p^2\sum_{i=1}^{k}\frac{1}{(2i-1)^2}\right) \pmod{p^4}.
\end{aligned}$$

To see the result, we note that

$$\frac{1^2 \cdot 3^2 \cdots (2k-1)^2}{2^{2k} \cdot (2k)!} = \frac{(2k)!^2}{(2 \cdot 4 \cdots (2k))^2 \cdot 2^{2k} \cdot (2k)!} = \frac{(2k)!}{2^{4k} \cdot k!^2} = \frac{\binom{2k}{k}}{16^k}. \qquad \square$$

Let p be an odd prime, and let $\{A(n)\}$ be the Apéry numbers given by

$$A(n) = \sum_{k=0}^{n} \binom{n+k}{k}^2 \binom{n}{k}^2.$$

It is well known that (see [1], [10]) $A(\frac{p-1}{2}) \equiv a(p) \pmod{p^2}$, where $a(n)$ is defined by

$$q \prod_{n=1}^{\infty} (1 - q^{2n})^4 (1 - q^{4n})^4 = \sum_{n=1}^{\infty} a(n) q^n.$$

By the fact $\binom{n+k}{k}\binom{n}{k} = \binom{n+k}{2k}\binom{2k}{k}$ and Lemma 2.2 we have

$$A\left(\frac{p-1}{2}\right) = \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}+k}{2k}^2 \binom{2k}{k}^2 \equiv \sum_{k=0}^{\frac{p-1}{2}} \left(\frac{\binom{2k}{k}}{(-16)^k}\right)^2 \binom{2k}{k}^2 \pmod{p^2}.$$

Hence

$$(2.2) \qquad a(p) \equiv A\left(\frac{p-1}{2}\right) \equiv \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^4}{4^{4k}} \pmod{p^2}.$$

Let $b(n)$ be given by $q \prod_{n=1}^{\infty}(1 - q^{4n})^6 = \sum_{n=1}^{\infty} b(n) q^n$. Then Mortenson [8] proved the following conjecture of Rodriguez-Villegas:

$$(2.3) \qquad \sum_{k=1}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^3}{4^{3k}} \equiv b(p) \pmod{p^2}.$$

Theorem 2.1. *Let p be an odd prime and let x be a variable. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{16^k}\left(x^k - (-1)^{\frac{p-1}{2}}(1-x)^k\right) \equiv \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{16^k}\left(x^k - (-1)^{\frac{p-1}{2}}(1-x)^k\right) \equiv 0 \pmod{p^2}.$$

Proof. For a variable t, by Lemmas 2.1 and 2.2 we have

$$(2.4) \quad P_{\frac{p-1}{2}}(t) = \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}+k}{2k}\binom{2k}{k}\left(\frac{t-1}{2}\right)^k \equiv \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{16^k}\left(\frac{1-t}{2}\right)^k \pmod{p^2}.$$

It is known that (see [6]) $P_n(t) = (-1)^n P_n(-t)$. Thus, by (2.4),

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{16^k}\left(\frac{1-t}{2}\right)^k \equiv (-1)^{\frac{p-1}{2}} \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{16^k}\left(\frac{1+t}{2}\right)^k \pmod{p^2}.$$

Now taking $t = 1 - 2x$ in the congruence we deduce that

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{16^k}\left(x^k - (-1)^{\frac{p-1}{2}}(1-x)^k\right) \equiv 0 \pmod{p^2}.$$

To complete the proof, we note that for $k \in \{\frac{p+1}{2}, \frac{p+3}{2}, \ldots, p-1\}$, $\binom{2k}{k} = 2k(2k-1)\cdots(k+1)/k! \equiv 0 \pmod{p}$. $\square$

Theorem 2.2. *Let p be an odd prime. Then*

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{32^k} \equiv \begin{cases} 0 \pmod{p^2} & \text{if } 4 \mid p - 3, \\ 2a - \frac{p}{2a} \pmod{p^2} & \text{if } 4 \mid p - 1 \text{ and } p = a^2 + b^2 \text{ with } 4 \mid a - 1. \end{cases}$$

Proof. When $p \equiv 3 \pmod 4$, taking $x = \frac{1}{2}$ in Theorem 2.1 we obtain the result. Now suppose $p \equiv 1 \pmod 4$ and so $p = a^2 + b^2$ with $a, b \in \mathbb{Z}$ and $a \equiv 1 \pmod 4$. It is well known that ([6])

$$(2.5) \qquad\qquad P_{2n+1}(0) = 0 \quad \text{and} \quad P_{2n}(0) = \frac{(-1)^n}{2^{2n}} \binom{2n}{n}.$$

Thus, by (2.4) and (2.5) we have

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{32^k} \equiv P_{\frac{p-1}{2}}(0) = \frac{(-1)^{\frac{p-1}{4}}}{2^{\frac{p-1}{2}}} \binom{\frac{p-1}{2}}{\frac{p-1}{4}} \pmod{p^2}.$$

According to the result due to Chowla, Dwork and Evans (see [2] or [3]), we have

$$\binom{\frac{p-1}{2}}{\frac{p-1}{4}} \equiv \frac{2^{p-1} + 1}{2} \left(2a - \frac{p}{2a}\right) \pmod{p^2}.$$

Set $q = (2^{\frac{p-1}{2}} - (-1)^{\frac{p-1}{4}})/p$. Then $2^{p-1} \equiv 1 + 2(-1)^{\frac{p-1}{4}} qp \pmod{p^2}$. Thus

$$\frac{2^{p-1} + 1}{2 \cdot 2^{\frac{p-1}{2}}} \equiv \frac{2 + 2(-1)^{\frac{p-1}{4}} qp}{2((-1)^{\frac{p-1}{4}} + qp)} = (-1)^{\frac{p-1}{4}} \pmod{p^2}.$$

Hence

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{32^k} \equiv \frac{(-1)^{\frac{p-1}{4}}}{2^{\frac{p-1}{2}}} \binom{\frac{p-1}{2}}{\frac{p-1}{4}} \equiv \frac{(-1)^{\frac{p-1}{4}}}{2^{\frac{p-1}{2}}} \cdot \frac{2^{p-1} + 1}{2} \left(2a - \frac{p}{2a}\right) \equiv 2a - \frac{p}{2a} \pmod{p^2}.$$

The proof is now complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Remark 2.1. Theorem 2.2 was conjectured by Zhi-Wei Sun ([13]), and the congruence for $\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{32^k} \pmod p$ was also proved by Zhi-Wei Sun in [13].

Theorem 2.3. *Let p be an odd prime and $r \in \{1, 2, \ldots, (p-1)/2\}$. Then*

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{32^k} k(k-1) \cdots (k - r + 1)$$

$$\equiv \begin{cases} 0 \pmod{p^2} & \text{if } 4 \mid (p + 1 - 2r), \\ (-1)^{\frac{p-1+2r}{4}} 2^{-\frac{p-1}{2}} \frac{(\frac{p-1}{2}+r)!}{\frac{p-1-2r}{4}! \frac{p-1+2r}{4}!} \pmod{p^2} & \text{if } 4 \mid (p - 1 - 2r). \end{cases}$$

Proof. By (2.4) we have

$$(2.6) \qquad \begin{aligned} \frac{d^r P_{\frac{p-1}{2}}(t)}{dt^r} &\equiv \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{(-32)^k} \cdot \frac{d^r (t-1)^k}{dt^r} \\ &= \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{(-32)^k} k(k-1) \cdots (k - r + 1)(t-1)^{k-r} \pmod{p^2}. \end{aligned}$$

Hence

$$\left.\frac{d^r\ P_{\frac{p-1}{2}}(t)}{dt^r}\right|_{t=0} = (-1)^r \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{32^k} k(k-1)\cdots(k-r+1).$$

By (1.3) we have

$$\frac{d^r}{dt^r}P_{\frac{p-1}{2}}(t) = \frac{1}{2^{(p-1)/2}} \cdot \frac{d^r}{dt^r} \sum_{m=0}^{[\frac{p-1}{4}]} \frac{(-1)^m(p-1-2m)!}{m!(\frac{p-1}{2}-m)!(\frac{p-1}{2}-2m)!} t^{\frac{p-1}{2}-2m}$$

$$= \frac{1}{2^{(p-1)/2}} \sum_{m=0}^{[\frac{p-1-2r}{4}]} \frac{(-1)^m(p-1-2m)!}{m!(\frac{p-1}{2}-m)!(\frac{p-1}{2}-2m)!}$$

$$\times (\frac{p-1}{2}-2m)(\frac{p-1}{2}-2m-1)\cdots(\frac{p-1}{2}-2m-r+1)t^{\frac{p-1}{2}-2m-r}.$$

Thus,

$$\left.\frac{d^r P_{\frac{p-1}{2}}(t)}{dt^r}\right|_{t=0} = \begin{cases} 0 & \text{if } r \not\equiv \frac{p-1}{2} \pmod 2, \\ \frac{(-1)^m(p-1-2m)!}{2^{(p-1)/2}\cdot m!(\frac{p-1}{2}-m)!} & \text{if } r = \frac{p-1}{2} - 2m. \end{cases}$$

Now combining all the above we obtain the result. $\square$

Corollary 2.1. *Let p be an odd prime. Then*

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{k^2\binom{2k}{k}^2}{32^k} \equiv \begin{cases} (-1)^{\frac{p+3}{4}} 2^{-\frac{p-1}{2}} \frac{\frac{p+3}{2}!}{\frac{p-5}{4}!\frac{p+3}{4}!} \pmod{p^2} & \text{if } p \equiv 1 \pmod 4, \\ (-1)^{\frac{p+1}{4}} 2^{-\frac{p-1}{2}} \frac{\frac{p+1}{2}!}{\frac{p-3}{4}!\frac{p+1}{4}!} \pmod{p^2} & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Proof. By Theorem 2.3 we have

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{k\binom{2k}{k}^2}{32^k} \equiv \begin{cases} 0 \pmod{p^2} & \text{if } p \equiv 1 \pmod 4, \\ (-1)^{\frac{p+1}{4}} 2^{-\frac{p-1}{2}} \frac{\frac{p+1}{2}!}{\frac{p-3}{4}!\frac{p+1}{4}!} \pmod{p^2} & \text{if } p \equiv 3 \pmod 4 \end{cases}$$

and

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{k(k-1)\binom{2k}{k}^2}{32^k} \equiv \begin{cases} (-1)^{\frac{p+3}{4}} 2^{-\frac{p-1}{2}} \frac{\frac{p+3}{2}!}{\frac{p-5}{4}!\frac{p+3}{4}!} \pmod{p^2} & \text{if } p \equiv 1 \pmod 4, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Observe that $k^2 = k(k-1) + k$. From the above we deduce the result. $\square$

Lemma 2.3. *Let p be a prime greater than 3 and let t be a variable. Then*

$$P_{[\frac{p}{3}]}(t) \equiv \sum_{k=0}^{[p/3]} \frac{(3k)!}{k!^3}\left(\frac{1-t}{54}\right)^k \pmod p.$$

Proof. Suppose $r = 1$ or 2 according to whether $3 \mid p-1$ or $3 \mid p-2$. Then clearly

$$\binom{\frac{p-r}{3}+k}{2k} = \frac{(\frac{p-r}{3}+k)(\frac{p-r}{3}+k-1)\cdots(\frac{p-r}{3}-k+1)}{(2k)!}$$

$$= \frac{(p+3k-r)(p+3k-r-3)\cdots(p-(3k+r-3))}{3^{2k}\cdot(2k)!}$$

$$\equiv (-1)^k\frac{(3k-r)(3k-r-3)\cdots(3-r)\cdot r(r+3)\cdots(3k+r-3)}{3^{2k}\cdot(2k)!}$$

$$= \frac{(-1)^k\cdot(3k)!}{3\cdot6\cdots3k\cdot3^{2k}\cdot(2k)!} = \frac{(-1)^k\cdot(3k)!}{3^k\cdot k!\cdot3^{2k}\cdot(2k)!} \pmod{p}.$$

Hence, by Lemma 2.1 we have

$$P_{[\frac{p}{3}]}(t) = \sum_{k=0}^{[p/3]}\binom{[\frac{p}{3}]+k}{2k}\binom{2k}{k}(\frac{t-1}{2})^k \equiv \sum_{k=0}^{[p/3]}\frac{(-1)^k\cdot(3k)!}{3^{3k}\cdot k!(2k)!}\cdot\frac{(2k)!}{k!^2}\Big(\frac{t-1}{2}\Big)^k$$

$$= \sum_{k=0}^{[p/3]}\frac{(3k)!}{27^k\cdot k!^3}\Big(\frac{1-t}{2}\Big)^k \pmod{p}.$$

This proves the lemma. $\qquad\square$

Theorem 2.4. *Let p be a prime greater than 3 and let x be a variable. Then*

$$\sum_{k=0}^{[p/3]}\frac{(3k)!}{27^k\cdot k!^3}\big(x^k - (-1)^{[\frac{p}{3}]}(1-x)^k\big) \equiv 0 \pmod{p}.$$

Proof. As $P_n(t) = (-1)^n P_n(-t)$, using Lemma 2.3 we deduce that

$$\sum_{k=0}^{[p/3]}\frac{(3k)!}{27^k\cdot k!^3}\Big(\Big(\frac{1-t}{2}\Big)^k - (-1)^{[p/3]}\Big(\frac{1+t}{2}\Big)^k\Big) \equiv 0 \pmod{p}.$$

Now putting $t = 1 - 2x$ in the congruence we obtain the result. $\qquad\square$

Corollary 2.2. *Let p be a prime greater than 3. Then*

$$\sum_{k=0}^{[p/3]}\frac{(3k)!}{27^k\cdot k!^3} \equiv \Big(\frac{p}{3}\Big) \pmod{p}.$$

Proof. Taking $x = 1$ in Theorem 2.4 and noting that $(-1)^{[p/3]} = (\frac{p}{3})$ we deduce the result. $\qquad\square$

Remark 2.2. By [8] or [10, p. 204] we have the stronger supercongruence $\sum_{k=0}^{p-1}\frac{(3k)!}{27^k\cdot k!^3} \equiv (\frac{p}{3}) \pmod{p^2}$.

Lemma 2.4. *Let p be an odd prime and $k \in \{1, 2, \ldots, \frac{p-1}{2}\}$. Then*

$$\binom{(p-1)/2}{k} \equiv \frac{1}{(-4)^k}\binom{2k}{k}\Big(1 - p\sum_{i=1}^{k}\frac{1}{2i-1}\Big) \pmod{p^2}.$$

Proof. It is clear that

$$\binom{\frac{p-1}{2}}{k} = \frac{\frac{p-1}{2}(\frac{p-1}{2}-1)\cdots(\frac{p-1}{2}-k+1)}{k!} = \frac{(p-1)(p-3)\cdots(p-(2k-1))}{2^k \cdot k!}$$

$$\equiv \frac{(-1)(-3)\cdots(-(2k-1))}{2^k \cdot k!}\Big(1 - p\sum_{i=1}^{k}\frac{1}{2i-1}\Big)$$

$$= \frac{(-1)^k \cdot (2k)!}{(2^k \cdot k!)^2}\Big(1 - p\sum_{i=1}^{k}\frac{1}{2i-1}\Big) \pmod{p^2}.$$

This yields the result. $\square$

Theorem 2.5. *Let p be a prime greater than 5. Then*

$$\sum_{k=0}^{[p/3]}\frac{(3k)!}{54^k \cdot k!^3} \equiv \begin{cases} 0 \pmod{p} & \text{if } 6 \mid p-5, \\ 2A \pmod{p} & \text{if } 6 \mid p-1 \text{ and } p = A^2 + 3B^2 \text{ with } 3 \mid A-1 \end{cases}$$

and

$$\sum_{k=0}^{[p/3]}\frac{k \cdot (3k)!}{54^k \cdot k!^3} \equiv \begin{cases} 0 \pmod{p} & \text{if } 6 \mid p-1, \\ \frac{1}{3}(-1)^{\frac{p+1}{6}}2^{-\frac{p+1}{3}}\binom{(p+1)/3}{(p+1)/6} \pmod{p} & \text{if } 6 \mid p-5. \end{cases}$$

Proof. Taking $t = 0$ in Lemma 2.3 and applying (2.5) and Lemma 2.4 we deduce that

$$\sum_{k=0}^{[p/3]}\frac{(3k)!}{54^k \cdot k!^3} \equiv \begin{cases} 0 \pmod{p} & \text{if } p \equiv 5 \pmod{6}, \\ \frac{(-1)^{(p-1)/6}}{2^{(p-1)/3}}\binom{(p-1)/3}{(p-1)/6} \equiv \binom{(p-1)/2}{(p-1)/6} \pmod{p} & \text{if } p \equiv 1 \pmod{6}. \end{cases}$$

Now suppose $p \equiv 1 \pmod{6}$, and so $p = A^2 + 3B^2$ with $A, B \in \mathbb{Z}$ and $A \equiv 1 \pmod{3}$. By [2, Theorem 9.4.4] we have $\binom{(p-1)/2}{(p-1)/6} \equiv 2A \pmod{p}$. Thus the first part follows.

By Lemma 2.3 we have

$$\frac{d}{dt}P_{[\frac{p}{3}]}(t) \equiv -\sum_{k=0}^{[p/3]}\frac{(3k)!}{54^k \cdot k!^3}\cdot k(1-t)^{k-1} \pmod{p}.$$

Thus, $\frac{d}{dt}P_{[\frac{p}{3}]}(t)\big|_{t=0} \equiv -\sum_{k=0}^{[p/3]}\frac{k \cdot (3k)!}{54^k \cdot k!^3} \pmod{p}$. From (1.3) we know that

$$\frac{d}{dt}P_{[\frac{p}{3}]}(t)\big|_{t=0} = \begin{cases} 0 & \text{if } p \equiv 1 \pmod{6}, \\ 2^{-\frac{p-2}{3}}\cdot(-1)^{\frac{p-5}{6}}\frac{\frac{p+1}{3}!}{\frac{p-5}{6}!\frac{p+1}{6}!} & \text{if } p \equiv 5 \pmod{6}. \end{cases}$$

Thus the second part is true. $\square$

Lemma 2.5. *Let p be an odd prime and $k \in \{1, 2, \ldots, \frac{p-1}{2}\}$. Then*

$$\frac{(-1)^k\binom{(p-1)/2+k}{k}}{\binom{(p-1)/2}{k}} \equiv 1 + 2p\sum_{i=1}^{k}\frac{1}{2i-1} \equiv 3 - 2(-4)^k\frac{\binom{(p-1)/2}{k}}{\binom{2k}{k}} \pmod{p^2}.$$

Proof. It is clear that

$$\frac{(-1)^k \binom{(p-1)/2+k}{k}}{\binom{(p-1)/2}{k}} = \frac{(\frac{p-1}{2}+k)(\frac{p-1}{2}+k-1)\cdots(\frac{p-1}{2}+1)}{(-1)^k\frac{p-1}{2}(\frac{p-1}{2}-1)\cdots(\frac{p-1}{2}-k+1)}$$

$$= \frac{(p+2k-1)(p+2k-3)\cdots(p+1)}{(-1)^k(p-1)(p-3)\cdots(p-(2k-1))}$$

$$\equiv \frac{1\cdot 3\cdots(2k-1)(1+p\sum_{i=1}^k \frac{1}{2i-1})}{1\cdot 3\cdots(2k-1)(1-p\sum_{i=1}^k \frac{1}{2i-1})}$$

$$\equiv \Big(1+p\sum_{i=1}^k \frac{1}{2i-1}\Big)^2 \equiv 1+2p\sum_{i=1}^k \frac{1}{2i-1} \pmod{p^2}.$$

This together with Lemma 2.4 yields the result. $\qquad\square$

Theorem 2.6. *Let p be an odd prime, $x \in \mathbb{Z}_p$ and $x \not\equiv -1 \pmod p$. Then*

$$P_{\frac{p-1}{2}}(x) \equiv \Big(\frac{2(x+1)}{p}\Big) P_{\frac{p-1}{2}}\Big(\frac{3-x}{1+x}\Big) \pmod p.$$

Proof. It is known that (see [4, (3.134)])

$$P_n(x) = \sum_{k=0}^n \binom{n}{k}^2 \Big(\frac{x+1}{2}\Big)^{n-k} \Big(\frac{x-1}{2}\Big)^k.$$

Thus, using Lemma 2.5 and (2.1) we see that

$$P_{\frac{p-1}{2}}(x) = \Big(\frac{x+1}{2}\Big)^{\frac{p-1}{2}} \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 \Big(\frac{x-1}{x+1}\Big)^k$$

$$\equiv \Big(\frac{2(x+1)}{p}\Big) \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}\binom{\frac{p-1}{2}+k}{k}(-1)^k \Big(\frac{x-1}{x+1}\Big)^k$$

$$= \Big(\frac{2(x+1)}{p}\Big) P_{\frac{p-1}{2}}\Big(1+2\cdot\frac{1-x}{1+x}\Big) = \Big(\frac{2(x+1)}{p}\Big) P_{\frac{p-1}{2}}\Big(\frac{3-x}{1+x}\Big) \pmod p.$$

This proves the theorem. $\qquad\square$

Corollary 2.3. *Let p be a prime of the form $4k+3$. Then $P_{\frac{p-1}{2}}(3) \equiv 0 \pmod p$.*

Proof. By Theorem 2.6 and (2.5) we have $P_{\frac{p-1}{2}}(3) \equiv (\frac{2}{p})P_{\frac{p-1}{2}}(0) = 0 \pmod p$. $\quad\square$

Theorem 2.7. *Let p be an odd prime, $x \in \mathbb{Z}_p$ and $x \not\equiv 0 \pmod p$. Then*

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{16^k}\Big(x^k - \Big(\frac{x}{p}\Big)x^{-k}\Big) \equiv 0 \pmod p.$$

Proof. Clearly the result is true for $x \equiv 1 \pmod p$. Now assume $x \not\equiv 1 \pmod p$. As $P_n(t) = (-1)^n P_n(-t)$ (see [6]), using Theorem 2.6 we see that for $t \in \mathbb{Z}_p$ with $t \not\equiv \pm 1 \pmod p$,

$$\Big(\frac{2(t+1)}{p}\Big) P_{\frac{p-1}{2}}\Big(\frac{3-t}{1+t}\Big) \equiv (-1)^{\frac{p-1}{2}} \Big(\frac{2(-t+1)}{p}\Big) P_{\frac{p-1}{2}}\Big(\frac{3+t}{1-t}\Big) \pmod p.$$

Thus,

$$P_{\frac{p-1}{2}}\left(\frac{3-t}{1+t}\right) \equiv \left(\frac{t^2-1}{p}\right)P_{\frac{p-1}{2}}\left(\frac{3+t}{1-t}\right) \pmod{p}.$$

Now applying (2.4) we deduce that

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{16^k}\left(\frac{1-\frac{3-t}{1+t}}{2}\right)^k \equiv \left(\frac{t^2-1}{p}\right)\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{16^k}\left(\frac{1-\frac{3+t}{1-t}}{2}\right)^k \pmod{p}$$

and so

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{16^k}\left(\left(\frac{t-1}{t+1}\right)^k - \left(\frac{(t-1)/(t+1)}{p}\right)\left(\frac{t+1}{t-1}\right)^k\right) \equiv 0 \pmod{p}.$$

Set $t = (1+x)/(1-x)$. Then $t \not\equiv \pm 1 \pmod{p}$ and $x = (t-1)/(t+1)$. Hence the result follows. $\qquad\square$

Let p be an odd prime, and let $x \in \mathbb{Z}_p$ with $x \not\equiv 0, 1 \pmod{p}$. By Theorems 2.1 and 2.7 we have

$$(2.7) \qquad \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{(16x)^k} \equiv \left(\frac{x(x-1)}{p}\right)\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{(16(1-x))^k} \pmod{p}.$$

Theorem 2.8. *Let p be an odd prime, $x \in \mathbb{Z}_p$ and $x \not\equiv 0 \pmod{p}$. Then*

$$\sum_{k=0}^{\frac{p-1}{2}} \binom{2k}{k}^2 \left(\frac{x}{4}\right)^{2k} \equiv \left(\frac{-x}{p}\right)P_{\frac{p-1}{2}}\left(\frac{x+x^{-1}}{2}\right) \pmod{p}.$$

Proof. From [4, (3.138)] we have the following result due to Kelisky:

$$(2.8) \qquad \sum_{k=0}^{n} \binom{2n-2k}{n-k}\binom{2k}{k}x^{2k} = 2^{2n}x^n P_n\left(\frac{x+x^{-1}}{2}\right).$$

Taking $n = (p-1)/2$ in (2.8) we have

$$\sum_{k=0}^{\frac{p-1}{2}} \binom{p-1-2k}{\frac{p-1}{2}-k}\binom{2k}{k}x^{2k} = 2^{p-1}x^{\frac{p-1}{2}}P_{\frac{p-1}{2}}\left(\frac{x+x^{-1}}{2}\right) \equiv \left(\frac{x}{p}\right)P_{\frac{p-1}{2}}\left(\frac{x+x^{-1}}{2}\right) \pmod{p}.$$

To see the result, using Lemma 2.2 we note that for $0 \le k \le \frac{p-1}{2}$,

$$(2.9) \qquad \binom{p-1-2k}{\frac{p-1}{2}-k} = \frac{(p-1-2k)(p-2-2k)\cdots(p-(\frac{p-1}{2}+k))}{(\frac{p-1}{2}-k)!}$$

$$\equiv (-1)^{\frac{p-1}{2}-k}\frac{(2k+1)(2k+2)\cdots(\frac{p-1}{2}+k)}{(\frac{p-1}{2}-k)!}$$

$$= (-1)^{\frac{p-1}{2}-k}\binom{\frac{p-1}{2}+k}{2k} \equiv (-1)^{\frac{p-1}{2}}\frac{\binom{2k}{k}}{16^k} \pmod{p}.$$

$$\square$$

Theorem 2.9. *Let p be a prime of the form $4k + 1$ and $p = a^2 + b^2$ with $a, b \in \mathbb{Z}$ and $a \equiv 1 \pmod 4$. Then*

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{8^k} \equiv \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{(-16)^k} \equiv P_{\frac{p-1}{2}}(3) \equiv (-1)^{\frac{p-1}{4}}\left(2a - \frac{p}{2a}\right) \pmod{p^2}.$$

Proof. By Theorem 2.1 and (2.4) we have $\sum_{k=0}^{\frac{p-1}{2}} \binom{2k}{k}^2 8^{-k} \equiv \sum_{k=0}^{\frac{p-1}{2}} \binom{2k}{k}^2 (-16)^{-k} \equiv P_{\frac{p-1}{2}}(3) \pmod{p^2}$. From Theorem 2.6, (2.5) and Gauss' congruence $\binom{(p-1)/2}{(p-1)/4} \equiv 2a \pmod p$ (see [2]) we have

$$P_{\frac{p-1}{2}}(3) \equiv \left(\frac{2}{p}\right) P_{\frac{p-1}{2}}(0) = \left(\frac{2}{p}\right) \frac{(-1)^{(p-1)/4}}{2^{(p-1)/2}} \binom{(p-1)/2}{(p-1)/4} \equiv (-1)^{\frac{p-1}{4}} \cdot 2a \pmod p.$$

Write $P_{\frac{p-1}{2}}(3) = (-1)^{\frac{p-1}{4}} \cdot 2a + qp$. Then $P_{\frac{p-1}{2}}(3)^2 \equiv 4a^2 + (-1)^{\frac{p-1}{4}} \cdot 4aqp \pmod{p^2}$. By a result due to Van Hamme [15], we have $(\sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}\binom{\frac{p-1}{2}+k}{k})^2 \equiv 4a^2 - 2p \pmod{p^2}$. This together with (2.1) yields $P_{\frac{p-1}{2}}(3)^2 \equiv 4a^2 - 2p \pmod{p^2}$. Hence $(-1)^{\frac{p-1}{4}} \cdot 4aq \equiv -2 \pmod p$, and so $P_{\frac{p-1}{2}}(3) \equiv (-1)^{\frac{p-1}{4}} \cdot 2a - \frac{p}{2(-1)^{(p-1)/4}a} \pmod{p^2}$. Now combining all the above we obtain the result. $\square$

Remark 2.3. For a prime $p = 4k + 1 = a^2 + b^2$ with $a \equiv 1 \pmod 4$, the congruence $\sum_{k=0}^{(p-1)/2} \binom{2k}{k}^2 8^{-k} \equiv \sum_{k=0}^{(p-1)/2} \binom{2k}{k}^2 (-16)^{-k} \equiv (-1)^{\frac{p-1}{4}} \cdot 2a \pmod p$ was proved by Zhi-Wei Sun in [13], and he also conjectured

$$\sum_{k=0}^{(p-1)/2} \binom{2k}{k}^2 8^{-k} \equiv \sum_{k=0}^{(p-1)/2} \binom{2k}{k}^2 (-16)^{-k}$$
$$\equiv (-1)^{\frac{p-1}{4}}\left(2a - \frac{p}{2a}\right) \pmod{p^2}.$$

Theorem 2.10. *Let p be a prime of the form $4k + 1$, and so $p = a^2 + b^2$ with $a, b \in \mathbb{Z}$ and $a \equiv 1 \pmod 4$. Then*

$$\sum_{k=0}^{\frac{p-1}{4}} \frac{\binom{4k}{2k}^2}{16^{2k}} \equiv \frac{1}{2} + (-1)^{\frac{p-1}{4}}a - (-1)^{\frac{p-1}{4}}\frac{p}{4a} \pmod{p^2}.$$

Proof. Since $\sum_{k=0}^{\frac{p-1}{2}} \binom{2k}{k}^2 16^{-k} + \sum_{k=0}^{\frac{p-1}{2}} \binom{2k}{k}^2 (-16)^{-k} = 2\sum_{k=0}^{\frac{p-1}{4}} \binom{4k}{2k}^2 16^{-2k}$, by (1.1) and Theorem 2.9 we deduce the result. $\square$

For a prime $p > 3$ and $A, B, C \in \mathbb{Z}_p$, and let $\#E_p(y^2 = x^3 + Ax^2 + Bx + C)$ be the number of points on the curve $E_p: y^2 = x^3 + Ax^2 + Bx + C$ over the field $\mathbb{F}_p$ of p elements.

Lemma 2.6 ([5]). *Let $p > 3$ be a prime and $\lambda \in \mathbb{Z}_p$ with $\lambda \not\equiv 0, 1 \pmod p$. Then*

$$p + 1 - \#E_p(y^2 = x(x-1)(x-\lambda)) \equiv (-1)^{\frac{p-1}{2}} \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}+k}{k}\binom{\frac{p-1}{2}}{k}(-\lambda)^k \pmod p.$$

Theorem 2.11. *Let $p > 3$ be a prime and $t \in \mathbb{Z}_p$. Then*

$$P_{\frac{p-1}{2}}(t) \equiv \sum_{k=0}^{\frac{p-1}{2}} \binom{2k}{k}^2\left(\frac{1-t}{32}\right)^k \equiv -\left(\frac{-6}{p}\right)\sum_{x=0}^{p-1}\left(\frac{x^3-3(t^2+3)x+2t(t^2-9)}{p}\right) \pmod p.$$

Proof. For $t \equiv \pm 1 \pmod{p}$ we have

$$\sum_{x=0}^{p-1} \left(\frac{x^3 - 3(t^2+3)x + 2t(t^2-9)}{p} \right)$$

$$= \sum_{x=0}^{p-1} \left(\frac{x^3 - 12x \mp 16}{p} \right) = \sum_{x=0}^{p-1} \left(\frac{(2x)^3 - 12(2x) \mp 16}{p} \right) = \left(\frac{2}{p} \right) \sum_{x=0}^{p-1} \left(\frac{x^3 - 3x \mp 2}{p} \right)$$

$$= \left(\frac{2}{p} \right) \sum_{x=0}^{p-1} \left(\frac{(x \pm 1)^2 (x \mp 2)}{p} \right) = \left(\frac{2}{p} \right) \left(\sum_{x=0}^{p-1} \left(\frac{x \mp 2}{p} \right) - \left(\frac{\mp 3}{p} \right) \right) = -\left(\frac{\mp 6}{p} \right).$$

Thus applying (2.4) and the fact $P_n(\pm 1) = (\pm 1)^n$ (see [6]) we deduce the result.

Now assume $t \not\equiv \pm 1 \pmod{p}$. For $A, B, C \in \mathbb{Z}_p$, it is easily seen that (see for example [12, pp. 221-222])

$$\# E_p(y^2 = x^3 + Ax^2 + Bx + C) = p + 1 + \sum_{x=0}^{p-1} \left(\frac{x^3 + Ax^2 + Bx + C}{p} \right).$$

Taking $\lambda = (1-t)/2$ in Lemma 2.6 and applying the above and (2.1) we see that

$$P_{\frac{p-1}{2}}(t) = \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2} + k}{k} \binom{\frac{p-1}{2}}{k} \left(\frac{t-1}{2} \right)^k$$

$$\equiv (-1)^{\frac{p-1}{2}} \left(p + 1 - \# E_p(y^2 = x(x-1)(x-(1-t)/2)) \right)$$

$$= (-1)^{\frac{p+1}{2}} \sum_{x=0}^{p-1} \left(\frac{x(x-1)(x-(1-t)/2)}{p} \right) \pmod{p}.$$

Since

$$\sum_{x=0}^{p-1} \left(\frac{x(x-1)(x-(1-t)/2)}{p} \right)$$

$$= \sum_{x=0}^{p-1} \left(\frac{\frac{x}{2}(\frac{x}{2}-1)(\frac{x}{2}-\frac{1-t}{2})}{p} \right) = \left(\frac{2}{p} \right) \sum_{x=0}^{p-1} \left(\frac{x(x-2)(x+t-1)}{p} \right)$$

$$= \left(\frac{2}{p} \right) \sum_{x=0}^{p-1} \left(\frac{x^3 + (t-3)x^2 - 2(t-1)x}{p} \right)$$

$$= \left(\frac{2}{p} \right) \sum_{x=0}^{p-1} \left(\frac{(x-\frac{t-3}{3})^3 + (t-3)(x-\frac{t-3}{3})^2 - 2(t-1)(x-\frac{t-3}{3})}{p} \right)$$

$$= \left(\frac{2}{p} \right) \sum_{x=0}^{p-1} \left(\frac{x^3 - \frac{t^2+3}{3}x + \frac{2t^3-18t}{27}}{p} \right) = \left(\frac{2}{p} \right) \sum_{x=0}^{p-1} \left(\frac{(\frac{x}{3})^3 - \frac{t^2+3}{3} \cdot \frac{x}{3} + \frac{2t^3-18t}{27}}{p} \right)$$

$$= \left(\frac{6}{p} \right) \sum_{x=0}^{p-1} \left(\frac{x^3 - 3(t^2+3)x + 2t(t^2-9)}{p} \right),$$

by the above and (2.4) we obtain the result. The proof is now complete. $\qquad \square$

Theorem 2.12. *Let $p > 3$ be a prime. Then*

$$P_{\frac{p-1}{2}}(-31) \equiv \sum_{k=0}^{\frac{p-1}{2}} \binom{2k}{k}^2 \equiv \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{256^k} \equiv -\left(\frac{p}{3}\right) \sum_{x=0}^{p-1} \left(\frac{x^3 - 723x - 7378}{p}\right) \pmod{p},$$

$$P_{\frac{p-1}{2}}(33)$$

$$\equiv \sum_{k=0}^{\frac{p-1}{2}} (-1)^k \binom{2k}{k}^2 \equiv (-1)^{\frac{p-1}{2}} \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{(-256)^k} \equiv (-1)^{\frac{p+1}{2}} \sum_{x=0}^{p-1} \left(\frac{x^3 - 91x + 330}{p}\right) \pmod{p},$$

$$P_{\frac{p-1}{2}}(-15) \equiv \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{2^k} \equiv \left(\frac{2}{p}\right) \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{128^k} \equiv (-1)^{\frac{p+1}{2}} \sum_{x=0}^{p-1} \left(\frac{x^3 - 19x - 30}{p}\right) \pmod{p},$$

$$P_{\frac{p-1}{2}}(9) \equiv \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{(-4)^k} \equiv (-1)^{\frac{p-1}{2}} \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{(-64)^k} \equiv (-1)^{\frac{p+1}{2}} \sum_{x=0}^{p-1} \left(\frac{x^3 - 7x + 6}{p}\right) \pmod{p},$$

$$P_{\frac{p-1}{2}}(5) \equiv \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{(-8)^k} \equiv \left(\frac{-2}{p}\right) \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{(-32)^k} \equiv -\left(\frac{p}{3}\right) \sum_{x=0}^{p-1} \left(\frac{x^3 - 21x + 20}{p}\right) \pmod{p}.$$

Proof. Taking $t = -31$ in Theorem 2.11 and applying Theorem 2.7 we see that

$$P_{\frac{p-1}{2}}(-31) \equiv \sum_{k=0}^{\frac{p-1}{2}} \binom{2k}{k}^2 \equiv \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{256^k} \equiv -\left(\frac{-6}{p}\right) \sum_{x=0}^{p-1} \left(\frac{x^3 - 3(31^2 + 3)x - 62(31^2 - 9)}{p}\right)$$

$$= -\left(\frac{-6}{p}\right) \sum_{x=0}^{p-1} \left(\frac{(2x)^3 - 4 \cdot 723 \cdot 2x - 8 \cdot 7378}{p}\right)$$

$$= -\left(\frac{-3}{p}\right) \sum_{x=0}^{p-1} \left(\frac{x^3 - 723x - 7378}{p}\right) \pmod{p}.$$

Taking $t = 33$ in Theorem 2.11 and applying Theorem 2.7 we see that

$$P_{\frac{p-1}{2}}(33)$$

$$\equiv \sum_{k=0}^{\frac{p-1}{2}} (-1)^k \binom{2k}{k}^2 \equiv \left(\frac{-16}{p}\right) \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2}{(-256)^k} \equiv -\left(\frac{-6}{p}\right) \sum_{x=0}^{p-1} \left(\frac{x^3 - 3276x + 71280}{p}\right)$$

$$= -\left(\frac{-6}{p}\right) \sum_{x=0}^{p-1} \left(\frac{(6x)^3 - 36 \cdot 91 \cdot 6x + 216 \cdot 330}{p}\right)$$

$$= -\left(\frac{-1}{p}\right) \sum_{x=0}^{p-1} \left(\frac{x^3 - 91x + 330}{p}\right) \pmod{p}.$$

The remaining congruences can be proved similarly. $\qquad\square$

For $a, b, n \in \mathbb{N}$, if $n = ax^2 + by^2$ for some $x, y \in \mathbb{Z}$, we say that $n = ax^2 + by^2$. In 2003, Rodriguez-Villegas [11] posed many conjectures on supercongruences. In particular, he conjectured that for any prime $p > 3$,

$$\sum_{k=0}^{p-1} \frac{(4k)!}{256^k k!^4} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 3 \pmod 8, \text{ and so } p = x^2 + 2y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod 8 \end{cases}$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{108^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } 3 \mid p-1, \text{ and so } p = x^2 + 3y^2, \\ 0 \pmod{p^2} & \text{if } 3 \mid p-2. \end{cases}$$

Recently Zhi-Wei Sun ([13, 14]) made a lot of conjectures on supercongruences. In particular, he conjectured that for a prime $p \neq 2, 7$,

$$\sum_{k=0}^{p-1} \binom{2k}{k}^3 \equiv \sum_{k=0}^{p-1} \frac{(4k)!}{81^k k!^4} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } (\frac{p}{7}) = 1 \text{ and so } p = x^2 + 7y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{p}{7}) = -1. \end{cases}$$

Inspired by his work, we pose the following conjectures.

Conjecture 2.1. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{(4k)!}{648^k k!^4} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod 4 \text{ and } p = x^2 + y^2 \text{ with } 2 \nmid x, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Conjecture 2.2. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{(4k)!}{(-144)^k k!^4} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod 3, \text{ and so } p = x^2 + 3y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

Conjecture 2.3. *Let $p \neq 2, 3, 7$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{(4k)!}{(-3969)^k k!^4} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 2, 4 \pmod 7, \text{ and so } p = x^2 + 7y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 3, 5, 6 \pmod 7. \end{cases}$$

Conjecture 2.4. *Let $p \neq 2, 3, 11$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{(6k)!}{66^{3k}(3k)! k!^3} \equiv \begin{cases} (\frac{p}{33})(4x^2 - 2p) \pmod{p^2} & \text{if } 4 \mid p-1 \text{ and } p = x^2 + y^2 \text{ with } 2 \nmid x, \\ 0 \pmod{p^2} & \text{if } 4 \mid p-3. \end{cases}$$

Conjecture 2.5. *Let $p > 5$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{(6k)!}{20^{3k}(3k)! k!^3} \equiv \begin{cases} (\frac{-5}{p})(4x^2 - 2p) \pmod{p^2} & \text{if } p \equiv 1, 3 \pmod 8 \text{ and } p = x^2 + 2y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod 8. \end{cases}$$

Conjecture 2.6. *Let $p > 5$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{(6k)!}{54000^k(3k)! k!^3} \equiv \begin{cases} (\frac{p}{5})(4x^2 - 2p) \pmod{p^2} & \text{if } 3 \mid p-1, \text{ and so } p = x^2 + 3y^2, \\ 0 \pmod{p^2} & \text{if } 3 \mid p-2. \end{cases}$$

Conjecture 2.7. *Let $p > 5$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{(6k)!}{(-12288000)^k(3k)! k!^3}$$
$$\equiv \begin{cases} (\frac{10}{p})(L^2 - 2p) \pmod{p^2} & \text{if } p \equiv 1 \pmod 3, \text{ and so } 4p = L^2 + 27M^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

Conjecture 2.8. *Let $p > 7$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{(6k)!}{(-15)^{3k}(3k)! k!^3} \equiv \begin{cases} (\frac{p}{15})(4x^2 - 2p) \pmod{p^2} & \text{if } (\frac{p}{7}) = 1, \text{ and so } p = x^2 + 7y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{p}{7}) = -1. \end{cases}$$

Conjecture 2.9. *Let* $p \neq 2, 3, 5, 7, 17$ *be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{(6k)!}{255^{3k}(3k)!k!^3} \equiv \begin{cases} (\frac{p}{255})(4x^2 - 2p) \pmod{p^2} & \textit{if } (\frac{p}{7}) = 1, \textit{ and so } p = x^2 + 7y^2, \\ 0 \pmod{p^2} & \textit{if } (\frac{p}{7}) = -1. \end{cases}$$

Conjecture 2.10. *Let* $p > 3$ *be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{1458^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \textit{if } p \equiv 1 \pmod 3, \textit{ and so } p = x^2 + 3y^2, \\ 0 \pmod{p^2} & \textit{if } p \equiv 2 \pmod 3. \end{cases}$$

Conjecture 2.11. *Let* $p > 5$ *be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{15^{3k}} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \textit{if } p \equiv 1, 4 \pmod{15}, \textit{ and so } p = x^2 + 15y^2, \\ 2p - 12x^2 \pmod{p^2} & \textit{if } p \equiv 2, 8 \pmod{15}, \textit{ and so } p = 3x^2 + 5y^2, \\ 0 \pmod{p^2} & \textit{if } p \equiv 7, 11, 13, 14 \pmod{15}. \end{cases}$$

Conjecture 2.12. *Let* $p > 5$ *be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{(-8640)^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \textit{if } 3 \mid p - 1, \ p = x^2 + 3y^2 \textit{ and } 5 \mid xy, \\ p - 2x^2 \pm 6xy \pmod{p^2} & \textit{if } 3 \mid p - 1, \ p = x^2 + 3y^2, \ 5 \nmid xy \\ & \qquad \textit{and } x \equiv \pm y, \pm 2y \pmod 5, \\ 0 \pmod{p^2} & \textit{if } 3 \mid p - 2. \end{cases}$$

Conjecture 2.13. *Let* $p > 3$ *be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{(3k)!}{54^k \cdot k!^3} \equiv \begin{cases} (\frac{x}{3})(2x - \frac{p}{2x}) \pmod{p^2} & \textit{if } 3 \mid p - 1, \textit{ and so } p = x^2 + 3y^2, \\ 0 \pmod{p^2} & \textit{if } 3 \mid p - 2. \end{cases}$$

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