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CONGRUENCES FOR THE SECOND-ORDER CATALAN NUMBERS

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ABSTRACT. Let p be any odd prime. We mainly show that

$$\sum_{k=1}^{p-1}\frac{2^k}{k}\binom{3k}{k}\equiv 0 \pmod p$$

and

$$\sum_{k=1}^{p-1}2^{k-1}C_k^{(2)}\equiv(-1)^{(p-1)/2}-1 \pmod p,$$

where $C_k^{(2)}=\binom{3k}{k}/(2k+1)$ is the kth Catalan number of order 2.

1. INTRODUCTION

The well-known Catalan numbers are those integers

$$C_n=\frac{1}{n+1}\binom{2n}{n}=\binom{2n}{n}-\binom{2n}{n-1}\quad(n=0,1,2,\ldots).$$

(As usual we regard $\binom{x}{-k}$ as 0 for $k=1,2,\ldots$.) There are many combinatorial interpretations for these important numbers (see, e.g., [St, pp. 219-229]). With the help of a sophisticated binomial identity, H. Pan and Z. W. Sun [PS] obtained some congruences on sums of Catalan numbers; in particular, by [PS, (1.16) and (1.8)], for any prime $p>3$ we have

$$(1.0)\qquad \sum_{k=0}^{p-1}C_k\equiv\frac{3(\frac{p}{3})-1}{2}\pmod p\quad\text{and}\quad\sum_{k=1}^{p-1}\frac{C_k}{k}\equiv\frac{3}{2}\left(1-\left(\frac{p}{3}\right)\right)\pmod p,$$

where the Legendre symbol $(\frac{a}{3})\in\{0,\pm1\}$ satisfies the congruence $a\equiv(\frac{a}{3}) \pmod 3$. Recently Z. W. Sun and R. Tauraso [ST1, ST2] obtained some further congruences concerning sums involving Catalan numbers.

For $m,n\in\mathbb{N}=\{0,1,2,\ldots\}$, we define

$$C_n^{(m)}=\frac{1}{mn+1}\binom{mn+n}{n}=\binom{mn+n}{n}-m\binom{mn+n}{n-1}$$

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and call it the nth Catalan number of order m. Clearly

$$C_n^{(1)} = C_n \quad \text{and} \quad C_n^{(2)} = \frac{1}{2n+1}\binom{3n}{n}.$$

In contrast with (1.0), we have the following result involving the second-order Catalan numbers.

Theorem 1.1. *Let p be an odd prime. Then*

$$\sum_{k=1}^{p-1} 2^k C_k^{(2)} \equiv 2\left((-1)^{(p-1)/2} - 1\right) \pmod p \tag{1.1}$$

and

$$\sum_{k=1}^{p-1} \frac{2^k C_k^{(2)}}{k} \equiv 4\left(1 - (-1)^{(p-1)/2}\right) \pmod p. \tag{1.2}$$

Actually Theorem 1.1 follows from our next two theorems.

Theorem 1.2. *Let $p > 5$ be a prime. Then*

$$\sum_{k=0}^{p-1} 2^k \binom{3k}{k} \equiv \frac{6(-1)^{(p-1)/2} - 1}{5} \pmod p, \tag{1.3}$$

$$\sum_{k=0}^{p-1} 2^k \binom{3k+1}{k} \equiv \frac{4(-1)^{(p-1)/2} + 1}{5} \pmod p. \tag{1.4}$$

Theorem 1.3. *For any prime p we have*

$$\sum_{k=1}^{p-1} \frac{2^k}{k}\binom{3k}{k} \equiv 0 \pmod p. \tag{1.5}$$

For any odd prime p we can also prove the following congruences:

$$5\sum_{k=1}^{p-1} 2^k \binom{3k+2}{k} \equiv (-1)^{(p-1)/2} - 1 \pmod p,$$

$$\sum_{k=1}^{p-1} \frac{2^{k-1}}{k}\binom{3k+1}{k} \equiv (-1)^{(p-1)/2} - 1 \pmod p,$$

$$\sum_{k=1}^{p-1} \frac{2^{k-1}}{k}\binom{3k+2}{k} \equiv \frac{3}{2}\left((-1)^{(p-1)/2} - 1\right) \pmod p.$$

We omit their proofs, which are similar to those of Theorems 1.2 and 1.3.

With the help of Theorems 1.2 and 1.3, we can easily deduce Theorem 1.1.

Proof of Theorem 1.1 *via Theorems* 1.2 *and* 1.3. Clearly (1.1) and (1.2) hold for $p = 3, 5$. Assume $p > 5$. By (1.3) and (1.4),

$$\begin{aligned}\sum_{k=0}^{p-1} \frac{2^k}{2k+1}\binom{3k}{k} &= 3\sum_{k=0}^{p-1} 2^k \binom{3k}{k} - 2\sum_{k=0}^{p-1} 2^k \binom{3k+1}{k} \\ &\equiv 2(-1)^{(p-1)/2} - 1 \pmod p.\end{aligned}$$

This proves (1.1). For (1.2) it suffices to note that

$$\sum_{k=1}^{p-1}\frac{2^k}{k(2k+1)}\binom{3k}{k}=\sum_{k=1}^{p-1}\frac{2^k}{k}\binom{3k}{k}-2\sum_{k=1}^{p-1}\frac{2^k}{2k+1}\binom{3k}{k}.$$

This concludes the proof. □

We are going to provide two lemmas in the next section. Theorems 1.2 and 1.3 will be proved in Sections 3 and 4 respectively.

2. Some lemmas

Lemma 2.1. *For $m,n\in\mathbb{N}$ we have*

$$\begin{aligned}&2^n\sum_{k=0}^{\lfloor m/3\rfloor}(-2)^k\binom{n}{m-3k}\binom{3k-m+n}{k}\\ &=(-1)^m\sum_{j=0}^{n}\binom{n}{j}\sum_{k=0}^{m}(-2)^k\binom{n}{m-k}\binom{2j}{k}.\end{aligned}\tag{2.1}$$

Proof. Let $P(x)=(2+2x-4x^3)^n$, and denote by $[x^k]P(x)$ the coefficient of x^k in the expansion of $P(x)$. Then

$$\begin{aligned}2^{-n}[x^m]P(x)&=[x^m]((1+x)-2x^3)^n\\ &=\sum_{k=0}^{\lfloor m/3\rfloor}\binom{n}{k}(-2)^k[x^{m-3k}](1+x)^{n-k}\\ &=\sum_{k=0}^{\lfloor m/3\rfloor}(-2)^k\binom{n}{k}\binom{n-k}{m-3k}\\ &=\sum_{k=0}^{\lfloor m/3\rfloor}(-2)^k\binom{n}{m-3k}\binom{3k-m+n}{k}.\end{aligned}$$

Since

$$P(x)=(1-x)^n((2x+1)^2+1)^n=\sum_{j=0}^{n}\binom{n}{j}(1-x)^n(2x+1)^{2j},$$

we also have

$$[x^m]P(x)=\sum_{j=0}^{n}\binom{n}{j}\sum_{k=0}^{m}2^k\binom{2j}{k}(-1)^{m-k}\binom{n}{m-k}.$$

Therefore (2.1) is valid.

For any prime p, if $n,k\in\mathbb{N}$ and $s,t\in\{0,1,\ldots,p-1\}$, then we have the well-known Lucas congruence (cf. [Gr] or [HS]), $\binom{pn+s}{pk+t}\equiv\binom{n}{k}\binom{s}{t}\pmod p$. This will be used in the proof of the following lemma. □

Lemma 2.2. *Let $p>5$ be a prime. Then we have*

$$\sum_{s=0}^{p-1}(-1)^s\sum_{t=0}^{p-1}2^t\binom{2s}{t}\equiv\frac{3(-1)^{(p-1)/2}+2}{5}\pmod p\tag{2.2}$$

and

$$\sum_{s=0}^{p-1}(-1)^s\sum_{t=0}^{p-1}2^t\binom{2s}{p+t}\equiv\frac{3}{10}\left(1-(-1)^{(p-1)/2}\right)\pmod p. \tag{2.3}$$

Proof. Observe that

$$\begin{aligned}&\sum_{s=0}^{p-1}(-1)^s\sum_{t=0}^{p-1}2^t\binom{2s}{t}\\
=&\sum_{s=0}^{(p-1)/2}(-1)^s\sum_{t=0}^{2s}2^t\binom{2s}{t}+\sum_{s=(p+1)/2}^{p-1}(-1)^s\sum_{t=0}^{p-1}2^t\binom{2s}{t}\\
=&\sum_{s=0}^{(p-1)/2}(-1)^s3^{2s}+\sum_{s=(p+1)/2}^{p-1}(-1)^s\sum_{t=0}^{p-1}2^t\binom{2s}{t}\\
=&\sum_{s=0}^{(p-1)/2}(-1)^s3^{2s}+\sum_{s=(p+1)/2}^{p-1}(-1)^s\left(\sum_{t=0}^{2s}2^t\binom{2s}{t}-\sum_{t=p}^{2s}2^t\binom{2s}{t}\right)\\
=&\sum_{s=0}^{p-1}(-1)^s3^{2s}-\sum_{s=(p+1)/2}^{p-1}(-1)^s\sum_{t=p}^{2s}2^t\binom{2s}{t}\\
=&\sum_{s=0}^{p-1}(-9)^s-\sum_{s=(p+1)/2}^{p-1}(-1)^s\sum_{r=0}^{2s-p}2^{p+r}\binom{2s}{p+r}.\end{aligned}$$

For $s=(p+1)/2,\dots,p-1$, by Lucas' congruence we have

$$\sum_{r=0}^{2s-p}2^r\binom{p+(2s-p)}{p+r}\equiv\sum_{r=0}^{2s-p}2^r\binom{2s-p}{r}=3^{2s-p}\pmod p.$$

Thus, with the help of Fermat's little theorem, we get

$$\begin{aligned}\sum_{s=0}^{p-1}(-1)^s\sum_{t=0}^{p-1}2^t\binom{2s}{t}&\equiv\frac{1-(-9)^p}{10}-\sum_{s=(p+1)/2}^{p-1}(-1)^s\frac{2}{3}\cdot 9^s\\
&\equiv 1-\frac{2}{3}(-9)^{\frac{p+1}{2}}\frac{1-(-9)^{(p-1)/2}}{10}\\
&\equiv\frac{3(-1)^{(p-1)/2}+2}{5}\pmod p.\end{aligned}$$

This proves (2.2).

In view of Lucas' congruence and Fermat's little theorem, we also have

$$\begin{aligned}
&\sum_{s=0}^{p-1}(-1)^s\sum_{t=0}^{p-1}2^t\binom{2s}{p+t}\\
&\equiv\sum_{s=(p+1)/2}^{p-1}(-1)^s\sum_{t=0}^{p-1}2^t\binom{2s-p}{t}=\sum_{s=(p+1)/2}^{p-1}(-1)^s3^{2s-p}\\
&=3^{-p}(-9)^{(p+1)/2}\frac{1-(-9)^{(p-1)/2}}{10}=(-1)^{(p+1)/2}\frac{3}{10}\left(1+(-1)^{(p+1)/2}3^{p-1}\right)\\
&\equiv\frac{3}{10}\left(1-(-1)^{(p-1)/2}\right)\pmod p.
\end{aligned}$$

So (2.3) is also valid. We are done. □

3. Proof of Theorem 1.2

In order to prove Theorem 1.2, we first present an auxiliary result.

Theorem 3.1. *Let $p>5$ be a prime, and let $d,\delta\in\{0,1\}$. Then*

$$\begin{aligned}
&\frac{(-1)^{d+\delta}}{2^\delta}\sum_{\delta p-d\leqslant 3k\leqslant\delta p+p-1-d}2^k\binom{3k+d}{k}\\
&\equiv\frac{4-\delta}{10}+\frac{(3\delta-2)(5d-3)}{10}(-1)^{(p-1)/2}\pmod p.
\end{aligned}\tag{3.1}$$

Proof. Applying (2.1) with $n=p-1$ and $m=\delta p+p-1-d$, we get

$$\begin{aligned}
&2^{p-1}\sum_{k=0}^{\lfloor(\delta p+p-1-d)/3\rfloor}(-2)^k\binom{p-1}{\delta p+p-1-d-3k}\binom{3k+d-\delta p}{k}\\
&=(-1)^{\delta p+p-1-d}\sum_{j=0}^{p-1}\binom{p-1}{j}\sum_{k=0}^{\delta p+p-1-d}(-2)^k\binom{p-1}{\delta p+p-1-d-k}\binom{2j}{k}.
\end{aligned}$$

Observe that

$$\begin{aligned}
&\sum_{k=0}^{\lfloor(\delta p+p-1-d)/3\rfloor}(-2)^k\binom{p-1}{\delta p+p-1-d-3k}\binom{3k+d-\delta p}{k}\\
&=\sum_{\delta p-d\leqslant 3k\leqslant\delta p+p-1-d}(-2)^k\binom{p-1}{p+\delta p-1-d-3k}\binom{3k+d-\delta p}{k}\\
&\equiv\sum_{\delta p-d\leqslant 3k\leqslant\delta p+p-1-d}(-2)^k(-1)^{\delta p+p-1-d-3k}\binom{3k+d}{k}\\
&\equiv(-1)^{d+\delta}\sum_{\delta p-d\leqslant 3k\leqslant\delta p+p-1-d}2^k\binom{3k+d}{k}\pmod p
\end{aligned}$$

and

$$\begin{aligned}&(-1)^{\delta p+p-1-d}\sum_{j=0}^{p-1}\binom{p-1}{j}\sum_{k=0}^{\delta p+p-1-d}(-2)^k\binom{p-1}{\delta p+p-1-d-k}\binom{2j}{k}\\ \equiv&\sum_{j=0}^{p-1}(-1)^j\sum_{\delta p-d\leqslant k<\delta p+p-d}2^k\binom{2j}{k}=\sum_{j=0}^{p-1}(-1)^j\sum_{t=0}^{p-1}2^{\delta p-d+t}\binom{2j}{\delta p-d+t}\\ \equiv&\ 2^{\delta-d}\sum_{s=0}^{p-1}(-1)^s\sum_{t=0}^{p-1}2^t\binom{2s}{\delta p-d+t}\pmod p.\end{aligned}$$

Therefore

$$\begin{aligned}&\sum_{\delta p-d\leqslant 3k\leqslant\delta p+p-1-d}2^k\binom{3k+d}{k}\\ &\equiv(-2)^{\delta-d}\sum_{s=0}^{p-1}(-1)^s\sum_{t=0}^{p-1}2^t\binom{2s}{\delta p-d+t}\pmod p.\end{aligned}$$

Recall that $d\in\{0,1\}$. We have

$$\begin{aligned}&\sum_{s=0}^{p-1}(-1)^s\sum_{t=0}^{p-1}2^t\binom{2s}{\delta p-d+t}\\ =&\sum_{s=0}^{p-1}(-1)^s\sum_{t=-d}^{p-1-d}2^{d+t}\binom{2s}{\delta p+t}\\ =&\sum_{s=0}^{p-1}(-1)^s\left(\sum_{t=0}^{p-1}2^{d+t}\binom{2s}{\delta p+t}+d\left(\binom{2s}{\delta p-1}-2^p\binom{2s}{\delta p+p-1}\right)\right)\\ =&\ 2^d\sum_{s=0}^{p-1}\sum_{t=0}^{p-1}(-1)^s2^t\binom{2s}{\delta p+t}+d\sum_{s=0}^{p-1}(-1)^s\left(\binom{2s}{\delta p-1}-2^p\binom{2s}{\delta p+p-1}\right)\end{aligned}$$

and hence

$$\begin{aligned}&(-1)^{d+\delta}\sum_{\delta p-d\leqslant 3k\leqslant\delta p+p-1-d}2^k\binom{3k+d}{k}-2^\delta\sum_{s=0}^{p-1}\sum_{t=0}^{p-1}(-1)^s2^t\binom{2s}{\delta p+t}\\ \equiv&\ d2^{\delta-d}\sum_{s=0}^{p-1}(-1)^s\left(\binom{2s}{\delta p-1}-2\binom{2s}{\delta p+p-1}\right)\\ \equiv&\ d2^{\delta-1}(3\delta-2)\sum_{s=0}^{p-1}(-1)^s\binom{2s}{p-1}\equiv d(3\delta-2)2^{\delta-1}(-1)^{(p-1)/2}\pmod p.\end{aligned}$$

Since

$$\sum_{s=0}^{p-1}(-1)^s\sum_{t=0}^{p-1}2^t\binom{2s}{\delta p+t}\equiv\frac{4-\delta}{10}+\frac{3}{10}(2-3\delta)(-1)^{(p-1)/2}\pmod p$$

by Lemma 2.2, we finally get

$$\begin{aligned}&\frac{(-1)^{d+\delta}}{2^\delta}\sum_{\delta p-d\leqslant 3k\leqslant \delta p+p-1-d}2^k\binom{3k+d}{k}\\ &\equiv \frac{4-\delta}{10}+\frac{3}{10}(2-3\delta)(-1)^{(p-1)/2}+\frac{d}{2}(3\delta-2)(-1)^{(p-1)/2}\\ &\equiv \frac{4-\delta}{10}+\frac{(3\delta-2)(5d-3)}{10}(-1)^{(p-1)/2} \pmod p.\end{aligned}$$

This proves (3.1). □

Proof of Theorem 1.2. Let $d\in\{0,1\}$. If $(2p-d)/3\leqslant k\leqslant p-1$, then $2k+d+1\leqslant 2k+2\leqslant 2p\leqslant 3k+d$ and hence

$$\binom{3k+d}{k}=\frac{(3k+d)\cdots(2k+d+1)}{k!}\equiv 0 \pmod p.$$

Therefore

$$\sum_{2p-d\leqslant 3k\leqslant 3p-3}2^k\binom{3k+d}{k}\equiv 0 \pmod p.$$

With the help of Theorem 3.1, we have

$$\begin{aligned}\sum_{k=0}^{p-1}2^k\binom{3k+d}{k}&\equiv \sum_{-d\leqslant 3k\leqslant 2p-1-d}2^k\binom{3k+d}{k}\\ &\equiv \sum_{\delta=0}^{1}\sum_{\delta p-d\leqslant 3k\leqslant \delta p+p-1-d}2^k\binom{3k+d}{k}\\ &\equiv \sum_{\delta=0}^{1}(-1)^d(-2)^\delta\left(\frac{4-\delta}{10}+\frac{(3\delta-2)(5d-3)}{10}(-1)^{(p-1)/2}\right)\\ &\equiv \frac{(-1)^{d-1}}{5}\left(1+(10d-6)(-1)^{(p-1)/2}\right) \pmod p.\end{aligned}$$

This yields (1.3) and (1.4). We are done. □

4. Proof of Theorem 1.3

Proof of Theorem 1.3. Obviously (1.5) holds for $p=2,3$. Below we assume $p>3$.

Let $\delta\in\{0,1\}$. Applying (2.1) with $m=p+\delta p$ and $n=p$ we get

$$\begin{aligned}&2^p\sum_{k=0}^{p}(-2)^k\binom{p}{p+\delta p-3k}\binom{3k-\delta p}{k}\\ &=(-1)^{\delta+1}\sum_{j=0}^{p}\binom{p}{j}\sum_{k=0}^{p+\delta p}(-2)^k\binom{p}{p+\delta p-k}\binom{2j}{k}.\end{aligned}\tag{4.1}$$

Observe that

$$\begin{aligned}&\sum_{k=0}^{p}(-2)^k\binom{p}{p+\delta p-3k}\binom{3k-\delta p}{k}\\&=\sum_{\delta p\leqslant 3k\leqslant p+\delta p-1}(-2)^k\binom{p}{3k-\delta p}\binom{3k-\delta p}{k}\\&=1-\delta+\sum_{\delta p<3k<p+\delta p}(-2)^k\binom{p}{3k-\delta p}\binom{3k-\delta p}{k}.\end{aligned}$$

For $j=1,\dots,p-1$ clearly

$$\binom{p}{j}=\frac{p}{j}\binom{p-1}{j-1}\equiv p\frac{(-1)^{j-1}}{j}\pmod{p^2}.$$

Thus

$$\begin{aligned}&\sum_{\delta p<3k<p+\delta p}(-2)^k\binom{p}{3k-\delta p}\binom{3k-\delta p}{k}\\&\equiv\sum_{\delta p<3k<p+\delta p}(-2)^k p\frac{(-1)^{3k-\delta p-1}}{3k-\delta p}\binom{3k-\delta p}{k}\\&\equiv(-1)^{\delta+1}\sum_{\delta p<3k<p+\delta p}(-2)^k p\frac{(-1)^k}{3k}\binom{(3k-\delta p)+\delta p}{k}\\&\qquad\qquad\qquad\qquad\text{(by Lucas' congruence)}\\&\equiv(-1)^{\delta+1}\ \frac{p}{3}\sum_{\delta p<3k<p+\delta p}\frac{2^k}{k}\binom{3k}{k}\pmod{p^2}.\end{aligned}$$

Notice that

$$\begin{aligned}&\sum_{j=0}^{p}\binom{p}{j}\sum_{k=0}^{p+\delta p}(-2)^k\binom{p}{p+\delta p-k}\binom{2j}{k}\\&=\sum_{\delta p\leqslant 2j\leqslant 2p}\binom{p}{j}\sum_{k=\delta p}^{p+\delta p}(-2)^k\binom{p}{k-\delta p}\binom{2j}{k}\\&=\sum_{\delta p<2j<2p}\binom{p}{j}\sum_{k=\delta p}^{p+\delta p}(-2)^k\binom{p}{k-\delta p}\binom{2j}{k}\\&\quad+\sum_{2j\in\{\delta p,2p\}}\binom{p}{j}\sum_{k=\delta p}^{p+\delta p}(-2)^k\binom{p}{k-\delta p}\binom{2j}{k}.\end{aligned}$$

Clearly

$$\begin{aligned}
&\sum_{\delta p<2j<2p}\binom{p}{j}\sum_{k=\delta p}^{p+\delta p}(-2)^k\binom{p}{k-\delta p}\binom{2j}{k}\\
\equiv&\sum_{\delta p<2j<2p}\binom{p}{j}\left((-2)^{\delta p}\binom{p}{0}\binom{2j}{\delta p}+(-2)^{p+\delta p}\binom{p}{p}\binom{2j}{p+\delta p}\right)\\
\equiv&\sum_{\delta p<2j<2p}\binom{p}{j}(-2)^{\delta p}\binom{2j-\delta p}{0}\\
&+(1-\delta)\sum_{p<2j<2p}\binom{p}{j}(-2)^{p+\delta p}\binom{2j-p}{p-p}\text{(by Lucas' congruence)}\\
\equiv&(-2)^{\delta}2^{1-\delta}(2^{p-1}-1)+(1-\delta)(-2)^{1+\delta}(2^{p-1}-1)\\
\equiv&(-1)^{\delta}\delta(2^p-2)=-\delta(2^p-2)\ (\mathrm{mod}\ p^2).
\end{aligned}$$

(Note that $\delta\in\{0,1\}$ and $2\sum_{p/2<j<p}\binom{p}{j}=\sum_{j=1}^{p-1}\binom{p}{j}=2^p-2$.) Also,

$$\sum_{2j=\delta p}\binom{p}{j}\sum_{k=\delta p}^{p+\delta p}(-2)^k\binom{p}{k-\delta p}\binom{2j}{k}=(1-\delta)\sum_{k=0}^{p}(-2)^k\binom{p}{k}\binom{0}{k}=1-\delta$$

and

$$\begin{aligned}
&\sum_{2j=2p}\binom{p}{j}\sum_{k=\delta p}^{p+\delta p}(-2)^k\binom{p}{k-\delta p}\binom{2j}{k}\\
\equiv&\sum_{k\in\{\delta p,p+\delta p\}}(-2)^k\binom{p}{k-\delta p}\binom{2p}{k}\\
\equiv&(-2)^{\delta p}\binom{2}{\delta}+(-2)^{p+\delta p}\binom{2}{1+\delta}=4^{\delta p}-2^{p+1}\ (\mathrm{mod}\ p^2).
\end{aligned}$$

(Recall that $\frac{1}{2}\binom{2p}{p}=\binom{2p-1}{p-1}\equiv 1\ (\mathrm{mod}\ p^3)$ by the Wolstenholme congruence (cf. [Gr] or [HT]).)

Combining the above with (4.1), we have

$$\begin{aligned}
&2^p\left(1-\delta+(-1)^{\delta+1}\ \frac{p}{3}\sum_{\delta p<3k<p+\delta p}\frac{2^k}{k}\binom{3k}{k}\right)\\
\equiv&(-1)^{\delta+1}\left(\delta(2-2^p)+1-\delta+4^{\delta p}-2^{p+1}\right)\ (\mathrm{mod}\ p^2).
\end{aligned}$$

Setting $\delta=0$ and $\delta=1$ respectively, we obtain

$$2^p-2^p\frac{p}{3}\sum_{0<3k<p}\frac{2^k}{k}\binom{3k}{k}\equiv 2^{p+1}-2\ (\mathrm{mod}\ p^2)$$

and

$$2^p\frac{p}{3}\sum_{p<3k<2p}\frac{2^k}{k}\binom{3k}{k}\equiv 2-2^p+4^p-2^{p+1}\ (\mathrm{mod}\ p^2).$$

It follows that

$$\frac{2}{3}p\sum_{0<3k<2p}\frac{2^k}{k}\binom{3k}{k}\equiv 4^p-4\cdot 2^p+4=(2^p-2)^2\equiv 0\ (\mathrm{mod}\ p^2).$$

If $2p \leqslant 3k < 3p$, then

$$\binom{3k}{k} = \frac{3k \cdots (2k+1)}{k!} \equiv 0 \pmod{p}.$$

Therefore

$$\sum_{k=1}^{p-1} \frac{2^k}{k}\binom{3k}{k} = \sum_{0<3k<2p} \frac{2^k}{k}\binom{3k}{k} + \sum_{2p\leqslant 3k<3p} \frac{2^k}{k}\binom{3k}{k} \equiv 0 \pmod{p}.$$

This completes the proof of Theorem 1.3. □

References

[Gr] A. Granville, *Arithmetic properties of binomial coefficients. I. Binomial coefficients modulo prime powers*, in: Organic Mathematics (Burnaby, BC, 1995), 253–276, CMS Conf. Proc., 20, Amer. Math. Soc., Providence, RI, 1997. MR1483922 (99h:11016)

[HT] C. Helou and G. Terjanian, *On Wolstenholme's theorem and its converse*, J. Number Theory **128** (2008), 475–499. MR2389852 (2008k:11003)

[HS] H. Hu and Z.-W. Sun, *An extension of Lucas' theorem*, Proc. Amer. Math. Soc. **129** (2001), 3471–3478. MR1860478 (2002i:11019)

[PS] H. Pan and Z.-W. Sun, *A combinatorial identity with application to Catalan numbers*, Discrete Math. **306** (2006), 1921–1940. MR2251572 (2007d:05018)

[St] R. P. Stanley, *Enumerative Combinatorics*, Vol. 2, Cambridge Univ. Press, Cambridge, 1999. MR1676282 (2000k:05026)

[ST1] Z.-W. Sun and R. Tauraso, *On some new congruences for binomial coefficients*, Acta Arith., to appear. `http://arxiv.org/abs/0709.1665`.

[ST2] Z.-W. Sun and R. Tauraso, *New congruences for central binomial coefficients*, preprint, `http://arxiv.org/abs/0805.0563`.

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