CONGRUENCES ON SIMPLE ω-SEMIGROUPS

by MARIO PETRICH

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1. Introduction and summary. An inverse semigroup whose idempotents form an ω-chain $e_0 > e_1 > e_2 > \ldots$ is called briefly an ω-semigroup. A structure theorem for simple ω-semigroups was established by Kočin [7]; a related structure theorem for simple, and also general, ω-semigroups was proved by Munn [10]. These results represent an extension of the structure theorem for bisimple ω-semigroups due to Reilly [14].

Congruences on a bisimple ω-semigroup were studied by Munn and Reilly [12]; in particular, they constructed all idempotent-separating congruences and the quotient relative to the least group congruence on such a semigroup. Group congruences on a bisimple ω-semigroup were characterized by Ault [1]. Conditions for modularity of the lattice of all congruences on a bisimple ω-semigroup were given by Munn [9]. Based on the work of Munn, Baird [2] established conditions for modularity of the lattice of all idempotent-separating and group congruences on a simple ω-semigroup. In another communication, Baird [3] described all congruences on a simple ω-semigroup contained in $\sigma \vee \mathcal{H}$. Kočin [8] announced certain properties of congruences on a simple ω-semigroup as well as of the corresponding homomorphic images.

We determine here all congruences on a simple ω-semigroup by using a representation for it which is essentially that of Kočin [7]. Determination of congruences throughout this paper is based on their description in [13].

By sections, this work is divided as follows. Section 2 contains a minimum of preliminaries concerning simple ω-semigroups and congruences on inverse semigroups. The content of Section 3 is a determination of (the lattice of) all normal congruences on the semilattice of idempotents of a simple ω-semigroup. A description of normal subsemigroups of a simple ω-semigroup contained in the centralizer of idempotents is the subject of Section 4. The results of these two sections make it possible to determine all non-group congruences on a simple ω-semigroup; group congruences are characterized in Section 5 by reducing this case to that of a bisimple ω-semigroup.

We find here an expression for all congruences on a simple ω-semigroup. In order to keep the length of the paper within reasonable bounds, we refrain from considering the lattice of congruences, that is, we omit discussing inclusion relations, meets, joins, etc. Neither do we discuss the representations of homomorphic images induced by these congruences. Congruences on non-simple ω-semigroups can be deduced from our results in a standard manner.

2. Preliminaries. For an extensive discussion of inverse semigroups, as well as congruences, we refer the reader to the book Howie [6]. We generally follow the terminology and notation therein.

The following facts can be extracted from [12] and [13]. Let S be an inverse

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1

semigroup with the semillatice E of idempotents. Further, let Λ be the lattice of congruences on S. For any $\rho \in \Lambda$, the restriction $\operatorname{tr} \rho = \rho \mid E$ is the *trace* of ρ, the set

$$\ker \rho = \{a \in S \mid a \,\rho\, e \text{ for some } e \in E\}$$

is the *kernel* of ρ. A congruence ξ on E is *normal* if for any $e, f \in E$, $a \in S$, $e \,\xi\, f$ implies $a^{-1}ea \,\xi\, a^{-1}fa$. The lattice Φ of normal congruences on E coincides with the lattice of traces of congruences on S.

The relation θ defined on Λ by

$$\rho \,\theta\, \tau \Leftrightarrow \operatorname{tr} \rho = \operatorname{tr} \tau$$

is a congruence on Λ, and $\rho \to \operatorname{tr} \rho$ is a complete lattice homomorphism of Λ onto Φ, so that $\Lambda/\theta \cong \Phi$. For each normal congruence ξ on E, the relation on S defined by

$$a \,\xi^{\max}\, b \Leftrightarrow a^{-1}ea \,\xi\, b^{-1}eb \quad \text{for all} \quad e \in E,$$

$$a \,\xi^{\min}\, b \Leftrightarrow ae = be \quad \text{for some} \quad e \in E, e \,\xi\, a^{-1}a \,\xi\, b^{-1}b,$$

are the greatest and the least elements of the θ-class of any congruence on S whose trace is ξ. In particular

$$\ker \xi^{\max} = \{a \in S \mid a^{-1}ea \,\xi\, a^{-1}ae \text{ for all } e \in E\}.$$

A subsemigroup K of S is *full* if $E \subseteq K$, *self-conjugate* if $a^{-1}Ka \subseteq K$ for all $a \in S$. A full, self-conjugate inverse subsemigroup is a *normal subsemigroup* of S. If ξ is a normal congruence on E and K is a normal subsemigroup of S such that

(i) $ae \in K, e \,\xi\, a^{-1}a \Rightarrow a \in K$,

(ii) $a \in K \Rightarrow a^{-1}ea \,\xi\, a^{-1}ae$,

for all $a \in S$, $e \in E$, then (ξ, K) is a *congruence pair* for S. Note that condition (ii) is equivalent to $K \subseteq \ker \xi^{\max}$.

For a congruence pair (ξ, K) for S, define a relation $\kappa_{(\xi,K)}$ on S by

$$a\kappa_{(\xi,K)}b \Leftrightarrow a^{-1}a \,\xi\, b^{-1}b, ab^{-1} \in K.$$

Then $\kappa_{(\xi,K)}$ is a congruence on S. Conversely, if ρ is a congruence on S, then $(\operatorname{tr} \rho, \ker \rho)$ is a congruence pair for S and $\rho = \kappa_{(\operatorname{tr} \rho, \ker \rho)}$.

The method of determining all congruences on a simple ω-semigroup S used here consists of first finding all normal congruences on E. We then observe that except for the universal relation on E, for all normal congruences ξ on E, we have $\ker \xi^{\max} \subseteq E\zeta$, the centralizer of E in S, which is easy to determine. This makes it possible to find all the corresponding normal subsemigroups. Finally, matching various normal congruences with these normal subsemigroups, we get all the congruence pairs. Group congruences are treated separately.

The structure of a semigroup G which is a chain of groups $G_0 > G_1 > \ldots > G_{s-1}$ for a positive integer s, is given as follows. A system of homomorphisms $\varphi_i : G_i \to G_{i+1}$ for $0 \leq i < s-1$ is given. We introduce the notation

$$\varphi_{ij} = \varphi_i \varphi_{i+1} \cdots \varphi_{j-1} \qquad (0 \leq i < j \leq s-1)$$

and let φ_{ii} be the identity mapping on G_i $(0 \leq i \leq s-1)$. Then for $g \in G_i$, $h \in G_j$, $0 \leq i, j \leq s-1$, the product in G is given by

$$ab = (a\varphi_{i,k})(b\varphi_{i,k})$$

where $k = \max\{i, j\}$. For the structure of a semilattice of groups see [**4**, §3; **5**, Chapter 4].

We now state a construction of simple ω-semigroups which amounts to a slight modification of that given by Kočin [**7**].

STRUCTURE THEOREM. *Let G be a semigroup which is a chain of s groups $G_0 > G_1 > \ldots > G_{s-1}$, and let α be a homomorphism of G into G_0. Let $\mathbb{N}$ be the set of all nonnegative integers. On $S = \mathbb{N} \times G \times \mathbb{N}$ define an operation by*

$$(m, g, n)(p, h, q) = (m + p - r, (g\alpha^{p-r})(h\alpha^{n-r}), n + q - r)$$

where $r = \min\{n, p\}$ and α^0 is the identity mapping on G. Then S is a simple ω-semigroup having s $\mathcal{D}$-classes. Conversely, every semigroup having these properties is isomorphic to one so constructed.

Munn [**10**] gave a different construction of simple ω-semigroups. His construction is based on a chain of groups G as in the above Structure Theorem, but instead of a homomorphism $\alpha: G \to G_0$, he takes a homomorphism $\varphi_{s-1}: G_{s-1} \to G_0$. Given such a homomorphism φ_{s-1}, we may define α on G by

$$\alpha: g \mapsto g\varphi_{i,s-1}\varphi_{s-1} \qquad (g \in G_i, 0 \leq i \leq s-1).$$

One verifies readily that α is a homomorphism of G into G_0. Conversely, if α is a homomorphism of G into G_0, then $\varphi_{s-1} = \alpha \mid G_{s-1}$ is evidently a homomorphism of G_{s-1} into G_0. The connection between Munn's and Kočin's representations was established by Munn [**11**].

The semigroup constructed in the Structure Theorem will be denoted by $S = \mathcal{B}(G, \alpha)$ and fixed throughout the paper. Observe that in S, $(m, g, n)^{-1} = (n, g^{-1}, m)$ and that the idempotents are given by

$$E = \{(m, e_i, m) \mid m \in \mathbb{N}, e_i \text{ is the identity of } G_i, 0 \leq i < s\}$$

with the ordering

$$(m, e_i, m) \leq (n, e_j, n) \Leftrightarrow m > n \text{ or } (m = n \text{ and } i \geq j).$$

3. Normal congruences. Congruences on E are evidently all equivalence relations with convex classes. They can be represented by increasing sequences over $\mathbb{N}$ indexed by $0, 1, 2, \ldots, n$ or by $\mathbb{N}$ if to each congruence ρ on $E = \{f_0 > f_1 > \ldots\}$ we associate the various indices of the greatest elements of the ρ-classes in the descending order.

On this model, but using the least element in the first few classes of the congruence ρ, we construct two types of normal congruences on E, and then prove that there are no other normal congruences on E.

For any $(p, g, q) \in S$, $g \in G_k$, and $(m, e_i, m) \in E$, by straightforward multiplication, we find

$$(p, g, q)^{-1}(m, e_i, m)(p, g, q) = (q+m-r, (e_k\alpha^{m-r})(e_i\alpha^{p-r}), q+m-r),$$

where $r = \min\{m, p\}$, since the idempotents are central in G. To check normality of a congruence on E, we thus need only consider expressions of the form $(p, e_k, q)^{-1}(m, e_i, m)(p, e_k, q)$ which will be used below without reference. We call the transition $e \mapsto a^{-1}ea$ *conjugation by a*.

3.1 Notation. Let $k_1, k_2, \ldots, k_t$ be a sequence of integers, possibly empty, satisfying $0 \leq k_1 < k_2 < \ldots < k_t < s-1$. For convenience, let $k_0 = -1$ and $k_{t+1} = s-1$. Define a relation $\rho = \rho(k_1, k_2, \ldots, k_t)$ on E by

$$(m, e_i, m)\ \rho\ (n, e_j, n) \Leftrightarrow m = n \quad \text{and} \quad k_{p-1} < i, j \leq k_p \quad \text{for some} \quad 1 \leq p \leq t+1.$$

3.2 Lemma. *With the notation introduced, $\rho = \rho(k_1, k_2, \ldots, k_t)$ is a normal congruence on E.*

Proof. It is evident that ρ is an equivalence relation on E with convex classes, and thus a congruence. Let

$$(m, e_i, m)\ \rho\ (n, e_j, n). \tag{1}$$

Then $m = n$. The group part of $(p, e_k, q)^{-1}(m, e_i, m)(p, e_k, q)$ is

$$(e_k\alpha^{m-r})(e_i\alpha^{p-r}) = \begin{cases} e_i & \text{if} \quad p < m, \\ e_{\max\{k,i\}} & \text{if} \quad p = m, \\ e_k & \text{if} \quad p > m, \end{cases} \tag{2}$$

where $r = \min\{p, m\}$. The group part of $(p, e_k, q)^{-1}(m, e_j, m)(p, e_k, q)$ is obtained by substituting j for i in (2). Now using hypothesis (1), the definition of ρ, and (2) with its counterpart for e_j, by considering several cases, we deduce easily that

$$(p, e_k, q)^{-1}(m, e_i, m)(p, e_k, q)\ \rho\ (p, e_k, q)^{-1}(n, e_j, n)(p, e_k, q),$$

which proves that ρ is normal.

3.3 Notation. Let $k_1, \ldots, k_t$ be a nonempty sequence of integers satisfying $0 \leq k_1 < \ldots < k_t < s-1$. For convenience, let $k_0 = -1$ and $k_{t+1} = s-1$. Define a relation $\tau = \tau(k_1, k_2, \ldots, k_t)$ on E by

$$(m, e_i, m)\ \tau\ (n, e_j, n) \Leftrightarrow \begin{cases} \text{either } m = n \text{ and } k_{p-1} < i, j \leq k_p \\ \text{for some } 1 \leq p \leq t+1, \\ \text{or } m = n+1,\ i \leq k_1,\ j > k_t, \\ \text{or } n = m+1,\ j \leq k_1,\ i > k_t. \end{cases}$$

3.4 Lemma. *With the notation introduced, $\tau = \tau(k_1, k_2, \ldots, k_t)$ is a normal congruence on E.*

Proof. It is easy to see that τ is an equivalence relation on E with convex classes and thus a congruence. Using the proof of Lemma 3.2, by symmetry, it suffices to assume

$$(m, e_i, m)\,\tau\,(m+1, e_j, m+1)$$

and apply conjugation by (p, e_k, q). By the definition of τ, we have $i>k_t$, $j \leq k_1$. We obtain

$$(p, e_k, q)^{-1}(m, e_i, m)(p, e_k, q) = (q+m-r, (e_k\alpha^{m-r})(e_i\alpha^{p-r}), q+m-r), \tag{3}$$

$$(p, e_k, q)^{-1}(m+1, e_j, m+1)(p, e_k, q) = (q+m+1-v, (e_k\alpha^{m+1-v})(e_j\alpha^{p-v}), q+m+1-v), \tag{4}$$

where $r = \min\{m, p\}$, $v = \min\{m+1, p\}$. For the group element on the right-hand side of (3) we have relations (2), and for the group element on the right-hand side of (4), we have

$$(e_k\alpha^{m+1-v})(e_j\alpha^{p-v}) = \begin{cases} e_j & \text{if } p<m+1, \\ e_{\max\{k,j\}} & \text{if } p=m+1, \\ e_k & \text{if } p>m+1. \end{cases} \tag{5}$$

Using (2) and (5), and considering several cases, we list below in the first column the relationship of p and m, in the second column the first entry on the right-hand side of (3), in the third column the second entry on the right-hand side of (3), in the fourth column the first entry on the right-hand side of (4), and in the fifth column the second entry on the right-hand side of (4).

$p<m$	$q+m-p$	e_i	$(q+m-p)+1$	e_j
$p=m$	q	$e_{\max\{k,i\}}$	$q+1$	e_j
$p=m+1$	q	e_k	q	$e_{\max\{k,j\}}$
$p>m+1$	q	e_k	q	e_k

Analysing each of these cases, we conclude that

$$(p, e_k, q)^{-1}(m, e_i, m)(p, e_k, q)\,\tau\,(p, e_k, q)^{-1}(m+1, e_j, m+1)(p, e_k, q),$$

which proves that τ is a normal congruence.

For the converse, we have

3.5 Lemma. *Let ξ be a non-universal normal congruence on E. For any $m, n \in \mathbb{N}$ and $0 \leq i, j < s$, we have the following statements.*

(i) $(m, e_i, m)\,\xi\,(m, e_j, m) \Rightarrow (n, e_i, n)\,\xi\,(n, e_j, n)$.

(ii) $(m, e_i, m)\,\xi\,(m+1, e_j, m+1) \Rightarrow (n, e_i, n)\,\xi\,(n+1, e_j, n+1)$ and $i>j$.

Proof. (i) This follows immediately from

$$(m, e_0, n)^{-1}(m, e_i, m)(m, e_0, n) = (n, e_i, n),$$
$$(m, e_0, n)^{-1}(m, e_j, m)(m, e_0, n) = (n, e_j, n).$$

(ii) Conjugation by the element (m, e_0, n) yields the first part of (ii). Assume that $(m, e_i, m)\,\xi\,(m+1, e_j, m+1)$ and $i \leq j$. By convexity of the ξ-classes, we deduce that

$(m, e_i, m)\xi(m+1, e_i, m+1)$. The first part of the conclusion in (ii) yields (n, e_i, n) $\xi(n+1, e_i, n+1)$ for all n. This means that ξ is the universal relation, a contradiction.

It is clear that every non-universal normal congruence on E uniquely determines the sequence $k_1, k_2, \ldots, k_t$ as in Notation 3.1 or 3.3. If we let $\tau_\varnothing$ denote the universal relation on E, we have a unique representation for all normal congruences on E. Hence the preceding three lemmas easily imply

3.6 THEOREM. *For all sequences* $0 \leq k_1 < k_2 < \ldots < k_t < s-1$, *the relations* $\rho(k_1, k_2, \ldots, k_t)$ *and* $\tau(k_1, k_2, \ldots, k_t)$ *exhaust the lattice* Φ *of all normal congruences on* E.

If A and B are two such sequences, one verifies easily that

$$\begin{aligned}
&\rho(A) \cap \rho(B) = \rho(A) \cap \tau(B) = \rho(C),\\
&\tau(A) \cap \tau(B) = \tau(C),\\
&\rho(A) \vee \rho(B) = \rho(A \cap B),\\
&\tau(A) \vee \tau(B) = \rho(A) \vee \tau(B) = \tau(A \cap B),
\end{aligned}$$

where C is obtained from A and B by writing the elements of $A \cup B$ in increasing order without repetitions.

Let $\mathcal{B}_d$ denote the Boolean lattice of all subsets of the set $[d] = \{0, 1, 2, \ldots, d-1\}$ under inclusion, and $\mathcal{B}_d^*$ its dual (that is, with inverted order). On Φ define a mapping χ by

$$\chi : \begin{cases} \rho(k_1, k_2, \ldots, k_t) \mapsto (\{k_1, k_2, \ldots, k_t\}, \varnothing), \\ \tau(k_1, k_2, \ldots, k_t) \mapsto (\{k_1, k_2, \ldots, k_t\}, \{0\}). \end{cases}$$

Then the above formulae for the meet and the join immediately imply that χ is a lattice isomorphism of Φ onto $\mathcal{B}_{s-1}^* \times \mathcal{B}_1$. Since $\mathcal{B}_{s-1}$ has an obvious involution, we have $\mathcal{B}_{s-1}^* \cong \mathcal{B}_{s-1}$. Further, the mapping φ defined on $\mathcal{B}_{s-1} \times \mathcal{B}_1$ by

$$\varphi : \begin{cases} (A, \varnothing) \mapsto A \\ (A, \{0\}) \mapsto A \cup \{s-1\} \end{cases}$$

is an isomorphism of $\mathcal{B}_{s-1} \times \mathcal{B}_1$ onto $\mathcal{B}_s$. We deduce

3.7 THEOREM. *For any simple ω-semigroup S with s $\mathcal{D}$-classes, the lattice Λ/θ is isomorphic to the Boolean lattice $\mathcal{B}_s$ of all subsets of a set with s elements.*

Proof. It suffices to recall that our S has exactly s $\mathcal{D}$-classes and that $\Phi \cong \Lambda/\theta$.

4. Non-group congruences. In order to characterize the congruences on S, we must still find all the possible kernels. For a congruence ρ on S, we know that $\ker \rho \subseteq \ker \xi^{\max}$ where $\xi = \operatorname{tr} \rho$. If ρ is a group congruence, then $\operatorname{tr} \rho$ is the universal congruence on E and conversely. Hence it remains to find $\ker \xi^{\max}$ where $\xi = \rho(k_1, k_2, \ldots, k_t)$ with $k_1, k_2, \ldots, k_t$ arbitrary and $\xi = \tau(k_1, k_2, \ldots, k_t)$ with $k_1, k_2, \ldots, k_t$ nonempty as in Theorem 3.6. Recall from Section 2 that

$$\ker \xi^{\max} = \{a \in S \mid a^{-1}ea \; \xi \; a^{-1}ae \text{ for all } e \in E\}. \tag{1}$$

In particular, if $a \in \ker \xi^{\max}$, then

$$a^{-1}(aa^{-1})a \,\xi\, (a^{-1}a)(aa^{-1})$$

so that

$$a^{-1}a \,\xi\, (a^{-1}a)(aa^{-1}).$$

Since we can interchange the roles of a and a^{-1} in (1), we also have $aa^{-1} \,\xi\, (aa^{-1})(a^{-1}a)$. We have thus proved

$$a \in \ker \xi^{\max} \Rightarrow aa^{-1} \,\xi\, a^{-1}a. \tag{2}$$

Recall that $E\zeta$ denotes the centralizer of E in S.

4.1 Lemma. *For any non-universal normal congruence ξ on E, we have*

$$\ker \xi^{\max} \subseteq \{(m, g, m) \mid m \in \mathbb{N}, g \in G\} = E\zeta.$$

Proof. Let $(m, g, n) \in \ker \xi^{\max}$. Consider first $\xi = \rho(k_1, k_2, \ldots, k_t)$. Then (2) yields $(m, e_i, m) \,\xi\, (n, e_i, n)$ where $g \in G_i$. According to the definition of $\rho(k_1, k_2, \ldots, k_t)$ we must have $m = n$.

Next take $\xi = \tau(k_1, k_2, \ldots, k_t)$, where the sequence $k_1, k_2, \ldots, k_t$ is nonempty. Then again $(m, e_i, m) \,\xi\, (n, e_i, n)$. Hence $m = n$, for if e.g. we have $m = n + 1$, then $i \leq k_1 < k_t < i$, which is impossible.

This proves the inclusion in the statement of the lemma; the equality follows by straightforward calculation.

It will follow from the results below that the inclusion in this lemma is actually an equality. Lemma 4.1 narrows considerably the choice of normal subsemigroups we are to find. The next task is to find all normal subsemigroups of S contained in $E\zeta$.

Recall that $S = \mathcal{B}(G, \alpha)$ where G is the chain of s groups $G_0 > G_1 > \ldots > G_{s-1}$. As usual, the multiplication in G is given in terms of homomorphisms

$$G_0 \xrightarrow{\varphi_0} G_1 \xrightarrow{\varphi_1} G_2 \longrightarrow \ldots \xrightarrow{\varphi_{s-2}} G_{s-1}.$$

4.2 Lemma. *For $0 \leq i \leq s-1$, let H_i be a normal subgroup of G_i such that $H_i\varphi_i \subseteq H_{i+1}$ for $0 \leq i \leq s-2$, and let $H = \bigcup_{i=0}^{s-1} H_i$. Then H is a normal subsemigroup of G. Conversely, every normal subsemigroup of G can be so constructed.*

Proof. The proof of the direct part follows by a straightforward verification and may be omitted. For the converse, let H be a normal subsemigroup of G. Then $H_i = H \cap G_i$, for $0 \leq i \leq s-1$, is evidently a normal subgroup of G_i. For any $g \in H_i$ with $i \leq s-2$, we have $ge_{i+1} \in H \cap G_{i+1} = H_{i+1}$ which shows that $H_i\varphi_i \subseteq H_{i+1}$.

Note that $\varphi_{s-1} = \alpha \mid G_{s-1}$ is a homomorphism of G_{s-1} into G_0 and that for any $0 \leq i < s-1$, we have

$$\alpha \mid G_i = \varphi_i\varphi_{i+1} \ldots \varphi_{s-1}. \tag{3}$$

4.3 LEMMA. *Let H be a normal subsemigroup of G as in Lemma 4.2, assume that $H_{s-1}\varphi_{s-1} \subseteq H_0$, and let*

$$K = \{(m, g, m) \mid m \in \mathbb{N}, g \in H\}. \tag{4}$$

Then K is a normal subsemigroup of S contained in $E\zeta$. Conversely, any such can be so constructed.

Proof. It follows from (3) and the hypotheses on H that $H\alpha \subseteq H$. This immediately implies that K is closed under multiplication. Since H is an inverse subsemigroup of G, K must be an inverse subsemigroup of S. Self-conjugacy of K in S follows easily from self-conjugacy of H in G. Since K is evidently full, it is a normal subsemigroup of S. It follows from Lemma 4.1 that $K \subseteq E\zeta$.

Conversely, let K be a normal subsemigroup of S contained in $E\zeta$. Let

$$H = \{g \in G \mid (0, g, 0) \in K\}. \tag{5}$$

A straightforward argument shows that H is a normal subsemigroup of G. Using the notation of Lemma 4.2, we let $g \in H_{s-1}$. Then

$$(1, e_0, 0)^{-1}(0, g, 0)(1, e_0, 0) = (0, g\alpha, 0) \in K$$

and hence $g\alpha \in H$. This shows that $H_{s-1}\varphi_{s-1} \subseteq H_0$. Since

$$(m, e_0, n)^{-1}(m, g, m)(m, e_0, n) = (n, g, n),$$

it follows at once that K has the form (4) where H is given by (5).

We will now identify those congruence pairs which involve the congruences $\rho(k_1, k_2, \ldots, k_t)$. Recall the notation introduced in Section 2.

4.4 LEMMA. *Let $k_1, k_2, \ldots, k_t$ be a sequence of integers, possibly empty, satisfying $0 \leq k_1 < k_2 < \ldots < k_t < s-1$. For convenience let $k_0 = -1$ and $k_{t+1} = s-1$. For each $1 \leq v \leq t+1$, let H_{k_v} be a normal subgroup of G_{k_v}. For $1 \leq v \leq t$, assume that $H_{k_v}\varphi_{k_v, k_{v+1}} \subseteq H_{k_{v+1}}$ and $H_{s-1}\varphi_{s-1}\varphi_{0,k_1} \subseteq H_{k_1}$. For $1 \leq v \leq t+1$ and $k_{v-1} < i < k_v$, let $H_i = H_{k_v}\varphi^{-1}_{i,k_v}$. Then the system $H_0, H_1, \ldots, H_{s-1}$ satisfies the conditions in Lemmas 4.2 and 4.3. Denoting K in formula (4) by $K(H_{k_1}, H_{k_2}, \ldots, H_{k_t}, H_{s-1})$, we have that*

$$(\rho(k_1, k_2, \ldots, k_t), K(H_{k_1}, H_{k_2}, \ldots, H_{k_t}, H_{s-1}))$$

is a congruence pair for S. Conversely, for fixed $\rho(k_1, k_2, \ldots, k_t)$, any such can be so constructed.

Proof. The hypothesis on H_{k_v} and the definition of H_i easily imply that the system $H_0, H_1, \ldots, H_{s-1}$ satisfies the requirements in Lemmas 4.2 and 4.3. Let H be as in Lemma 4.2 and $K = K(H_{k_1}, H_{k_2}, \ldots, H_{k_t}, H_{s-1})$ be as in formula (4). According to Lemma 4.3, K is a normal subsemigroup of S. Let $\rho = \rho(k_1, k_2, \ldots, k_t)$. Assume that

$$(m, g, n)(d, e_k, d) \in K, \quad (d, e_k, d)\rho(m, g, n)^{-1}(m, g, n).$$

The second relation yields $d = n$ and $k_{v-1} < i, k \leq k_v$ for some $1 \leq v \leq t+1$, where $g \in G_i$.

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Hence the first relation gives $ge_k \in H_{\max\{i,k\}}$. If $i \geqslant k$, this implies $g \in H_i$. Let $i < k$; then $g\varphi_{i,k} \in H_k$ and thus

$$g\varphi_{i,k_v} = g\varphi_{i,k}\varphi_{k,k_v} \in H_{k_v}.$$

But then the definition of H_i clearly shows that $g \in H_i$. The first relation above also implies that $m = n$. Consequently $(m, g, n) \in K$.

Next let $(m, g, m) \in K$ where $g \in H_i$ and $(n, e_j, n) \in E$. Letting $r = \min\{m, n\}$, we have

$$(m, g, m)^{-1}(n, e_j, n)(m, g, m) = (m+n-r, (g\alpha^{n-r})^{-1}(e_j\alpha^{m-r})(g\alpha^{n-r}), m+n-r),$$
$$(m, g, m)^{-1}(m, g, m)(n, e_j, n) = (m+n-r, (e_i\alpha^{n-r})(e_j\alpha^{m-r}), m+n-r).$$

The group elements on the right-hand sides of these two expressions are idempotents. A simple inspection shows that

(1) for $m > n$, they are both equal to e_i,

(2) for $m < n$, they are both equal to e_j,

(3) for $m = n$, they are both equal to $e_{\max\{i,j\}}$.

It follows that

$$(m, g, m)^{-1}(n, e_j, n)(m, g, m)\ \rho\ (m, g, m)^{-1}(m, g, m)(n, e_j, n).$$

Consequently (ρ, K) is a congruence pair for S.

Conversely, let $\rho = \rho(k_1, k_2, \ldots, k_t)$ and assume that (ρ, K) is a congruence pair for S. Then K is of the form (4) with the conditions as in Lemma 4.3. Let $g \in G_i$, $k_{v-1} < i < k_v$, and $g\varphi_i \in H_{i+1}$. Then for any $m \in \mathbb{N}$, we have

$$(m, g, m)(m, e_{i+1}, m) = (m, g\varphi_i, m) \in K,$$
$$(m, e_{i+1}, m)\ \rho\ (m, g, m)^{-1}(m, g, m),$$

which by hypothesis implies that $(m, g, m) \in K$, so that $g \in H_i$. It is easy to see that the established property

$$g\varphi_i \in H_{i+1} \Rightarrow g \in H_i$$

implies that the H_i have the form specified for them in the first part of this lemma, so we have $K = K(H_{k_1}, H_{k_2}, \ldots, H_{k_t}, H_{s-1})$.

For the congruences $\tau(k_1, k_2, \ldots, k_t)$ we have a similar situation.

4.5 Lemma. *Let $k_1, k_2, \ldots, k_t$ be a nonempty sequence of integers satisfying $-1 = k_0 < k_1 < k_2 < \ldots < k_t < s-1$. For each $1 \leqslant v \leqslant t$, let H_{k_v} be a normal subgroup of G_{k_v}. For $1 \leqslant v < t$, assume that $H_{k_v}\varphi_{k_v,k_{v+1}} \subseteq H_{k_{v+1}}$ and $H_{k_t}\varphi_{k_t,s-1}\varphi_{s-1}\varphi_{0,k_1} \subseteq H_{k_1}$. For $1 \leqslant v \leqslant t$ and $k_{v-1} < i < k_v$, let $H_i = H_{k_v}\varphi_{i,k_v}^{-1}$; for $i > k_t$, let $H_i = H_{k_1}(\varphi_{i,s-1}\varphi_{s-1}\varphi_{0,k_1})^{-1}$. Then the system $H_0, H_1, \ldots, H_{s-1}$ satisfies the conditions in Lemmas 4.2 and 4.3. Denoting K in formula (4) by $K(H_{k_1}, H_{k_2}, \ldots, H_{k_t})$, we have that*

$$(\tau(k_1, k_2, \ldots, k_t), K(H_{k_1}, H_{k_2}, \ldots, H_{k_t}))$$

is a congruence pair for S. Conversely, for a fixed $\tau(k_1, k_2, \ldots, k_t)$, any such can be so constructed.

Proof. The proof of Lemma 4.4 goes through in this case with minor modifications. We show only that

$$g \in G_{s-1},\ g\varphi_{s-1} \in H_0 \Rightarrow g \in H_{s-1}.$$

Assume that $g \in G_{s-1}$ and $g\varphi_{s-1} \in H_0$. Then, for any $m \in \mathbb{N}$, we have

$$\begin{aligned}(m, g, m)(m+1, e_0, m+1) &= (m+1, g\alpha, m+1)\\ &= (m+1, g\varphi_{s-1}, m+1) \in K(H_{k_1}, H_{k_2}, \ldots, H_{k_t})\end{aligned}$$

and also

$$(m+1, e_0, m+1) \quad \tau(k_1, k_2, \ldots, k_t) \quad (m, g, m)^{-1}(m, g, m).$$

These imply that $(m, g, m) \in K(H_{k_1}, H_{k_2}, \ldots, H_{k_t})$, so that $g \in H_{s-1}$.

Note that $K(H_{k_1}, H_{k_2}, \ldots, H_{k_t}, H_{s-1})$ in Lemma 4.4 depends on one more parameter than $K(H_{k_1}, H_{k_2}, \ldots, H_{k_t})$ in Lemma 4.5.

Letting ρ be the congruence on S whose trace is $\rho(k_1, k_2, \ldots, k_t)$ and whose kernel is $K(H_{k_1}, H_{k_2}, \ldots, H_{k_t}, H_{s-1})$ we obtain, for $g \in G_i$, $h \in G_j$,

$$\begin{aligned}(m, g, n)\rho(p, h, q) &\Leftrightarrow \begin{cases}(m, g, n)^{-1}(m, g, n)\ \rho(k_1, k_2, \ldots, k_t)\ (p, h, q)^{-1}(p, h, q),\\ (m, g, n)(p, h, q)^{-1} \in K(H_{k_1}, H_{k_2}, \ldots, H_{k_t}, H_{s-1})\end{cases}\\ &\Leftrightarrow \begin{cases}(n, e_i, n)\rho(k_1, k_2, \ldots, k_t)(q, e_j, q),\\ (m, g, n)(q, h^{-1}, p) \in K(H_{k_1}, H_{k_2}, \ldots, H_{k_t}, H_{s-1})\end{cases}\\ &\Leftrightarrow \begin{cases}m = p,\ n = q,\ k_{v-1} < i, j \leq k_v \quad \text{for some} \quad 1 \leq v \leq t+1,\\ gh^{-1} \in H_{\max\{i,j\}}.\end{cases}\end{aligned}$$

Now letting τ be the congruence on S whose trace is $\tau(k_1, k_2, \ldots, k_t)$ and whose kernel is $K(H_{k_1}, H_{k_2}, \ldots, H_{k_t})$, in view of the above, it suffices to consider the case $n = q+1$. In this case, again taking $g \in G_i$, $h \in G_j$, we have

$$\begin{aligned}(m, g, n)\tau(p, h, q) &\Leftrightarrow \begin{cases}(q+1, e_i, q+1)\ \tau(k_1, k_2, \ldots, k_t)\ (q, e_j, q),\\ (m, g, q+1)(q, h^{-1}, p) \in K(H_{k_1}, H_{k_2}, \ldots, H_{k_t})\end{cases}\\ &\Leftrightarrow m = p+1,\ n = q+1,\ i \leq k_1,\ j > k_t,\ g(h\alpha)^{-1} \in H_i.\end{aligned}$$

We have a similar result in the case $q = n+1$.

We can now summarize our discussion as follows. First $S = \mathcal{B}(G, \alpha)$ can be given by the sequence of groups and their homomorphisms

$$G_0 \xrightarrow{\varphi_1} G_1 \xrightarrow{\varphi_2} G_2 \longrightarrow \ldots \longrightarrow G_{s-1} \xrightarrow{\varphi_{s-1}} G_0.$$

For $i<j$ we let $\varphi_{ij}=\varphi_i\varphi_{i+1}\dots\varphi_{j-1}$ and take φ_{ii} to be the identity mapping on $G_i (0\leq i, j\leq s-1)$. This determines the multiplication in $G=\bigcup_{i=0}^{s-1} G_i$ and specifies $\alpha: G\rightarrow G_0$ by

$$g\alpha = g\varphi_{i,s-1}\varphi_{s-1} \qquad (g\in G_i).$$

Let $k_1, k_2, \dots, k_t$ be a sequence of integers, possibly empty, satisfying $0\leq k_1<k_2<\dots<k_t<s-1$. Let $k_0=-1$ and $k_{t+1}=s-1$. For $1\leq v\leq t+1$, let H_{k_v} be a normal subgroup of G_{k_v} and assume that

$$H_{k_v}\varphi_{k_v,k_{v+1}}\subseteq H_{k_{v+1}}\ (1\leq v\leq t),\ H_{s-1}\varphi_{s-1}\varphi_{0,k_1}\subseteq H_{k_1}. \tag{6}$$

Define a relation $\rho=\rho(H_{k_1}, H_{k_2}, \dots, H_{k_t}, H_{s-1})$ on S by

$$(m, g, n)\ \rho\ (p, h, q)$$
$$\Leftrightarrow\begin{cases} m=p,\ n=q,\ k_{v-1}<i, j\leq k_v, \\ (g\varphi_{i,k_v})(h\varphi_{j,k_v})^{-1}\in H_{k_v} \quad \text{for some} \quad 1\leq v\leq t+1, \end{cases} \tag{7}$$

where $\quad g\in G_i,\ h\in G_j$.

We assume now that the above sequence $k_1, k_2, \dots, k_t$ is nonempty. Let H_{k_v} be as above for $1\leq v\leq t$, and assume the first part of (6) for $1\leq v<t$ and also that

$$H_{k_t}\varphi_{k_t,s-1}\varphi_{s-1}\varphi_{0,k_1}\subseteq H_{k_1}.$$

Define a relation $\tau=\tau(H_{k_1}, H_{k_2}, \dots, H_{k_t})$ on S by

$$(m, g, n)\ \tau\ (p, h, q)$$
$$\Leftrightarrow\begin{cases} \text{either (7) occurs for } v\neq t+1, \\ \text{or} \\ m=p+1,\ n=q+1,\ i\leq k_1,\ j>k_t,\ (g(h\alpha)^{-1})\varphi_{i,k_1}\in H_{k_1}, \\ \text{or} \\ p=m+1,\ q=n+1,\ j\leq k_1,\ i>k_t,\ ((g\alpha)h^{-1})\varphi_{j,k_1}\in H_{k_1}. \end{cases}$$

We can thus announce

4.6 Theorem. *The relations $\rho(H_{k_1}, H_{k_2}, \dots, H_{k_t}, H_{s-1})$ and $\tau(H_{k_1}, H_{k_2}, \dots, H_{k_t})$ are congruences on S. Conversely, every non-group congruence on S can be uniquely written in the form of one of these congruences.*

We can now easily deduce a result of Kočin [**8**, Theorem 1]. In his terminology, a congruence ρ on a semigroup T is *locally idempotent-separating* if no two distinct $\mathcal{D}$-equivalent idempotents of T are ρ-equivalent.

4.7 Corollary. *Every non-group congruence on S is locally idempotent-separating.*

For any $\rho\in\Lambda$, let $\rho_{\max}$ and $\rho_{\min}$ denote the greatest and the least congruences θ-related to ρ, respectively.

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For $\rho=\rho(H_{k_1}, H_{k_2}, \ldots, H_{k_t}, H_{s-1})$, we see at once that

$$\rho_{\max}=\rho(G_{k_1}, G_{k_2}, \ldots, G_{k_t}, G_{s-1})$$

which implies that ker $\rho_{\max}=E\zeta$, a fact mentioned earlier. The same is true for $\tau=\tau(H_{k_1}, H_{k_2}, \ldots, H_{k_t})$. We thus deduce

4.8 COROLLARY. *If ρ is any non-group congruence on S, then* ker $\rho_{\max}=E\zeta$.

With the ρ before Corollary 4.8, we obviously have

$$\rho_{\min}=\rho(\{e_{k_1}\}, \{e_{k_2}\}, \ldots, \{e_{k_t}\}, \{e_{s-1}\}),$$

where e_i is the identity of G_i. The same type of statement holds for τ.

5. Group congruences. For bisimple ω-semigroups group congruences were explicitly constructed by Ault [1]. We could adapt her construction to the present situation. Instead, we will reduce our case to hers. Indeed, we will show that our S has essentially the same group congruences as one of its bisimple ω-subsemigroups, and will then apply her result.

Let

$$S_0=\{(m, g, n)\in S \mid m, n\in \mathbb{N}, g\in G_0\}.$$

It is clear that S_0 is a subsemigroup of S isomorphic to $\mathcal{B}(G_0, \alpha_0)$ where $\alpha_0=\alpha \mid G_0$. Group congruences on inverse semigroups are uniquely determined by their kernels. These are normal subsemigroups K of S satisfying: $ae\in K$, $e\in E \Rightarrow a\in K$. It is easy to see that this condition, for a normal subsemigroup of S, is equivalent to K being (right) unitary in S. Note that the group congruence ρ determined by K is given by: $x\,\rho\,y \Leftrightarrow xy^{-1}\in K$.

If K is a unitary normal subsemigroup of S, then it follows at once that $K_0=K\cap S_0$ is a unitary normal subsemigroup of S_0. Conversely, we have

5.1 LEMMA. *Let K_0 be a unitary normal subsemigroup of S_0, and let*

$$K=\{(m, g, n)\in S \mid (m+1, g\alpha, n+1)\in K_0\}.$$

Then K is the unique unitary normal subsemigroup of S for which $K\cap S_0=K_0$.

Proof. First note that $g\alpha\in G_0$ for any $g\in G$, so that indeed $(m+1, g\alpha, n+1)\in S_0$. Further,

$$(m+1, g\alpha, n+1)(p+1, h\alpha, q+1)=(m+p+2-t, (g\alpha^{p+2-t})(h\alpha^{n+2-t}), n+q+2-t), \quad (1)$$

where $t=\min\{n+1, p+1\}=1+\min\{n, p\}$. Letting $r=\min\{n, p\}$, we obtain that the expression in (1) is equal to

$$((m+p-r)+1, [(g\alpha^{p-r})(h\alpha^{n-r})]\alpha, (n+q-r)+1).$$

This shows that K is closed under multiplication. It is obvious that K is closed under the taking of inverses and that it is full.

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Let $r=\min\{n,p\}$. We assume that $r\leq m$ and compute

$$(p,h,q)^{-1}(m,g,n)(p,h,q)=(m+q-r,(h^{-1}\alpha^{m-r})(g\alpha^{p-r})(h\alpha^{n-r}),n+q-r) \tag{2}$$

where $r=\min\{n,p\}$, and on the other hand,

$$\begin{aligned}(p+1,h\alpha,q+1)^{-1}&(m+1,g\alpha,n+1)(p+1,h\alpha,q+1)\\ &=(m+q+2-t,(h^{-1}\alpha^{m+2-t})(g\alpha^{p+2-t})(h\alpha^{n+2-t}),n+q+2-t)\\ &=((m+q-r)+1,[(h^{-1}\alpha^{m-r})(g\alpha^{p-r})(h\alpha^{n-r})]\alpha,(n+q-r)+1)\end{aligned} \tag{3}$$

since $t=\min\{n+1,p+1\}=r+1$. A comparison of (2) and (3) shows that if $(m,g,n)\in K$, then also

$$(p,h,q)^{-1}(m,g,n)(p,h,q)\in K \quad \text{for all} \quad (p,h,q)\in S.$$

The case $r>m$ is treated similarly. Hence K is self-conjugate.

Next let $(m,g,n)(p,e_k,p)\in K$. It follows that

$$\begin{aligned}(m+1,g\alpha,n+1)(p+1,e_0,p+1)&=(m+p+2-t,g\alpha^{p+2-t},n+p+2-t)\\ &=((m+p-r)+1,(g\alpha^{p-r})\alpha,(n+p-r)+1)\in K_0,\end{aligned}$$

where $t=\min\{n+1,p+1\}$ and $r=\min\{n,p\}$. Hence $(m+1,g\alpha,n+1)\in K_0$ and thus $(m,g,n)\in K$. Consequently K is unitary in S.

Now observe that for $m,n\in\mathbb{N}$ and $g\in G$, we have

$$(m+1,g\alpha,n+1)=(m,g,n)(n+1,e_0,n+1), \tag{4}$$

where $(n+1,e_0,n+1)$ is an idempotent in S_0. If $(m,g,n)\in K_0$, then (4) yields that $(m+1,g\alpha,n+1)\in K_0$ and thus $(m,g,n)\in K$. Conversely, let $(m,g,n)\in K\cap S_0$. Then by definition, $(m+1,g\alpha,n+1)\in K_0$. Hence (4) gives $(m,g,n)\in K_0$ since K_0 is unitary in S_0. We have proved that $K_0=K\cap S_0$.

Finally let K' be a unitary normal subsemigroup of S such that $K'\cap S_0=K_0$. A simple argument involving equation (4) shows that $K=K'$, establishing uniqueness.

We can now apply the description of (right) unitary normal subsemigroups of Ault [**1**, Theorem 1] to $S_0=\mathcal{B}(G_0,\alpha_0)$, which we state as follows.

5.2 Theorem. *Let N be a normal subgroup of G_0, x be an element of G_0, and k be a nonnegative integer satisfying*

(i) $g\in N\Leftrightarrow g\alpha_0\in N$,

(ii) $N(x\alpha_0)=Nx=g^{-1}Nx(g\alpha_0^k)$,

for all $g\in G_0$. Let

$$K_0=\{(m,g,n)\in S_0 \mid n-m=ak \text{ for some integer } a,\ g\in Nx^a\},$$

where if $k=0$, then $a=0$. Then K_0 is a unitary normal subsemigroup of S_0. Conversely, any such can be so obtained for some N, x, k.

With this notation, we immediately obtain from Lemma 5.1 the following characterization.

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5.3 LEMMA. *For N, x, k satisfying the conditions of Theorem* 5.2, *the set*

$$K=\{(m, g, n)\in S \mid n-m=ak \text{ for some integer } a,\ g\alpha\in Nx^a\}$$

is a unitary normal subsemigroup of S. Conversely, any such can be so constructed for some N, x, k.

Keeping the same notation, we can finally state the characterization of group congruences on S.

5.4 THEOREM. *For N, x, k satisfying the conditions in Theorem* 5.2, *define a relation $\sigma=\sigma(N, x, k)$ by*

$$(m, g, n)\ \sigma\ (p, h, q) \Leftrightarrow \begin{cases} (n-m)-(q-p)=ak \text{ for some integer } a, \\ \qquad (g\alpha^{q+1})(h\alpha^{n+1})^{-1}\in Nx^a. \end{cases}$$

Then $\sigma(N, x, k)$ is a group congruence on S. Conversely, every group congruence on S can be so constructed for some N, x, k.

Proof. This follows from Lemma 5.3 using [**1**, Lemma].

In light of Lemma 5.1, we have that group congruences on S_0 are precisely the restrictions to S_0 of group congruences on S. Furthermore, each group congruence on S_0 extends uniquely to a group congruence on S. Because of this relationship, we can use all the results of Ault [**1**] and adapt them easily to the congruences on S. In particular, we get this way the inclusion relation, the join and the meet of group congruences on S in the above representation.

We have characterized the meet and the join of normal congruences on E after Theorem 3.6. For the congruences on S, we have

$$\rho\subseteq\tau \Leftrightarrow \operatorname{tr}\rho\subseteq\operatorname{tr}\tau,\ \ker\rho\subseteq\ker\tau.$$

Using the description of congruences $\rho(H_{k_1}, H_{k_2}, \ldots, H_{k_t}, H_{s-1})$ and $\tau(H_{k_1}, H_{k_2}, \ldots, H_{k_t})$ in the preceding section and $\sigma(N, x, k)$ in this section, it is easy to establish the inclusion relation for arbitrary congruences on S. The meet and the join of these congruences is somewhat more intricate but, in principle, does not involve serious obstacles.

Using Munn's [**10**] description of ω-semigroups with a proper kernel and the results obtained above, we can describe congruences also on such semigroups. This is notationally somewhat involved, but can be done without great difficulties. Congruences on the remaining ω-semigroups, that is, ω-chains of groups, can be characterized in a straightforward manner.

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Université de Montpellier
France