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CONIC APPROXIMATIONS AND COLLINEAR SCALINGS FOR OPTIMIZERS*

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Abstract. Many optimization algorithms update quadratic approximations to their objective functions. This paper suggests a generalization from quadratic to conic approximations, defined as ratios of quadratics whose denominators are squares, $(\alpha + a^T x)^2$. These can better match the values and gradients of typical objective functions, and hence give better estimates for their minimizers. Equivalently, affine scalings, $S(w) = x_0 + Jw$, of the domain of objective functions f are generalized to collinear scalings, $S(w) = x_0 + Jw/(1 + h^T w)$, to make the Hessian of the composition fS more nearly constant as well as better conditioned. Certain general features of optimization algorithms using conic approximations and collinear scalings are presented. These are not only invariant under affine scalings, along with Newton–Raphson and variable metric algorithms, but they are also invariant under the larger group of invertible collinear scalings.

1. Introduction. Many optimization algorithms update quadratic approximations to their objective function f, and use the minimizers of successive approximations to estimate minimizers for f. In steepest descent algorithms, each quadratic approximation has a unit Hessian and matches the gradient of f at one point. In Newton–Raphson algorithms [9], each approximation matches the Hessian as well as the gradient of f at one point. In variable metric algorithms [4], each approximation matches the gradient of f at two points. But nonquadratic approximations are needed to match function values $f_{\pm}$, as well as gradients $g_{\pm}$ at points $x_{\pm}$, whenever $f_+ - f_-$ does not equal $\frac{1}{2}(g_+ + g_-)^T(x_+ - x_-)$. This paper generalizes from quadratic to *conic* approximating functions, defined in § 2 as those ratios of quadratics whose denominators are squares.

Some reasons for suggesting this particular generalization are:

1) Under appropriate conditions, a conic interpolation can be determined by successive function and gradient evaluations at the $n+1$ vertices of an n dimensional simplex, using $O(n^2)$ numerical operations after each, much as a quadratic interpolation can be determined, to within an additive constant, by just its gradients at these vertices.

2) Optimization algorithms using conic approximations can be made invariant under the group of collinear transformations characteristic of projective geometry. Newton–Raphson and variable metric algorithms are invariant only under the proper subgroup of affine transformations, while steepest descent and conjugate gradient algorithms are invariant only under the still smaller subgroup of isometries of Euclidean space. While affine transformations can improve the conditioning of the Hessian of the objective function at any point, collinear transformations can also make the transformed Hessian more nearly constant, since the Jacobian of collinear transformations, unlike that of affine ones, need not be constant.

3) Conic functions, like most of the objective functions they are to approximate, need not be symmetric about their minimizers. They can also better fit exponential, penalty, or other functions which share with conics the property of increasing rapidly near some $n-1$ dimensional hyperplane in $\mathbb{R}^n$.

4) The minimizer of a conic function, like that of a typical objective function, need not be in the direction of a Newton step. In contrast, each minimizer of the nonquadratic approximations considered by Fried [5], Jacobson and Oxman [6], and others [2], [3], [7] is always in the direction of a Newton step since their approximating functions satisfy

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$f(x)=f_*+\phi(x-x_*)$ for some homogeneous function ϕ of degree $\nu>0$; i.e., $\phi(\lambda s)=\lambda^\nu\phi(s)$ for all $\lambda>0$ and $s\in\mathbb{R}^n$.

5) Theoretical concepts and computational methods of linear algebra are applicable to algorithms using conic functions when points in the n dimensional domain of these functions are specified by the n ratios among $n+1$ homogeneous coordinates, essentially because the group of invertible collinear transformations is isomorphic to the group of invertible $(n+1)\times(n+1)$ matrices modulo multiples of the unit matrix.

Section 2 introduces some basic terms and concepts, and § 3 applies these to the study of conic functions with given values and gradients at the vertices of a simplex. Section 4 shows how these conic interpolations can be obtained using $O(n^2)$ numerical operations after each function and gradient evaluation. Section 5 gives an algorithm schemata from which specific optimization algorithms can be derived; it can be read first by those primarily interested in computation since it makes few references to the rest of this paper.

The ellipsis "iff" is used for "if and only if". Lower case Greek letters denote real numbers which need not be integers; lower case Latin letters denote integers, functions, or column vectors; and upper case letters denote more general maps, matrices, or spaces. The transpose of any matrix A is A^T, and $A^{-T}=(A^T)^{-1}=(A^{-1})^T$.

2. Definitions and explications.

$\mathbb{R}^n$ is the n dimensional space of real $n\times 1$ column vectors;
$\mathbb{R}^{m\times n}$ is the mn dimensional space of real $m\times n$ matrices;
$\mathbb{R}^{n\vee n}$ is the $\frac{1}{2}n(n+1)$ dimensional space of real symmetric $n\times n$ matrices, ordered by $A\geqq 0$ iff $v^TAv\geqq 0$ for all $v\in\mathbb{R}^n$;
I_n is the $n\times n$ unit matrix; and
X is an open convex subset in $\mathbb{R}^n$.

A smooth function $f:X\to\mathbb{R}$ is:
affine iff its gradient is constant;
quadratic iff its Hessian is constant;
collinear iff it is a ratio of affine functions;
conic iff it is a ratio of a quadratic to the square of an affine function;
positive iff $f(x)>0$ for all $x\in X$, and
cupped iff it has a minimizer, all its level sets are convex, and it has no smooth extension to a larger open convex domain.

A map $S:W\to X$ between convex sets W and X is *affine, quadratic, collinear*, or *conic* iff each affine $f:X\to\mathbb{R}$ makes the composition $fS:W\to\mathbb{R}$ affine, quadratic, collinear, or conic respectively.

A *gauge* for a function $f:X\to\mathbb{R}$ is a smooth positive function $p:X\to\mathbb{R}$ which makes the product $p^2f:X\to\mathbb{R}$ quadratic.

A *scaling* for a function $f:X\to\mathbb{R}$ is a smooth map $S:W\to X$ from an open set W in a Euclidean space to X which makes the composition $fS:W\to\mathbb{R}$ quadratic with unit Hessian.

Some basic consequences of these definitions are:

1) *Hierarchy of conic functions.* Each constant function is affine; a function is affine iff it is both quadratic and collinear, and each quadratic or collinear function is conic.

2) *Restrictions to lines.* A function is affine, quadratic, collinear, or conic iff its restriction to each line in its domain is affine, quadratic, collinear, or conic respectively.

3) *Maximal extensions.* A function has a (collinear) conic extension to all $\mathbb{R}^n$ iff it is (affine) quadratic. The largest convex domain for any other conic function is an open half space in $\mathbb{R}^n$.

(4) *Critical points.* The critical points of each conic function $f: X \to \mathbb{R}$ form an affine subspace in X, possibly null; i.e., if $x \neq y$ are critical points of f, then each point of X on the line through x and y is also a critical point of f. While Hessians at different critical points need not be the same, they share a common null space, consisting of multiples of the displacements $x - y$ between critical points. A collinear function has a critical point iff it is constant. If a conic function has a local minimizer x_*, then x_* is a global minimizer, each level set of f is convex, and f has just one cupped conic extension.

5) *Level sets.* The level sets of each conic function are conic sections, but these are similar and concentric iff the function is quadratic. To within an affine transformation of its domain, each conic function with a unique minimizer is equivalent to one with unit Hessian at its minimizer $0 \in \mathbb{R}^n$. For each such normalized conic $f: X \to \mathbb{R}$, there is just one $a \in \mathbb{R}^n$ with

$$a^T x < 1 \quad \text{and} \quad f(x) = f_* + \frac{1}{2}\frac{x^T x}{(1 - a^T x)^2}$$

for all $x \in X$. The level sets of f are convex conic sections which are invariant under rotations about a and have one focus at $0 \in \mathbb{R}^n$. This function is quadratic and its level sets are concentric spheres iff $a = 0$. For $a \neq 0$, there is just one level set for each eccentricity $e > 0$. It is

$$\{x \in X : a^T a x^T x \leqq e^2(1 - a^T x)^2\}$$

within which $f(x) \leqq f_* + \frac{1}{2}e^2/a^T a$. This is an ellipsoid for $e < 1$, a paraboloid for $e = 1$, and a lobe of a hyperboloid for $e > 1$. The function $f: X \to \mathbb{R}$ is convex only in the segment of a paraboloid

$$\{x \in X : a^T a x^T x \leqq (1 + a^T x)^2\}.$$

6) *A representation of collinear maps.* A map $S: W \to X$ is collinear and has the value $x_0 \in X$ and Jacobian $J \in \mathbb{R}^{n \times m}$ at $0 \in W \subseteq \mathbb{R}^m$ iff there is an $h \in \mathbb{R}^m$ with

$$1 + h^T w > 0 \quad \text{and} \quad S(w) = x_0 + \frac{Jw}{1 + h^T w} \tag{2.1}$$

for all $w \in W$. The Jacobian of this map at any $w \in W$ is

$$J(w) = J\frac{(I_m + wh^T)^{-1}}{1 + h^T w} = \frac{1}{1 + h^T w}\left(J - \frac{Jwh^T}{1 + h^T w}\right).$$

This map is invertible iff J is invertible and $SW = X$, in which case

$$S^{-1}(x) = \frac{J^{-1}(x - x_0)}{1 - h^T J^{-1}(x - x_0)}.$$

7) *A representation of conic functions.* A function $f: X \to \mathbb{R}$ is conic and has the value $f_0 \in \mathbb{R}$ and gradient $g_0 \in \mathbb{R}^n$ at $x_0 \in X$ iff there is an $a \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \vee n}$ with

$$a^T s < 1 \quad \text{and} \quad f(x_0 + s) = f_0 + \frac{g_0^T s}{1 - a^T s} + \frac{1}{2}\frac{s^T A s}{(1 - a^T s)^2} \tag{2.2}$$

for all $s \in \mathbb{R}^n$ with $x_0 + s \in X$. This function is collinear iff $A = 0$. Its gradient at any point $x = x_0 + s$ of X is

$$g = \frac{1}{\gamma^2}(I_n - as^T)^{-1}(\gamma g_0 + As) = \frac{1}{\gamma^3}(\gamma I_n + as^T)(\gamma g_0 + As),$$

where $\gamma = 1 - a^T s$ is the value at $x = x_0 + s$ of a gauge for f. This gradient vanishes at $x_* = x_0 + s_* \in X$ iff

$$(A - g_0 a^T) s_* = -g_0.$$

The value of f is the same at all critical points x_* and equals

$$f_* = f_0 + \frac{1}{2}\frac{g_0^T s_*}{\gamma_*} = f_0 - \frac{1}{2}\frac{s_*^T A s_*}{\gamma_*^2},$$

where $\gamma_* = 1 - a^T s_*$. The Hessian of f at x_* is

$$\frac{1}{\gamma_*^2}(A - a g_0^T - g_0 a^T + 2(f_0 - f_*) a a^T) = \frac{1}{\gamma_*^2}(I_n - a s_*^T)^{-1} A (I_n - s_* a^T)^{-1}$$
$$= \frac{1}{\gamma_*^4}(\gamma_* I_n + a s_*^T) A (\gamma_* I_n + s_* a^T).$$

A critical point is a minimizer iff $A \geqq 0$; it is unique iff A is invertible, in which case $\gamma_* = 1/(1 - a^T A^{-1} g_0)$, $s_* = -\gamma_* A^{-1} g_0$, $f_* = f_0 - \frac{1}{2} g_0^T A^{-1} g_0$, and the Hessian of f at x_* is

$$\frac{1}{\gamma_*^2}(A - a g_0^T) A^{-1} (A - g_0 a^T).$$

8) *Scalings.* The collinear map of (2.1) scales the conic function of (2.2) iff $h = J^T a$ and $J^T A J = I_m$. A scaling $S: W \to X$ of any function $f: X \to \mathbb{R}$ pairs the level sets of f with concentric spheres in W. If a function $f: X \to \mathbb{R}$ with a critical point x_* has an invertible scaling $S: W \to X$, then x_* is a unique minimizer; the Hessian of f at x_* is positive definite, and this Hessian equals $J^{-T} J^{-1}$, where J^{-1} is the Jacobian of $S^{-1}: X \to W$ at x_*. Morse's lemma [8] implies the converse: if a smooth function $f: \mathbb{R}^n \to \mathbb{R}$ has a positive definite Hessian at a critical point $x_* \in \mathbb{R}^n$, then there is an invertible scaling of the restriction of f to some neighborhood X of x_*.

9) *Homogeneous coordinates.* A function $f: X \to \mathbb{R}$ is conic iff for each $x_0 \in X$ there is a $c \in \mathbb{R}^{n+1}$ and $\Lambda \in \mathbb{R}^{n+1 \vee n+1}$ with

$$c^T \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 \quad \text{and} \quad f(x_0 + s) = \frac{1}{2}\frac{w^T \Lambda w}{(c^T w)^2} \tag{2.3}$$

for all $s \in \mathbb{R}^n$ with $x_0 + s \in X$ and all positive multiples $w \in \mathbb{R}^{n+1}$ of $\begin{pmatrix} s \\ 1 \end{pmatrix}$. The gradient $g \in \mathbb{R}^n$ of f at $x = x_0 + s$ is uniquely determined by

$$\frac{1}{\gamma}\Lambda \begin{pmatrix} s \\ 1 \end{pmatrix} = \gamma \begin{pmatrix} g \\ -g^T s \end{pmatrix} + 2 f(x) c, \tag{2.4}$$

where $\gamma = c^T \begin{pmatrix} s \\ 1 \end{pmatrix}$ is the value at $x = x_0 + s$ of a gauge for f. Equations (2.2) and (2.3) specify the same conic function iff

$$c = \begin{pmatrix} -a \\ 1 \end{pmatrix} \quad \text{and} \quad \Lambda = \begin{pmatrix} A - g_0 a^T - a g_0^T & g_0 \\ g_0^T & 0 \end{pmatrix} + 2 f_0 c c^T. \tag{2.5}$$

10) *Miscellaneous.* An affine function over all $\mathbb{R}^n$ is positive iff it is a positive constant, and these are the only gauges for nonconstant quadratic functions. There are nonconstant positive affine functions over any proper open convex subset X in $\mathbb{R}^n$, and

while all these gauge each constant function over X, the gauges for each nonconstant conic function are positive multiples of each other.

The sum of two (collinear) conic maps is (collinear) conic iff they share a gauge. The set of those collinear maps $S: W \to X$, from an m dimensional W to an n dimensional X which share a gauge, is an $(m+1)n$ dimensional vector subspace in the $\frac{1}{2}(m+2)(m+1)n$ dimensional vector space of all conic maps $S: W \to X$ which share this same gauge.

Collinear maps preserve collinearity, convexity, and cross ratios; i.e., if $S: W \to X$ is collinear, then

(i) S pairs collinear points u, v, and w of W with collinear points $S(u)$, $S(v)$, and $S(w)$ of X;

(ii) S pairs each convex subset U in W with a convex subset SU in X; and

(iii) if t, u, v, and w are collinear points of W, then for any norms on the spaces W and X,

$$\frac{\|t-u\|\,\|v-w\|}{\|t-v\|\,\|u-w\|}=\frac{\|St-Su\|\,\|Sv-Sw\|}{\|St-Sv\|\,\|Su-Sw\|}$$

whenever the denominators are positive. Note that the choice of norms does not affect these ratios since $\|\lambda s\| = |\lambda|\,\|s\|$ for any norm.

3. Conic interpolations. While each iteration of the algorithms to be considered uses the values and gradients of an objective function at just two points to update a conic interpolation, it is instructive first to derive a necessary and sufficient condition for there to be a conic interpolation to given function and gradient values at the vertices of any simplex.

THEOREM 1. *There is a conic function $f: X \to \mathbb{R}$ with values $f_k \in \mathbb{R}$ and gradients $g_k \in \mathbb{R}^n$ at the $m+1$ vertices x_k of an m dimensional simplex in X iff there are positive numbers γ_k with*

$$f_j - f_i = \frac{1}{2}\left(\frac{\gamma_i}{\gamma_j}\,g_i + \frac{\gamma_j}{\gamma_i}\,g_j\right)^T (x_j - x_i) \tag{3.1}$$

for all i and j.

Proof. First assume that f is a conic function with values f_k and gradients g_k at x_k, and let γ_k be the value at x_k of a gauge for f. Since a gauge is positive, $\gamma_k > 0$, and since it is affine, its value at any point $x(\tau) = x_i + (x_j - x_i)\tau$ on the line through x_i and x_j is $\gamma_i + (\gamma_j - \gamma_i)\tau$. Use the definition of a gauge to show that the function $q: \mathbb{R} \to \mathbb{R}$ defined by

$$q(\tau) = (\gamma_i + (\gamma_j - \gamma_i)\tau)^2 f(x(\tau))$$

is quadratic. Evaluate q and its derivative q' at 0 and 1, and then use $q(1) - q(0) = \frac{1}{2}(q'(0) + q'(1))$ to get (3.1).

Now assume that some $f_k \in \mathbb{R}$, $g_k \in \mathbb{R}^n$, and positive numbers γ_k satisfy (3.1) for the $m+1$ vertices x_k of a simplex. Since only the ratios among the γ_k enter in (3.1), choose $\gamma_0 = 1$. Define $s_k = x_k - x_0$ and use the linear independence of the m vectors s_k for $k \neq 0$ to show there is an $a \in \mathbb{R}^n$ with

$$a^T s_k = 1 - \gamma_k \tag{3.2}$$

for all k. For each k, define the vector $r_k = \gamma_k^2(g_k - a s_k^T g_k) - \gamma_k g_0$, and for each i and j, define the number

$$\rho_{ij} = \tfrac{1}{2}\left(\frac{\gamma_j}{\gamma_i}\,g_j - \frac{\gamma_i}{\gamma_j}\,g_i\right)^T (x_j - x_i). \tag{3.3}$$

Use (3.1) and (3.2), together with $(f_j - f_i) + (f_i - f_0) + (f_0 - f_j) = 0$, to get

$$r_i^T s_j = \gamma_i \gamma_j (\rho_{0i} + \rho_{0j} - \rho_{ij}), \tag{3.4}$$

and hence $r_i^T s_j = r_j^T s_i$. Use this and the linear independence of the m vectors s_k for $k \neq 0$ to show there is an $A \in \mathbb{R}^{n \vee n}$ with

$$A s_k = r_k \tag{3.5}$$

for all k. In (2.2), use any $a \in \mathbb{R}^n$ satisfying (3.2) and $A \in \mathbb{R}^{n \vee n}$ satisfying (3.5) to obtain a conic function with the given values and gradients. □

COROLLARY 1. *There is an affine function $f: X \to \mathbb{R}$ with values $f_k \in \mathbb{R}$ and gradients $g_k \in \mathbb{R}^n$ at the $m+1$ vertices x_k of a simplex in X iff for all i and j,*

$$f_j - f_i = g_i^T (x_j - x_i) = g_j^T (x_j - x_i),$$

a quadratic interpolation iff

$$f_j - f_i = \tfrac{1}{2}(g_i + g_j)^T (x_j - x_i),$$

a collinear interpolation iff there are positive numbers γ_k with

$$f_j - f_i = \frac{\gamma_i}{\gamma_j} g_i^T (x_j - x_i) = \frac{\gamma_j}{\gamma_i} g_j^T (x_j - x_i),$$

and a conic interpolation iff there are positive numbers γ_k with

$$f_j - f_i = \frac{1}{2}\left(\frac{\gamma_i}{\gamma_j} g_i + \frac{\gamma_j}{\gamma_i} g_j\right)^T (x_j - x_i).$$

Proof. This last condition simply repeats the theorem and is included here only to facilitate comparisons. Obtain the other conditions from this one by showing that a conic interpolation is collinear iff the ρ_{ij} defined by (3.3) are all zero, that it is quadratic iff it has a constant gauge, with $\gamma_i = \gamma_j$ for all i and j, and that it is affine iff it is both quadratic and collinear. □

COROLLARY 2. *If $f: X \to \mathbb{R}$ is a conic function with values $f_k \in \mathbb{R}$ and gradients $g_k \in \mathbb{R}^n$ at points $x_k \in X$, then for each i and j there is just one $\rho_{ij} = \rho_{ji} \in \mathbb{R}$ for which the values γ_k at x_k of each gauge for f satisfy* (3.3). *This ρ_{ij} also satisfies*

$$\rho_{ij}^2 = (f_j - f_i)^2 - g_i^T (x_j - x_i) g_j^T (x_j - x_i), \tag{3.6}$$

$$\rho_{ij} = f_i - f_j + \frac{\gamma_j}{\gamma_i} g_j^T (x_j - x_i) = f_j - f_i - \frac{\gamma_i}{\gamma_j} g_i^T (x_j - x_i) \tag{3.7}$$

and

$$\rho_{ij} = \frac{1}{2} \gamma_0^2 \left(\frac{s_j}{\gamma_j} - \frac{s_i}{\gamma_i}\right)^T A \left(\frac{s_j}{\gamma_j} - \frac{s_i}{\gamma_i}\right) \tag{3.8}$$

for the $A \in \mathbb{R}^{n \vee n}$ of (2.2), *where $s_k = x_k - x_0$. If f is cupped, then $\rho_{ij} \geqq 0$. For each point x of the affine subspace in X spanned by the x_k, there are $\omega_k \in \mathbb{R}$ with $\sum \omega_k > 0$ and $x = \sum \omega_k x_k / \sum \omega_k$. Each gauge for f with values γ_k at x_k has the value $\sum \omega_k \gamma_k / \sum \omega_k > 0$ at x, and the value of f at x is*

$$f(x) = \frac{\sum \omega_k \gamma_k f_k}{\sum \omega_k \gamma_k} - \frac{1}{2} \frac{\sum \omega_i \omega_j \gamma_i \gamma_j \rho_{ij}}{(\sum \omega_k \gamma_k)^2}. \tag{3.9}$$

Proof. Define ρ_{ij} by (3.3), use (3.1) to get (3.6) and (3.7), and use (3.4) and (3.5) to get (3.8). If f is cupped, get $\rho_{ij} \geqq 0$ from $A \geqq 0$ and (3.8). Since each gauge is affine, its value at $\sum \omega_k x_k / \sum \omega_k$ is $\sum \omega_k \gamma_k / \sum \omega_k$, and since each gauge is positive, $\sum \omega_k \gamma_k > 0$. Multiply the function f specified in (3.9) by the square of this gauge to show that f is conic, and evaluate f and its directional derivatives at x_k to show that it is the required interpolation. □

COROLLARY 3. *There are at most 2^n conic functions with values $f_k \in \mathbb{R}$ and gradients $g_k \in \mathbb{R}^n$ at the $n+1$ vertices x_k of an n dimensional simplex in $\mathbb{R}^n$. Of these, at most one is cupped. If there is a cupped conic interpolation f and if $f_0 > f_k$ for each $k > 0$, then there is just one gauge for f whose value at x_0 is 1; its value at each x_k is*

$$\gamma_k = \frac{-g_0^T s_k}{f_0 - f_k + \rho_{0k}},$$

where $s_k = x_k - x_0$ and $\rho_{0k} = ((f_0 - f_k)^2 - g_0^T s_k g_k^T s_k)^{1/2}$.

Proof. Use Corollary 2 to show that there is at most one interpolating conic for each choice of signs for the n numbers ρ_{0k} with $k > 0$, and that these signs are all positive for a cupped conic. □

COROLLARY 4. *There is a conic function over the interval $[x_-, x_+] \subseteq \mathbb{R}$ with values $f_\pm \in \mathbb{R}$ and slopes $g_\pm \in \mathbb{R}$ at points $x_+ > x_-$ of $\mathbb{R}$ iff there are roots $\gamma > 0$ to the quadratic equation*

$$g_+ \gamma^2 - 2 \frac{f_+ - f_-}{x_+ - x_-} \gamma + g_- = 0. \tag{3.10}$$

For each root γ, there is just one conic interpolation gauged by an affine function whose values $\gamma_\pm$ at $x_\pm$ have the ratio $\gamma_+/\gamma_- = \gamma$. The value and slope of this interpolation at $x(\tau) = x_- + (x_+ - x_-)\tau \in X$ are

$$f(x(\tau)) = \frac{(1-\tau) f_- + \gamma \tau f_+}{1 - \tau + \gamma \tau} - \frac{\gamma \tau (1-\tau) \rho}{(1 - \tau + \gamma \tau)^2}$$

and

$$g(x(\tau)) = \frac{(1-\tau) g_- + \gamma^3 \tau g_+}{(1 - \tau + \gamma \tau)^3},$$

where

$$\begin{aligned} \rho &= \frac{1}{2}\left(\gamma g_+ - \frac{1}{\gamma} g_-\right)(x_+ - x_-) \\ &= f_- - f_+ + \gamma g_+ (x_+ - x_-) \\ &= f_+ - f_- - \frac{1}{\gamma} g_- (x_+ - x_-) \quad \text{and} \\ \rho^2 &= (f_+ - f_-)^2 - g_+ g_- (x_+ - x_-)^2. \end{aligned}$$

There is a cupped conic extension of f iff either f is constant, or else $\gamma^2 g_+ > g_-$ and $\gamma^3 g_+ > g_-$, when the minimizer x_* and minimum f_* of f are

$$x_* = \frac{\gamma^3 g_+ x_- - g_- x_+}{\gamma^3 g_+ - g_-}$$

and

$$f_* = f_\pm - \frac{(f_+ - f_- \pm \rho)^2}{4\rho}.$$

Proof. Use (3.1) with $\gamma = \gamma_+/\gamma_-$ to get (3.10). Use Theorem 1, Corollary 2, and algebra to verify the other conclusions. $\square$

This corollary provides a basis for one dimensional optimization algorithms such as those developed by Bjørstad and Nocedal [1]. The following simple example assumes that for each $k>0$, there is a cupped conic interpolation to the values and slopes of the objective function at x_k and x_{k-1} with a unique minimizer at x_{k+1}, and that $f_{k-1} > f_k$.

ALGORITHM 1.

Input: A point x_0 in the domain $X \subseteq \mathbb{R}$ of an objective function $f: X \to \mathbb{R}$, an initial nonzero step $s_1 \in \mathbb{R}$ to be taken from x_0, and a subalgorithm for calculating the value $f_k \in \mathbb{R}$ and slope $g_k \in \mathbb{R}$ of f at any $x_k \in X$.

Calculate f_0 and g_0 at x_0.

For each integer $k>0$ until some convergence criterion is met, calculate f_k and g_k at $x_k = x_{k-1} + s_k$, and set

$$\rho_k = ((f_{k-1} - f_k)^2 - g_{k-1} g_k s_k^2)^{1/2},$$

$$\gamma_k = \frac{-g_{k-1} s_k}{f_{k-1} - f_k + \rho_k} \quad \text{and}$$

$$s_{k+1} = \frac{-\gamma_k^3 g_k s_k}{\gamma_k^3 g_k - g_{k-1}}.$$

4. Recursive interpolations and scalings. A minimizer of a cupped conic function f can be calculated from the values and gradients of f at $n+1$ points, using $O(n^3)$ operations. This section shows how $O(n^2)$ operations can be used after each function and gradient evaluation to update a conic interpolation and locate its minimizer, or equivalently, to update a collinear scaling of its domain.

The main idea is to use (2.3) to replace the conic function $f: X \to \mathbb{R}$ by the quadratic function $w \to \frac{1}{2} w^T \Lambda w$ over the n dimensional hyperplane in $\mathbb{R}^{n+1}$ with $c^T w = 1$, and to replace each point $x_k = x_0 + s_k \in X$ by a corresponding point

$$w_k = \frac{1}{\gamma_k} \begin{pmatrix} s_k \\ 1 \end{pmatrix}$$

on this hyperplane. The next algorithm calculates a $V \in \mathbb{R}^{n+1 \vee n+1}$ for a cupped conic function $f: X \to \mathbb{R}$ and n dimensional simplex in X, which is then used by the following theorem to specify a minimizer and minimum of f.

ALGORITHM 2.

Input: For each integer k from 0 through n, the value $f_k \in \mathbb{R}$ and gradient $g_k \in \mathbb{R}^n$ of a cupped conic function $f: X \to \mathbb{R}$ at the kth vertex x_k of an n dimensional simplex in X, with $f_0 > f_k$ for $k>0$.

Set $V_0 = 0 \in \mathbb{R}^{n+1 \vee n+1}$. For each k from 1 through n, set

$$s_k = x_k - x_0, \quad \rho_k = ((f_0 - f_k)^2 - g_0^T s_k g_k^T s_k)^{1/2}, \quad \gamma_k = \frac{-g_0^T s_k}{f_0 - f_k + \rho_k},$$

$$z_k = \begin{pmatrix} \dfrac{1}{\gamma_k} s_k \\ \dfrac{1}{\gamma_k} - 1 \end{pmatrix}, \quad y_k = \begin{pmatrix} \gamma_k g_k - g_0 \\ -\gamma_k g_k^T s_k \end{pmatrix} \quad \text{and} \quad v_k = z_k - V_{k-1} y_k.$$

$$\text{If } y_k^T v_k > 0, \quad \text{set } V_k = V_{k-1} + \frac{v_k v_k^T}{y_k^T v_k};$$

$$\text{else,} \quad \text{set } V_k = V_{k-1}.$$

Set $V = V_n$.

THEOREM 2. *For each cupped conic function $f: X \to \mathbb{R}$ whose values $f_k \in \mathbb{R}$ at the $n+1$ vertices x_k of an n dimensional simplex in X satisfy $f_0 > f_k$ for all $k > 0$, define $V \in \mathbb{R}^{n+1 \vee n+1}$ by Algorithm 2. If $\gamma_* > 0$ and $s_* \in \mathbb{R}^n$ are defined by*

$$\frac{1}{\gamma_*}\begin{pmatrix} s_* \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - V\begin{pmatrix} g_0 \\ 0 \end{pmatrix},$$

then f has its minimum value

$$f_* = f_0 + \frac{1}{2}\frac{1}{\gamma_*} g_0^T s_* = f_0 - \frac{1}{2}[g_0, 0] V \begin{pmatrix} g_0 \\ 0 \end{pmatrix}$$

at $x_0 + s_*$, where $g_0 \in \mathbb{R}^n$ is the gradient of f at x_0. Furthermore,

(4.1) $$V \geqq 0, \quad Vc = 0, \quad V\Lambda V = V \quad \textit{and} \quad \Lambda V \Lambda = \Lambda - 2f_* cc^T,$$

where $c \in \mathbb{R}^{n+1}$ and $\Lambda \in \mathbb{R}^{n+1 \vee n+1}$ are defined by (2.3).

Proof. Use Corollary 3 of Theorem 1 to show that the γ_k defined in Algorithm 2, together with $\gamma_0 = 1$, are the values at x_k of a gauge for f. Use (2.4) to show that for each k from 1 through n, the y_k and z_k defined in Algorithm 2 satisfy

(4.2) $$c^T z_k = 0 \quad \text{and} \quad y_k = \Lambda z_k + 2(f_0 - f_k)c.$$

Since f is cupped, there is a minimum value $f_* \in \mathbb{R}$ of f and a $w_* \in \mathbb{R}^{n+1}$ with $c^T w_* = 1$ and $\Lambda w_* = 2f_* c$. For each k from 0 through n, define $Z_k \in \mathbb{R}^{n+1 \times k+1}$ and $U_k \in \mathbb{R}^{n+1 \vee n+1}$ by

$$Z_k = [w_*, z_1, z_2, \cdots, z_k] \quad \text{and} \quad U_k = \Lambda - \Lambda V_k \Lambda - 2f_* cc^T,$$

and define the induction hypothesis

($\mathcal{P}_k$) $$V_k \geqq 0, \quad V_k c = 0, \quad V_k \Lambda V_k = V_k, \quad U_k \geqq 0 \quad \text{and} \quad U_k Z_k = 0.$$

To prove $\mathcal{P}_0$, note that $V_0 = 0$, that $f \geqq f_*$ in (2.3) implies $U_0 \geqq 0$, and that $U_0 Z_0 = U_0 w_* = \Lambda w_* - 2f_* c = 0$. The following proof of $\mathcal{P}_k$ from $\mathcal{P}_{k-1}$ is similar to one for variable metric algorithms.

Assume $\mathcal{P}_{k-1}$ for some $k > 0$ and show that $\Lambda v_k = U_{k-1} z_k$, and hence $y_k^T v_k = z_k^T U_{k-1} z_k \geqq 0$, with equality iff $U_{k-1} z_k = 0$. If $y_k^T v_k = 0$, the algorithm sets $V_k = V_{k-1}$, so that $U_k = U_{k-1}$ and $U_k Z_k = U_{k-1} Z_k = [U_{k-1} Z_{k-1}, U_{k-1} z_k] = 0$. Now assume $y_k^T v_k > 0$ and get $V_k \geqq V_{k-1} \geqq 0$. Use $V_{k-1} c = 0$ and $c^T z_k = 0$ to get $V_k c = 0$. Use $V_k \Lambda z_k = V_k y_k = z_k$ and $V_{k-1} \Lambda V_{k-1} = V_{k-1}$ to get $V_k \Lambda V_k = V_k$. Use the definitions of U_k and V_k to get

$$U_k = U_{k-1} - U_{k-1} \frac{z_k z_k^T}{z_k^T U_{k-1} z_k} U_{k-1}.$$

Then use $U_{k-1} \geqq 0$ to get $U_k \geqq 0$, and use $U_{k-1} Z_{k-1} = 0$ and $U_k z_k = 0$ to get $U_k Z_k = 0$. This completes the proof of $\mathcal{P}_k$ from $\mathcal{P}_{k-1}$, and hence of $\mathcal{P}_n$.

Use the linear independence of the vectors s_k for $k \neq 0$, together with $c^T z_k = 0$ and $c^T w_* = 1$, to show that rank $(Z_k) = k+1$, which with $U_k Z_k = 0$ gives rank $(U_k) \leqq n - k$; in particular, $U_n = 0$. Use this, $\mathcal{P}_n$, and (2.3) and (2.4) to obtain the conclusions of the theorem. □

While each iteration in Algorithm 2 refers back to the starting point x_0, a linear mapping in each iteration, using $O(n^2)$ operations, can update this reference point so that the kth iteration uses only the step $x_k - x_{k-1}$ instead of $x_k - x_0$. The next algorithm differs from Algorithm 2 just by this mapping.

ALGORITHM 3.

Input: For each integer k from 0 through n, the value $f_k \in \mathbb{R}$ and gradient $g_k \in \mathbb{R}^n$ of a cupped conic function $f: X \to \mathbb{R}$ at the kth vertex x_k of an n dimensional simplex in X, with $f_{k-1} > f_k$ for each $k > 0$.

Set $V_0 = 0 \in \mathbb{R}^{n+1 \vee n+1}$. For each k from 1 through n, set

$$s_k = x_k - x_{k-1}, \quad \rho_k = ((f_{k-1} - f_k)^2 - g_{k-1}^T s_k g_k^T s_k)^{1/2}, \quad \gamma_k = \frac{-g_{k-1}^T s_k}{f_{k-1} - f_k + \rho_k},$$

$$z_k = \begin{pmatrix} \dfrac{1}{\gamma_k} s_k \\ \dfrac{1}{\gamma_k} - 1 \end{pmatrix}, \quad y_k = \begin{pmatrix} \gamma_k g_k - g_{k-1} \\ -\gamma_k g_k^T s_k \end{pmatrix}, \quad v_k = z_k - V_{k-1} y_k,$$

and

$$D_k = \gamma_k \begin{pmatrix} I_n & -s_k \\ 0 & 1 \end{pmatrix}.$$

$$\text{If } y_k^T v_k > 0, \quad \text{set } V_k = D_k \left(V_{k-1} + \frac{v_k v_k^T}{y_k^T v_k} \right) D_k^T;$$

$$\text{else,} \quad \text{set } V_k = D_k V_{k-1} D_k^T.$$

Set $V = V_n$.

COROLLARY 1. *For each cupped conic function $f: X \to \mathbb{R}$ whose values $f_k \in \mathbb{R}$ at the $n+1$ vertices x_k of an n dimensional simplex in X satisfy $f_{k-1} > f_k$ for all $k > 0$, define $V \in \mathbb{R}^{n+1 \vee n+1}$ by Algorithm 3. If $\gamma_* > 0$ and $s_* \in \mathbb{R}^n$ are defined by*

$$\frac{1}{\gamma_*} \begin{pmatrix} s_* \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - V \begin{pmatrix} g_n \\ 0 \end{pmatrix},$$

then f has its minimum value

$$f_* = f_n + \frac{1}{2} \frac{1}{\gamma_*} g_n^T s_* = f_n - \frac{1}{2} [g_n^T, 0] V \begin{pmatrix} g_n \\ 0 \end{pmatrix}$$

at $x_n + s_*$, where $g_n \in \mathbb{R}^n$ is the gradient of f at x_n. Furthermore,

$$V \geqq 0, \quad V c_n = 0, \quad V \Lambda_n V = V \quad \text{and} \quad \Lambda_n V \Lambda_n = \Lambda_n - 2 f_* c_n c_n^T,$$

where $c_n \in \mathbb{R}^{n+1}$ and $\Lambda_n \in \mathbb{R}^{n+1 \vee n+1}$ are defined by replacing x_0 by x_n in (2.3).

Proof. For each k from 0 through n, define $c_k \in \mathbb{R}^{n+1}$ and $\Lambda_k \in \mathbb{R}^{n+1 \vee n+1}$ by replacing x_0 by x_k in (2.3). Show that for each k from 1 through n, the y_k and z_k defined in Algorithm 3 satisfy

$$c_{k-1}^T z_k = 0 \quad \text{and} \quad y_k = \Lambda_{k-1} z_k + 2(f_{k-1} - f_k) c_{k-1}$$

instead of (4.2). As in the proof of Theorem 2, define $f_* \in \mathbb{R}$ and $w_* \in \mathbb{R}^{n+1}$ by $c_0^T w_* = 1$ and $\Lambda_0 w_* = 2 f_* c_0$. Define $Z_0 = w_*$ and for each k from 1 through n, define $Z_k \in \mathbb{R}^{n+1 \times k+1}$ by $Z_k = D_k [Z_{k-1}, z_k]$, where $D_k \in \mathbb{R}^{n+1 \times n+1}$ is defined in Algorithm 3. Replace c and Λ by c_k and Λ_k in the definition of U_k and the induction hypothesis $\mathcal{P}_k$ of the proof of Theorem 2, and replace c and Λ by c_{k-1} and Λ_{k-1} in the proof of $\mathcal{P}_k$ from $\mathcal{P}_{k-1}$. Use $c_k = D_k^{-T} c_{k-1}$ and $\Lambda_k = D_k^{-T} \Lambda_{k-1} D_k^{-1}$ to complete the proof of this corollary. □

The symmetric matrices $V_k \geqq 0$ can be factored as $V_k = L_k L_k^T$ for an $L_k \in \mathbb{R}^{n+1 \times m_k}$ of rank $m_k = \operatorname{rank}(V_k) \leqq n$. These L_k are determined only to within a right multiplication by an orthogonal $m_k \times m_k$ matrix. Each L_k determines an $h_k \in \mathbb{R}^{m_k}$ and $J_k \in \mathbb{R}^{n \times m_k}$ by

$$\begin{pmatrix} J_k \\ h_k^T \end{pmatrix} = L_k, \tag{4.3}$$

and these, with $x_k \in X$, determine a collinear scaling $S_k : W_k \to X$ of $f : X \to \mathbb{R}$ by (2.1). The composite function $fS_k : W_k \to \mathbb{R}$ is quadratic with unit Hessian since $\mathcal{P}_k$ implies $L_k^T \ \Lambda \ L_k = I_m$ and $L_k^T c = 0$, or equivalently, that h_k and J_k satisfy $h_k = J_k^T a$ and $J_k^T A J_k = I_m$.

The only changes needed in Algorithm 3 to update L_k, and hence the collinear map $S_k : W_k \to X$, instead of $V_k = L_k L_k^T$, are to define an integer $m_0 = 0$, and then for each $k > 0$, to replace the update for V_k by the following update for m_k and $L_k \in \mathbb{R}^{n+1 \times m_k}$:

$$\text{If } y_k^T v_k > 0, \quad \text{set } m_k = m_{k-1} + 1 \quad \text{and} \quad L_k = D_k[L_{k-1}, v_k/(y_k^T v_k)^{1/2}];$$

$$\text{else}, \quad \text{set } m_k = m_{k-1} \quad \text{and} \quad L_k = D_k L_{k-1}.$$

5. An algorithm schemata. Instead of discarding all previous interpolations in each iteration, as in Algorithm 1 for one dimensional problems, or saving all previous interpolations, as in Algorithms 2 and 3 for cupped conic objective functions, more general algorithms need to selectively replace some old information about the objective function with new. The choice of what is to be replaced needs to take into account at least two factors. For one, information gathered in regions far from a minimizer is usually better discarded than that from nearby regions. For another, information from steps which are nearly in the same direction as the current step is usually better discarded than that from steps in quite different directions. Conflict between these desiderata arise when steps near the minimizer lie in a subspace of few dimensions, and various strategies for balancing them need to be considered. However since these considerations are not unique to algorithms using conic approximations or collinear scalings, the following schemata leaves open this choice of strategy, and only suggests how any one could be implemented.

Input: m and n, positive integers specifying the dimension m of the region in $\mathbb{R}^n$ consistent with any linear equality constraints; if there are no constraints, $m = n$;

$x_0 \in \mathbb{R}^n$, a point consistent with any constraints where the first value and gradient of the objective function will be computed;

$J_0 \in \mathbb{R}^{n \times m}$, a matrix whose m columns span all steps consistent with any constraints. If the initial conic approximation is quadratic, then the columns of J_0 are conjugate steps which if taken from the minimizer would each increase the quadratic approximation by $\frac{1}{2}$;

$h_0 \in \mathbb{R}^m$, a column vector equal to zero if the initial conic approximation is quadratic;

$\varepsilon \in \mathbb{R}$, a positive number, used in a test which stops the calculation when the minimum of the current conic approximation is within ε of the current function value; and a subalgorithm for calculating the value $f_k \in \mathbb{R}$ and gradient $g_k \in \mathbb{R}^n$ of the objective function at x_k.

Step 0. Call f_0 and g_0 at x_0.

Comment. The initial conic approximation to the objective function satisfies

$$f\left(x_0+\frac{J_0 w}{1+h_0^T w}\right)=f_0+g_0^T J_0 w+\frac{1}{2}w^T w \tag{5.1}$$

for all $w \in \mathbb{R}^m$ with $1+h_0^T w>0$. This approximation is singular at those $x \in \mathbb{R}^n$ equal to $x_0+J_0 w/h_0^T w$ for some $w \in \mathbb{R}^m$ with $h_0^T w>0$. If $g_0^T J_0 h_0<1$, it has the minimum value $f_0-\frac{1}{2}\|J_0^T g_0\|^2$ at

$$x_0-\frac{J_0 J_0^T g_0}{1-h_0^T J_0^T g_0}.$$

For each $k>0$:

Step 1_k. Set $w_k=-J_{k-1}^T g_{k-1}$. If $\frac{1}{2}w_k^T w_k<\varepsilon$, then stop.
Else, go to Step 2_k.

Comment. This simple convergence test is invariant under collinear mappings; it may be supplemented with others.

Step 2_k. Find a step $s_k=\lambda_k J_{k-1} w_k$, typically with $\lambda_k=1/(1+h_{k-1}^T w_k)$, for which the function value f_k and gradient g_k at $x_k=x_{k-1}+s_k$ satisfy

$$0>g_{k-1}^T s_k,$$
$$f_{k-1}>f_k \quad \text{and} \quad (f_{k-1}-f_k)^2>g_{k-1}^T s_k g_k^T s_k.$$

Comment. Some step s_k will satisfy the stated conditions provided the objective function has a lower bound.

Step 3_k. Set $\rho_k=((f_{k-1}-f_k)^2-g_{k-1}^T s_k g_k^T s_k)^{1/2}$,

$$\gamma_k=-g_{k-1}^T s_k/(f_{k-1}-f_k+\rho_k), \quad \text{and}$$
$$r_k=J_{k-1}^T(\gamma_k g_k-g_{k-1})-h_{k-1}\gamma_k g_k^T s_k.$$

Comment. This $\gamma_k \in \mathbb{R}$ and $r_k \in \mathbb{R}^m$ are used in the update for h_k and J_k. The γ_k is the ratio of the value of a gauge at x_k to that at x_{k-1}, and r_k is the change in the gradient of the composite function $fS_{k-1}: W \to \mathbb{R}$ resulting from the step from x_{k-1} to x_k, where S_{k-1} satisfies (2.1). If differences in function values are used to estimate gradients, then the components of r_k can be estimated directly from function differences, rather than first estimating g_k and then using it to calculate r_k.

Step 4_k. Choose a vector $v_k \in \mathbb{R}^m$ with $v_k^T v_k=2\rho_k \neq v_k^T r_k$.
Set $u_k=(v_k-r_k)/(v_k-r_k)^T v_k$,

$$h_k=\gamma_k h_{k-1}+(1-\gamma_k-\gamma_k h_{k-1}^T v_k)u_k, \text{ and}$$
$$J_k=\gamma_k J_{k-1}+(s_k-\gamma_k J_{k-1} v_k)u_k^T-s_k h_k^T.$$

Comment. The choice of the vector $v_k \in \mathbb{R}^m$ determines what old information about the objective function is to be replaced by new information. If $u_k^T v_j=0$ for some $j<k$, then all information from the jth step is kept. If s_k is nearly in the same direction as some s_j, for which information is to be kept, then v_k should be nearly in the same direction as v_j.

THEOREM 3. *If an algorithm of this type is used with a cupped conic objective function $f: X \to \mathbb{R}$, and if for each $k \leqq m$, the v_k chosen in Step 4_k makes $u_k^T v_j=0$ for all $j<k$, then the $m+1$ points x_k for $k \leqq m$ span the m dimensional affine subspace in X of*

those x for which $x - x_0$ is in the column space of J_0. The restriction of f to this affine subspace has a minimum iff $1 + h_m^T w_{m+1} > 0$, and if this is the case, this restriction has its minimum value

$$f_* = f_m - \tfrac{1}{2}\|w_{m+1}\|^2$$

at

$$x_* = x_m + \frac{J_m w_{m+1}}{1 + h_m^T w_{m+1}}.$$

Furthermore, $L_m \in \mathbb{R}^{n+1 \times m}$ defined by (4.3) *satisfies*

$$L_m^T \Lambda_m L_m = I_m \quad \text{and} \quad L_m^T c_m = 0,$$

where $c_m \in \mathbb{R}^{n+1}$ and $\Lambda_m \in \mathbb{R}^{n+1 \vee n+1}$ are defined by replacing x_0 by x_m in (2.3).

Proof. To simplify this argument, keep x_0 as a reference point, as in Algorithm 2, instead of shifting the reference point from x_{k-1} to x_k in the kth iteration, as in this algorithm and Algorithm 3; i.e., make these changes in the kth iteration:

$$s_k = x_k - x_0, \quad \rho_k = ((f_0 - f_k)^2 - g_0^T s_k g_k^T s_k)^{1/2}, \quad \gamma_k = -g_0^T s_k/(f_0 - f_k + \rho_k),$$

$$r_k = J_{k-1}^T(\gamma_k g_k - g_0) - h_{k-1}\gamma_k g_k^T s_k,$$

$$h_k = h_{k-1} + \left(\frac{1}{\gamma_k} - 1 - h_{k-1}^T v_k\right) u_k$$

and

$$J_k = J_{k-1} + \left(\frac{1}{\gamma_k} s_k - J_{k-1}^T v_k\right) u_k^T.$$

Use Corollary 3 of Theorem 1 to show that these γ_k, together with $\gamma_0 = 1$, are the values at x_k of a gauge for f. Define y_k and z_k as in Algorithm 2 and define L_k by (4.3). Show that $r_k = L_{k-1}^T y_k$,

$$L_k = L_{k-1} + (z_k - L_{k-1} v_k) u_k^T,$$

$z_k = L_k v_k$, and $v_k = L_k^T y_k$. Define $c \in \mathbb{R}^{n+1}$ and $\Lambda \in \mathbb{R}^{n+1 \vee n+1}$ by (2.3) so that y_k and z_k satisfy (4.2). Show that if y_j and z_j also satisfy (4.2), as well as $z_j = L_{k-1} v_j$, $v_j = L_{k-1}^T y_j$, and $u_k^T v_j = 0$, then L_k inherits from L_{k-1} the property that $z_j = L_k v_j$ and $v_j = L_k^T y_j$. Then use induction as in the proof of Theorem 2. Lastly, verify that the update for h_k and J_k given in the algorithm differs from that given here by just the linear mapping D_k defined in Algorithm 3. □

6. Summary and conclusions. Three ways to specify conic functions have been introduced; one given by (2.2) and used in Theorem 1, one given by (2.3) using homogeneous coordinates and used in Theorem 2, and one given by (5.1) in terms of a collinear scaling and used in the algorithm schemata of § 5. Equations (2.2) and (2.3) specify the same conic function iff their parameters satisfy (2.5), and (2.2) and (5.1) specify the same conic function iff $h_0 = J_0^T a$ and $J_0^T A J_0 = I$.

Theorem 1 and its corollaries in § 3 summarize basic properties of conic interpolations with given values and gradients at the vertices of a simplex. Corollary 4 specializes Theorem 1 to conic interpolations over line intervals, and Algorithm 1 suggests how these can be used for line searches, though safeguards must be added to make this a general purpose algorithm. Theorem 2 gives the properties of Algorithm 2, which uses $O(n^2)$ operations to update a conic interpolation after each evaluation of a function and its gradient. Algorithm 3 differs from Algorithm 2 in using only the steps $x_k - x_{k-1}$ rather than $x_k - x_0$ for making these updates. Algorithm 4 updates collinear scalings for the objective function rather than conic approximations to it, though these are equivalent, in the absence of rounding, for approximations with unique minimizers. Theorem 3 states the basic properties of the algorithm schemata of § 5 for updating collinear scalings. Specific algorithms can be obtained from this by adding rules for choosing the factor λ_k determining the magnitude of $x_k - x_{k-1}$ in Step 2, and for choosing the direction of the vector v_k in Step 4, which determines what information from previous iterations is to be replaced in the current one.

P. Bjørstad and J. Nocedal [1] have developed an algorithm for one dimensional minimization which improves upon Algorithm 1 of § 3. Theirs includes safeguards to insure stability. They prove it has an R quadratic convergence rate, when there is a neighborhood of the minimum in which f'' is positive and f''' is Lipschitz continuous, and that it has a faster convergence rate than algorithms making cubic interpolations when $f''f'''' > 2(f''')^2$.

D. Sorensen [10] has shown that an algorithm updating collinear scalings for n dimensional problems has a Q superlinear convergence rate and that it compares favorably with a widely used BFGS variable metric algorithm on a variety of standard test problems. His algorithm updates a matrix C_k corresponding to $J_k J_k^T = A_k^{-1}$ for the J of (2.1) and A of (2.2).

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