Conic Distributions and Accessible Sets

B. Langerock*
Department of Architecture, St.-Lucas
Institute for Higher Education
in the Sciences & the Arts (W&K)
Hoogstraat 51, B9000 Ghent Belgium

Abstract

Motivated by nonlinear control theory, we introduce the notion of conic distributions on a smooth manifold. We study topological and smoothness aspects of the set of accessible points associated to a conic distribution. We introduce the notion of abnormal paths and we study its relation to boundary points of the accessible set. Among others we provide sufficient conditions for the accessible set to be a maximal integral of the smallest integrable vector distribution containing the conic distribution. Under rather strong conditions, we are able to prove that the accessible set has the structure of a 'manifold with corners'.
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1 Introduction, basic definitions and motivation

The main goal of this paper is to consider topological and smoothness properties of accessible sets associated to a family of vector fields. The concept of an accessible set of a family of vector fields has been studied, amongst others, in [1, 4, 15]. We retrieve some known results in particular on the topology of accessible sets (see [4]) and we develop new techniques for studying smoothness properties of accessible sets. For instance, we are able to provide accessible sets with a smooth structure (a manifold with corners) for a 'simple polyhedral conic distribution' satisfying additional integrability conditions. Other approaches to the study of the structure of accessible sets in control systems can be found in [2, 3] (and references therein). Throughout this paper we consider families of smooth (local) vector fields. We will not adopt

*bavo.langerock@architectuur.sintlucas.wenk.be

the common assumption that controls are measurable functions. However, we hope that the relevance and generality of the framework presented becomes apparent by observing the rather wild singularities encountered in accessible sets with respect to the topological and smooth structure, even for families of smooth vector fields (see the examples below).

In this section we start with the general concept of a conic distribution and give the elementary definitions associated to such structures. The main objective of this section is to show that conic distributions are closely related to nonlinear control theory, being a major research topic in engineering sciences. Some familiarity with geometric control theory will be helpful for a better understanding of the concepts that we associate to a conic distribution, for instance the definition of abnormal paths, controllability and accessibility. Throughout this paper N is always assumed to be a smooth n-dimensional manifold (Hausdorff, second countable, C^∞) and smooth always means of class C^∞. In this paper we study families of vector fields on N and, without further mentioning, a vector field is always assumed to be smooth (possibly only locally defined and not necessarily complete). We refer the reader to [6, 14] for more details regarding definitions on compositions of (local) flows of members of such a family and to [9] for general aspects of convex cones in a linear space.

A family $\mathcal{F}$ of vector fields on N is said to be *everywhere defined* if, given any point $x \in N$, there exists an element $X \in \mathcal{F}$ such that x is contained in the domain of X. Following [6], we say that the *flow* of an ordered family of ℓ vector fields $\mathcal{X} = (X_\ell, \dots, X_1)$ is given by the map

$$(T, x) \mapsto \mathcal{X}_T(x) = \phi^\ell_{t_\ell} \circ \cdots \circ \phi^1_{t_1}(x),$$

where $\{\phi^i_t\}$ denotes the (local) flow of X_i. This map is defined for all pairs (x, T), with $x \in N$ and $T = (t_\ell, \dots, t_1) \in \mathbb{R}^\ell$, for which the composition on the right-hand side is defined. It is implicitly assumed that the composition of (local) diffeomorphisms $\phi^i_{t_i}$ is such that, for a given $T \in \mathbb{R}^\ell$, the domain of the composition $\mathcal{X}_T$ is a non-empty open subset of N. However, for notational convenience, we allow the domain to be the empty set (see [6, p. 387] for further details). For each appropriate T, the map $\mathcal{X}_T$ is a diffeomorphism from an open subset of N to another open subset of N. It is not hard to see that, if x is in the domain of all X_i, then the map $T \mapsto \mathcal{X}_T(x)$ is smooth and is defined on an open neighbourhood of $0 \in \mathbb{R}^\ell$. The subset $\mathbb{R}^\ell_+$ of $\mathbb{R}^\ell$ is assumed to be the set of all $(t_\ell, \dots, t_1)$ with $t_i > 0$.

Before introducing conic distributions, we now recall some definitions from [6] concerning ordinary linear (generalised) distributions. These definitions are then generalised in a straightforward way to 'conic distributions'.

Definition 1.1.

1. A distribution on a manifold N is a subset D of TN such that for all $x \in N$ the set $D_x = D \cap T_xN$ has the structure of a linear subspace of T_xN.

2. A distribution is differentiable if for any $v \in D_x$, there exists a (local) vector field X such that $X(x) = v$ and $X(y) \in D_y$ for all y in the domain of X (vector fields satisfying this condition are called 'vector fields in D').

3. The differentiable distribution $D(\mathcal{F})$ generated by an everywhere defined family of vector fields $\mathcal{F}$ on N is defined by

$$D_x(\mathcal{F}) = \left\{ \sum_{i=1}^{\ell} \lambda^i X_i(x) \mid \ell \in \mathbb{N}, (\lambda_\ell, \ldots, \lambda_1) \in \mathbb{R}^\ell, X_i \in \mathcal{F}, i = 1, \ldots, \ell \right\}.$$

4. The *orbit* $L_x(\mathcal{F})$ through x of the everywhere defined family of (local) vector fields $\mathcal{F}$ is the subset of N defined by

$$L_x(\mathcal{F}) = \{\mathcal{X}_T(x) \mid \ell \in \mathbb{N}, \mathcal{X} = (X_\ell, \cdots, X_1), X_i \in \mathcal{F}, i = 1, \ldots, \ell, T \in \mathbb{R}^\ell\}.$$

The results in [12, 13, 14] on integrability of generalised distributions state, among others, that the orbit $L_x(\mathcal{F})$ can be characterised as the leaf through x of the foliation determined by the smallest integrable distribution containing $D(\mathcal{F})$. In particular, this result implies that the orbits through x of two families of vector fields $\mathcal{F}$ and $\mathcal{F}'$ are identical if $D(\mathcal{F}) = D(\mathcal{F}')$. Furthermore, the smallest integrable distribution $I(\mathcal{F})$ containing $D(\mathcal{F})$ was proven to be equal to

$$I_x(\mathcal{F}) = \operatorname{span}\{(\mathcal{X}_T)_*(Y)(x) \mid x \in N, \ell \in \mathbb{N}, \mathcal{X} = (X_\ell, \ldots, X_1), T \in \mathbb{R}^\ell, Y \in \mathcal{F}\}.$$

Note that $I(\mathcal{F})$ contains the closure of $D(\mathcal{F})$ under the Lie bracket operation, i.e. it contains all possible finite iterations of Lie brackets of vector fields in $D(\mathcal{F})$ (see [14] for further details). The above definition is now repeated but replacing the linear subspaces of T_nN everywhere by convex cones.

Definition 1.2.

1. A *conic distribution* on a manifold N is a subset C of TN such that for all $x \in N$ the set $C_x = C \cap T_xN$ has the structure of a convex cone in T_xN.

2. A conic distribution is differentiable if for any $v \in C_x$, there exists a (local) vector field X such that $X(x) = v$ and $X(y) \in C_y$ for all y in the domain of X (vector fields satisfying this condition are called 'vector fields in C').

3. The differentiable conic distribution $C(\mathcal{F})$ generated by an everywhere defined family of vector fields $\mathcal{F}$ is defined as

$$C_x(\mathcal{F}) = \left\{ \sum_{i=1}^{\ell} \lambda^i X_i(x) \mid \ell \in \mathbb{N}, (\lambda_\ell, \ldots, \lambda_1) \in \mathbb{R}_+^\ell, X_i \in \mathcal{F}, i = 1, \ldots, \ell \right\}.$$

4. The *accessible set* $\mathrm{Acc}_x(\mathcal{F})$ from x of the everywhere defined family of (local) vector fields $\mathcal{F}$ is the subset of N defined by

$$\mathrm{Acc}_x(\mathcal{F}) = \{\mathcal{X}_T(x) \mid \ell \in \mathbb{N}, \mathcal{X} = (X_\ell, \cdots, X_1), X_i \in \mathcal{F}, i = 1, \ldots, \ell, T \in \mathbb{R}_+^\ell\}.$$

Keeping these definitions in mind, we first show how a geometric control problem gives rise to a family of vector fields on its configuration space. The accessible set of such a family is proven to be closely related to the concept of accessibility from control theory. Throughout this paper we always assume that the conic distributions are differentiable without further mentioning.

1.1 Nonlinear control theory as a motivating example to study conic distributions

In control theory one is interested in studying dynamical systems that admit an external (read: human) influence. To make this more precise, assume that the state space of the system whose behaviour we wish to study is represented by an n-dimensional manifold N. In standard dynamical systems theory, the 'law of motion' determining the behaviour of the system is expressed in terms of a vector field X on N. The motions $x(t)$ of the system are solutions of the system of ODE's (in a local coordinate chart (U, x^i) of N):

$$\dot{x}^i(t) = X^i(x(t)), \quad i = 1, \dots, n,$$

where X^i are the local components of X. In control theory one assumes in addition that the vector field, determining the allowable motions, depends on some additional variables $u = (u^1, \dots, u^k) \in V \subset \mathbb{R}^k$, called control variables; V is called the control domain and is an arbitrary subset of $\mathbb{R}^k$. These control variables can be modified (discontinuously) at all time: they represent the external input to the system. Typically, the control function $t \mapsto u(t)$ is allowed to be a measurable function of time [10]. Summarising, we have that the motions of the control system are solutions of the system of ODE's that, for a given (measurable) control $t \mapsto u(t) \in V$, takes the form

$$\dot{x}^i(t) = X^i(x(t), u(t)), \quad i = 1, \dots, n. \tag{1}$$

The accessible set from $x \in N$ is then defined as the set of points $y \in N$ for which there exists a control $u : [a, b] \to V$ such that $x(b) = y$, where $x(t)$ solves (1). For our purposes however, it is not necessary to consider measurable controls. In the following we will assume that a control $u(t)$ is a *smooth* function, admitting a finite number of *discontinuous jumps*, i.e. there exits ℓ increasing instants in time $t_i \in [a, b]$, $i = 1, \dots, \ell$ such that u is discontinuous at t_i, but smooth elsewhere and such that u restricted to $]t_i, t_{i+1}[$ admits a smooth extension to $[t_i, t_{i+1}]$, for all $i = 0, \dots, \ell - 1$. The existence of jumps in the control is essential from the engineering point of view. Furthermore we assume that the control domain V is an open subset of $\mathbb{R}^k$ (it should be noted however that the following arguments equally apply to the case where V is an arbitrary manifold with or without boundary).

Now, in a differential geometric context the system of ODE's in (1) can be interpreted as being equivalent to a family of (time-dependent) vector fields on N, parameterised by the set of all allowable controls $u(t)$. Then, admissible motions of the control system are concatenations of (segments of) integral curves of members of this family of vector fields (see e.g. [10, 16]). In some cases the controlled curves can be realised as concatenations of integral curves of vector fields that have no time-dependence. This is the case if all solutions $x(t)$ to (1) are immersed in N (see below). We will constrain ourselves to control systems that are determined by a family of vector fields whose members do not dependent on time. (Note that this is not an essential restriction, since any time dependent vector field induces a non-vanishing vector field on the manifold $\mathbb{R} \times N$). A possible differential geometric framework for studying control systems is given by anchored bundles [11].

Definition 1.3. An *anchored bundle* (U, ρ) consists of a vector bundle $\nu : U \to N$ with typical fibre V, and a a smooth bundle mapping $\rho : U \to TN$ which is fibred over the identity on N.

The structure of an anchored bundle allows for the definition of *ρ-admissible curve*, being the analogue for a control. Assume that we have fixed an anchored bundle (U, ρ) on N.

Definition 1.4. Let $c : [a, b] \to U$ denote a smooth curve in U, and let $\tilde{c} = \nu \circ c$ denote the projected curve in N, called the base curve of c. Then, c is called a *smooth ρ-admissible curve* if $\tilde{c}$ is an immersion and $\rho \circ c = \dot{\tilde{c}}$.

Local coordinates on N will be denoted by (x^i) and corresponding bundle adapted coordinates on U by (x^i, u^a), with $i = 1, \dots, n$ and $a = 1, \dots, k$ (k being the dimension of the typical fibre V of U). The coordinate expression of the bundle map ρ reads

$$\rho(x, u) = \gamma^i(x, u) \frac{\partial}{\partial x^i} \ . \tag{2}$$

A smooth ρ-admissible curve $c(t) = (x^i(t), u^a(t))$ locally satisfies a system of ODE's of the form (1):

$$\dot{x}^i(t) = \gamma^i(x^j(t), u^a(t)).$$

The typical fibre of the bundle $U \to N$ is to be interpreted as the domain of the control variables V.

The structure of an anchored bundle is encountered in many area's of differential geometry:

- A Poisson structure Λ on a manifold P determines a mapping $\sharp_\Lambda : T^*P \to TP$ such that the image is precisely the generalised integrable distribution whose leafs are symplectic submanifolds of P. The pair $(T^*N, \sharp_\Lambda)$ is an example of an anchored bundle.
- A Lie algebroid is, by definition, an anchored bundle with the additional property that the module of sections of $U \to N$ is equipped with a real Lie algebra, which satisfies a Leibniz condition with respect to multiplication by functions on N.

- Any regular distribution D on N is a subbundle of TN. The natural injection of this subbundle into TN makes D into an anchored bundle, whose admissible curves are precisely the set of curves tangent to the distribution.
- A sub-Riemannian structure on a manifold N is a regular distribution $i : D \hookrightarrow TN$ which is equipped with a Riemannian bundle metric, say h. The mapping $g : T^*N \to TN$, defined by $g = i \circ \sharp_h \circ i^*$ makes T^*N into an anchored bundle.

Similar to control curves, the class of ρ-admissible curves should be further extended to curves admitting (a finite number of) discontinuities in the form of certain 'jumps' in the fibres of U, such that the corresponding base curve is piecewise smooth. In order to define these "piecewise" ρ-admissible curves we first consider the composition of smooth ρ-admissible curves.

The *concatenation* of a finite number of, say ℓ, smooth ρ-admissible curves $c_i : [a_{i-1}, a_i] \to U$ for $i = 1, \ldots, \ell$, satisfying the conditions $\tilde{c}_i(a_i) = \tilde{c}_{i+1}(a_i)$ for $i = 1, \ldots, \ell - 1$, is the map $c_\ell \diamond \cdots \diamond c_1 : [a_0, a_\ell] \to U$ defined by

$$(c_\ell \diamond \cdots \diamond c_1)(t) = \begin{cases} c_1(t) & t \in [a_0, a_1], \\ \vdots & \\ c_\ell(t) & t \in \,]a_{\ell-1}, a_\ell]. \end{cases} \tag{3}$$

Note that the base curve of $c_\ell \diamond \cdots \diamond c_1$ is a piecewise smooth curve. However, in general $c_\ell \diamond \cdots \diamond c_1$ is discontinuous at $t = a_i$, for $i = 1, \ldots, \ell - 1$. The composition $c = c_\ell \diamond \cdots \diamond c_1$ is called *a piecewise ρ-admissible curve*, or simply a *ρ-admissible curve*.

We now arrive to the important notion of accessibility in control theory. Given an anchored bundle (U, ρ), the *accessible set* $\mathrm{Acc}_x(\rho)$ from a point $x \in N$, is the set of points in N that can be reached by following the base curve of a ρ-admissible curve starting in x, i.e. $y \in \mathrm{Acc}_x(\rho)$ if there exists a ρ-admissible curve $c : [a, b] \to U$ such that $\tilde{c}(a) = x$ and $\tilde{c}(b) = y$ (this corresponds to the notion of accessible set of the control system $\dot{x} = \gamma(x, u)$ as mentioned in the beginning of this section). Now, consider the family of vector fields $\mathcal{F}(\rho) = \{\rho \circ \sigma \mid \sigma \in \Gamma(\nu)\}$ on N, associated to an anchored bundle (U, ρ) (where $\Gamma(\nu)$ denotes the module of sections of $\nu : U \to N$). We prove that $\mathrm{Acc}_x(\rho) = \mathrm{Acc}_x(\mathcal{F}(\rho))$. This equality says that the notion of accessibility in geometric control theory is related to the notion of accessibility of families of vector fields.

We first show that any integral curve of a vector field in the family $\mathcal{F}(\rho)$ is the base curve of smooth ρ-admissible curve. Note that the integral curve of any vector field of the form $\rho \circ \sigma$, with σ a section of ν, is the base curve of a (smooth) ρ-admissible curve. Indeed, if $\tilde{c} : [a, b] \to N$ is an integral curve of $\rho \circ \sigma$, then $\sigma \circ \tilde{c} : [a, b] \to U$ is a ρ-admissible curve with base $\tilde{c}$. Assume that we fix $\mathcal{X} = (\rho \circ \sigma_\ell, \ldots, \rho \circ \sigma_1)$, with $\sigma_i \in \Gamma(\nu)$. Using the above correspondence, it is not hard to see that $\mathcal{X}_T(x) \in \mathrm{Acc}_x(\rho)$ for arbitrary $T \in \mathbb{R}^\ell_+$ in the domain of $\mathcal{X}$. Indeed, this follows from the fact that $\mathcal{X}_T(x)$ can be reached by concatenating ℓ smooth base curves of ρ-admissible curves, namely the integral curves of the members of $\mathcal{X}$. More explicitly, these ℓ integral curves can be defined inductively as follows, for $i = 1, \ldots, \ell$

$$\tilde{c}_i(t) := \phi^i_{\left(t - \sum_{s=1}^{i-1} t_s\right)} \left(\tilde{c}_{i-1} \left(\sum_{s=1}^{i-1} t_s \right) \right) \text{ for } t \in \left[\sum_{s=1}^{i-1} t_s, \sum_{s=1}^{i} t_s \right]. \tag{4}$$

The concatenation of these $\tilde{c}_i$'s is defined similarly as in (3). It is essential that $T \in \mathbb{R}^\ell_+$. Indeed, if $t_i < 0$ for some $i = 1, \ldots, \ell$, then the above definition would fail. One might alter the definition by allowing this situation, i.e. if $t_i < 0$ then follow the integral curve of $-\rho \circ \sigma_i$ during time $-t_i$. However, an integral curve of a vector field in $-\mathcal{F}(\rho)$ is not necessarily the base curve of a ρ-admissible curve (this might occur if ρ is a non-linear bundle map). This remark should justify Definition 1.2. Thus we have that $\mathrm{Acc}_x(\mathcal{F}(\rho)) \subset \mathrm{Acc}_x(\rho)$. To prove the reverse inclusion, remark that the base curve of any smooth ρ-admissible curve is locally an integral curve of an element of $\mathcal{F}(\rho)$ (this follows from the immersion condition), and therefore the base curve can be written as a concatenation of integral curves of vector fields in $\mathcal{F}(\rho)$.

1.2 Outline of the paper

The structure of the paper is as follows. In Section 2 we provide some examples of conic distributions, revealing the general structure of an accessible set of a conic distribution. We make several basic observations concerning the accessible set of a family of vector fields. Section 3, which is rather technical, contains the main theorems of the paper. In that section we construct a new convex cone at a point in the accessible set, having the property that it is our best 'approximation' to what could be regarded as the tangent space to the accessible set. In Section 4 we introduce the concept of abnormal paths and, using the theorems from Section 3, we prove some topological and smoothness properties of the accessible set. To conclude the paper we prove in Section 5, under rather restrictive conditions on the conic distribution, that the accessible set admits the structure of a 'manifold with corners'.

2 Examples and elementary observations

Throughout this section we assume that a family $\mathcal{F}$ is fixed on N. From the definition of $\mathrm{Acc}_x(\mathcal{F})$ one can deduce that the point x itself in general is not contained in $\mathrm{Acc}_x(\mathcal{F})$. Furthermore, the accessible sets induce a partial order relation $<_\mathcal{F}$ on N: we say that $x <_\mathcal{F} y$ if $y \in \mathrm{Acc}_x(\mathcal{F})$. The transitivity condition follows from the following straightforward property: if $y \in \mathrm{Acc}_x(\mathcal{F})$, then $\mathrm{Acc}_y(\mathcal{F}) \subset \mathrm{Acc}_x(\mathcal{F})$. We now use the construction in (4) to define an *admissible path.*

Definition 2.1. Fix an ordered family $\mathcal{X} = (X_\ell, \ldots, X_1)$ with $X_i \in \mathcal{F}$ for $i = 1, \ldots, \ell$ and an element $T = (t_\ell, \ldots, t_1) \in \mathbb{R}^\ell_+$. The piecewise smooth curve in N obtained from $\mathcal{X}_T$ by concatenating the integral curves of $X_1, \dots, X_\ell$, for times $t_1, \dots, t_\ell$, respectively, is called an *admissible path* through x associated to $\mathcal{X}_T$ and is denoted by $(\mathcal{X}, T)_x(t)$. More explicitly, the smooth pieces of the path are defined inductively, for $i = 1, \dots, \ell$, by:

$$\tilde{c}_i(t) := \phi^i_{\left(t - \sum_{s=1}^{i-1} t_s\right)} \left(\tilde{c}_{i-1} \left(\sum_{s=1}^{i-1} t_s \right) \right) \text{ for } t \in \left[\sum_{s=1}^{i-1} t_s, \sum_{s=1}^{i} t_s \right],$$

with $\{\phi^i_t\}$ the flow of X_i. Using the notations from the preceding section, we may write $(\mathcal{X}, T)_x(t) := (\tilde{c}_\ell \diamond \cdots \diamond \tilde{c}_1)(t)$.

The endpoint of the path associated to $\mathcal{X}_T$ is simply $\mathcal{X}_T(x)$. Thus, with this notation, any point in $\mathrm{Acc}_x(\mathcal{F})$ is obtained by following an admissible path starting at x.

Definition 2.2.

1. A conic distribution C is *open (resp. closed)* if, for all $x \in N$, the set C_x is an open (resp. closed) subset of $\operatorname{span} C_x$ with respect to the subset topology induced by T_xN, where $\operatorname{span} C_x$ denotes the linear subspace of T_xN generated by C_x.
2. The *rank* $\operatorname{rk} C_x$ of a conic distribution C at a point x is defined as $\operatorname{rk} C_x = \dim(\operatorname{span} C_x)$. A conic distribution is said to be of constant rank if $\operatorname{rk} C_x = \operatorname{rk} C_y$ for all $x, y \in N$.
3. A conic distribution C is called *polyhedral* if there exists a locally finite family $\mathcal{F}$ such that $C = C(\mathcal{F})$.
4. With a conic distribution we can consider the family of all vector fields $\mathcal{C}$ in C, i.e. $X \in \mathcal{C}$ iff $X(x) \in C_x$ for all x in the domain of X. The accessible set $\mathrm{Acc}_x(\mathcal{C})$ of the family $\mathcal{C}$ of vector fields in C is also denoted by $\mathrm{Acc}_x(C)$.

We now make some elementary observations. Let $\mathcal{C}$ denote the family of vector fields in the conic distribution $C(\mathcal{F})$. It should be clear that, since $\mathcal{F}$ is contained in $\mathcal{C}$, the accessible sets satisfy $\mathrm{Acc}_x(\mathcal{F}) \subset \mathrm{Acc}_x(\mathcal{C})$. In Section 4 we study the reverse of this inclusion.

From the definition of the orbit of a family of vector fields in the introduction, it follows that $\mathrm{Acc}_x(\mathcal{F})$ is a subset of the leaf $L_x(\mathcal{F})$ through x of the foliation determined by the integrable distribution $I(\mathcal{F})$. It is easily seen that, under the additional assumption that $\mathcal{F} = -\mathcal{F}$ (i.e. if the family is invariant under multiplication by -1), then for all x we have $\mathrm{Acc}_x(\mathcal{F}) = L_x(\mathcal{F})$ (note that the point x can be accessed by a nontrivial path). For instance, if we return to the anchored bundle setting, it is clear that $\mathcal{F}(\rho) = -\mathcal{F}(\rho)$ if ρ is a linear bundle mapping. In this case, the accessible set from x of the linear control system is precisely the orbit through x of the family $\mathcal{F}(\rho)$. This set is a smooth submanifold of N, i.e. it is a maximal integral submanifold of the distribution $I(\mathcal{F}(\rho))$.

Assume that we fix some $x \in N$ and that we wish to study $\mathrm{Acc}_x(\mathcal{F})$. In view of the above remarks, we only need to consider the case where N is connected and the family of vector fields $\mathcal{F}$ is such that the entire manifold N is the leaf of the corresponding integrable distribution $I(\mathcal{F})$. Throughout the remaining of this paper we therefore assume that $N = L_x(\mathcal{F})$. Summarising we then have that

$$\mathrm{Acc}_x(\mathcal{F}) \subset \mathrm{Acc}_x(\mathcal{C}) \subset N = L_x(\mathcal{F}).$$

Before giving examples, we first mention some topological issues. Let $\overline{\mathbb{R}}_+^\ell$ denote the closure of $\mathbb{R}_+^\ell$ in $\mathbb{R}^\ell$. Consider the finest topology on the set $\mathrm{Acc}_x(\mathcal{F})$ such that all maps of the form

$$\overline{\mathbb{R}}_+^\ell \setminus \{0\} \to \mathrm{Acc}_x(\mathcal{F}) : T \mapsto \mathcal{X}_T(x),$$

with $\ell \in \mathbb{N}$ and $\mathcal{X}$ an arbitrary path of ℓ vector fields in $\mathcal{F}$, are continuous. This topology is called the *topology on $Acc_x(\mathcal{F})$ generated by $\mathcal{F}$*. It is in general finer then the subset topology of $\mathrm{Acc}_x(\mathcal{F})$ w.r.t. the topology of $N = L_x(\mathcal{F})$ (note that here the topology on N coincides with the topology of N as a leaf of $I(\mathcal{F})$).

We now consider some examples of conic distributions and their accessible sets. The examples given below should give some insight in the special structure that an accessible set may have. We conclude with an example of an accessible set where the topology generated by a family of vector fields is finer than the subset topology, see Example 2.2.

Example 2.1. The first four examples are constructed on the plane: $N = \mathbb{R}^2$.

1. This example shows that an accessible set need not be of constant dimension. Consider on $\mathbb{R}^2$ the family $\mathcal{F}$ consisting of the vector fields $\partial/\partial x$, defined on the whole of $\mathbb{R}^2$, and $\partial/\partial y$ restricted to the right half plane, i.e. on $V_+ = \{(x, y) \in \mathbb{R}^2 \mid x > 0\}$. The picture shows $\mathrm{Acc}_{(x,y)}(\mathcal{F})$ for some $(x, y) \in \mathbb{R}^2 \setminus V_+$. It is not hard to see that any point in the shaded area can be accessed by concatenating the flows of these two vector fields.

2. Consider the polyhedral conic distribution defined by $\partial/\partial x$ with domain $\mathbb{R}^2$ and $\partial/\partial y$ restricted to $V_- = \{(x, y) \in \mathbb{R}^2 \mid x < 0\}$. The accessible sets are drawn in the picture for a point with $x < 0$ and for $x > 0$.

3. Here we consider the conic distribution generated by $\partial/\partial x$ and $\partial/\partial x + x\partial/\partial y$, both defined on $\mathbb{R}^2$. This example is to show that the boundary of $\mathrm{Acc}_{(x,y)}(\mathcal{F})$ need not be a smooth submanifold. The conic distribution $C(\mathcal{F})$ is full rank, except at points on the axis $x = 0$, where the conic distribution has rank 1. It is precisely at these points that the boundary is not smooth. For instance, the upper boundary of $\mathrm{Acc}_{(x,y)}(\mathcal{F})$, where (x, y) is a point in V_-, consists of the concatenation of the following integral curves: $t \mapsto (t + x, y)$ and $t \mapsto (t + x, \frac{1}{2}(t + x)^2 + y)$, respectively. The boundary is the piecewise smooth curve (recall that $x < 0$ on V_-)

$$t \mapsto \begin{cases} (t + x, y) & \text{for } t \in [0, -x] \\ (t + x, \frac{1}{2}(t + x)^2 + y) & \text{for } t \in [-x, +\infty[. \end{cases}$$

It determines a C^1 submanifold of $\mathbb{R}^2$.

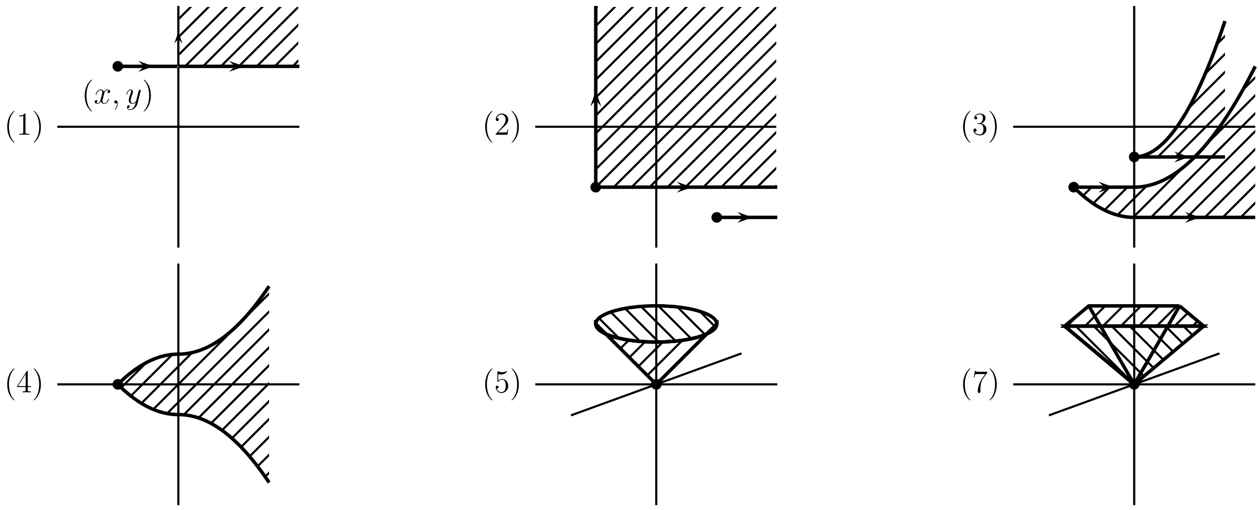
4. Another example on the plane is given by the family consisting of the vector fields $\partial/\partial x + x\partial/\partial y$ and $\partial/\partial x - x\partial/\partial y$. The accessible set through a point $(x, 0) \in V_-$ is drawn in figure (4). Again boundaries are C^1 submanifolds.

5. Consider on $\mathbb{R}^3$ the family of vector fields

$$\mathcal{F} = \{X_\theta = \partial/\partial z + \cos\theta\partial/\partial x + \sin\theta\partial/\partial y \mid \theta \in [0, 2\pi[\}.$$

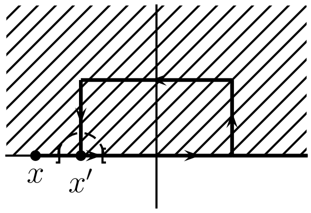
   The accessible set from the origin is drawn in figure (5).

6. Let (N, g) be a Lorentz manifold (i.e. g is a metric with signature $(+ - --)$) and such that N is equipped with a global time direction. The subset C of TN consisting of all time-like future oriented tangent vectors v such that $g(v, v) > 0$ defines an open conic distribution on N. The points in the accessible set through x of this conic distribution are precisely the points that can be reached by following a (non-singular) worldline (i.e. a curve with tangent vector $0 \neq \dot\gamma \in C$) respecting the time direction.



7. Consider a finite family of globally defined vector fields $\mathcal{F} = \{X_1, \ldots, X_\ell\}$. Figure (7) gives an example in $N = \mathbb{R}^3$, and where $\ell = 4$ and all X_i are constant. Note that polyhedral conic distributions are closed.

Example 2.2. To conclude we give an example where the subset topologyand the topology generated by the family of vector fields do not coincide. Consider the following polyhedral family $\mathcal{F}$ of four vector fields: $X_0 = \partial_x$, defined on $\mathbb{R}^2$; $X_1 = \partial_y$, defined on $]0, \infty[\times\mathbb{R}$; $X_2 = -\partial_x$ on $\mathbb{R}\times]0, \infty[$ and $X_3 = -y\partial_y$ on $\mathbb{R}^2$. The accessible set from $(x, 0)$, with $x < 0$, is the union of the upper half plane $\mathbb{R}\times]0, \infty[$ and the half straightline $I = \{(r, 0) \mid x < r \in \mathbb{R}\}$. The points in I can only be accessed by following the flow of X_0, while the points in upper half plane are accessible by combinations of integral curves of X_0, X_1, X_2 and X_3. The main issue here is that the entire plane is a leaf of the foliation induced by this family and that the topology on $\mathrm{Acc}_{(x,0)}(\mathcal{F})$ induced by the family is finer then the subset topology. Indeed, in the topology generated by $\mathcal{F}$, the set $\{(r,0) \mid |r-x'|<\delta, \delta>0\}$ is a neighbourhood of $(x',0)$ in $\mathrm{Acc}_{(x,0)}(\mathcal{F})$, with $x<x'<0$, whereas in the subset topology, any neighbourhood of $(x',0)$ should contain a half-circle with centre at $(x',0)$.



3 The variational cone and smooth submanifolds of the accessible set

Similar to the theory of generalised distributions, a naive guess for the characterisation of the infinitesimal structure of the orbit space $L_x(\mathcal{F})$ would be the subspace of TN determined by the distribution $D(\mathcal{F})$. Indeed, any tangent vector in $D_x(\mathcal{F})$ is a tangent vector to a curve in $L_x(\mathcal{F})$ through x. However, in order to find the entire set of tangent vectors to curves in the orbit through x, one has to extend the set $D_x(\mathcal{F})$ to $I_x(\mathcal{F})$. Recall that the latter is the linear subspace defined by (cf. the introduction):

$$\mathrm{span}\{(\mathcal{X}_T)_*(Y)(x) \mid \ell \in \mathbb{N}, \mathcal{X}=(X_\ell,\ldots,X_1), T\in\mathbb{R}^\ell, Y\in\mathcal{F}\}.$$

In this section we will try to extend this idea to the accessible set $\mathrm{Acc}_x(\mathcal{F})$ of a given family $\mathcal{F}$. Let $y\in\mathrm{Acc}_x(\mathcal{F})$. We start by showing that any element in $C_y(\mathcal{F})$ is a tangent vector to a curve through y contained in $\mathrm{Acc}_x(\mathcal{F})$. Subsequently we will introduce the variational cone at any point $y\in\mathrm{Acc}_x(\mathcal{F})$. This cone also consists of vectors tangent to curves in $\mathrm{Acc}_x(\mathcal{F})$, however it contains $C(\mathcal{F})$ as a subcone. Its definition is inspired on the notion of approximating or variational cones encountered in [5, 10, 16]. These cones were introduced as sets of tangent vectors to variations to the curves under investigation. However, below we follow a slightly different approach, which will be more convenient for further discussions.

Recall that throughout this paper we have assumed that a connected manifold N and a family of vector fields $\mathcal{F}$ are given such that the integrable distribution $I(\mathcal{F})$ generated by $\mathcal{F}$ equals TN, i.e. for any point $x\in N$, we have $L_x(\mathcal{F})=N$. As noted in the previous section, this is not an essential restriction.

We first show that any element of $C_y(\mathcal{F})$ is a tangent vector to a curve in $\mathrm{Acc}_x(\mathcal{F})\ni y$. Fix an element v in $C_y(\mathcal{F})$. By definition of $C(\mathcal{F})$, we know that there exists a finite, say $\ell\in\mathbb{N}$, number of vector fields $X_1, \ldots, X_\ell$ in $\mathcal{F}$, such that v can be written as $v = \sum_i \lambda_i X_i(y)$, with $\Lambda = (\lambda_\ell, \ldots, \lambda_1) \in \mathbb{R}^\ell_+$. Of course, all these vector fields are assumed to have the point y in their domain. Fix an ordering of these vector fields, say $\mathcal{X} = (X_\ell, \ldots, X_1)$, and consider the curve in $\mathrm{Acc}_x(\mathcal{F})$ given by $\epsilon \mapsto \mathcal{X}_{\epsilon\Lambda}(y)$ with $\epsilon \geq 0$. It is not hard to see that for sufficiently small ϵ, the map $\mathcal{X}_{\epsilon\Lambda}$ contains the point y in its domain. The tangent vector at $\epsilon = 0$ to this curve is precisely v, what we wanted to prove.

We now arrive at the point where we introduce the notion of variational cone to a point $y \in \mathrm{Acc}_x(\mathcal{F})$.

Fix a family $\mathcal{X}$ of ℓ vector fields in $\mathcal{F}$ such that $y = \mathcal{X}_T(x)$ for some $T = (t_\ell, \ldots, t_1) \in \mathbb{R}^\ell_+$. The flows of the vector fields X_i, $i = 1, \ldots, \ell$, constituting $\mathcal{X}$ are denoted by $\{\phi^i_t\}$. Next, consider the following family of vector fields, each member being defined on a neighbourhood of y:

$$\begin{aligned}\mathcal{V}_{\mathcal{X},T} \;&=\; \big\{(\mathcal{X}_{T'_i})_*(Y) \mid 1 \leq i \leq \ell,\ 0 \leq t'_i < t_i,\\ &\qquad Y \in \mathcal{C} \cup \{-X_i\},\ T'_i = (t_\ell, \ldots, t_{i+1}, t'_i, 0, \ldots, 0)\big\}.\end{aligned}$$

Recall that the maps $\mathcal{X}_{T'_i}$ are local diffeomorphisms. We used the notation $\mathcal{V}_{\mathcal{X},T}$ to emphasise that this family depends on the admissible path through x associated to $\mathcal{X}_T$ starting at x and with endpoint y. Take any element in $\mathcal{V}_{\mathcal{X},T}$, say $(\mathcal{X}_{T'_i})_*(Y)$, consider its flow $\{\mathcal{X}_{T'_i} \circ \psi_s \circ \mathcal{X}^{-1}_{T'_i}\}$ with $\{\psi_s\}$ the flow of Y, and evaluate it at $y = \mathcal{X}_T(x)$. Then, we have that

$$\mathcal{X}_{T'_i} \circ \psi_s \circ \mathcal{X}^{-1}_{T'_i}(y) = \phi^\ell_{t_\ell} \circ \cdots \circ \phi^i_{t'_i} \circ \psi_s \circ \phi^i_{t_i - t'_i} \circ \cdots \circ \phi^1_{t_1}(x).$$

From the above expression it follows that $s \mapsto \mathcal{X}_{T'_i} \circ \psi_s \circ \mathcal{X}^{-1}_{T'_i}(y)$ is entirely contained in $\mathrm{Acc}_x(\mathcal{F})$ for all $s \geq 0$ such that $\psi_s(\mathcal{X}^{-1}_{T'_i}(y))$ is in the domain of $\mathcal{X}_{T'_i}$. The tangent vector at $s = 0$ then equals $(\mathcal{X}_{T'_i})_*(Y)_y$. In fact, we can show that any element in the convex cone $C_y(\mathcal{V}_{\mathcal{X},T})$ in T_yN can be regarded as the tangent vector to a curve in $\mathrm{Acc}_x(\mathcal{F})$. In view of this property, which we are now going to prove, the convex cone $C_y(\mathcal{V}_{\mathcal{X},T})$ is called *the variational cone at y associated to the admissible path* $(\mathcal{X}, T)_x$. The variational cone determines an extension of $C_y(\mathcal{F})$, in the sense that it contains more vectors tangent to curves in $\mathrm{Acc}_x(\mathcal{F})$.

So, consider an arbitrary element v of $C_y(\mathcal{V}_{\mathcal{X},T})$. We wish to construct a curve in $\mathrm{Acc}_x(\mathcal{F})$ through y whose tangent at y is precisely v. By definition of $C_y(\mathcal{V}_{\mathcal{X},T})$, the tangent vector v can be written as a finite linear combination of say p elements in $\mathcal{V}_{\mathcal{X},T}$

$$(\mathcal{X}_{T'_{i_\alpha}})_*(Y_\alpha) \text{ with } T_{i_\alpha} = (t_\ell, \ldots, t_{i_\alpha+1}, t'_{i_\alpha}, 0, \ldots, 0),$$

for $\alpha = 1, \ldots, p$, with strictly positive coefficients, say $\Lambda = (\lambda_p, \ldots, \lambda_1)$. We now consider a *time-ordering* of these p vector fields, i.e. we assume that the vector fields have been rearranged such that $i_1 \leq i_2 \leq \cdots \leq i_p$ and if for some α, $i_\alpha = i_{\alpha+1}$, then we chose $t'_{i_{\alpha+1}} \leq t'_{i_\alpha}$. Such an arrangement is always possible. Assuming that this ordering is carried out, we now consider the flow of the following time-ordered family of vector fields

$$\mathcal{X}' = ((\mathcal{X}_{T'_{i_p}})_*(Y_p), \ldots, (\mathcal{X}_{T'_{i_1}})_*(Y_1)).$$

The composite flow $\{\mathcal{X}'_S\}$ is well-defined on a neighbourhood of $y = \mathcal{X}_T(x)$. It takes a tedious but straightforward analysis to see that the time-ordering guarantees that the points $\mathcal{X}'_S(\mathcal{X}_T(x))$ all belong to $\mathrm{Acc}_x(\mathcal{F})$ provided $S \in U \subset \overline{\mathbb{R}}^p_+$, with U a sufficiently small neighbourhood of $0 \in \overline{\mathbb{R}}^p_+$. Now consider the curve $\epsilon \mapsto (\epsilon\lambda_p, \ldots, \epsilon\lambda_1) \in \overline{\mathbb{R}}^p_+$ for $\epsilon \geq 0$ small enough such that $(\epsilon\lambda_p, \ldots, \epsilon\lambda_1)$ is in the domain of $\{\mathcal{X}'_S\}$. Then the tangent vector to the curve $\epsilon \mapsto \mathcal{X}'_{(\epsilon\lambda_p,\ldots,\epsilon\lambda_1)}(y)$ is precisely the tangent vector

$$\lambda_p(\mathcal{X}_{T'_{i_p}})_*(Y_p)_y + \cdots + \lambda_1(\mathcal{X}_{T'_{i_1}})_*(Y_1)_y.$$

Therefore, we conclude that $C_y(\mathcal{V}_{\mathcal{X},T})$ can indeed be regarded as a *variational cone* or an *approximating cone* to the accessible set from x at y. Under the condition that the variational cone has maximal rank, the following important property plays a fundamental role in a proof of the maximum principle (see [5]).

Lemma 3.1. *Assume that $C_y(\mathcal{V}_{\mathcal{X},T})$ has maximal rank, i.e. the cone has a non-empty interior w.r.t. the topology of T_yN. Then, given any curve $\gamma : [0,1] \to N$ whose tangent vector at $t = 0$ is in the interior of $C_y(\mathcal{V}_{\mathcal{X},T})$, there exists an $\epsilon > 0$ such that $\gamma(t) \in Acc_x(\mathcal{F})$ for all $t \in [0, \epsilon[$. Moreover, for any $t \in]0, \epsilon[$ there exists an admissible path taking x to $\gamma(t)$ such that the associated variational cone at $\gamma(t)$ equals the entire tangent space $T_{\gamma(t)}N$.*

Proof. Consider the tangent vector $v = \dot{\gamma}(0)$. Since v is in the interior of $C_y(\mathcal{V}_{\mathcal{X},T})$ by assumption, there exist n independent elements in $C_y(\mathcal{V}_{\mathcal{X},T})$ such that v is in the interior of the polyhedral cone generated by these n elements. In general, each of these n vectors can be written as a finite linear combination of elements in $\mathcal{V}_{\mathcal{X},T}(y)$ with strictly positive coefficients. We first consider the specific case where $v = \sum_i \lambda_i \tilde{Y}_i(y)$ with $\lambda_i > 0$ and $\tilde{Y}_i \in \mathcal{V}_{(\mathcal{X},T)}$, for $i = 1, \ldots, n$. Consider a time-ordering $\mathcal{Y}$ of these n vector fields $\tilde{Y}_i$. The diffeomorphism $S \in \mathbb{R}^n \mapsto \mathcal{Y}_S(y)$ is well defined on a neighbourhood of 0 in $\mathbb{R}^n$, and therefore determines a coordinate chart on a neighbourhood of y. Using similar arguments as before, the flow of the time-ordered family $\mathcal{Y}$ satisfies the condition that $\mathcal{Y}_S(y) \in \mathrm{Acc}_x(\mathcal{F})$, for S in $\overline{\mathbb{R}}^n_+$. Moreover, it is not hard to see that the image of the natural basis of $\mathbb{R}^n$ under the tangent to $S \mapsto \mathcal{Y}_S(y)$ at $0 \in \mathbb{R}^n$ is precisely the basis formed by $\tilde{Y}_i(y)$. Thus, since all components of $\dot{\gamma}(0)$ are strictly positive in this induced coordinate chart, they will remain so for sufficiently small t. Therefore, there exists a curve $t \mapsto S(t)$ in $\mathbb{R}^n_+$ such that $\gamma(t) = \mathcal{Y}_{S(t)}(y) \in \mathrm{Acc}_x(\mathcal{F})$ for $0 < t < \epsilon$, with ϵ sufficiently small.

Note that, if we fix the path through x associated to $\mathcal{Y}_S \circ \mathcal{X}_T$, then the variational cone $C_{\gamma(t)}(\mathcal{V}_{\mathcal{Y}\circ\mathcal{X},(S,T)})$ equals the entire tangent space. This follows from the fact that the path $\mathcal{Y}_S$ is generated by n independent vector fields. From the definition of the variational cone as the convex hull of $\mathcal{V}_{\mathcal{Y}\circ\mathcal{X},(S,T)}(\gamma(t))$, it then follows that the variational cone equals the entire tangent space. In the next lemma we show that this implies that $\gamma(t)$ is an interior point of the accessible set from $\gamma(t)$.

In the more general case where $\tilde{Y}_i \notin \mathcal{V}_{\mathcal{X},T}$, the vector field $\tilde{Y}_i$ can be written as a linear combination (with positive coefficients) of vector fields in $\mathcal{V}_{\mathcal{X},T}$. The above arguments can be repeated in this situation with only minor modifications, however, the notation becomes rather involved. Therefore we only consider a simple specific case to show the changes that may be necessary. Assume that $\tilde{Y}_1 = \zeta^1 \tilde{Z}_1 + \zeta^2 \tilde{Z}_2$ and $\tilde{Y}_i = \tilde{Z}_{i+1} \in \mathcal{V}_{\mathcal{X},T}$, for $i = 2, \ldots, n$. In order to apply the above idea, one has to construct a coordinate chart. For that purpose we use the flow of a time-ordering of the family $\mathcal{Z} = (\tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3, \ldots, \tilde{Z}_{n+1})$. By definition this flow takes its arguments in $\mathbb{R}^{n+1} \times N$. One obtains a map from a neighbourhood of 0 in $\mathbb{R}^n$ to N by evaluating this flow in y and by assuming that the time-arguments of the flows of $\tilde{Z}_1$ and $\tilde{Z}_2$ in this time-ordering are identical. It is not hard to see that this map is a diffeomorphism (the tangent to this map at $0 \in \mathbb{R}^n$ is non degenerate and maps the standard basis onto the basis spanned by $\tilde{Y}^i(y)$ in T_yN) and satisfies all properties needed to repeat the above arguments. This should be sufficient to prove the general situation. □

Lemma 3.2. *If $C_y(\mathcal{V}_{\mathcal{X},T}) = T_yN$, then y is an interior point of $Acc_x(\mathcal{F})$ w.r.t. the topology of N, i.e. y has a neighbourhood in N, entirely contained in $Acc_x(\mathcal{F})$.*

Proof. Fix n independent elements $\tilde{Y}^i(y)$ of $C_y(\mathcal{V}_{\mathcal{X},T})$ such that $(\mathcal{X}_T)_* X_1(y)$ is in the interior of the polyhedral cone generated by these vector fields $\tilde{Y}_i(y)$. (Again we will assume that $\tilde{Y}_i \in \mathcal{V}_{\mathcal{X},T}$ for notational convenience. All arguments in this proof are easily extended to the more general situation). Let $\mathcal{Y}$ denote the composite flow associated to the time-ordered composition of these vector fields $\tilde{Y}_i$. Again, since all $\tilde{Y}_i$ are defined on a neighbourhood of y, the map $\mathbb{R}^n \to N : S \mapsto \mathcal{Y}_S(y)$ is a diffeomorphism on a neighbourhood U of the origin $0 \in \mathbb{R}^n$.

Now, consider the map $S \mapsto \overline{\mathcal{Y}}_S(y)$ defined by $\overline{\mathcal{Y}}_S(y) = (\mathcal{Y}_{-S})^{-1}(y)$. This map is well defined for any S in a neighbourhood of the origin $0 \in \mathbb{R}^n$ and therefore determines a coordinate chart. Indeed, $\{\overline{\mathcal{Y}}_S\}$ is the flow associated to, what one might call, a reversed time-ordering of the family of vector fields in the ordered family $\mathcal{Y}$. We will now repeat the arguments used in the previous lemma on the diffeomorphism given by $S \mapsto \overline{\mathcal{Y}}_S(y)$.

Let $T' = (t_\ell, \ldots, t'_1) \in \mathbb{R}^n$ for some $0 \leq t'_1 < t_1$, and compute the tangent vector to the curve through y given by $\epsilon \mapsto \mathcal{X}_{T'} \circ \phi^1_{-\epsilon} \circ \mathcal{X}^{-1}_{T'}(y)$ at $\epsilon = 0$. It is not hard to see that this vector equals $-((\mathcal{X}_{T'})_* X_1)(y) = -((\mathcal{X}_T)_* X_1)(y)$. Since $((\mathcal{X}_T)_* X_1)(y)$ is in the interior of the polyhedral cone generated by $\tilde{Y}_i(y)$, we may conclude that in the coordinate chart determined by the diffeomorphism $S \mapsto \overline{\mathcal{Y}}_S(y)$, the components of the tangent vector $-((\mathcal{X}_{T'})_* X_1)(y)$ are all strictly negative. Using similar argument as in the previous lemma, we have that, for any ϵ small enough, there exists an $S' \in \mathbb{R}^n_+$ such that $\mathcal{X}_{T'} \circ \phi^1_{-\epsilon} \circ \mathcal{X}^{-1}_{T'}(y) = \overline{\mathcal{Y}}_{-S'}(y)$. Now, some elementary manipulations imply that $\mathcal{Y}_{S'} \circ \mathcal{X}_{T'} \circ \phi^1_{-\epsilon} \circ \mathcal{X}^{-1}_{T'}(y) = y$. By restricting ϵ if necessary such that the condition $0 < \epsilon < t_1 - t'_1$ is fulfilled, we have that the composite $\mathcal{Y}_S \circ \mathcal{X}_{T'} \circ \phi^1_{-\epsilon} \circ \mathcal{X}^{-1}_{T'}(y)$ is in $\mathrm{Acc}_x(\mathcal{F})$ for all S in a neighbourhood U of S'. From the fact that $S \mapsto \mathcal{Y}_S \circ \mathcal{X}_{T'} \circ \phi^1_{-\epsilon} \circ \mathcal{X}^{-1}_{T'}(y)$ is (locally) a diffeomorphism, we thus found a neighbourhood of y entirely contained in $\mathrm{Acc}_x(\mathcal{F})$. □

Let P denote a submanifold of N containing x and consider the subfamily $\mathcal{F}_P$ of $\mathcal{F}$ consisting of vector fields in $\mathcal{F}$ that are everywhere tangent to P. The set $\mathrm{Acc}_x(\mathcal{F}_P)$ is a subset of $\mathrm{Acc}_x(\mathcal{F})$. Applying the previous theorem on the submanifold P, equipped with the family $\mathcal{F}_P$, we may conclude that if the variational cone at $y \in P$ equals TP, then a neighbourhood of y in P is entirely contained in $\mathrm{Acc}_x(\mathcal{F})$. This fact might allow one to study boundary aspects of accessible sets. This is left for future work however. Note that this condition, namely that the variational cone at $y \in P$ of the family $\mathcal{F}_P$ equals T_yP, implies that T_yP is entirely contained in $C_y(\mathcal{V}_{\mathcal{X},T})$.

Remark 3.3. Intuitively one could define the 'tangent cone' $C_y\mathrm{Acc}_x(\mathcal{F})$ to the accessible set $\mathrm{Acc}_x(\mathcal{F})$ at the point y as the following union of variational cones:

$$C_y\mathrm{Acc}_x(\mathcal{F}) := \bigcup_{\mathcal{X},T} C_y(\mathcal{V}_{X,T}),$$

with the union taken over all admissible paths from x to y. At this point, we were not able to introduce some kind of smooth manifold structure with 'singular points' that is broad enough to allow the 'tangent space' at a point to be an arbitrary convex cone. In Section 5 we provide rather restrictive conditions on the family $\mathcal{F}$ such that the accessible set can be given the structure of a manifold with corners. The tangent cone at a corner point then coincides with $C_y\mathrm{Acc}_x(\mathcal{F})$.

4 Abnormal paths and duality

We use the same assumptions on N and $\mathcal{F}$ as in the preceding section. At every point x we can consider the dual cone to C_x, which is denoted by C_x^* (see [9] for a definition of duality). Roughly speaking, it contains all 'hyperplanes' in T_xN such that the cone C_x is entirely contained in one half-space determined by the hyperplane. Since the rank of a conic distribution is clearly a lower semicontinuous function, the dual cone $C^* \subset T^*N$ is in general not 'differentiable'. However, if C is a constant rank conic distribution, C^* determines what could be called a 'Pfaffian conic distribution'. Sections along admissible paths that are contained in C^* will play an important role when considering abnormal paths and regularity of the accessible set. We start with a definition of an abnormal path, which is inspired on the notion of abnormal extremals as encountered in the study of the maximum principle [5]. This correspondence is made more precise later on, see Remark 4.3.

Definition 4.1. An admissible path $(\mathcal{X}, T)_x$ through x and with endpoint y, is called *abnormal* if $C_y(\mathcal{V}_{\mathcal{X},T}) \neq T_yN$.

Note that a point y may be accessed from a given point x both by normal and by abnormal paths. From Lemma 3.2 we know that points accessible by normal paths are interior points of the accessible set (w.r.t. the topology on N). So the endpoints of abnormal paths are possible candidates for boundary points of the set $\mathrm{Acc}_x(\mathcal{F})$ (the boundary of set A in N is defined as $A \setminus \mathrm{int}(A)$).

Throughout the next paragraph we show that a path is abnormal iff there exists a one-form along that path satisfying a certain system of ODE's. In order to determine this system of ODE's, we first consider the specific case of an integral curve $t \mapsto \phi_t(x)$ through x of a vector field X with flow $\{\phi_t\}$ and let $\eta(t)$ denote a section of T^*N along this integral curve, i.e. $\eta(t) \in T^*_{\phi_t(x)}N$. We define

$$\mathcal{L}_X \eta(t) = \left.\frac{d}{ds}\right|_0 T^*\phi_s(\eta(t+s)) \ ,$$

where $T^*\phi_s : T^*_xN \to T^*_{\phi_{-s}(x)}N$ denotes the dual map to the tangent of ϕ_s. We say that $\eta(t)$ is *Lie transported* if $\mathcal{L}_X\eta(t) = 0$ for all t. This is equivalent to saying that $\eta(t) = T^*\phi_s(\eta(t+s))$ for all s for which the right-hand side is well defined. The system of differential equations on the components of $\eta(t)$ is locally expressed by:

$$\dot{\eta}_i(t) = -\eta_j(t)\frac{\partial X^j}{\partial x^i}(x(t)) \ . \tag{5}$$

From this it should be clear that this Lie derivation along an integral curve of a vector field essentially depends on the Jacobian of the vector field, i.e. on its first jet prolongation (cf. [5] where this operator was introduced as a 'parallel transport operator' of a generalised connection). Now, the above definition can be extended to concatenations of integral curves, i.e. to paths associated to ordered families of vector fields. In particular, let $\mathcal{X} = (X_\ell, \ldots, X_1)$ denote such an ordered family of vector fields, and fix some $T = (t_\ell, \ldots, t_1) \in \mathbb{R}^\ell_+$. Consider the path $(\mathcal{X}, T)_x : [0, \sum_i t_i] \to N$ through x. Consider an arbitrary element $\eta_i \in T^*_xN$ and Lie transport it along the integral curve of X_1 through x. The endpoint of this smooth curve in T^*N is now taken as the initial point for the Lie transportation along the second part of the path associated to $\mathcal{X}_T$ through x, namely the integral curve determined by X_2. Continuing this procedure until we reach the endpoint of the path, yields a *piecewise smooth* curve $\eta(t)$ in T^*N along the path associated with $\mathcal{X}_T$. We say that $\eta(t)$ is *Lie transported* along the path $\mathcal{X}_T$ through x and we formally denote this by $\mathcal{L}_\mathcal{X}\eta(t) = 0$. Next, consider two piecewise smooth curves $T(t)$ and $T'(t)$ in $\mathbb{R}^\ell$ with $t \in [0, \sum_{i=1}^\ell t_i]$, defined as follows

$$\begin{aligned}&\text{if } t = t'_i + t_{i-1} + \cdots + t_1, \text{ with } 0 < t'_i \leq t_i, \text{ then } T(t) = (0, \ldots, 0, t'_i, t_{i-1}, \ldots, t_1)\\ &\text{if } t = t_\ell + \cdots + t_{i+1} + t'_i, \text{ with } 0 < t'_i \leq t_i, \text{ then } T'(t) = (t_\ell, \ldots, t_{i+1}, t'_i, 0, \ldots, 0).\end{aligned}$$

Using these functions, it is easily seen that a piecewise smooth section $\eta(t)$ of T^*N is *Lie transported* along the path $(\mathcal{X}, T)_x(t)$ iff we can write: $\eta(t) = T^*\mathcal{X}_{T'(t)}(\eta(\sum_i t_i))$ or $\eta(0) = T^*\mathcal{X}_{T(t)}(\eta(t))$ for any t.

The following theorem gives a characterisation of abnormal paths in terms of solutions to Lie transported one-forms. We refer to [9] for a definition of support hyperplanes of a convex cone.

Theorem 4.1. *A path $(\mathcal{X}, T)_x(t)$ taking x to y is abnormal iff there exists a non-trivial piecewise smooth section $\eta(t)$ of T^*N along the path $(\mathcal{X}, T)_x(t)$, such that,*

1. *$\mathcal{L}_{\mathcal{X}}\eta(t) = 0$, i.e. $\eta(t)$ is Lie transported along $(\mathcal{X}, T)_x(t)$ in the sense described above;*
2. *$\eta(t) \in C^*_{(\mathcal{X},T)_x(t)}$ determines a support hyperplane for the tangent to the path at time t of the cone $C(\mathcal{F})$.*

Proof. We first prove that the abnormality of the path implies (1) and (2).

From the definition of abnormality, we know that $C_y(\mathcal{V}_{\mathcal{X},T}) \neq T_yN$, i.e. there exists a non-zero element η_f in the dual cone $C^*_y(\mathcal{V}_{\mathcal{X},T})$. We consider its Lie transportation along the path $(\mathcal{X}, T)_x(t)$, i.e. using the notations from above, we have a non-trivial Lie transported one-form $\eta(t) = T^*\mathcal{X}_{T'(t)}(\eta_f)$. It remains to be checked that it satisfies (2). The elements, generating the variational cone at y take the following form: $T\mathcal{X}_{T'(t)}(v)$ with $v = Y(\mathcal{X}^{-1}_{T'(t)}(y))$ for $Y \in \mathcal{F}$ or v equals the negative of the tangent vector to the path at time t. Therefore, $0 \geq \langle \eta_f, T\mathcal{X}_{T'(t)}(v)\rangle$. The right-hand side of this inequality equals

$$\langle T^*\mathcal{X}_{T'(t)}(\eta_f), v\rangle = \langle \eta(t), v\rangle.$$

From this, one easily deduces (2). By reversing these arguments, it is not hard to see that (1) and (2) imply abnormality of the path. □

Remark 4.2. Condition (2) can be replaced by

3. $\eta(t) \in C^*_{(\mathcal{X},T)_x(t)}(\mathcal{V}_{\mathcal{X},T_t})$, where $(\mathcal{X}, T_t)$ is the path associated to $\mathcal{X}$ and $T_t = T(t)$.

Note that (3) implies (2), since C is a subcone of the variational cone and since the variational cone contains the negative of the vector fields last followed along the path (this implies that $\eta(t)$ annihilates this vector field, i.e. $\eta(t)$ is a support one-form). The other direction follows from the techniques used to prove that (2) implies (3), but this time applied not on the entire path, but on the piece $(\mathcal{X}, T_t)_x$ of the path $(\mathcal{X}, T)_x$.

Remark 4.3. If the conic distribution is related to a control system, the second condition is equivalent to saying that a certain function (usually referred to as the Hamiltonian $u \mapsto H = \eta_i\rho^i(x, u)$) attains a global maximum at $u(t)$ when letting the control variable vary. Together with the system of ODE's in (5) we retrieve the necessary condition for abnormal extremals provided by the maximum principle, see [10]. This observation should justify Definition 4.1 (typically, paths for which the variational cone is not degenerate are called *extremal* instead of abnormal extremal). We refer to [5] where a more detailed characterisation is given for abnormal extremals from optimal control theory in terms of variational cones. The proof that there actually exist non-trivial abnormal minimisers can be found in [8]. Below we are interested in these paths since they are closely related to paths taking us to boundary points of the accessible set.

In the remaining part of this section we will use Lemma 3.2 and Theorem 4.1 to show some general results on the structure of the accessible set.

We can conclude from the above that if any point in $\mathrm{Acc}_x(\mathcal{F})$ can be accessed by a normal path, then $\mathrm{Acc}_x(\mathcal{F})$ is an open submanifold of N. In particular, this implies that in the more general case, where the integrable distribution $I(\mathcal{F})$ does not equal TN, the accessible set $\mathrm{Acc}_x(\mathcal{F})$ through x is a maximal integral submanifold of the distribution $I(\mathcal{F})$ through x. A sufficient condition for any path to be normal is formulated in the next corollary. A *conic distribution of maximal rank* is a conic distribution for which the rank equals the dimension of N at each point.

Corollary 4.4. *A family of vector fields $\mathcal{F}$ generating an open conic distribution $C(\mathcal{F})$ of maximal rank does not admit abnormal paths. Therefore, its accessible set is an open submanifold of N.*

Proof. This follows easily from the fact that along an abnormal path, the tangent vector at each point belongs to the boundary of $C(\mathcal{F})$ (i.e. from Theorem 4.1 we know that there exists a supporting hyperplane for that tangent vector in $C(\mathcal{F})$, i.e. the tangent vector is not in the interior). This is impossible since $C(\mathcal{F})$ is assumed to be open and of maximal rank. □

The subsequent corollary provides information on the boundary points of the accessible set.

Corollary 4.5. *Each path reaching a boundary point of the accessible set $Acc_x(\mathcal{F})$ from x is abnormal. Such a path is constructed by concatenating integral curves of vector fields in $\mathcal{F}$ that are in the boundary of $C(\mathcal{F})$ (w.r.t. the topology of TN).*

Proof. Assume that y is a boundary point of $\mathrm{Acc}_x(\mathcal{F})$. This implies that every admissible path from x to y is abnormal, i.e. the variational cones are proper cones in T_yN. Fix a Lie transported one-form $\eta(t)$ along such an abnormal path satisfying condition (2) from Theorem 4.1. We know that $\eta(t)$ determines a supporting hyperplane for the tangent vector to the path at time t. In particular, this implies that the tangent vector is in the boundary of $C(\mathcal{F})$. In fact Theorem 4.1 implies that the tangent vector at time t is in the boundary of $C(\mathcal{V}_{\mathcal{X},T_t})$, where $T_t = T(t)$. □

In the next theorem we give a partial answer to the question whether the accessible sets of two different families of vector fields generating the same conic distribution coincide.

Theorem 4.6. *Consider a conic distribution $C(\mathcal{F})$ of maximal rank. Then we have that*

1. *$cl(Acc_x(\mathcal{F})) = cl(Acc_x(C(\mathcal{F})))$;*
2. *$\mathrm{int}(Acc_x(\mathcal{F})) = \mathrm{int}(Acc_x(C(\mathcal{F}))) = Acc_x(\mathrm{int}\ C(\mathcal{F}))$.*

(where the interior and closure are taken w.r.t. the topology of N)

Proof. Note that int $C(\mathcal{F})$ is a smooth conic distribution of maximal rank. For notational convenience we write $C = \text{int } C(\mathcal{F})$. Consider any point y in $\text{Acc}_x(C)$ and fix an admissible path determined by a family of vector fields in C, taking x to y. The first vector field being followed is, as usual denoted by X_1. Since X_1 is in the interior of $C(\mathcal{F})$, we know from Lemma 3.1 that $\phi_t^1(x)$ is in the interior of $\text{Acc}_x(\mathcal{F})$ for t sufficiently small. Next, we show that the entire path is contained in $\text{Acc}_x(\mathcal{F})$. Consider the restriction of X_1 to the open submanifold int $(\text{Acc}_x(\mathcal{F}))$. Assume that $\phi_t^1(x) \in \text{int } (\text{Acc}_x(\mathcal{F}))$ for all $t < \epsilon \leq t_1$ and that $z = \phi_\epsilon^1(x) \notin \text{int Acc}_x(\mathcal{F})$. Using similar arguments as in Lemma 3.2, we may conclude from the fact that $-X_1(z)$ is in the interior of $C_z(-\mathcal{F})$, there exists a time $t < \epsilon$ such that z is in the interior of the accessible set from $\phi_t^1(x)$ w.r.t. the family $\mathcal{F}$. This contradicts the previous assumption. Thus the entire integral curve of X_1 starting at x is contained in $\text{Acc}_x(\mathcal{F})$. Continuing this way with the other vector fields inducing the path from x to y, allows us to conclude that $\text{Acc}_x(C) \subset \text{int } (\text{Acc}_x(\mathcal{F}))$.

The inclusions $\text{Acc}_x(C) \subset \text{int}(\text{Acc}_x(\mathcal{F})) \subset \text{Acc}_x(\mathcal{F}) \subset \text{Acc}_x(C(\mathcal{F}))$ are shown to hold. On the other hand, one can prove that any point $y \in \text{Acc}_x(C(\mathcal{F}))$ which is reached by a path associated to vector fields in $C(\mathcal{F})$ (i.e. vector fields that might be in the boundary of $C(\mathcal{F})$), belongs to the closure of $\text{Acc}_x(C)$. To show this, we consider the case where the path under consideration is induced by one vector field, say X_1 with flow $\{\phi_t^1\}$ and $y = \phi_{t_1}^1(x)$ for some $t_1 > 0$. Fix any vector field Y in C whose domain coincides with that of X_1 (this is not an essential restriction), and consider the flow $\{\phi_t^s\}$ of $(1-s)X_1 + sY$. For s sufficiently small the point x is in the domain of $\phi_{t_1}^s$ and the point $\phi_{t_1}^s(x)$ is contained in $\text{Acc}_x(C)$ since $(1-s)X_1+sY$ is a vector field in the interior of $C(\mathcal{F})$. Therefore $\lim_{s\to 0} \phi_t^s(x) \in \text{cl}(\text{Acc}_x(C))$. Thus we also have $\text{Acc}_x(C(\mathcal{F})) \subset \text{cl}(\text{Acc}_x(C))$. This proves (1).

We now prove that any point in the interior of $\text{Acc}_x(C(\mathcal{F}))$ is in $\text{Acc}_x(C)$. Let y be an interior point of $\text{Acc}_x(C(\mathcal{F}))$. Consider n (ordered) independent vector fields in C defined on a neighbourhood of y. The map $T \mapsto \mathcal{Y}_T(y)$ defines a diffeomorphism on a neighbourhood V of 0 in $\mathbb{R}^n$ to N. The intersection of $\mathcal{Y}_{V\cap\mathbb{R}^n_-}(y)$ with $\text{Acc}_x(C)$ is nonempty, since if this were the case, the set $\mathcal{Y}_{V\cap\mathbb{R}^n_-}(y)$, which is an open subset of $\text{Acc}_x(C(\mathcal{F}))$, would not be contained in the closure of $\text{Acc}_x(C)$, contradicting the previous result. Therefore, fix such a point $z = \mathcal{Y}_S(y) \in \mathcal{Y}_{V\cap\mathbb{R}^n_-}(y)\cap\text{Acc}_x(C)$. We may then write that $\overline{\mathcal{Y}}_{-S}(z) = y$ where $-S \in \mathbb{R}^n_+$ and $\overline{\mathcal{Y}}$ is the composite flow of the vector fields in $\mathcal{Y}$ in a reversed order, which shows that $y \in \text{Acc}_x(C)$. □

The above theorem guaranties that, roughly speaking, families of vector fields generating the same conic distribution have equal accessible sets (up to boundaries). This can be seen as an analogue of what is known from [12, 13, 14], namely that two families of vector fields have equal orbits if their associated distributions are equal. We will come back to the implications of this result in the discussion at the end of this paper.

We also want to mention here that the above results are proven under the conditions that $C(\mathcal{F})$ has maximal rank in N, where N equals the orbit of $\mathcal{F}$ through some point in N. In the more general where $N \neq L_x(\mathcal{F})$, this condition says that the distribution $D(\mathcal{F})$ generated by the family $\mathcal{F}$ is assumed to be integrable.

5 Simple polyhedral conic distributions

The main purpose of this section is to show that the set $\mathrm{Acc}_x(\mathcal{F})$ can be given the structure of a submanifold with corners if we impose rather strong conditions on the family of vector fields $\mathcal{F}$. We were not able to give the accessible set a smooth structure in the case of a general polyhedral conic distribution (it should be clear from Section 3 that more general structures may appear). We leave this aspect for future work. Below we will introduce some constraints on the family $\mathcal{F}$ that allow us to prove that the variational cone is independent of the admissible path used to define it. This independence allows us to regard the variational cone as the tangent space to $\mathrm{Acc}_x(\mathcal{F})$ with respect to some smooth structure. The structure of the variational cone at a boundary point will allow us to define a notion of "depth" of this boundary point, i.e. the depth of this point as a corner w.r.t. $\mathrm{Acc}_x(\mathcal{F})$ as a manifold with corners. We first give a brief description of the notion of manifold with corners, and we refer to [7] for a more detailed discussion on this matter.

We first fix some notations. Let $\mathbb{R}^n_k$ denote the cartesian product of $n-k$ copies of $\mathbb{R}$ and k copies of $\overline{\mathbb{R}}_+ = [0,\infty[$, i.e. $\mathbb{R}^n_k = \overline{\mathbb{R}}^k_+ \times \mathbb{R}^{n-k}$. On $\mathbb{R}^n_k$ we consider the subset topology relative to the standard topology on $\mathbb{R}^n$. A function f on an open subset U of $\mathbb{R}^n_k$ is called smooth if there exists an extension of f to a smooth function F on an open subset U' of $\mathbb{R}^n$ such that $U' \cap \mathbb{R}^n_k = U$ and $F|_U = f$.

Consider two open subsets U_1 and U_2 of $\mathbb{R}^n_{k_1}$ and $\mathbb{R}^n_{k_2}$ respectively. A map $f : U_1 \to U_2$ is a diffeomorphism if there exists open subsets U_1' and U_2' of $\mathbb{R}^n$ with $U_i' \cap \mathbb{R}^n_{k_i} = U_i$, $i = 1, 2$ and a diffeomorphism $F : U_1' \to U_2'$ with $F|_{U_1} = f$. We are now ready to define a manifold with corners.

Let N be a paracompact Hausdorff topological space. A chart (U, ϕ) on N is a map $\phi : U \to \mathbb{R}^n_k$ which is a homeomorphism from an open set $U \subset N$ to an open subset of $\mathbb{R}^n_k$. Two charts (ϕ_i, U_i) are said to be compatible if $\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$ is a diffeomorphism. Similarly to the standard definition of a manifold, an atlas of N is a family of compatible charts covering N. A smooth structure is a maximal atlas.

The tangent space to a manifold with corners can be defined similarly as in the standard manifold setting. One distinguishes between inward and outward pointing tangent vectors, i.e. tangent vector are inward pointing if, in a local coordinate neighbourhood (U, ϕ), they are contained in $\mathbb{R}^n_k \supset U$. An inward pointing vector field is then defined as a section of the tangent bundle, whose values at every point are inward pointing. It can be proven that the flow of such an inward pointing vector field is well-defined on a manifold with corners. It should be clear from the above definition how the notion of a smooth map between two manifolds with corners should be defined.

We are now ready to formulate the conditions referred to in the beginning of this section. We consider polyhedral conic distributions of full rank defined by a family $\mathcal{F}$ of $n = \dim N$ independent global vector fields. We impose, in addition, some 'integrability' conditions on these vector fields in $\mathcal{F}$:

1. for any two vector fields $X_i, X_j \in \mathcal{F}$ we have that $[X_i, X_j]$ is contained in the cone $C(-X_i, X_j)$ and
2. for any point $y \in \mathrm{Acc}_x(\mathcal{F})$ there exists a 'maximal' path $(\mathcal{X}, T)_x$ to y, in the sense that given any other path $(\mathcal{X}', T')_x$ to y then the family of vector fields in $\mathcal{X}'$ is contained in the family of vector fields in $\mathcal{X}$.

Note that (1) in particular implies that the distribution associated with any subset of $\mathcal{F}$ is integrable and that (2) does not imply that a 'maximal' path $(\mathcal{X}, T)_x$ is unique, it is only determined up to an ordering of $\mathcal{X}$. We then say the conic distribution $C(\mathcal{F})$ is a simple polyhedral conic distribution.

Theorem 5.1. *The accessible set of a simple polyhedral conic distribution $C(\mathcal{F})$ can be given the structure of a submanifold of N with corners. The topology of $Acc_x(\mathcal{F})$ as a manifold with corners is equal to the topology on $Acc_x(\mathcal{F})$ generated by $\mathcal{F}$.*

Proof. The proof of this theorem is based on the ideas developed in the theory of maximal integral submanifolds of an integrable distribution [6]. It is our goal to prove that through any point $y \in \mathrm{Acc}_x(\mathcal{F})$ there exists a smooth 'submanifold with corners' such that its image set is entirely contained in $\mathrm{Acc}_x(\mathcal{F})$. Given any two such submanifolds with corners, we then show that they are locally diffeomorphic at points of intersection. This implies that chart of the different submanifolds are compatible and that they ultimately define a smooth structure on the accessible set, in the sense that it is a submanifold with corners of N.

We start by constructing a submanifold with corners through a point $y \in \mathrm{Acc}_x(\mathcal{F})$. For that purpose we have to compute the variational cone at y. Denote a maximal path $(\mathcal{X}, T)_x$ through x with $\mathcal{X} = (X_{i_1}, \ldots, X_{i_k})$ such that $y = \mathcal{X}_T(x)$ and $T \in \mathbb{R}^k_+$. We first show the variational cone associated to two different maximal paths coincide. From condition (2) it follows that y is only accessible by following admissible paths constructed by concatenating integral curves of vector fields in $\mathcal{F}_I = \{X_{i_1}, \ldots, X_{i_k}\}$ where $I = \{i_1, \ldots, i_k\}$. Indeed, assume that the subfamily $\mathcal{F}_J$ with $J = \{j_1, \ldots, j_\ell\}$ induces another maximal path taking x to y. Then (2) implies that $I = J$. Next, we show that the variational cone $C_y(\mathcal{V}_{\mathcal{X},T})$ is 'path independent' by showing that, in a specific basis of T_yN constructed independently of the ordering of the vector fields $\mathcal{F}_I$, it equals $\mathbb{R}^n_{n-k}$ for some fixed k. Let us make this statement more precise: if we can show that $C_y(\mathcal{F} \cup -\mathcal{F}_I) = C_y(\mathcal{V}_{\mathcal{X},T})$, then the basis

$$\mathcal{F} = \{X_{i_{k+1}}, \ldots, X_{i_n}, X_{i_1}, \ldots, X_{i_k}\}$$

which identifies T_yN with $\mathbb{R}^n$, will identify the subset $C_y(\mathcal{V}_{\mathcal{X},T})$ of T_yN with $\mathbb{R}^n_{n-k}$. We first show that $C^*_y(\mathcal{F} \cup -\mathcal{F}_I) = C^*_y(\mathcal{V}_{\mathcal{X},T})$. From the definition of the variational cone, we immediately have that $C_y(\mathcal{F} \cup -\mathcal{F}_I) \subset C_y(\mathcal{V}_{\mathcal{X},T})$, implying that $C^*_y(\mathcal{V}_{\mathcal{X},T}) \subset C^*_y(\mathcal{F} \cup -\mathcal{F}_I)$. Indeed, using the notations from Section 3, we know that $-(\mathcal{X}_{T_{i_\alpha}})_*(X_{i_\alpha})$ is in $\mathcal{V}_{\mathcal{X},T}$ for all $\alpha = 1, \ldots, k$ by definition of $\mathcal{V}_{\mathcal{X},T}$. From the integrability conditions on $\mathcal{F}$, it follows that $\mathrm{span}(\mathcal{F}_I) = \mathrm{span}\{(\mathcal{X}_{T_{i_\alpha}})_*(X_{i_\alpha}) \mid \alpha = 1, \ldots, k\}$. We now prove the reverse inclusion $C^*_y(\mathcal{F} \cup -\mathcal{F}_I) \subset C^*_y(\mathcal{V}_{\mathcal{X},T})$. Consider any $\eta \in C^*_y(\mathcal{F} \cup -\mathcal{F}_I)$. We have to prove that $\langle \eta, (\mathcal{V}_{\mathcal{X},T})_y \rangle \leq 0$. Recall that $T'(t) \in \mathbb{R}^k$ was defined in Section 4 as the piecewise smooth curve $T'(t) = (t_k, \ldots, t_{\alpha+1}, t'_\alpha, 0, \ldots, 0)$ with $t'_\alpha = t - (t_k + \cdots + t_{\alpha+1}) \leq t_\alpha$. In order to show that $\langle \eta, (\mathcal{V}_{\mathcal{X},T})_y \rangle \leq 0$ it suffices to prove that $\langle \eta, (\mathcal{X}_{T'(t)})_*(Y)(y) \rangle \leq 0$ for any vector field $Y \in \mathcal{F} \setminus \mathcal{F}_I$ (it is easily seen that η annihilates $\mathrm{span}\, \mathcal{F}_I$). Let $Y \in \mathcal{F} \setminus \mathcal{F}_I$ and consider the real function $t \mapsto f(t) = \langle \eta, (\mathcal{X}_{T'(t)})_*(Y)(y) \rangle$, with $t \in [0, t_f = \sum_{\alpha=1}^k t_\alpha]$. We know that $f(0) \leq 0$ by definition of η and that $\dot{f}(t) = \langle \eta, (\mathcal{X}_{T'(t)})_*([X_{i_\alpha}, Y])(y) \rangle$ if X_{i_α} denotes the vector field whose the path at time t, i.e. $T(t) = (t_k, \ldots, t_{\alpha+1}, t'_\alpha, 0, \ldots, 0)$. From condition (1) on the family $\mathcal{F}$ we know that $[X_{i_\alpha}, Y] = -CX_{i_\alpha} + DY$ with C, D non-negative functions. In particular, we can write $\dot{f}(t) = D(t) f(t)$, with $D(t) \geq 0$. In this way we find a trapping region for the function values of f: if $f(0) \leq 0$ then $f(t) \leq 0$ for all $t \in [0, t_f]$. This implies that $\eta \in C^*_y(\mathcal{V}_{\mathcal{X},T})$. Since $C_y(\mathcal{F} \cup -\mathcal{F}_I)$ is a closed cone and since $C_y(\mathcal{F} \cup -\mathcal{F}_I) \subset C_y(\mathcal{V}_{\mathcal{X},T})$, it finally follows that $C_y(\mathcal{F} \cup -\mathcal{F}_I) = C_y(\mathcal{V}_{\mathcal{X},T})$ (see [9]).

We define a smooth submanifold with corners in N through $y = \mathcal{X}_T(x) \in \mathrm{Acc}_x(\mathcal{F})$ which is entirely contained in $\mathrm{Acc}_x(\mathcal{F})$. Now, consider *the* variational cone $C_y(\mathcal{V}_{\mathcal{X},T})$. We introduce the following family of n vector fields $\tilde{Y}_\alpha = (\mathcal{X}_{T_{i_\alpha}})_*(X_{i_\alpha})$ for $\alpha = 1, \ldots, k$ and the remaining $n-k$ vector fields $\tilde{Y}_i$ are precisely $\mathcal{F} \setminus \mathcal{F}_I$ for $i = k+1, \ldots, n$ (i.e. we have $C(\mathcal{V}_{\mathcal{X},T}) = C(\pm\tilde{Y}_1, \ldots, \pm\tilde{Y}_k, \tilde{Y}_{k+1}, \ldots, \tilde{Y}_n)$). Considering a time ordering of these n vector fields $\tilde{Y}_i$, say $\mathcal{X}'$. The flow of $\mathcal{X}'$ is well-defined on a neighbourhood A of the origin in $\mathbb{R}^n_{n-k}$:

$$A \subset \mathbb{R}^n_{n-k} \to N : S \mapsto \mathcal{X}'_S(x).$$

Note that $\{\mathcal{X}'_S(y) | S \in A\} \subset \mathrm{Acc}_x(\mathcal{F})$. The manifold with corners A is thus embedded into N and has the important property that at every point of A, the cone of inward tangent vectors defined at a point S of A coincides with the variational cone at $\mathcal{X}'_S(y)$.

Assume that we have two such submanifolds with corners, say A and A', both having the property that at each point the cone of inward pointing tangent vectors equals the variational cone. We now show that at any point of intersection $y \in A \cap A'$ both sets are locally diffeomorphic (in the sense of manifolds with corners). Consider the variational cone at y and fix a map $S \mapsto \Psi(S) := \mathcal{X}'_S(y)$ on a neighbourhood U of 0 in $\mathbb{R}^n_{n-k}$, as it was constructed above. Any of the vector fields in $\mathcal{X}'$ is tangent to both A and A' and moreover, they are inward pointing. The restriction of the vector fields in $\mathcal{X}'$ to A and A' implies that we can consider the restriction of Ψ to A and A': $\Psi_A : U \to A$ and $\Psi_{A'} : U \to A'$ with U chosen small enough such that $\Psi(U) \subset A \cap A'$, and that these mappings are diffeomorphisms. It is clear that, by construction, $\Psi_A \circ \Psi^{-1}_{A'} : \Psi_{A'}(U) \to \Psi_A(U)$ is the identity map, clearly defining a local diffeomorphism on a neighbourhood of y in A' to A. This implies that the charts on $A \cup A'$ induced by the smooth structures of A and A' are compatible, implying that $A \cup A'$ can be given the structure of a submanifold of N with corners such that the set of inward pointing tangent vectors at a point y is precisely the variational cone at y. The set $\mathrm{Acc}_x(\mathcal{F})$ thus inherits a smooth structure, if we take the union of all such submanifolds with corners contained in $\mathrm{Acc}_x(\mathcal{F})$.

It remains to check that the topology on $\mathrm{Acc}_x(\mathcal{F})$ as a manifold with corners is precisely the topology on $\mathrm{Acc}_x(\mathcal{F})$ generated by $\mathcal{F}$. It suffices to note that a neighbourhood of y as a submanifold is a neighbourhood of the topology generated by $\mathcal{F}$, since any vector field in $\mathcal{F}$ is inward pointing. The other direction is straightforward from the definition of the smooth structure on $\mathrm{Acc}_x(\mathcal{F})$. □

Example 5.2. Consider the family of vector fields $\{\partial/\partial\theta, \partial/\partial z, \partial/\partial r\}$, where (r, θ, z) denote cylinder coordinates on $\mathbb{R}^3/\{(0,0,z)|z \in \mathbb{R}\}$. It is clear that the accessible set for a given point (x, y, z) equals the half space in $\mathbb{R}^3$ determined by the hyperplane orthogonal to $\partial/\partial z$ minus the interior of the half-cylinder through (x, y, z) in the direction of the positive z-axis.

The purpose of this is to provide a non-trivial example where not all paths accessing a point are maximal (cf. condition (2)). It can be verified that any point of the form $(x, y, z+h)$ with $h > 0$ can be accessed by two paths. The integral curve through (x, y, z) of the vector field $\partial/\partial z$ determines a path which is not 'maximal'. Indeed, if we concatenate to this path the integral curve of $\partial/\partial\theta$ through $(x, y, z+h)$, followed during time $t = 2\pi$. The endpoint of this path is $(x, y, z+h)$ and it is a maximal path.

Example 5.3. Consider the family of vector fields $\{X_1 = \partial/\partial x, X_2 = \partial/\partial x + e^x\partial/\partial y, X_3 = \partial/\partial x + e^x\partial/\partial z\}$. It is easily seen that

$$\begin{aligned}
[X_1, X_2] &= e^x\partial/\partial y \in C(-X_1, X_2),\\
[X_1, X_3] &= e^x\partial/\partial z \in C(-X_1, X_3),\\
[X_2, X_3] &= e^x(\partial/\partial z - \partial/\partial y) \in C(-X_2, X_3).
\end{aligned}$$

By drawing the orbits of the vector fields in the family the maximal path condition is easily seen to be fulfilled. The accessible sets are manifolds with corners.

Discussion and outlook

Main results

Throughout this paper we provide some first results on convex conic distributions. Our motivation to study these distributions finds its origin in geometric non-linear control theory and the associated theory on accessible sets. Corollary 4.4 gives sufficient conditions for the accessible set to be a maximal integral of the smallest integrable distribution associated to the family $\mathcal{F}$ of control vector fields. The implication of Theorem 4.6 from the point of view of non-linear control theory can be stated as follows: given a control system such that control vector fields $\mathcal{F}(\rho)$ determine an integrable distribution, then the accessible set of any family of vector fields $\mathcal{F}'$, satisfying $C(\mathcal{F}(\rho)) = C(\mathcal{F}')$, equals the accessible set of the control system (up to boundary points). This results provides an alternative way for computing the accessible set of a system: it is sufficient to compute the accessible set of any family of vector fields that generates the same cone as the family of control vector fields. From an engineering point of view this further raises the following question: given a conic distribution generating a desirable accessible set, what control laws (i.e. families of vector fields $\mathcal{F}(\rho)$ with variable ρ) are possible so that I recover the desired accessible set. A more profound study of these aspects is left for future work.

Controllability

The notion of state controllability of a system expresses the idea that, given any point (or state) x in N, then for any point y in a neighbourhood of x there exists a control that takes the system from y to x. Within the framework presented above, state controllability at x is equivalent to saying that x is an interior point of $\mathrm{Acc}_x(-\mathcal{F})$. Indeed, if this is the case there exists a neighbourhood of x in $\mathrm{Acc}_x(\mathcal{F})$ such that for any point y in this neighbourhood there exists an ordered family $\mathcal{X}$ of vector fields in $-\mathcal{F}$ such that $\mathcal{X}_T(x) = y$. It is not hard to see that this is equivalent to saying that $x \in \mathrm{Acc}_y(\mathcal{F})$. We also leave for future work to study the notion of state controllability (and accessibility) within the setting of conic distributions. In particular, we wish to apply the strong results from [15] within the framework of conic distributions.

Manifolds with conic singularities

It should be clear that the notion of a smooth manifold admitting conic singularities is needed in order to study the more general structure that accessible sets may acquire. This is also left for future work.

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