$2(m \uparrow x)$

$CR 134281$

SPACE SHUTTLE GN&C EQUATION DOCUMENT

No. 25

Conic State Extrapolation

By

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Charles Stark Draper Laboratory
June 1973

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for

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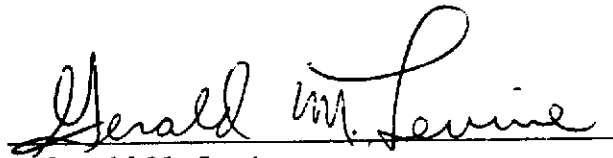The publication of this report does not constitute approval by the
National Aeronautics and Space Administration of the findings or the
conclusions contained therein. It is published only for the exchange
and stimulation of ideas.

PREFACE

This document contains the new Unified Conic formulation of R. H. Battin, and is intended to replace the Space Shuttle GN&C Equation Document No. 3 by W. M. Robertson.

FOREWORD

This document is one of a series of candidates for inclusion in a future revision of JSC -04217, "Space Shuttle Guidance, Navigation and Control Design Equations." The enclosed has been prepared under NAS9-10268, Task No. 15-A, "GN& C Flight Equation Specification Support", and applies to function 1 of the Orbital Coast Navigation Module (ON2) as defined in MSC-03690, Rev. D, "Space Shuttle Orbiter Guidance, Navigation and Control Software Functional Requirements", dated January 1973.

Gerald M. Levine
Division Leader, Guidance Analysis
NASA Programs Department

iv

TABLE OF CONTENTS

NOMENCLATURE

a   Semi-major axis

A   *Value of $\frac{1}{2} U_2(\tilde{x}; \tilde{\alpha})$ function

b, b_1, b_2
b_3, b_5 Intermediate variables

b_4 Reciprocal of normalized final position magnitude ($b_4 = r_0/r$)

B   *Value of $2 \left(U_0(\frac{\tilde{x}}{2}; \tilde{\alpha}) + \tilde{\sigma}_0 U_1(\frac{\tilde{x}}{2}; \tilde{\alpha}) \right)$ function

D   *Value of $U_1(\frac{\tilde{x}}{2}; \tilde{\alpha})$ function

e_{flag} Error flag

E   *Value of $U_0(\frac{\tilde{x}}{2}; \tilde{\alpha})$ function .

f   True anomaly

f_0 Coefficient indicating sign of transfer

f_1 Coefficient to convert to normalized velocity $(\tilde{\underline{v}} = f_1 \underline{v})$

f_2 Inverse of f_1 $(\underline{v} = f_2 \tilde{\underline{v}})$

f_3 Coefficient to convert to normalized transfer time interval $(\tilde{\Delta t} = f_3 \Delta t)$

f_4 Inverse of f_3 $(\Delta t = f_4 \tilde{\Delta t})$

f_5 Coefficient to convert to normalized eccentric anomaly $(\tilde{x} = f_5 x)$

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* These capital letters do not indicate matrices.

f_6 Inverse of f_5 ($x = f_6 \; \tilde{x}$)

f_7 Intermediate variable

F [*]State extrapolation coefficient

$F_t, \tilde{F}_t$ [*]State extrapolation coefficients $\left(F_t = \sqrt{\dfrac{\mu}{r_0^3}} \; \tilde{F}_t \right)$

$G, \tilde{G}$ [*]State extrapolation coefficients $\left(G = \sqrt{\dfrac{r_0^3}{\mu}} \; \tilde{G} \right)$

G_t [*]State extrapolation coefficient

i Iteration counter

i_{max} Maximum permissible number of iterations

$\underline{i}_{r_0}$ Unit vector in $\underline{r}_0$ direction

j Counter for calculation of the continued fraction Q

k Counter for the summation of the U_1 series

k_{max} Maximum value for k

$\tilde{p}$ Normalized semi-latus rectum ($p = r_0 \, \tilde{p}$)

$\tilde{P}$ [*]Normalized orbital period   ($P = P_{circular} \; \tilde{P}$)

Q [*]Continued fraction function

$r, \tilde{r}$ Final position magnitude and its normalized analog ($r = r_0 \, \tilde{r}$)

$\underline{r}(t)$ Inertial position vector corresponding to time t

r_0 Magnitude of $\underline{r}_0$

---

[*] These capital letters do not indicate matrices.

$\underline{r}_0$ Inertial position vector corresponding to time t_0

s_{max} Maximum scalar representable on the computer

s_{min} Minimum positive scalar representable on the computer

u_1 Value of $U_1 (\frac{\tilde{x}}{4} ; \tilde{\alpha})$ function

u_{1old} Previous value of u_1

$U_n(x;\alpha)$ [*]Universal function of Battin

$\underline{v}(t)$ Inertial velocity vector corresponding to time t

$\underline{v}_0, \tilde{\underline{v}}_0$ Inertial velocity vector, corresponding to initial time t_0, and its normalized analog ($\underline{v}_0 = v_{circular} \tilde{\underline{v}}_0$)

w Value of $U_0 (\frac{\tilde{x}}{4} ; \tilde{\alpha})$ function

$x, \tilde{x}$ Eccentric anomaly difference and its normalized analog ($x = \sqrt{r_0} \ \tilde{x}$, independent variable used in the Kepler iteration scheme)

$\tilde{x}_{arg}$ Independent variable used as the input argument to the Transfer Time Interval Routine

x_c Value of x to which the Kepler iteration scheme converged

x'_c Previous value of x_c

$\tilde{x}_{guess}$ Guess of $\tilde{x}$

$\tilde{x}_{last}$ Previous value of $\tilde{x}$

---

[*]This capital letter does not indicate a matrix.

$x_{max}, \tilde{x}_{max}$ — Independent variable upper bound, and its normalized analog

$x_{min}, \tilde{x}_{min}$ — Independent variable lower bound, and its normalized analog

$\tilde{x}_{old}$ — Value of $\tilde{x}_{guess}$ from the previous iteration

$\tilde{x}_P$ — Value of $\tilde{x}$ corresponding to one period of the orbit

x_q, y — Intermediate variables

$z, \tilde{z}$ — Values of $U_1(\frac{x}{4}; \alpha)$, $U_1(\frac{\tilde{x}}{4}; \tilde{\alpha})$ functions $\left(z = \sqrt{r_0}\ \tilde{z}\right)$

$\alpha, \tilde{\alpha}$ — Reciprocal of the semi-major axis and its normalized analog $(\tilde{\alpha} = r_0 \alpha)$

$\Delta t, \Delta\tilde{t}$ — Transfer time interval and its normalized analog $(\Delta t = P_{circular}\ \Delta\tilde{t})$

$\Delta\tilde{t}_{arg}$ — Dependent variable used as the output argument from the Transfer Time Interval Routine

Δt_c — Value of the transfer time interval calculated in the Kepler Routine

$\Delta t'_c$ — Previous value of Δt_c

$\Delta\tilde{t}_{error}$ — Error between $\Delta\tilde{t}_{guess}$ and $\Delta\tilde{t}$ during any particular iteration

$\Delta\tilde{t}_{guess}$ — Transfer time interval corresponding to the most recent independent variable $\tilde{x}_{guess}$ during the Kepler iteration

$\Delta\tilde{t}_{last}$ — Previous value of $\Delta\tilde{t}$

$\Delta t'_{max}$ — Dependent variable boundary distance on the maximum side used in the Secant Iterator

$\Delta t'_{min}$ — Dependent variable boundary distance on the minimum side used in the Secant Iterator

Δt_{max}, $\Delta \tilde{t}_{max}$	Dependent variable upper bound and its normalized analog
Δt_{min}, $\Delta \tilde{t}_{min}$	Dependent variable lower bound and its normalized analog
$\Delta \tilde{t}_{old}$	Value of $\Delta \tilde{t}_{guess}$ from the previous iteration
Δu_1	Increment in u_1
$\Delta \tilde{x}$	Increment in $\tilde{x}_{guess}$
ϵ_α	Tolerance defining near parabolic orbits used in boundary selection
$\tilde{\epsilon}_t$	Primary convergence criterion: error in normalized transfer time interval
ϵ'_t	Secondary convergence criterion: minimum permissible difference between two transfer time intervals used in the Secant Iterator
ϵ_w	Tolerance defining area where the computer cannot handle the continued fraction Q calculation
θ	Transfer angle (true anomaly increment)
μ	Gravitational parameter of the earth
$\sigma_0, \tilde{\sigma}_0$	Conic parameter and its normalized analog ($\sigma_0 = \sqrt{r_0}\,\tilde{\sigma}_0$)

1. INTRODUCTION

The Conic State Extrapolation Routine provides the capability to conically extrapolate any spacecraft inertial state vector either backwards or forwards as a function of time or as a function of transfer angle. It is merely the coded form of two versions of the solution of the two-body differential equations of motion of the spacecraft center of mass. Because of its relatively fast computation speed and moderate accuracy, it serves as a preliminary navigation tool and as a method of obtaining quick solutions for targeting and guidance functions. More accurate (but slower) results are provided by the Precision State Extrapolation Routine.

2. FUNCTIONAL FLOW DIAGRAMS

The Conic State Extrapolation Routine basically consists of two parts - one for extrapolating in transfer time interval, and one for extrapolating in transfer angle. Some portions of the formulation are common to the two parts and may be arranged as subroutines on a computer. Similarly, there are several areas of overlap with the Conic Required Velocity Routine (Ref. 5), and these may be treated likewise.

2.1 Conic State Extrapolation as a Function of Transfer Time Interval - Kepler Routine

This routine involves a single loop iterative procedure, and hence is organized into three sections - initialization, iteration, and final computations. This scheme is shown in Figure 1, where the variable $"\tilde{x}_{guess}"$ is the independent variable used in the iteration procedure. For a given initial state, the variable $"\tilde{x}_{guess}"$ measures the amount of transfer along the extrapolated trajectory. Hence, the iteration involves adjusting $"\tilde{x}_{guess}"$ until the transfer time interval calculated from it agrees, to within some tolerance, with the specified transfer time interval. Then the extrapolated state vector is calculated from several intermediate variables already calculated in the transfer time interval computation.

2.2 Conic State Extrapolation as a Function of Transfer Angle - Theta Routine

This routine involves a direct calculation (i.e., does not have an iteration scheme), as shown in Figure 2. The extrapolated state vector is again calculated from the same intermediate variables used in the Kepler Routine. However, this time these variables are computed directly in terms of the initial state and the transfer angle.

ENTER

Initialization

- Normalize the input parameters
- Compute various normalized conic parameters
- Compute a starting guess for the iteration variable $"\tilde{x}_{guess}"$

Iteration

Adjust $"\tilde{x}_{guess}"$

- Compute the transfer time interval corresponding to $"\tilde{x}_{guess}"$

Converged ?

No

Yes

Final Computations

- Compute the extrapolated state vector corresponding to the converged value of $"\tilde{x}_{guess}"$

EXIT

Figure 1. Kepler Routine, Functional Flow Diagram

2-2

ENTER

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┌─────────────────────────────────────┐
│ • Normalize the input parameters   │
│   │
│ • Compute various normalized conic │
│   parameters   │
└─────────────────────────────────────┘

↓

┌─────────────────────────────────────┐
│ • Compute the transfer time interval│
│   corresponding to the specified │
│   transfer angle θ │
└─────────────────────────────────────┘

↓

┌─────────────────────────────────────┐
│ • Compute the extrapolated state   │
│   vector corresponding to the   │
│   specified transfer angle θ │
└─────────────────────────────────────┘

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EXIT
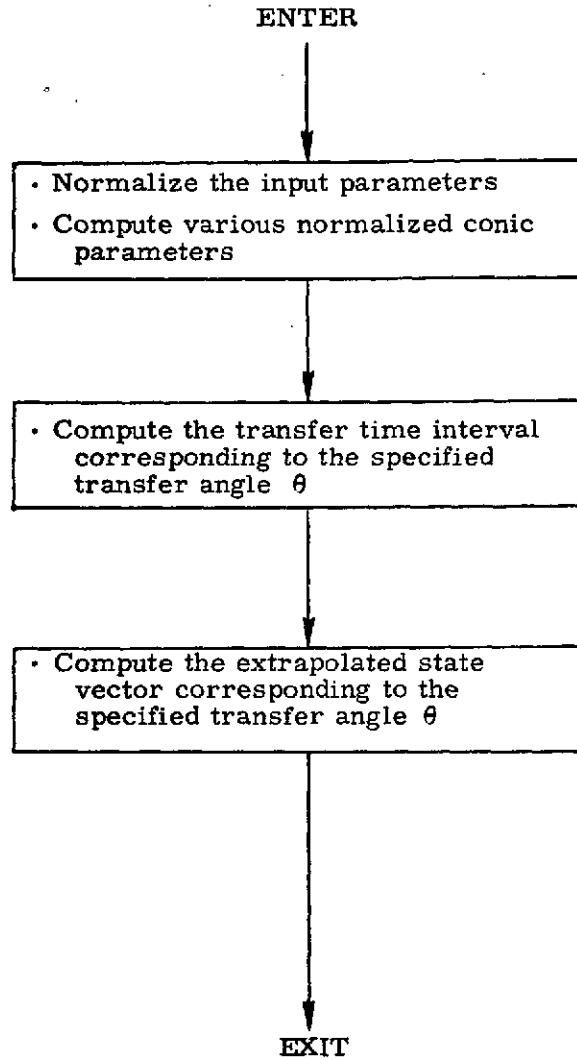
Figure 2. Theta Routine, Functional Flow Diagram

3.   INPUT AND OUTPUT VARIABLES

The Conic State Extrapolation Routine has only one universal constant: the gravitational parameter of the earth. Its principal input variables are the inertial state vector which is to be extrapolated and the transfer time interval or transfer angle through which the extrapolation is to be made. Several optional input variables may be supplied in the transfer time case in order to speed the computations. The principal output variable of both cases is the extrapolated inertial state vector.

3.1   Conic State Extrapolation as a Function of Transfer Time Interval - Kepler Routine

Input Variables

$(\underline{r}_0, \underline{v}_0)$ Inertial state vector which is to be extrapolated (corresponds to time t_0).

Δt Transfer time interval through which the extrapolation is to be made.

x Starting guess of the independent variable (used to speed convergence). If no guess is available, set $x = 0$, and the routine will generate its own guess.

$\Delta t'_c$ Value of dependent variable, the transfer time interval, to which the previous call to Kepler had converged.

x'_c Value of the independent variable, universal eccentric anomaly difference, to which the previous call to Kepler had converged.

Output Variables

$(\underline{r}, \underline{v})$ Extrapolated inertial state vector (corresponds to time t).

Δt_c Converged value of the dependent variable, the transfer time interval, from the Kepler iteration scheme (should agree closely with Δt).

x_c Converged value of the independent variable from the Kepler iteration scheme.

$$e_{flag} \quad \text{Error flag} = \begin{cases} 0 & \text{no error} \\ 1 & \text{solution failed to converge} \\ 2 & U_1 \text{ series failed to converge} \\ 3 & \text{both 1 and 2 occured} \end{cases}$$

3.2 Conic State Extrapolation as a Function of Transfer Angle - Theta Routine

$(\underline{r}_0, \underline{v}_0)$ Inertial state vector which is to be extrapolated.

θ Transfer angle through which the extrapolation is to be made.

$(\underline{r}, \underline{v})$ Extrapolated inertial state vector .

Δt_c Transfer Time Interval corresponding to the conic extrapolation through the transfer angle θ.

$$e_{flag} \quad \text{Error flag} = \begin{cases} 0 & \text{no error} \\ 1 & \text{orbit too nearly rectilinear} \\ 2 & \text{multi-revolution input for hyperbolic trajectory} \\ 3 & \text{transfer past asymptote on a hyperbolic trajectory} \\ 4 & \text{transfer required closure through infinity} \end{cases}$$

4. DESCRIPTION OF EQUATIONS

·The universal formulation of Battin in terms of the universal eccentric anomaly difference is used. It should be noted that the formulation given in the detailed flow diagrams of Section 5 is a normalized version of the formulation presented here. It is felt that the equation descriptions of this section are more clear when expressed in standard variables rather than in normalized variables.

The universal eccentric anomaly, usually denoted by x, is defined by the relations

$$x = \begin{cases} \sqrt{a}\,(E - E_0) & \text{ellipse} \\ \sqrt{p}\,(\tan f/2 - \tan f_0/2\,) & \text{parabola} \\ \sqrt{-a}\,(H - H_0\,) & \text{hyperbola} \end{cases}$$

Here, a is the semi-major axis, E and H are the eccentric anomaly and its hyperbolic analog, p is the semi-latus rectum and f the true anomaly. Using this definition, the generalized form of Kepler's equation expresses the transfer time interval as

$$\Delta t = \frac{1}{\sqrt{\mu}}\,[r_0\,U_1(x;\,\alpha) + \sigma_0\,U_2(x;\,\alpha) + U_3(x;\,\alpha)]$$

where

$$U_n(x;\,\alpha) = x^n\,(\frac{1}{n!} - \frac{\alpha\,x^2}{(n+2)!} + \frac{(\alpha\,x^2)^2}{(n+4)!} - \dots)$$

and

$$\alpha = \frac{1}{a} = \frac{2}{r_0} - \frac{v_0^2}{\mu}$$

and

$$\sigma_0 = \frac{r_0 \cdot v_0}{\sqrt{\mu}}$$

Furthermore, the expressions for the extrapolated position and velocity vectors $(\underline{r},\ \underline{v}\,)$ in terms of the initial position and velocity vectors $(\underline{r}_0,\ \underline{v}_0)$ are also given in terms of x by the following standard formulae

$$r = r_0\,U_0(x;\,\alpha) + \sigma_0\,U_1(x;\,\alpha) + U_2(x;\,\alpha)$$

$$\underline{r}(t) = [1 - \frac{1}{r_0}\,U_2(x;\,\alpha)]\,\underline{r}_0 + [\frac{r_0\,U_1(x;\,\alpha) + \sigma_0\,U_2(x;\,\alpha)}{\sqrt{\mu}}]\,\underline{v}_0$$

$$\underline{v}(t) = [-\frac{\sqrt{\mu}}{r\,r_0}\,U_1(x;\,\alpha)]\,\underline{r}_0 + [1 - \frac{1}{r}\,U_2(x;\,\alpha)]\,\underline{v}_0$$

However, in this document a slightly modified, but equivalent, formulation is used, namely

$$\Delta t = \frac{U_1(\frac{x}{2}; \alpha)}{\sqrt{\mu}} \, [2(r_0 \, U_0 (\tfrac{x}{2}; \alpha) + \sigma_0 \, U_1(\tfrac{x}{2}; \alpha)) + \tfrac{1}{2} \, U_2(x; \alpha) Q]$$

$$r = r_0 + (1 - \alpha \, r_0) \, U_2(x; \alpha) + 2 \, \sigma_0 \, U_0 (\tfrac{x}{2}; \alpha) \, U_1 (\tfrac{x}{2}; \alpha)$$

$$\underline{r}(t) = F \, \underline{r}_0 + G \, \underline{v}_0$$

$$\underline{v}(t) = F_t \, \underline{r}_0 + G_t \, \underline{v}_0$$

Here, the function Q is defined in Section 4.1, but the other variables are as follows:

$$F = 1 - \frac{1}{r_0} \, U_2(x; \alpha)$$

$$G = \frac{2 \, r_0 \, U_0 (\tfrac{x}{2}; \alpha) \, U_1 (\tfrac{x}{2}; \alpha) + \tilde{\sigma}_0 \, U_2 (x; \alpha)}{\sqrt{\mu}}$$

$$F_t = -\frac{2 \sqrt{\mu}}{r \, r_0} \, U_0 (\tfrac{x}{2}; \alpha) \, U_1 (\tfrac{x}{2}; \alpha)$$

$$G_t = 1 - \frac{1}{r} \, U_2 (x; \alpha)$$

The family of functions $U_n(x; \alpha)$ used in these equations can be related to some elementary functions by noting that

$$U_0(x; \alpha) = \begin{cases} \cos \sqrt{\alpha} \cdot x & \alpha > 0 \\ \cosh \sqrt{-\alpha} \, x & \alpha < 0 \end{cases}$$

$$U_1(x; \alpha) = \begin{cases} \dfrac{\sin \sqrt{\alpha} \cdot x}{\sqrt{\alpha}} & \alpha > 0 \\[3mm] \dfrac{\sinh \sqrt{-\alpha} \, x}{\sqrt{-\alpha}} & \alpha < 0 \end{cases}$$

Then, from these the higher order functions follow directly from the useful identity

$$U_n(x; \alpha) + \alpha \, U_{n+2}(x; \alpha) = \frac{x^n}{n!}$$

4.1   The Q Function

A very useful function employed in the evaluation of the universal formulation of Battin is the Q function which may be written

$$Q = \frac{U_3(x;\alpha)}{U_1^3(\frac{x}{2};\alpha)}$$

It can be shown that Q is a hypergeometric function of the variable

$$q = \alpha\, U_1^2(\frac{x}{4};\alpha)$$

From this, Battin has developed a continued fraction formulation for evaluating Q as a function of the variable

$$w = U_0(\frac{x}{4};\alpha)$$

to improve the convergence. The formulation is conveniently expressed by

$$Q = \frac{1}{w^2}\; [\,1 + \frac{2}{3}\; \frac{1}{w(w+1)}\; (1 - \frac{B_1}{4})\,]$$

where

$$B_{n-1} = \frac{\frac{w-1}{w+1}}{1 + (\frac{n-1}{n+2})(1 - B_n)}$$

Evaluation of this expression is started by setting $B_M = 0$ for a sufficiently large value of M, and calculating lower B_n 's recursively until B_1 is reached. This means that the number of levels M of the continued fraction necessary to achieve some specified accuracy must be precomputed as a function of w.

4.2   Conic State Extrapolation as a Function of Transfer
Time Interval - Kepler Routine

Since the transfer time interval Δt is given, it is desired to find the x corresponding to it in the generalized Kepler equation.   Then the extrapolated state vector (r, v) expression can be evaluated based on that value of x . Unfortunately, the generalized Kepler equation expresses Δt as a transcendental function of x , and no power series inversion of the equation is known which has good convergence properties for all orbits.   Therefore, it is necessary to solve the equation iteratively for x.

The actual evaluation of the Kepler equation is conveniently expressed in terms of the variable

$$z = 2 U_1 \left(\frac{x}{2}; \alpha\right)$$

From this it can be shown that

$$U_0 \left(\frac{x}{4}; \alpha\right) = \sqrt{1 - \frac{\alpha}{4} z^2}$$

$$U_0 \left(\frac{x}{2}; \alpha\right) = 1 - \frac{\alpha}{2} z^2$$

$$U_1 \left(\frac{x}{2}; \alpha\right) = w z$$

$$U_2 (x; \alpha) = 2 w^2 z^2$$

These quantities are needed to evaluate the transfer time interval corresponding to x during each iteration. Then, once convergence is achieved the latest values of these quantities are used to calculate the extrapolated state vectors.

The iteration scheme used to solve for x is a modified secant method (linear inverse interpolation/extrapolation). It merely finds a new guess for the independent variable x_{new} which attempts to adjust the dependent variable Δt to the desired value. This adjustment is based on a linear interpolation/extrapolation of the last guess of the independent variable x_{old}. The scheme requires a minimum point and a maximum point on the Δt vs. x curve between which the solution is known to be. These two points define two possible guesses for the new x, namely

$$x_{new} = x_{old} + \left(\frac{\Delta t - \Delta t_{old}}{\Delta t_{old} - \Delta t_{min}}\right) (x_{old} - x_{min})$$

$$x_{new} = x_{old} + \left(\frac{\Delta t - \Delta t_{old}}{\Delta t_{old} - \Delta t_{max}}\right) (x_{old} - x_{max})$$

one of which is an interpolation and the other an extrapolation. The modified scheme preferentially selects the extrapolated guess from the two possibilities. However, if this happens to extrapolate outside the upper or lower bounds, then the interpolated guess is used to ensure a solution. It should be noted that these bounds are continually reset during the iteration as more and more values of x are determined to be too small or too large. This modification has been found to improve the convergence when the desired transfer time interval is near a "knee" in the Δt vs. x curve.

4.3 Conic State Extrapolation as a Function of Transfer Angle - Theta Routine

As with the Kepler Routine, the universal formulation of Battin in terms of the universal eccentric anomaly difference x is used in the Theta Routine. However, in this case the functions $U_n(x; \alpha)$ may be expressed directly in terms of the specified transfer angle. The relationships used in this formulation are

$$U_0\left(\frac{x}{2}; \alpha\right) = \sqrt{\frac{r}{r_0}} \left[\cos\frac{\theta}{2} - \frac{\sigma_0}{\sqrt{p}}\sin\frac{\theta}{2}\right]$$

$$U_1\left(\frac{x}{2}; \alpha\right) = \sqrt{\frac{r\, r_0}{p}} \; \sin\frac{\theta}{2}$$

$$U_2(x; \alpha) = \frac{2\, r\, r_0}{p} \; \sin^2\frac{\theta}{2}$$

However, to evaluate these functions requires the final position magnitude, but from the position magnitude extrapolation equation it can be shown that

$$\frac{r_0}{r} = 2\left[\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} - \frac{\sigma_0}{\sqrt{p}}\sin\frac{\theta}{2}\right) + \frac{r_0}{p}\sin^2\frac{\theta}{2}\right] - 1$$

Thus, from these quantities the extrapolated state vectors are calculated using the formulae in Section 4. Furthermore, these quantities may also be used to evaluate the transfer time interval from the generalized Kepler equation.

5.   DETAILED FLOW DIAGRAMS

This section contains detailed flow diagrams of two Conic State Extrapolation Routines (Kepler and Theta) and the subroutines used by them.

Each input and output variable in the routine and subroutine call statements can be followed by a symbol in brackets. This symbol identifies the notation for the corresponding variable in the detailed description and flow diagrams of the called routine. When identical notation is used, the bracket symbol is omitted.

The Kepler routine utilizes the following subroutines, which are diagrammed in Section 5.3:

- Kepler Transfer Time Interval (Section 5.3.1)

  - U_1 Series Summation (Section 5.3.2)
  - Q Continued Fraction (Section 5.3.3)

- Kepler Iteration Loop (Section 5.3.4)

  - Secant Iterator (Section 5.3.5)
  - Kepler Transfer Time Interval (Section 5.3.1)

    - U_1 Series Summation (Section 5.3.2)
    - Q Continued Fraction (Section 5.3.3)

- Extrapolated State Vector (Section 5.3.6)

The Theta routine utilizes the following subroutines, which are diagrammed in Section 5.3:

- Q Continued Fraction (Section 5.3.3)
- Extrapolated State Vector (Section 5.3.6)

5.1 Conic State Extrapolation as a Function of Transfer Time - Kepler Routine

UNIVERSAL CONSTANTS

PROGRAM CONSTANTS

INPUT VARIABLES

$$\mu$$

$$i_{max}, \, k_{max}, \, s_{min}, \, s_{max}$$
$$\epsilon_\alpha, \, \epsilon_w, \, \tilde{\epsilon}_t, \, \epsilon'_t$$

$$\underline{r}_0, \underline{v}_0, \Delta t,$$
$$x, \Delta t'_c, x'_c$$

$$\Delta t \geq 0$$

No   Yes

$$f_0 = -1$$

$$f_0 = 1$$

$$n = 0$$
$$r_0 = |\underline{r}_0|$$

$$f_1 = \frac{f_0 \sqrt{r_0}}{\sqrt{\mu}} \qquad f_4 = f_1 \, r_0$$

$$f_2 = \frac{1}{f_1} \qquad f_5 = \frac{f_0}{\sqrt{r_0}}$$

$$f_3 = \frac{f_2}{r_0} \qquad f_6 = f_0 \sqrt{r_0}$$

$$\underline{i}_{r_0} = \frac{\underline{r}_0}{r_0}$$

$$\tilde{\underline{v}}_0 = f_1 \underline{v}_0$$

$$\tilde{\sigma}_0 = \underline{i}_{r_0} \cdot \tilde{\underline{v}}_0$$

$$b_0 = \tilde{\underline{v}}_0 \cdot \tilde{\underline{v}}_0 - 1$$

$$\tilde{\alpha} = 1 - b_0$$

1
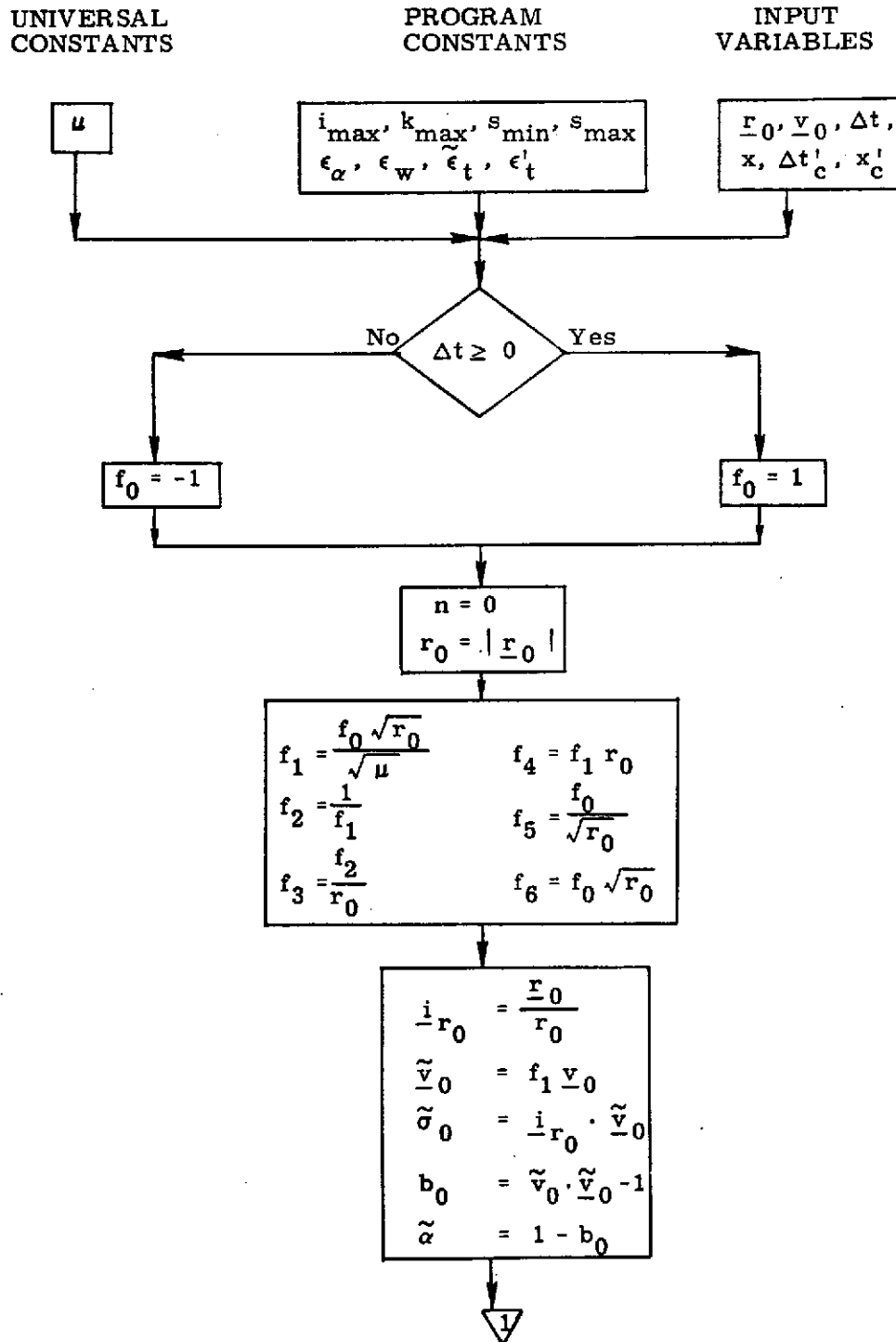
Figure 3a. Kepler Routine, Detailed Flow Diagram

○1

$$\tilde{x}_{guess} = f_5 x \qquad \Delta\tilde{t} = f_3 \Delta t$$
$$\tilde{x}_{last} = f_5 x'_c \qquad \Delta\tilde{t}_{last} = f_3 \Delta t'_c$$
$$\tilde{x}_{min} = 0 \qquad \Delta\tilde{t}_{min} = 0$$

$\sqrt{|\tilde{\alpha}|} < \epsilon_\alpha$ — Yes / No

Yes:
$$\tilde{x}_{max} = 2\pi s_{max}^{1/3}$$
$$\Delta\tilde{t}_{max} = 2\pi s_{max}$$

No:
$$\tilde{x}_{max} = \frac{2\pi}{\sqrt{|\tilde{\alpha}|}}$$

$\tilde{\alpha} > 0$ — No / Yes

Yes:
$$\Delta\tilde{t}_{max} = \frac{\tilde{x}_{max}}{\alpha}$$
$$\tilde{x}_P = \tilde{x}_{max}$$
$$\tilde{P} = \Delta\tilde{t}_{max}$$

No (Call Kepler):
Call Kepler Transfer Time Interval (Section 5.3.1)

Input: $\tilde{x}_{max}[\tilde{x}_{arg}], \tilde{\sigma}_0, \tilde{\alpha}, \epsilon_w, s_{max}, k_{max}$

Output: $\Delta\tilde{t}_{max}[\Delta\tilde{t}_{arg}]$

$\Delta\tilde{t}_{max} \geq \Delta\tilde{t}$ — Yes / No

No:
$$\Delta\tilde{t}_{min} = \Delta\tilde{t}_{max}$$
$$\tilde{x}_{min} = \tilde{x}_{max}$$
$$\tilde{x}_{max} = 2\tilde{x}_{max}$$

$\Delta\tilde{t} < \tilde{P}$ — Yes / No

No:
$$n = n + 1$$
$$\Delta\tilde{t} = \Delta\tilde{t} - \tilde{P}$$
$$\Delta\tilde{t}_{last} = \Delta\tilde{t}_{last} - \tilde{P}$$
$$\tilde{x}_{guess} = \tilde{x}_{guess} - \tilde{x}_P$$
$$\tilde{x}_{last} = \tilde{x}_{last} - \tilde{x}_P$$

▽2
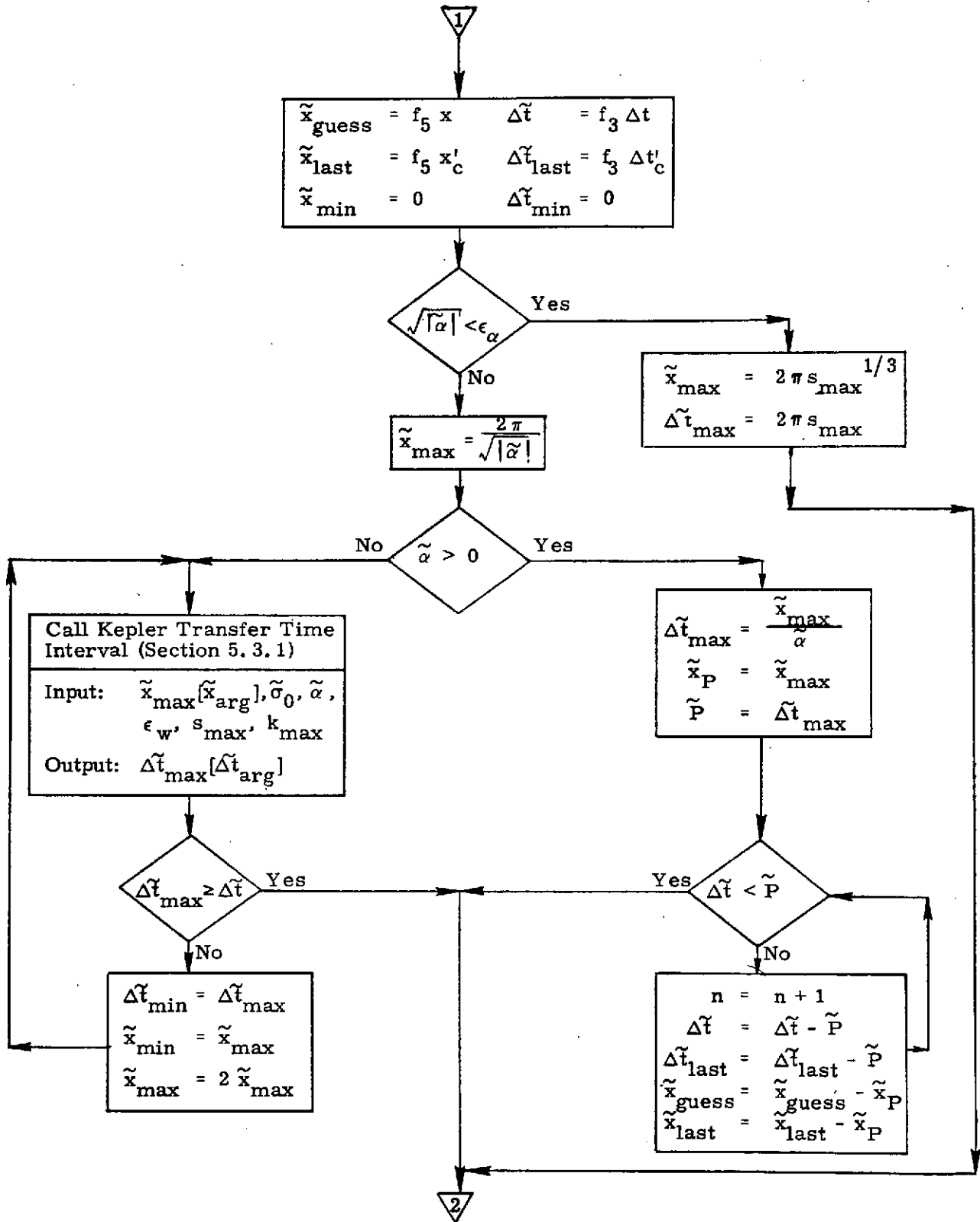
Figure 3b.   Kepler Routine, Detailed Flow Diagram

②

$$\tilde{x}_{min} < \tilde{x}_{guess} < \tilde{x}_{max}$$

No → $\tilde{x}_{guess} = \dfrac{(\tilde{x}_{min} + \tilde{x}_{max})}{2}$

Yes

Call Kepler Transfer Time Interval (Section 5.3.1)

Input: $\tilde{x}_{guess} [\tilde{x}_{arg}], \tilde{\sigma}_0, \tilde{\alpha}$
$\epsilon_w, s_{max}, k_{max}$

Output: $\Delta \tilde{t}_{guess} [\Delta \tilde{t}_{arg}]$

No ← $\Delta \tilde{t} < \Delta \tilde{t}_{guess}$ → Yes

$\tilde{x}_{min} < \tilde{x}_{last} < \tilde{x}_{guess}$ and $\Delta \tilde{t}_{min} < \Delta \tilde{t}_{last} < \Delta \tilde{t}_{guess}$

No → $\tilde{x}_{guess} < \tilde{x}_{last} < \tilde{x}_{max}$ and $\Delta \tilde{t}_{guess} < \Delta \tilde{t}_{last} < \Delta t_{max}$

$\tilde{x}_{min} = \tilde{x}_{last}$
$\Delta \tilde{t}_{min} = \Delta \tilde{t}_{last}$

$\tilde{x}_{max} = \tilde{x}_{last}$
$\Delta \tilde{t}_{max} = \Delta \tilde{t}_{last}$

Call Kepler Iteration Loop (Sec. 5.3.4)

Input: $i_{max}, \tilde{\epsilon}_t, \epsilon'_t, \Delta \tilde{t}, \tilde{x}_{guess}, \Delta \tilde{t}_{guess}$
$\tilde{x}_{min}, \Delta \tilde{t}_{min}, \tilde{x}_{max}, \Delta \tilde{t}_{max}, \tilde{\sigma}_0,$
$\tilde{\alpha}, s_{max}, k_{max}$

Output: $\tilde{x}_{guess}, \Delta \tilde{t}_{guess}, A, D, E, e_{flag}$

③

Figure 3c. Kepler Routine, Detailed Flow Diagram

5-4

$$\nabla 3$$

$$\widetilde{r} = 1 + 2(b_0 A + \widetilde{\sigma}_0 D E)$$

$$b_4 = 1/\widetilde{r}$$

$$x_c = f_6 (\widetilde{x}_{guess} + n \, \widetilde{x}_P)$$

$$\Delta t_c = f_4 (\Delta \widetilde{t}_{guess} + n \, \widetilde{P})$$

Call Extrapolated State Vector (Sec. 5.3.6)

Input: $f_2, b_4, \widetilde{\sigma}_0, r_0, A, D, E, \underline{i}_{r_0}, \widetilde{\underline{v}}_0$

Output: $\underline{r}, \underline{v}$

OUTPUT VARIABLES

$$\underline{r}, \underline{v}, \Delta t_c, x_c, e_{flag}$$
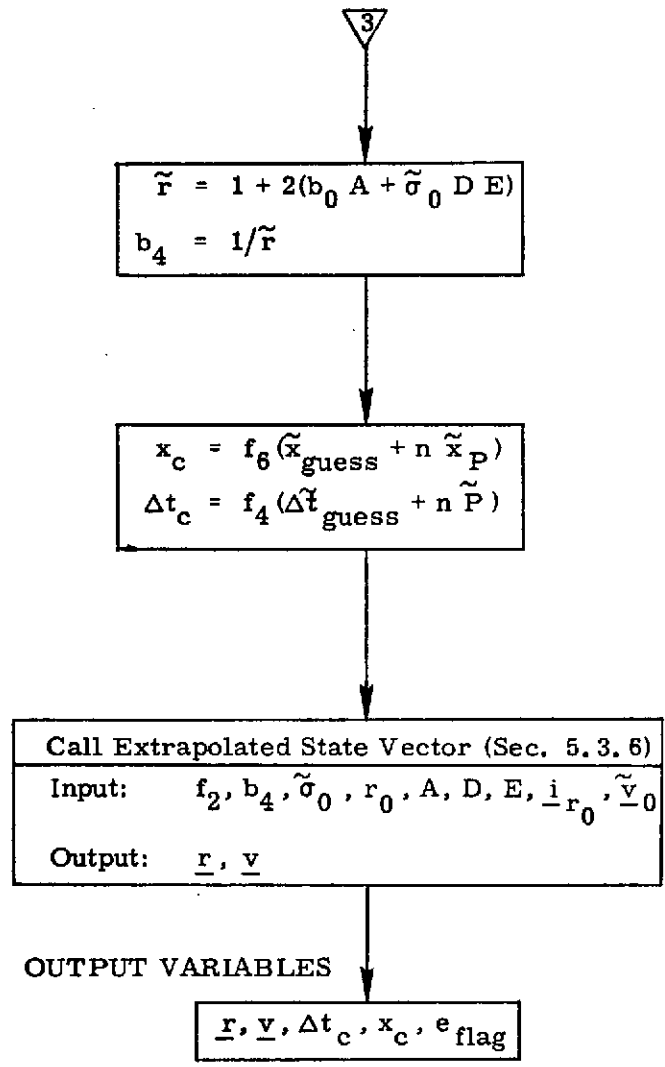
Figure 3d. Kepler Routine, Detailed Flow Diagram

5.2 Conic State Extrapolation as a Function of Transfer Angle - Theta Routine

UNIVERSAL CONSTANT	PROGRAM CONSTANTS	INPUT VARIABLES
μ	$\epsilon_\alpha,\ \epsilon_p,\ \epsilon_w,\ s_{max}$	$\underline{r}_0,\ \underline{v}_0,\ \theta$

$$\theta \geq 0$$

No → $f_0 = -1$

Yes → $f_0 = 1$

$$r_0 = |\underline{r}_0|$$
$$n = 0$$
$$e_{flag} = 0$$

$$f_1 = \frac{f_0 \sqrt{r_0}}{\sqrt{\mu}}$$
$$f_2 = \frac{1}{f_1}$$
$$f_4 = f_1 r_0$$

$$\underline{i}_{r_0} = \frac{\underline{r}_0}{r_0}$$
$$\underline{\tilde{v}}_0 = f_1 \underline{v}_0$$
$$\tilde{\alpha} = 2 - \underline{\tilde{v}}_0 \cdot \underline{\tilde{v}}_0$$

$$\theta = f_0\,\theta$$

$$\theta \geq 2\pi$$

Yes →
$$n = n + 1$$
$$\theta = \theta - 2\pi$$

No

$\bigtriangledown$ 1
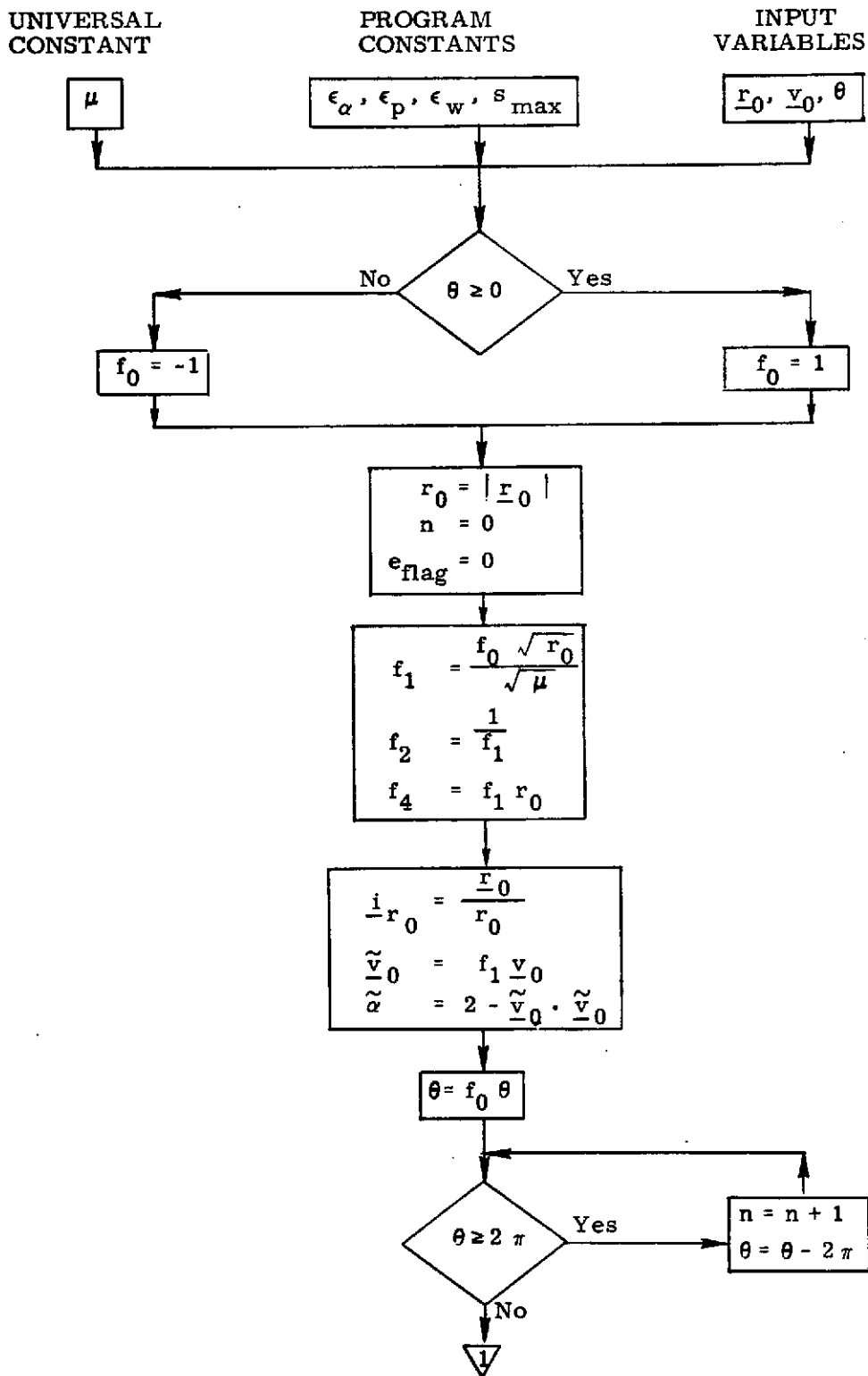
Figure 4a. Theta Routine, Detailed Flow Diagram
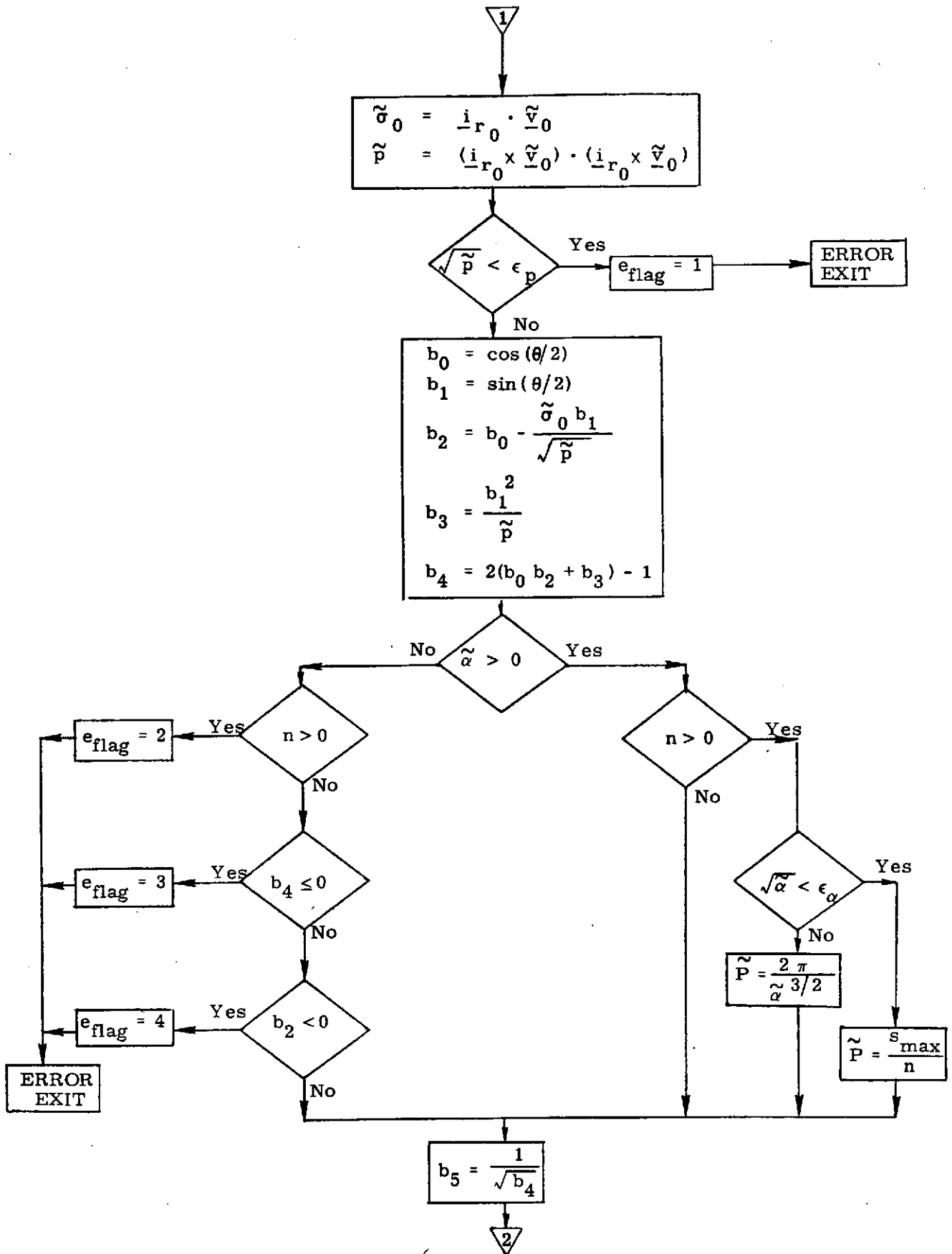
Figure 4b. Theta Routine, Detailed Flow Diagram

$\boxed{2}$

$$A = \frac{b_3}{b_4}$$

$$B = 2\,b_0\,b_5$$

$$D = \sqrt{A}$$

$$E = b_2\,b_5$$

$$w = \sqrt{\frac{1+E}{2}}$$

Call Q Continued Fraction
(Sec. 5.3.3)

Input: w, ϵ_w, s_{max}

Output: Q

$$\Delta t_c = f_4\,(D\,(B + AQ) + n\,\breve{P}\,)$$

Call Extrapolated State Vector
(Sec. 5.3.6)

Input: f_2, b_4, $\tilde{\sigma}_0$, r_0, A, D, E, $\underline{i}_{r_0}$, $\tilde{\underline{v}}_0$

Output: $\underline{r}$, $\underline{v}$

OUTPUT VARIABLES

$\underline{r}$, $\underline{v}$, Δt_c, e_{flag}

Figure 4c. Theta Routine, Detailed Flow Diagram

5.3 <u>Subroutines Used by the Transfer Time and/or the Transfer Angle Conic Extrapolation Routines</u>

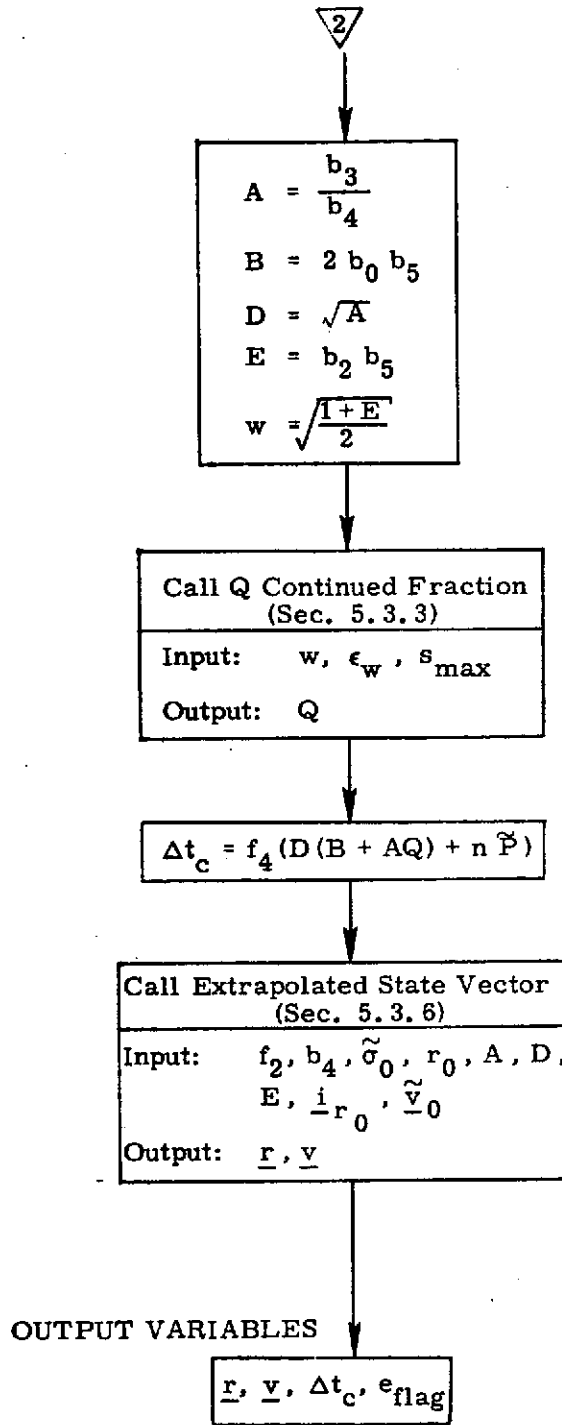**5.3.1 Kepler Transfer Time Interval Subroutine**

INPUT VARIABLES

$$\tilde{x}_{arg}, \tilde{\sigma}_0, \tilde{\alpha}, \epsilon_w, s_{max}, k_{max}$$

$$e_{flag} = 0$$

Call U_1 Series Summation (Sec. 5.3.2)

Input: $\tilde{x}_{arg}, \tilde{\alpha}, k_{max}$

Output: u_1, e_{flag}

$$\tilde{z} = 2u_1$$
$$E = 1 - \frac{1}{2}\tilde{\alpha}\,\tilde{z}^2$$
$$w = \sqrt{\frac{1+E}{2}}$$
$$D = w\,\tilde{z}$$
$$A = D^2$$
$$B = 2(E + \tilde{\sigma}_0 D)$$

Call Q Continued Fraction (Sec. 5.3.3)

Input: w, ϵ_w, s_{max}

Output: Q

$$\Delta\tilde{t}_{arg} = D(B + AQ)$$

OUTPUT VARIABLES

$$\Delta\tilde{t}_{arg}, A, D, E, e_{flag}$$
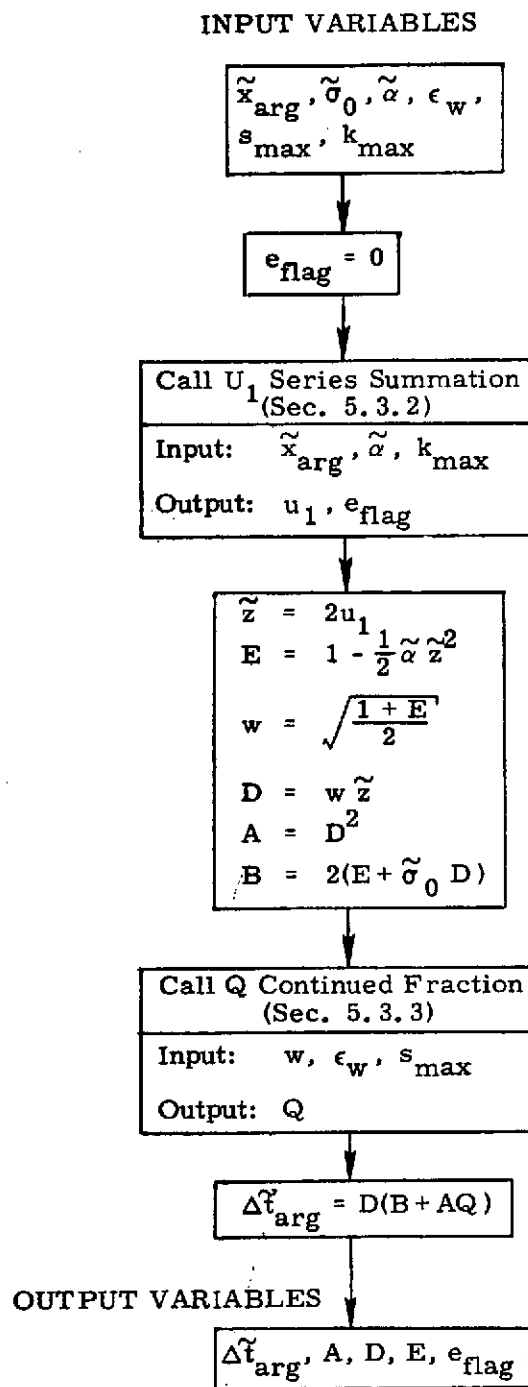
Figure 5. Kepler Transfer Time Interval, Detailed Flow Diagram

5.3.2 U_1 Series Summation Subroutine

INPUT VARIABLES

$$\boxed{\widetilde{x}_{arg}, \widetilde{\alpha}, k_{max}}$$

$$\Delta u_1 = \frac{\widetilde{x}_{arg}}{4}$$
$$u_1 = \Delta u_1$$
$$f_7 = -\widetilde{\alpha}\,\Delta u_1^2$$

$$\boxed{k = 3}$$

$$\boxed{k = k + 2}$$

$$\Delta u_1 = \frac{f_7\,\Delta u_1}{k(k-1)}$$
$$u_{1old} = u_1$$
$$u_1 = u_1 + \Delta u_1$$

$u_1 = u_{1old}$?   Yes

No

$k < k_{max}$?   Yes

No

$$\boxed{e_{flag} = 2}$$

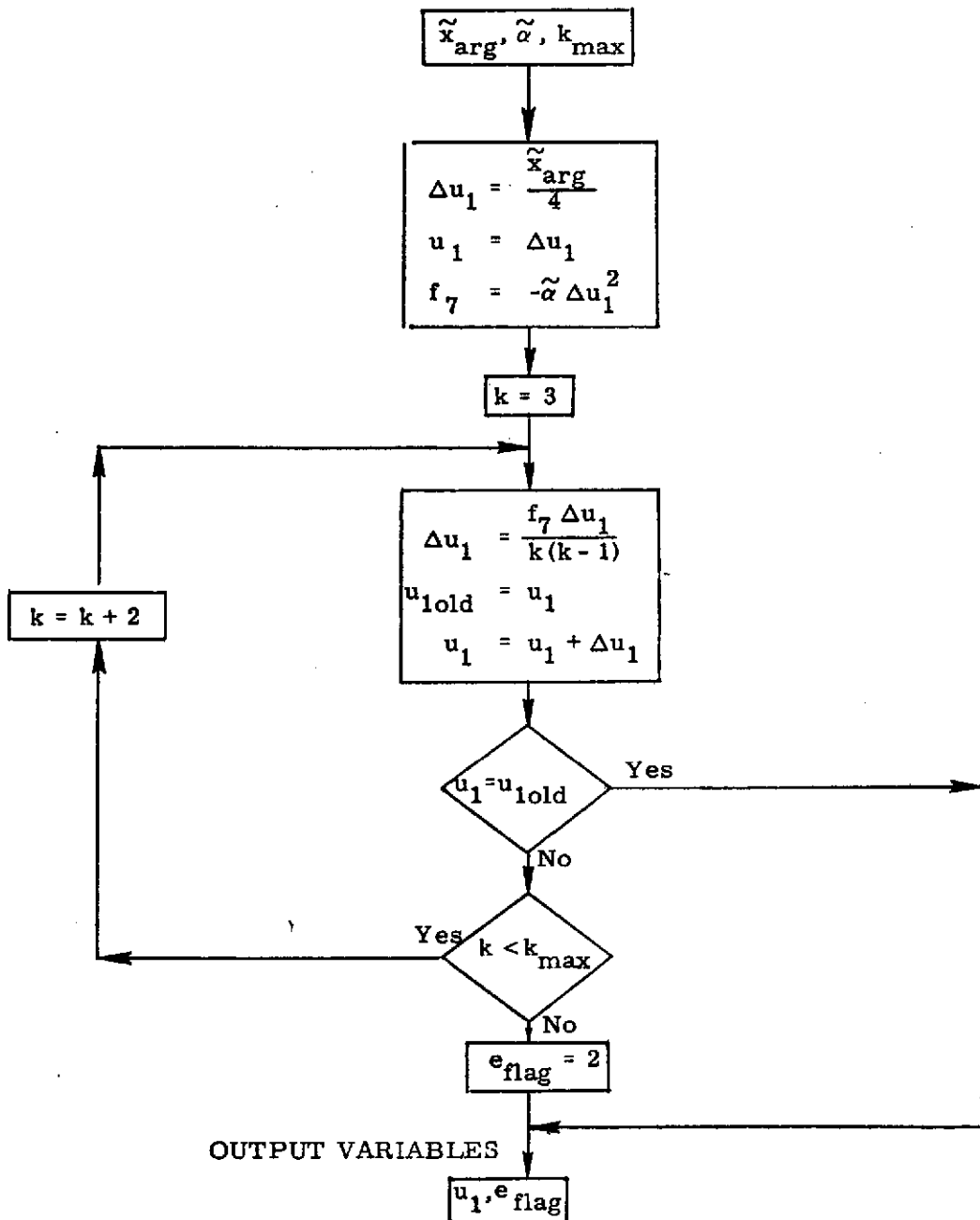OUTPUT VARIABLES

$$\boxed{u_1, e_{flag}}$$

Figure 6. U_1 Series Summation, Detailed Flow Diagram

5.3.3   Q   Continued Fraction Subroutine

This subroutine is identical to the one used in the Conic Required Velocity routines.   It should be noted that this routine has been designed to give double precision (16 digits) accuracy.
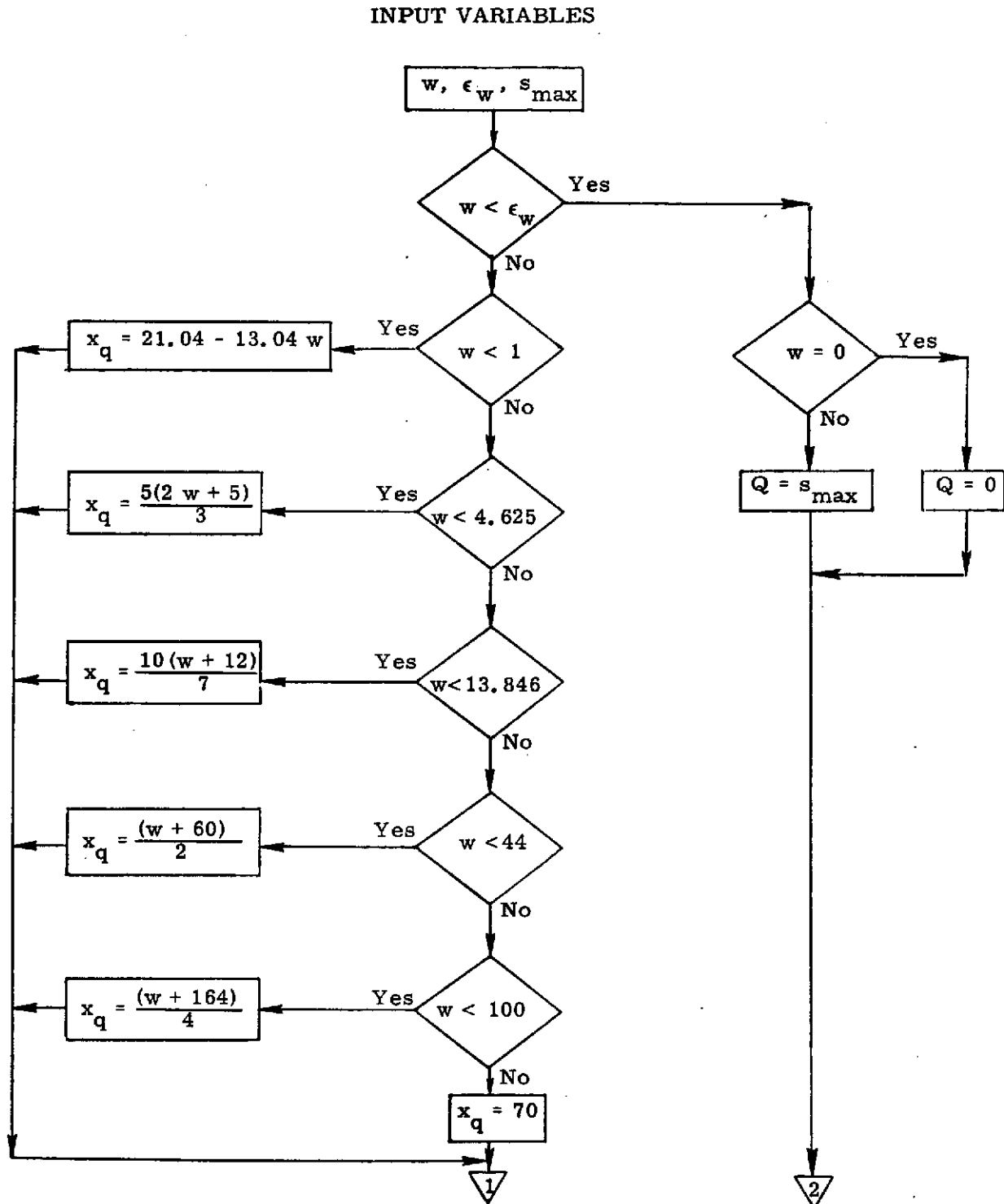
INPUT VARIABLES

Figure 7a.   Q Continued Fraction, Detailed Flow Diagram

$$b = 0$$
$$y = \left(\frac{w-1}{w+1}\right)$$

$$j = [x_q] \quad *$$

$$j = j - 1$$

$$b = \frac{y}{1 + \left(\frac{j-1}{j+2}\right)(1-b)}$$

$$j > 2 \quad \text{Yes} \quad \text{No}$$

$$Q = \frac{1}{w^2} \left[1 + \frac{2(1 - \frac{b}{4})}{3\ w(w+1)}\right]$$

OUTPUT VARIABLE

$$Q$$
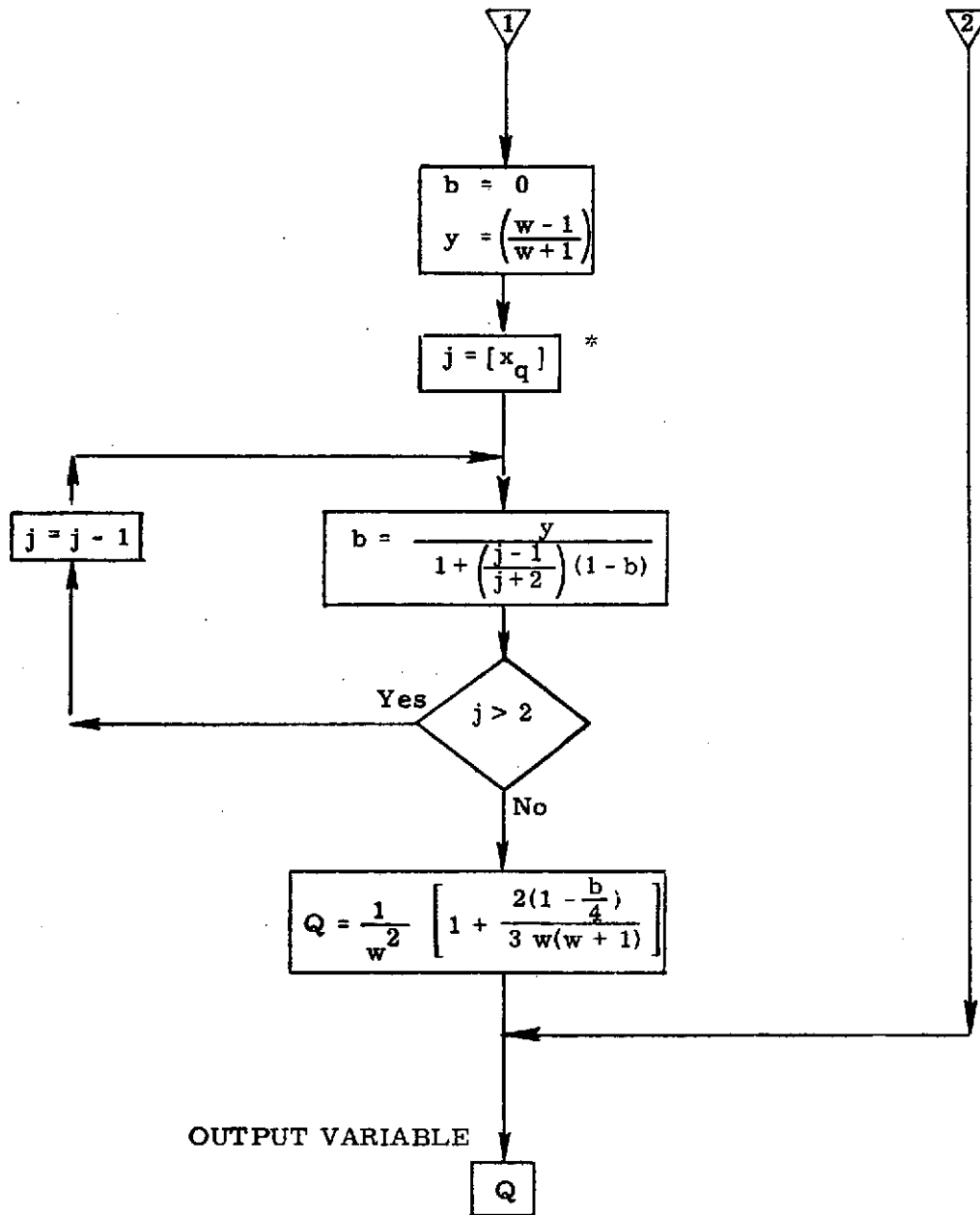
* The symbol $[x_q]$ denotes the largest integer whose magnitude does not exceed the magnitude of x_q.

Figure 7b. Q Continued Fraction, Detailed Flow Diagram

5.3.4 Kepler Iteration Loop Subroutine

This subroutine is similar, though not identical, to the iteration loops used in the Conic Required Velocity routines. All could easily be combined into one routine but have been diagrammed spearately for purposes of clarity.
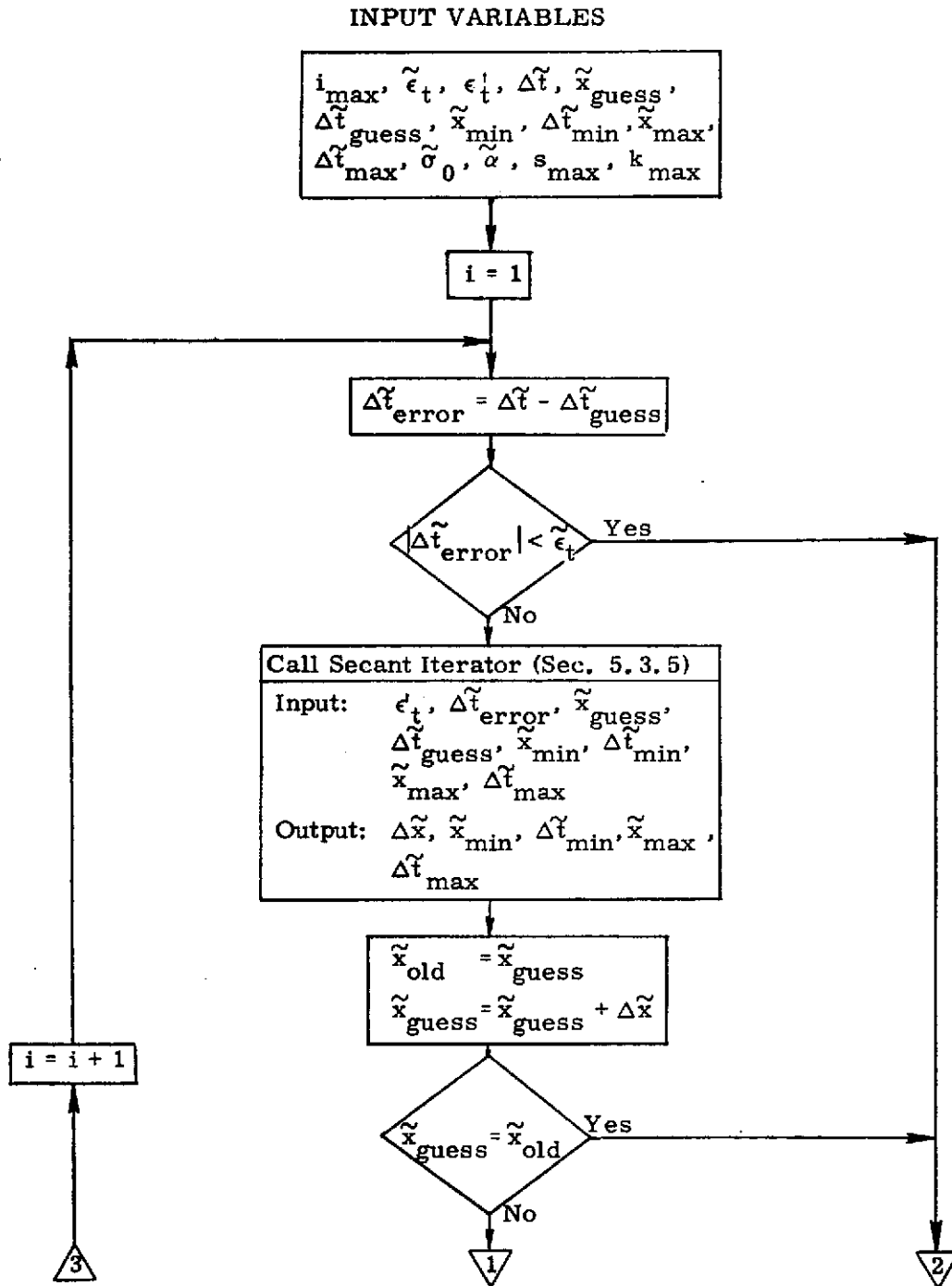
INPUT VARIABLES

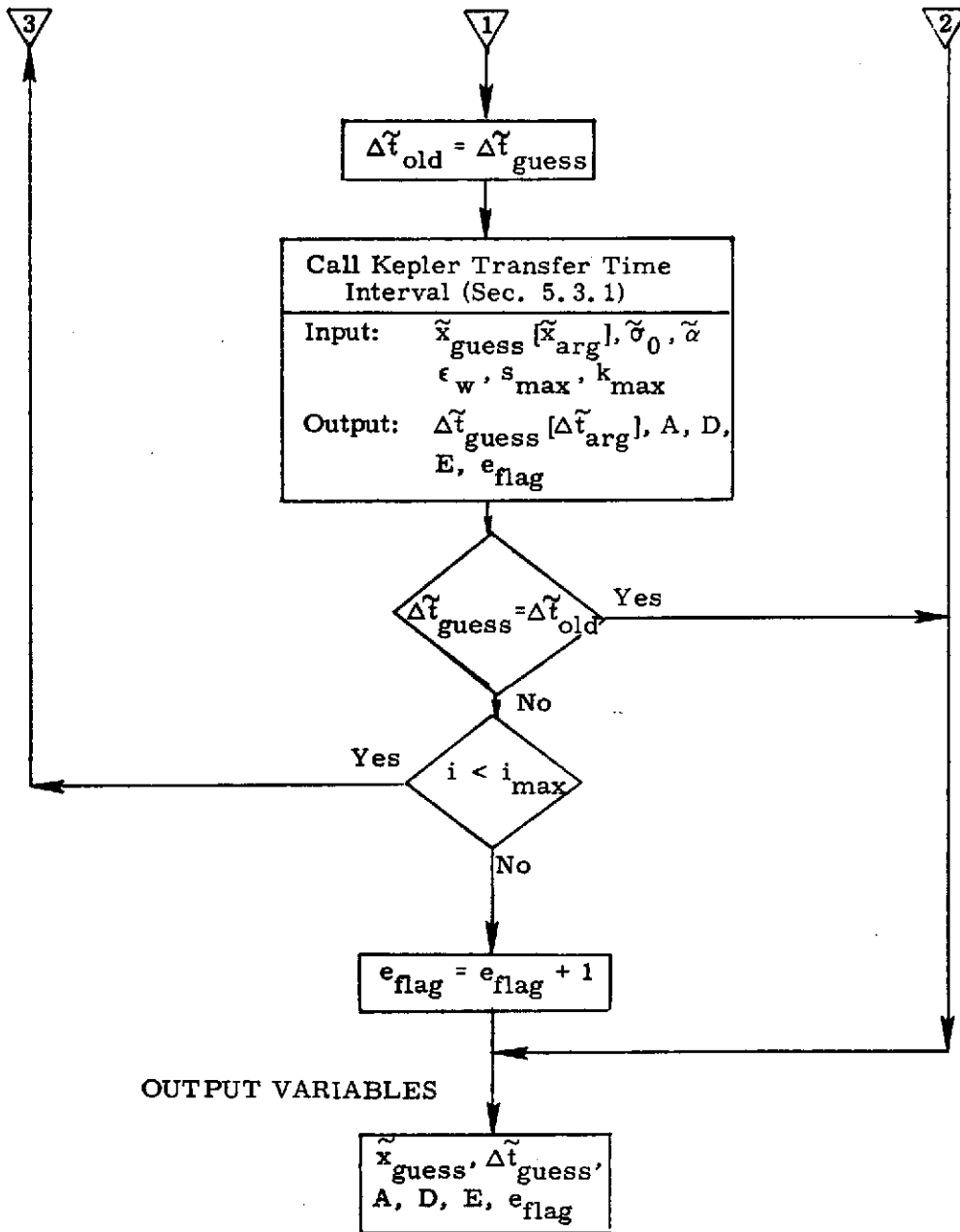Figure 8a. Kepler Iteration Loop, Detailed Flow Diagram

Figure 8b. Kepler Iteration Loop, Detailed Flow Diagram

5.3.5   Secant Iterator Subroutine

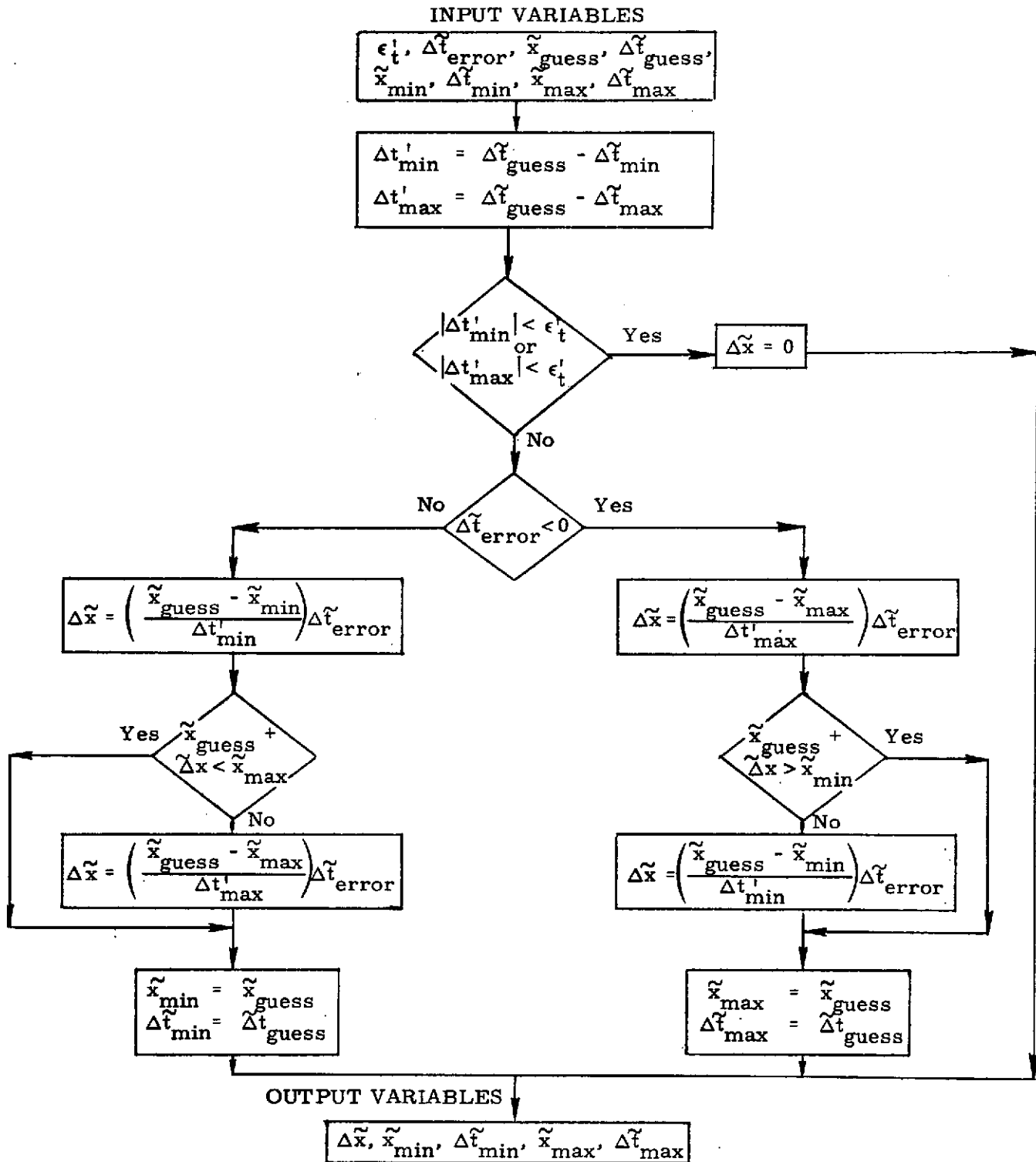This subroutine is identical to the one used in the Conic Required Velocity routines.

INPUT VARIABLES

$$\epsilon_t', \ \Delta \tilde{t}_{error}, \ \tilde{x}_{guess}, \ \Delta \tilde{t}_{guess},$$
$$\tilde{x}_{min}, \ \Delta \tilde{t}_{min}, \ \tilde{x}_{max}, \ \Delta \tilde{t}_{max}$$

$$\Delta t'_{min} = \Delta \tilde{t}_{guess} - \Delta \tilde{t}_{min}$$
$$\Delta t'_{max} = \Delta \tilde{t}_{guess} - \Delta \tilde{t}_{max}$$

$$|\Delta t'_{min}| < \epsilon_t' \ \text{or} \ |\Delta t'_{max}| < \epsilon_t' \quad \text{Yes} \quad \Delta \tilde{x} = 0$$

No

$$\Delta \tilde{t}_{error} < 0$$

No — Yes

$$\Delta \tilde{x} = \left(\frac{\tilde{x}_{guess} - \tilde{x}_{min}}{\Delta t'_{min}} \right) \Delta \tilde{t}_{error}$$

$$\Delta \tilde{x} = \left(\frac{\tilde{x}_{guess} - \tilde{x}_{max}}{\Delta t'_{max}} \right) \Delta \tilde{t}_{error}$$

$$\text{Yes} \quad \tilde{x}_{guess} + \Delta \tilde{x} < \tilde{x}_{max}$$

$$\tilde{x}_{guess} + \Delta \tilde{x} > \tilde{x}_{min} \quad \text{Yes}$$

No

No

$$\Delta \tilde{x} = \left(\frac{\tilde{x}_{guess} - \tilde{x}_{max}}{\Delta t'_{max}} \right) \Delta \tilde{t}_{error}$$

$$\Delta \tilde{x} = \left(\frac{\tilde{x}_{guess} - \tilde{x}_{min}}{\Delta t'_{min}} \right) \Delta \tilde{t}_{error}$$

$$\tilde{x}_{min} = \tilde{x}_{guess}$$
$$\Delta \tilde{t}_{min} = \Delta \tilde{t}_{guess}$$

$$\tilde{x}_{max} = \tilde{x}_{guess}$$
$$\Delta \tilde{t}_{max} = \Delta \tilde{t}_{guess}$$

OUTPUT VARIABLES

$$\Delta \tilde{x}, \ \tilde{x}_{min}, \ \Delta \tilde{t}_{min}, \ \tilde{x}_{max}, \ \Delta \tilde{t}_{max}$$

Figure 9.   Secant Iterator, Detailed Flow Diagram

5.3.6 Extrapolated State Vector Subroutine

INPUT VARIABLES

$$f_2, b_4, \tilde{\sigma}_0, r_0,$$
$$A, D, E, \underline{i}_{r_0}, \tilde{\underline{v}}_0$$

$$F = 1 - 2A$$
$$\tilde{G} = 2(D E + \tilde{\sigma}_0 A)$$
$$\tilde{F}_t = -2 b_4 D E$$
$$G_t = 1 - 2 b_4 A$$

$$\underline{r} = r_0 (F \underline{i}_{r_0} + \tilde{G} \tilde{\underline{v}}_0)$$
$$\underline{v} = f_2 (\tilde{F}_t \underline{i}_{r_0} + G_t \tilde{\underline{v}}_0)$$

OUTPUT VARIABLES

$$\underline{r}, \underline{v}$$

Figure 10. Extrapolated State Vector, Detailed Flow Diagram

6. SUPPLEMENTARY INFORMATION

The analytic expressions for the generalized Kepler equation and the extrapolated position and velocity vectors are given in Battin (Ref. 3). Battin also devised the normalized formulation and organized the overall flow diagram structure. However, the details of the structure and the testing were done by Shepperd and Robertson.

A Newton iteration scheme may be devised at some time in the future. This would result in faster convergence near the solution point, but its worth depends on the expense of the derivative evaluation. It may be shown that if the derivative evaluation by itself takes more than 44% of the computation time used for the other calculations in one pass through the loop, then it is more efficient timewise to use the secant method.

REFERENCES

1. Battin, R. H. , 1964, Astronautical Guidance, McGraw-Hill.

2. Battin, R. H. , and Fraser, D. C. , Notebook for Space Guidance and Navigation.

3. Battin, R. H. , and Fraser, D. C. , Space Guidance and Navigation, AIAA Professional Study Series.

4. Hildebrand, F. B. , 1956, Introduction to Numerical Analysis, McGraw-Hill.

5. Shepperd, S. W. , and Robertson, W. M. , June 1973, Conic Required Velocity, Draper Laboratory Space Shuttle GN&C Equation Document No. 26, to be published.