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CONJUGACY CLASSES IN THE WEYL GROUP

by

R. W. Carter

1. Introduction

The object of this paper is to describe the decomposition of the Weyl group of a simple Lie algebra into its classes of conjugate elements. By the Cartan-Killing classification of simple Lie algebras over the complex field [7] the Weyl groups to be considered are:

$$
\begin{array}{ll}
W(A_l) & l \geqq 1 \\
W(B_l) \cong W(C_l) & l \geqq 2 \\
W(D_l) & l \geqq 4 \\
W(G_2) & \\
W(F_4) & \\
W(E_6) & \\
W(E_7) & \\
W(E_8) &
\end{array}
$$

Now the conjugacy classes of all these groups have been determined individually, in fact it is also known how to find the irreducible complex characters of all these groups. $W(A_l)$ is isomorphic to the symmetric group S_{l+1}. Its conjugacy classes are parametrised by partitions of $l+1$ and its irreducible representations are obtained by the classical theory of Frobenius [6], Schur [9] and Young [14]. $W(B_l)$ and $W(C_l)$ are both isomorphic to the 'hyperoctahedral group' of order $2^l.l!$ Its conjugacy classes can be parametrised by pairs of partitions (λ, μ) with $|\lambda|+|\mu| = l$, and its irreducible representations have been described by Specht [10] and Young [15]. $W(D_l)$ is a subgroup of $W(B_l)$ of index 2 and its conjugacy classes and irreducible representations are also described by Young [15]. The exceptional Weyl group $W(G_2)$ is isomorphic to the dihedral group of order 12. $W(F_4)$ is a soluble group of order 1152, and is isomorphic to the orthogonal group $O_4(3)$ leaving invariant a quadratic form of maximal index in a 4-dimensional vector space over the field $GF(3)$. Its conjugacy classes can, for example, be obtained from the work of Wall [13] and its character table has been obtained by Kondo [16]. The other three exceptional Weyl groups $W(E_6)$, $W(E_7)$, $W(E_8)$ are rather more complicated, and their conjugacy classes and character tables have been determined by Frame [4], [5].

Although the conjugacy classes and irreducible characters are known for all the Weyl groups individually no unified approach has hitherto been obtained which makes use of the common structure of the groups as reflection groups. It is desirable to do this in view of the importance of the Weyl groups in the theories of Lie groups, Lie algebras and algebraic groups. We shall give such a unified description of the conjugacy classes in the present paper – for the irreducible representations the problem remains open.

As the conjugacy classes of $W(A_l)$ can be parametrised by partitions, the main problem is to find a suitable generalisation of the idea of a partition to groups of type other than A. The objects which we use to generalise partitions are certain graphs which we shall call 'admissible diagrams'. These are interpreted in terms of the underlying root system of the group, and are closely related to Dynkin diagrams. We shall prove a classification theorem for admissible diagrams which is somewhat reminiscent of the well known classification theorem for Dynkin diagrams (See, for example, [7]). In fact the admissible diagrams for Weyl groups of type A are graphs, all of whose connected components are Dynkin diagrams of type A. These admissible diagrams are therefore in a natural $1-1$ correspondence with partitions.

The admissible diagrams can also be used to parametrise (though not in a $1-1$ manner) the conjugacy classes of nilpotent elements in the corresponding simple Lie algebra over the complex field, and the conjugacy classes of unipotent elements in the corresponding algebraic group [17].

2. Products of reflections

Let V be a Euclidean space of dimension l. For each non-zero vector r in V, let w_r be the reflection in the hyperplane orthogonal to r. Thus

$$w_r(x) = x - 2\frac{(r, x)}{(r, r)}r.$$

Let Φ be a subset of V satisfying the following axioms

(i) Φ is a finite subset of non-zero vectors which span V.
(ii) If $r, s \in \Phi$ then $w_r(s) \in \Phi$.
(iii) If $r, s \in \Phi$ then $2(r, s)/(r, r)$ is a rational integer.
(iv) If r, $\lambda r \in \Phi$ where λ is real, then $\lambda = \pm 1$.

Then Φ is isomorphic to the root system of some semisimple Lie algebra

[7]. The Weyl group of this Lie algebra is isomorphic to the group W of orthogonal transformations of V generated by the reflections w_r for all $r \in \Phi$. The dimension l of V is called the rank of W.

LEMMA 1. *Let S be any set of vectors in V. Then an element $w \in W$ which fixes all vectors in S can be expressed as a product of reflections w_r, each of which fixes all vectors in S.*

PROOF. This is a well-known property of Weyl groups. See for example [12].

Now each element w in W can be expressed in the form

$$w = w_{r_1} w_{r_2} \cdots w_{r_k} \qquad r_i \in \Phi.$$

We denote by $\bar{l}(w)$ the smallest value of k in any such expression for w.

LEMMA 2. *$\bar{l}(w)$ is the number of eigenvalues of w on V which are not equal to 1. In particular $\bar{l}(w) \leqq l$.*

PROOF. Suppose $\bar{l}(w) = k$. Then w has an expression of the form

$$w = w_{r_1} w_{r_2} \cdots w_{r_k}.$$

Let H_{r_i} be the hyperplane orthogonal to r_i and let

$$U = H_{r_1} \cap H_{r_2} \cap \cdots \cap H_{r_k}.$$

Then w fixes every vector in U and $\dim U \geqq l-k$. Thus w has at least $l-k$ eigenvalues equal to 1, and so at most k not equal to 1.

Conversely, suppose w has k eigenvalues not equal to 1. Let V_1 be the set of elements of V fixed by w, and $V_1^\perp$ be the orthogonal subspace. Then $\dim V_1 = l-k$ and $\dim V_1^\perp = k$. w fixes every vector in V_1, so is a product of reflections w_r fixing each vector in V_1. Thus all the roots r occurring are in $V_1^\perp$. Suppose w fixes some vector in V. Then $k < l$ and so $\dim V_1^\perp < \dim V$. The roots in $V_1^\perp$ form a root system in the subspace they generate which has dimension less than l, and w is an element of the Weyl group of this root system. So by induction w is a product of at most k reflections and $\bar{l}(w) \leqq k$.

It is therefore sufficient to show that if w fixes no vector in V then w can be expressed as a product of at most l reflections. Let $r \in \Phi$. Since w fixes no vector, $w-1$ is non-singular. Thus there exists $v \in V$ such that $(w-1)v = r$. Thus $w(v) = v+r$. Now

$$(w(v), w(v)) = (v, v).$$

Thus $(v+r, v+r) = (v, v)$ and so

$$\frac{2(v, r)}{(r, r)} = -1.$$

It follows that $w_r(v) = v+r$. Hence $w(v) = w_r(v)$ and so $w_r w(v) = v$. By Lemma 1 $w_r w$ is a product of reflections which all fix v so is contained in a Weyl group of smaller rank. By induction $w_r w$ is a product of at most $l-1$ reflections. Thus w is a product of at most l reflections and the lemma is proved.

An expression $w_{r_1} w_{r_2} \cdots w_{r_k}$ will be called reduced if $l(w_{r_1} w_{r_2} \cdots w_{r_k}) = k$.

LEMMA 3. *Let $r_1, r_2 \cdots r_k \in \Phi$. Then $w_{r_1} w_{r_2} \cdots w_{r_k}$ is reduced if and only if $r_1, r_2 \cdots r_k$ are linearly independent.*

PROOF. Let $w = w_{r_1} w_{r_2} \cdots w_{r_k}$. Suppose the expression is reduced. Then w has k eigenvalues not equal to 1 and so

$$\dim (H_{r_1} \cap H_{r_2} \cap \cdots \cap H_{r_k}) = l-k.$$

(The dimension cannot be larger since w operates as the identity on this subspace). It follows that $r_1, r_2 \cdots r_k$ are linearly independent.

Now suppose conversely that $r_1, r_2 \cdots r_k$ are linearly independent. Consider the subspace $(w-1)V$. Choose a vector x such that

$$x \in H_{r_2} \cap \cdots \cap H_{r_k} \text{ but } x \notin H_{r_1}.$$

Then $w(x)-x$ is a non-zero multiple of r_1. Thus $r_1 \in (w-1)V$. Now choose an x with $x \in H_{r_3} \cap \cdots \cap H_{r_k}$ but $x \notin H_{r_2}$. Then $w(x)-x = \lambda r_2 + \mu r_1$ where λ, μ are real and $\lambda \neq 0$. Hence $r_2 \in (w-1)V$. Arguing in this way we see that $r_1, \cdots r_k \in (w-1)V$. Thus $\dim(w-1)V = k$. Now if w had a shorter expression $w = w_{s_1} w_{s_2} \cdots w_{s_h}$ with $h < k$, every element of $(w-1)V$ would be a linear combination of $s_1, s_2 \cdots s_h$, and so we would have $\dim (w-1)V < k$, a contradiction.

We now consider involuations in W. If $r_1, r_2 \cdots r_k$ are mutually orthogonal elements of Φ, the reflections $w_{r_1}, w_{r_2} \cdots w_{r_k}$ commute with one another and so $w_{r_1} w_{r_2} \cdots w_{r_k}$ is an involution (or the identity). We show that every involution in W has this form.

LEMMA 4. *If the transformation -1 is in the Weyl group of the root system Φ, then Φ contains l orthogonal roots.*

PROOF. Lemma 2 shows that $l(-1) = l$ and Lemma 3 shows that we may write

$$-1 = w_{r_1} w_{r_2} \cdots w_{r_l}$$

where $r_1, r_2 \cdots r_l$ is a linearly independent set of roots. Let v be a vector orthogonal to r_1. Then

$$w_{r_1} w_{r_2} \cdots w_{r_l}(v) = -v$$

and so

$$w_{r_2} \cdots w_{r_l}(v) = -w_{r_1}(v) = -v.$$

Thus we have

$$2v = v - w_{r_2} \cdots w_{r_l}(v)$$

and this is a linear combination of $r_2, \cdots r_l$. Thus the subspace orthogonal to r_1 is the subspace spanned by $r_2, \cdots r_l$. In particular r_1 is orthogonal to $r_2, \cdots r_l$. A repetition of this argument shows that r_2 is orthogonal to $r_3, \cdots r_l$ and eventually we see that $r_1, r_2, \cdots r_l$ are mutually orthogonal.

LEMMA 5. *Every involution w in W can be expressed as a product of $\bar{l}(w)$ reflections corresponding to mutually orthogonal roots.*

PROOF. Let V_1 be the set of vectors x such that $w(x) = x$ and V_{-1} be the set of vectors with $w(x) = -x$. Then $V = V_1 \oplus V_{-1}$ since w is an involution. If $\dim V_1 > 0$, w is an element of a Weyl group of smaller rank, so the result follows by induction. If $\dim V_1 = 0$ then $w = -1$ on V. Then there exist l orthogonal roots $r_1, r_2 \cdots r_l$ in Φ by Lemma 4. Hence $w = w_{r_1} w_{r_2} \cdots w_{r_l}$ and $\bar{l}(w) = l$.

3. The subset W_0

If w_i is an involution in W, the decomposition of V with respect to w_i will now be denoted by

$$V = V_1(w_i) \oplus V_{-1}(w_i)$$

where $w_i = 1$ on $V_1(w_i)$ and $w_i = -1$ on $V_{-1}(w_i)$. We denote by W_0 the subset of W of elements w expressible in the form $w = w_1 w_2$, where $w_1^2 = w_2^2 = 1$ and

$$V_{-1}(w_1) \cap V_{-1}(w_2) = 0.$$

(It will turn out eventually that $W_0 = W$.)

LEMMA 6. *Let $w = w_1 w_2$ where $w_1^2 = w_2^2 = 1$ and*

$$V_{-1}(w_1) \cap V_{-1}(w_2) = 0.$$

Then

$$\bar{l}(w_1) + \bar{l}(w_2) = \bar{l}(w).$$

PROOF. Suppose x is a vector in V such that $w(x) = x$. Then $w_1 w_2(x) = x$ and so $w_1(x) = w_2(x)$. Thus $w_1(x) - x = w_2(x) - x$. Now $w_1(x) - x \in V_{-1}(w_1)$ and $w_2(x) - x \in V_{-1}(w_2)$. Thus both these vectors are 0, and so

$$V_1(w) = V_1(w_1) \cap V_1(w_2).$$

Now $V_{-1}(w_1)^{\perp} = V_1(w_1)$ and $V_{-1}(w_2)^{\perp} = V_1(w_2)$. Since

$$V_{-1}(w_1) \cap V_{-1}(w_2) = 0$$

we have

$$V_1(w_1) + V_1(w_2) = V.$$

Thus

$$l(w) = l - \dim V_1(w) = 2l - \dim V_1(w_1) - \dim V_1(w_2) = l(w_1) + l(w_2).$$

LEMMA 7. *W_0 is a union of conjugacy classes of W.*

PROOF. Suppose $w \in W_0$ and consider $w'ww'^{-1}$. Let $w = w_1 w_2$ as above. Then

$$V_{-1}(w'w_1w'^{-1}) = w'(V_{-1}(w_1)),$$
$$V_{-1}(w'w_2w'^{-1}) = w'(V_{-1}(w_2)).$$

Thus

$$V_{-1}(w'w_1w'^{-1}) \cap V_{-1}(w'w_2w'^{-1}) = w'(V_{-1}(w_1) \cap V_{-1}(w_2)) = 0.$$

Hence $w'ww'^{-1} \in W_0$.

Let w be an element of W_0 and $w = w_1 w_2$ be a decomposition of w into a product of two involutions as described in Lemma 6. By Lemma 5 each of w_1, w_2 can be expressed as products of reflections corresponding to mutually orthogonal roots. Thus

$$w_1 = w_{r_1} w_{r_2} \cdots w_{r_k}, \qquad w_2 = w_{r_{k+1}} w_{r_{k+2}} \cdots w_{r_{k+h}}$$

and $w = w_{r_1} w_{r_2} \cdots w_{r_{k+h}}$ where $k+h = l(w)$. Corresponding to each such decomposition of w we define a graph Γ. Γ has $l(w)$ nodes, one corresponding to each root $r_1, r_2 \cdots r_{k+h}$. The nodes corresponding to distinct roots r, s are joined by a bond of strength $n_{rs} \cdot n_{sr}$, where

$$n_{rs} = \frac{2(r, s)}{(r, r)} \qquad n_{sr} = \frac{2(s, r)}{(s, s)}.$$

We observe that n_{rs}, n_{sr} are both integers, and that their product is either 0, 1, 2 or 3. Thus the number of bonds joining any two nodes is 0, 1, 2 or 3.

If $w \in W_0$ has a decomposition with graph Γ, any conjugate of w also has a decomposition with graph Γ. For if $w = w_{r_1} w_{r_2} \cdots w_{r_{k+h}}$, we have $w'ww'^{-1} = w_{s_1} w_{s_2} \cdots w_{s_{k+h}}$ where $s_i = w'(r_i)$. We say then the graph Γ is associated with this conjugacy class. It is possible that more than one graph may be associated with a given conjugacy class of W in W_0.

EXAMPLES. (i) The conjugacy class containing the unit element of W is represented by $\Gamma = \phi$, the empty set.

(ii) The conjugacy class consisting of the Coxeter elements of W is in W_0, and is represented by the graph Γ which is the Dynkin diagram of

W. (A Coxeter element is the product of reflections corresponding to a complete system of fundamental roots of Φ. It was shown by Coxeter that all such elements are conjugate [2], irrespective of the fundamental system chosen or the order in which the reflections occur). The graph Γ for a Coxeter element may be chosen as the Dynkin diagram of W, by definition of the latter. As it is possible to divide the nodes in any Dynkin diagram into two disjoint subsets, each corresponding to a set of mutually orthogonal roots, it follows that the class of Coxeter elements is in W_0.

(iii) A subsystem of a root system Φ is a subset of Φ which is itself a root system in the space which it spans. If W is the Weyl group of Φ, a Weyl subgroup of W is a subgroup generated by the reflections w_r corresponding to the roots $r \in \Phi'$, where Φ' is a subsystem of Φ.

Let W' be a Weyl subgroup of W. Then the conjugacy class of W containing the Coxeter elements of W' is in W_0. The graph Γ representing it may be taken to be the Dynkin diagram of W'.

4. Admissible diagrams

Each of the graphs Γ corresponding to conjugacy classes of W in W_0 satisfy the following conditions:

(a) The nodes of Γ correspond to a set of linearly independent roots.

(b) Each subgraph of Γ which is a cycle contains an even number of nodes. (A subgraph of a graph is a subset of the nodes, together with the bonds joining the nodes in the subset. A cycle is a graph in which each node is joined to just two others by bonds of multiplicity greater than 0).

A graph satisfying conditions (a) and (b) will be called an admissible diagram.

LEMMA 8. *Every admissible diagram without cycles associated to a conjugacy class of W is the Dynkin diagram of some Weyl subgroup of W.*

PROOF. Let Γ be such a graph. The nodes of Γ correspond to a set of roots in Φ. Any such root may be replaced by its negative without affecting the corresponding reflection. Since Γ contains no cycles we may choose a set of roots corresponding to the nodes of Γ which are mutually obtuse, i.e. $(r, s) \leqq 0$ for any pair. (This is evident using induction on the number of nodes). However, the diagram corresponding to a set of linearly independent roots which are mutually obtuse is a Dynkin diagram. Let Π' be this set of roots and W' be the group generated by the reflections w_r with $r \in \Pi'$. Let $\Phi' = W'(\Pi')$. Then Φ' is a subsystem of

Φ and W' is the Weyl group of Φ' .Γ is the Dynkin diagram of the Weyl subgroup W' of W.

The graphs which are Dynkin diagrams of Weyl subgroups may be obtained by a standard algorithm, due independently to Borel and de Siebenthal [1] and to Dynkin [3]. This involves the extended Dynkin diagram, which is obtained from the Dynkin diagram by the addition of one further node corresponding to the negative of the highest root. The Dynkin diagrams of all possible Weyl subgroups are obtained as follows. Take the extended Dynkin diagram of Φ and remove one or more nodes in all possible ways. Take also the duals of the diagrams obtained in this way from $\tilde{\Phi}$, the root system dual to Φ. (The dual of a root system is obtained by interchanging long and short roots). Then repeat the process with the diagrams obtained, and continue any number of times. It is a straightforward matter to enumerate the Weyl subgroups in the individual cases using this algorithm.

We now consider admissible diagrams which do contain cycles. The following lemma reduces the classification problem for such graphs to the case in which Γ is an admissible diagram associated with Φ but with no proper subsystem of Φ.

LEMMA 9. *Let Γ be an admissible diagram for Φ. Then there exists an admissible diagram $\bar{\Gamma}$ without cycles, whose connected components $\bar{\Gamma}_i$ are therefore Dynkin diagrams, such that Γ can be obtained from $\bar{\Gamma}$ by replacing certain $\bar{\Gamma}_i$ by admissible diagrams with cycles associated with W_i, the Weyl group of $\bar{\Gamma}_i$, but with no proper Weyl subgroup of W_i.*

PROOF. Let $r_1, r_2, \cdots$ be a set of roots corresponding to the nodes of the graph Γ. These roots split into disjoint subsets corresponding to the connected components of Γ, distinct subsets being orthogonal to one another. Let $r_1, r_2, \cdots r_k$ be one of these subsets. Let Φ' be the smallest root system in Φ containing $r_1, r_2, \cdots r_k$. (The intersection of all the subsystems of Φ containing these vectors). Φ' is an indecomposable root system of rank k. We thus obtain subsystems Φ', Φ'', $\cdots$ of Φ whose union has a Dynkin diagram $\bar{\Gamma}$ which is an admissible diagram for W. The diagram of $r_1, r_2, \cdots r_k$ is an admissible diagram for W', the Weyl group of Φ', but for no Weyl subgroup of W'. For no subsystem of Φ' contains all the roots $r_1, r_2, \cdots r_k$.

We must therefore concentrate on diagrams admissible for W but for no Weyl subgroup of W.

LEMMA 10. *Let Γ be a diagram admissible for W but for no Weyl subgroup of W. Then the type of W is uniquely determined by Γ.*

PROOF. Let S be a set of roots corresponding to the nodes of Γ. Let

W_s be the group generated by the reflections w_r for $r \in S$ and let $\Phi' = W_s(S)$. Φ' is completely determined by S. We show that Φ' is a root system. Then, since Γ is not admissible for any Weyl subgroup, it follows that $\Phi' = \Phi$. Thus Φ is determined by S and the type of W is determined by Γ.

To show that Φ' is a root system we must verify that if $r, s \in \Phi'$ then $w_r(s) \in \Phi'$. Now $r = w_1(\bar{r})$ where $w_1 \in W_S$, $\bar{r} \in S$, and $s = w_2(\bar{s})$ where $w_2 \in W_S$, $\bar{s} \in S$. Thus

$$w_r(s) = w_1 w_{\bar{r}} w_1^{-1} w_2(\bar{s}) \in W_S(S) = \Phi'$$

and the lemma is proved.

5. A classification theorem

In order to classify the admissible diagrams for a Weyl group W it is sufficient, by what has been said, to determine the admissible diagrams containing a cycle which are associated with W but with no Weyl subgroup of W. To this end it is useful to have detailed information about the indecomposable root systems, and this information is given in Table 1.

TABLE 1

Type	Roots			
A_l	$e_i - e_j$		$0 \leqq i, j \leqq l$	$i \neq j$
B_l	$\pm e_i \pm e_j$		$1 \leqq i, j \leqq l$	$i \neq j$
	$\pm e_i$		$1 \leqq i \leqq l$	
C_l	$\pm e_i \pm e_j$		$1 \leqq i, j \leqq l$	$i \neq j$
	$\pm 2e_i$		$1 \leqq i \leqq l$	
D_l	$\pm e_i \pm e_j$		$1 \leqq i, j \leqq l$	$i \neq j$
F_4	$\pm e_i \pm e_j$		$1 \leqq i, j \leqq 4$	$i \neq j$
	$\pm e_i$		$1 \leqq i \leqq 4$	
	$\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)$			
E_6	$\pm e_i \pm e_j$		$3 \leqq i, j \leqq 7$	$i \neq j$
	$\frac{1}{2} \sum_{i=1}^{8} \varepsilon_i e_i$		$\varepsilon_i = \pm 1, \quad \prod_{i=1}^{8} \varepsilon_i = 1$	
			$\varepsilon_1 = \varepsilon_2 = \varepsilon_8$	
E_7	$\pm e_i \pm e_j$	$\pm(e_1 + e_8)$	$2 \leqq i, j \leqq 7$	$i \neq j$
	$\frac{1}{2} \sum_{i=1}^{8} \varepsilon_i e_i$		$\varepsilon_i = \pm 1, \quad \prod_{i=1}^{8} \varepsilon_i = 1$	
			$\varepsilon_1 = \varepsilon_8$	
E_8	$\pm e_i \pm e_j$		$1 \leqq i, j \leqq 8$	$i \neq j$
	$\frac{1}{2} \sum_{i=1}^{8} \varepsilon_i e_i$		$\varepsilon_i = \pm 1, \quad \prod_{i=1}^{8} \varepsilon_i = 1$	

TABLE 2

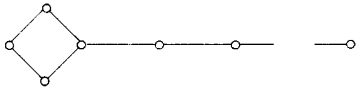
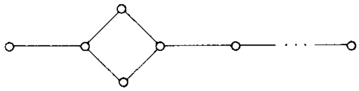
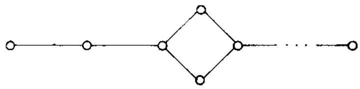
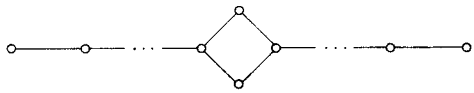
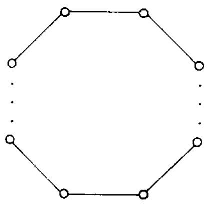
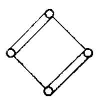
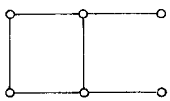
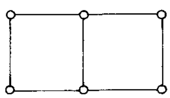
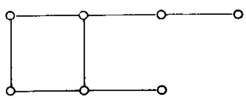
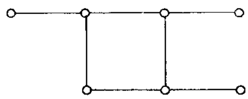
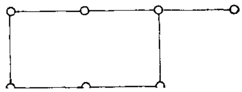
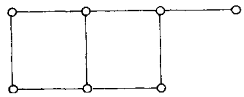
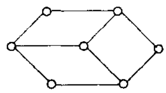
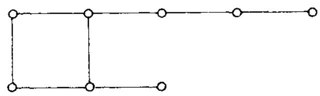
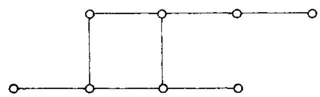
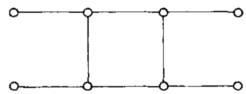
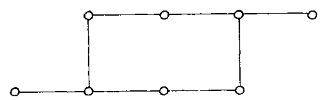
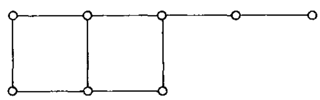
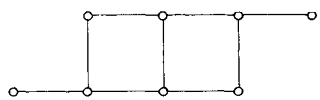
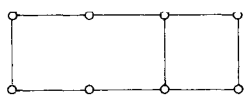
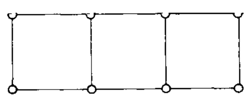
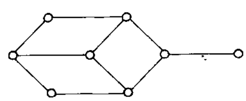
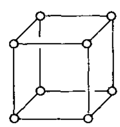
Φ	Γ	
A_l	None	
B_l	None	
C_l	None	
D_l	$D_l\ (a_1)$	
	$D_l\ (a_2)$	
	$D_l\ (a_3)$	
	$\vdots$	
	$D_l\ (a_{\frac{1}{2}l-1})$	 (if l even)
	$D_l\ (b_{\frac{1}{2}l-1})$	 (if l even)
G_2	None	
F_4	$F_4\ (a_1)$	
E_6	$E_6\ (a_1)$	
	$E_6\ (a_2)$	
E_7	$E_7\ (a_1)$	
	$E_7\ (a_2)$	

TABLE 2 (*continued*)

Φ	Γ	
	$E_7\ (b_2)$	
	$E_7\ (a_3)$	
	$E_7\ (a_4)$	
E_8	$E_8\ (a_1)$	
	$E_8\ (a_2)$	
	$E_8\ (a_3)$	
	$E_8\ (b_3)$	
	$E_8\ (a_4)$	
	$E_8\ (a_5)$	
	$E_8\ (b_5)$	
	$E_8\ (a_6)$	
	$E_8\ (a_7)$	
	$E_8\ (a_8)$	

The vectors e_i in this table form an orthonormal basis of a Euclidean space. It is convenient to describe the root system of A_l in an $(l+1)$ dimensional space, and the root systems of E_6 and E_7 as subsystems of the root system of E_8 in an 8-dimensional space. We have omitted the system G_2 as we shall not require its roots in this form.

THEOREM A. *Let Γ be an admissible diagram associated with the indecomposable root system Φ but with no proper subsystem of Φ, such that Γ contains a cycle. Then Γ is one of the graphs shown in Table 2. Moreover these graphs all have the property described.*

(The reason for choosing the notation as it is in Table 2 will become apparent later).

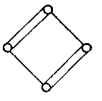
PROOF. It is necessary to use a case-by-case discussion. We make use of the facts that every subgraph of an admissible diagram is an admissible diagram, and that every subgraph without cycles of an admissible diagram is a Dynkin diagram. Note that Γ must be connected as otherwise it would be associated with a subsystem $\Phi_1 \cup \Phi_2 \cup \cdots$ where Φ_i consists of the intersection of Φ with the subspace spanned by the roots corresponding to the nodes of the i^{th} connected component of Φ.

Suppose Φ is of type A_l. Then no cycle can occur as an admissible diagram. For the corresponding roots (with a suitable numbering of the basis vectors e_i) would have to be

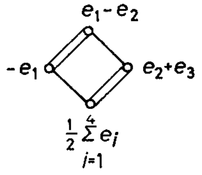
$$e_1-e_2,\, e_2-e_3,\, e_3-e_4, \cdots e_{k-1}-e_k,\, e_k-e_1$$

which are not linearly independent. Thus no admissible diagram can contain a cycle.

Suppose Φ is of type B_l, C_l or F_4. Then Φ contains roots of two different lengths. Now Γ must contain a double bond, as otherwise Γ would be associated with the Weyl subgroup corresponding to the subsystem of all short (or long) roots. Γ cannot contain a branch point, for if it did it would have a connected subgraph without cycles containing a double bond and a branch point, whereas no connected Dynkin diagram has this property. Hence Γ must be a cycle. It must therefore contain at least two double bonds. However no connected Dynkin diagram contains two double bonds. Thus Γ can have only 4 nodes, and must be



This graph does not occur in B_4 or C_4 since in B_4 any two short roots are orthogonal and in C_4 any two long roots are orthogonal. However it does occur in F_4, e.g. as shown below.

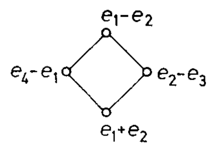


Now suppose Φ is of type D_l. Consider an admissible diagram which is a k-cycle. If $k > 4$ we see from Table 1 that, with a suitable choice of basis, the corresponding roots may be taken as

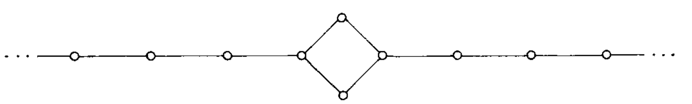
$$e_1 - e_2,\ e_2 - e_3,\ e_3 - e_4, \cdots e_{k-1} - e_k,\ e_k + e_1.$$

Now no vector $\pm e_i \pm e_j$ which is independent of these can be added to give a connected admissible diagram. For the additional node would be joined to two consecutive nodes, and the graph would contain a 3-cycle. It follows that if Γ is a graph of the required type and Γ contains a k-cycle where $k > 4$, then Γ must be a cycle. Therefore l is even and Γ is an l-cycle.

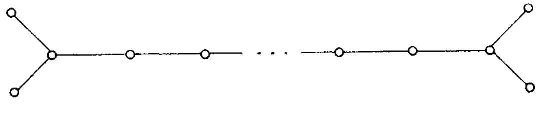
However if Γ contains a 4-cycle there is another possibility. Instead of the configuration described above, the roots corresponding to a 4-cycle can be taken, with a suitable choice of basis, as



Independent roots may then be added at either side, but not at the top or bottom, of this diagram. We then obtain graphs of the form:



These graphs cannot be extended further so that there is a branch point on one of the side chains. For if this were possible there would be a subgraph of form



which is not a Dynkin diagram. Thus the only connected admissible diagrams with a cycle and with l nodes are those of Table 2. It is easily verified that these actually occur.

Now suppose Φ is of type G_2. Then any admissible diagram has at most two nodes, so cannot contain a cycle.

In order to deal with the remaining cases E_6, E_7, E_8 we need some additional lemmas.

LEMMA 11. (i) *Any two sets of* 3 *orthogonal roots in* $\Phi(E_6)$ *are equivalent under* $W(E_6)$.

(ii) *Any two sets of* 3 *orthogonal roots in* $\Phi(E_8)$ *are equivalent under* $W(E_8)$.

(iii) *The sets of* 4 *orthogonal roots in* $\Phi(E_8)$ *fall into two classes under* $W(E_8)$.

PROOF. (i) Let r_1, r_2, r_3 be a set of orthogonal roots in $\Phi(E_6)$. Any root in $\Phi(E_6)$ can be transformed into r_1 by an element of $W(E_6)$. The roots of $\Phi(E_6)$ orthogonal to r_1 form a subsystem $\Phi(A_5)$. Thus any root orthogonal to r_1 can be transformed into r_2 by an element of $W(A_5)$. The roots of $\Phi(A_5)$ orthogonal to r_2 form a subsystem $\Phi(A_3)$. Thus any root orthogonal to r_1 and r_2 can be transformed into r_3 by an element of $W(A_3)$. Hence any set of 3 orthogonal roots can be transformed into r_1, r_2, r_3.

(ii) A similar argument applies for sets of 3 orthogonal roots in $\Phi(E_8)$, the sequence of subsystems in this case being

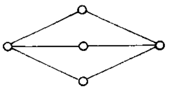
$$\Phi(E_8) \supset \Phi(E_7) \supset \Phi(D_6).$$

(iii) The argument fails for sets of 4 orthogonal roots in $\Phi(E_8)$, because the roots of $\Phi(D_6)$ orthogonal to a given root form a subsystem $\Phi(A_1+D_4)$. The roots of $\Phi(A_1+D_4)$ are not all equivalent under the group $W(A_1+D_4)$, but fall into two subsets. Thus the sets of 4 orthogonal roots fall into at most two classes under $W(E_8)$. However there are exactly two classes, because

$$e_8-e_1,\ e_1-e_3,\ e_4-e_5,\ e_6-e_7$$
$$e_8-e_1,\ e_2-e_3,\ e_4-e_5,\ e_6+e_7$$

are two sets of roots not equivalent under $W(E_8)$. For $\frac{1}{2}\sum_{i=1}^{4} r_i$ is a root for the first set but not the second.

LEMMA 12. *The graph*



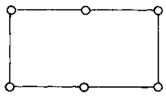
is not an admissible diagram in E_8.

PROOF. By Lemma 11 the three mutually orthogonal roots in this graph may be chosen as $e_2-e_3, e_4-e_5, e_6-e_7$. The two remaining roots must then have the form

$$\tfrac{1}{2}(\pm(e_2-e_3)\pm(e_4-e_5)\pm(e_6-e_7)\pm(e_1-e_8)).$$

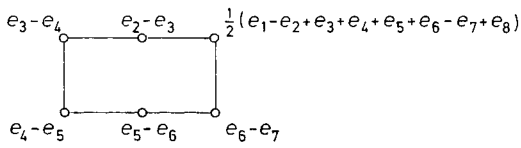
But then the 5 roots are linearly dependent.

LEMMA 13. *The graph*

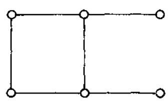


is an admissible diagram in E_7 but not in E_6.

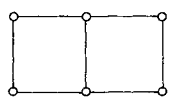
PROOF. In E_6 the argument is similar to Lemma 12. In E_7 the following set of roots will do.



Suppose Φ is a root system of type E_6 and Γ is an admissible diagram of the type being considered in Theorem A. By Lemma 10, Γ is not one of the graphs obtained already in the types considered so far. If Γ contains just one cycle, a 4-cycle, Γ must be the graph

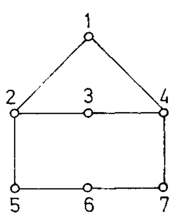


It is easily verified that this graph can be realized in E_6. Now suppose Γ has just two subgraphs which are cycles, and that these are both 4-cycles. Then Γ must be the graph



which can also be realized in E_6. If Γ contains three or more 4-cycles as subgraphs, Γ must contain a subgraph of the kind shown in Lemma 12, which is impossible. Finally, the case where Γ is a 6-cycle is ruled out by Lemma 13.

LEMMA 14. *The graph*



is not an admissible diagram in E_8.

PROOF. Numbering the roots as in the figure and using Lemma 11 we may choose r_1, r_3, r_7 as

$$r_1 = e_8 - e_1 \quad r_3 = e_2 - e_3 \quad r_7 = e_4 - e_5$$

r_5 may then be taken as either $e_6 + e_7$ or $e_6 - e_7$. Now r_4 must have form

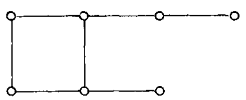
$$r_4 = \tfrac{1}{2}(\pm(e_8 - e_1) \pm (e_2 - e_3) \pm (e_4 - e_5) \pm (e_6 - e_7))$$

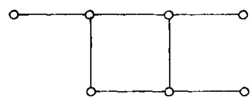
and so r_5, being orthogonal to r_4, must be $e_6 + e_7$. In order to be linearly independent of the rest, r_6 must have form $\frac{1}{2}\sum_{i=1}^{8} \varepsilon_i e_i$ where $\varepsilon_i = \pm 1$. Since r_6 is orthogonal to r_1, r_3 but not to r_5, r_7 we have

$$r_6 = \tfrac{1}{2}(\pm(e_8 + e_1) \pm (e_2 + e_3) \pm (e_4 - e_5) \pm (e_6 + e_7)).$$

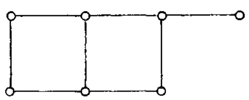
But this is not a root, so the graph cannot be realized.

Suppose Φ is a root system of type E_7 and Γ is an admissible diagram of the required kind. If Γ contains just one cycle, a 4-cycle, Γ must be one of the graphs

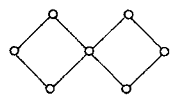




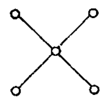
and both can be realised in E_7. If Γ has just two cycles as subgraphs, both 4-cycles, then Γ must be the graph



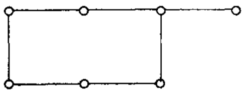
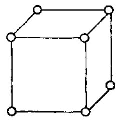
which can also be realised. (Observe that the graph



is not admissible as it contains a subgraph

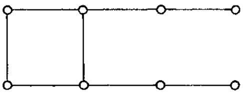
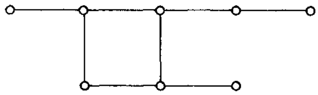


without cycles which is not a Dynkin diagram. In fact no node in an admissible diagram can be joined to more than 3 others). If Γ has three or more 4-cycles and no 6-cycles then there is no possibility, by Lemma 12. Suppose Γ has a 6-cycle. Then, by Lemma 14, Γ must be one of the graphs

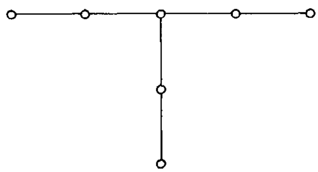
both of which can be realised in E_7.

LEMMA 15. *The graphs*

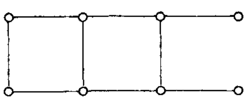
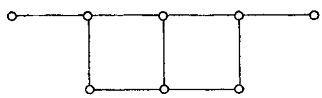
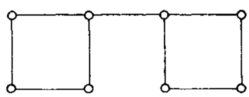
are not admissible diagrams in E_8.

PROOF. The first of these graphs is excluded by choosing a set of 4 orthogonal roots in each of the two ways described in Lemma 11, and showing that the remaining roots form with these a linearly dependent set in both cases. The second graph contains a subgraph



without cycles which is not a Dynkin diagram.

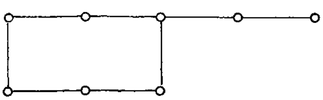
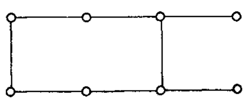
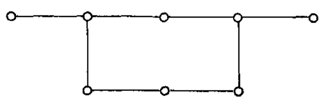
LEMMA 16. *The graphs*

are not admissible diagrams in E_8.

PROOF. The first of these is again excluded by using Lemma 11, and the second and third both contain subgraphs without cycles which are not Dynkin diagrams.

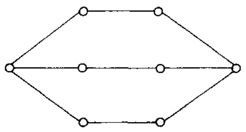
LEMMA 17. *The graphs*

are not admissible diagrams in E_8.

PROOF. All the graphs have subgraphs without cycles which are not Dynkin diagrams.

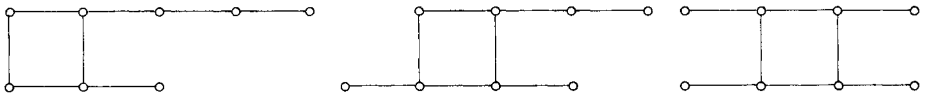
LEMMA 18. *The graph*



is not admissible in E_8.

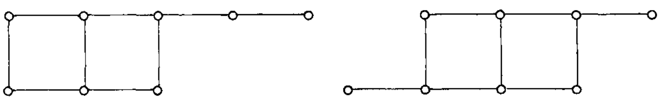
PROOF. This also has a subgraph without cycles which is not a Dynkin diagram.

Having proved these preliminary lemmas we can now find all the admissible diagrams of the required kind in E_8. Let Γ be such an admissible diagram. Suppose Γ contains just one cycle, a 4-cycle. Then, by Lemma 15, Γ must be one of the graphs



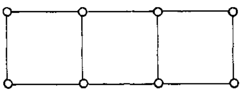
All of these can be realised in E_8.

Now suppose Γ contains just two subgraphs which are cycles, both 4-cycles. Then, by Lemma 16, Γ must be one of the graphs



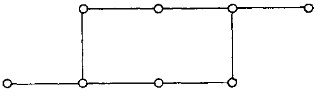
Both of these can be realised in E_8.

Now suppose Γ contains three or more subgraphs which are 4-cycles, and no 6-cycle. Lemma 12 shows that there cannot be two 4-cycles with three common nodes, and so Γ must be the graph



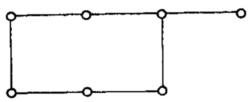
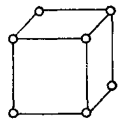
which can be realised in E_8.

Now suppose Γ contains just one cycle, a 6-cycle. Then Lemma 17 shows that Γ must be the graph

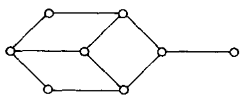
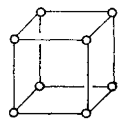


which can be realised in E_8.

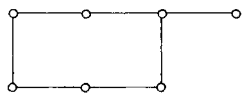
Finally suppose that Γ contains more than one cycle, including a 6-cycle. By Lemma 14, a single node added to this 6-cycle must give one of the graphs

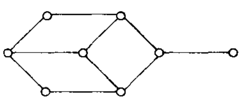
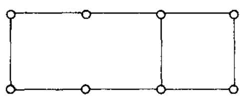
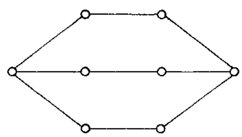
By Lemma 14 again, a single node added to the latter graph gives one of the graphs

which can both be realised in E_8. Using Lemma 14 once more, a single node added to the graph



to give an admissible diagram with at least two cycles must give one of the graphs

The first two of these can be realised in E_8, but not the third, by Lemma 18. Thus the only admissible diagrams of the kind considered are the ones in Table 2, and the proof of Theorem A is complete.

6. The characteristic polynomials

We may now obtain all the admissible diagrams associated with an indecomposable root system Φ by means of the algorithm using the extended Dynkin diagram, combined with the results of Lemma 9 and Theorem A. The algorithm gives all such graphs which have no cycles, and in any such graph we may replace any connected component by one of the appropriate graphs from Table 2 to give the diagrams with cycles.

We now turn to the question of how much can be said about an element w of W_0 given an admissible diagram corresponding to w, and the main result of the present section is that the characteristic polynomial of w on V is determined by the admissible diagram.

LEMMA 19. *Let Φ be a root system and $r_1, r_2, \cdots r_k$ be a set of linearly independent roots in Φ whose graph is a cycle. Then the number of acute angles between consecutive roots in the cycle is odd.*

PROOF. Let us remove one node from the cycle, thus leaving a graph which is a chain. By replacing certain of the roots r_i by their negatives we may reach the situation in which all angles between consecutive roots in this chain are obtuse. By replacing the omitted root by its negative if necessary we may assume that this final root is at an obtuse angle to one of its neighbours. If it were at an obtuse angle to the other, all the angles between this set of linearly independent roots would be obtuse and so the graph would be a Dynkin diagram. This is not so, as no cycle is a Dynkin diagram. Thus exactly one angle between consecutive roots in the cycle is acute. We now go back to the original set of roots. Each time a root is replaced by its negative two angles change their type (i.e. from obtuse to acute or conversely). Thus the number of acute angles between consecutive r_i is odd.

PROPOSITION 20. *Let Γ be an admissible diagram and $r_1, r_2, \cdots r_k$ be a set of roots corresponding to the nodes of Γ. Let A be the $k \times k$ matrix with coefficients*

$$A_{ij} = \frac{2(r_i, r_j)}{(r_i, r_i)}$$

Then A is determined by Γ, to within alterations obtained by replacing certain r_i by their negatives, provided we know (in the cases when Γ contains a multiple bond) which nodes of Γ correspond to long roots and which to short roots.

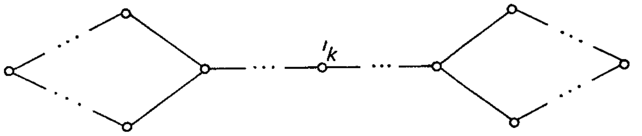
PROOF. We must show that A is independent of the choice of $r_1, r_2, \cdots r_k$ to the extent indicated. We use induction on k, the result being obvious if $k = 1$. Suppose $k > 1$ and let A, $\bar{A}$ be two matrices determined from Γ by different choices of the roots. By removing one node from Γ we obtain submatrices B, $\bar{B}$ of A, $\bar{A}$ such that $\bar{B}$ can be obtained from B by changing the signs of certain of $r_1, r_2, \cdots r_{k-1}$. By carrying out such sign changes we may assume that A, $\bar{A}$ have the form

$$A = \left(\begin{array}{c|c} B & * \\ \hline * & 2 \end{array}\right) \qquad \bar{A} = \left(\begin{array}{c|c} B & * \\ \hline * & 2 \end{array}\right)$$

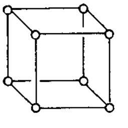
Now the absolute values

$$\left| \frac{2(r_i, r_j)}{(r_i, r_i)} \right|$$

are determined by a knowledge of Γ and of which nodes of Γ correspond to long roots. Thus we must consider the signs, in particular the sign of (r_i, r_k) for $i < k$. If there is a node of Γ joined to just one other node, we take the root corresponding to the former node as r_k and the result is clear. Suppose each node is joined to at least two others and that some node is joined to exactly two others. Choose r_k to correspond to this latter node. Then r_k must be part of a subgraph of Γ which is a cycle. For otherwise there would be a subgraph containing r_k of form



contrary to the classification theorem for admissible diagrams. Consider a cycle containing r_k. All the signs of (r_i, r_j) for consecutive roots of this cycle are determined except for the two involving r_k. One of these may be chosen arbitrarily (by replacing r_k by its negative if necessary) and the other is then determined by Lemma 19. Thus the matrix A is determined. Finally suppose that each node of Γ is joined to at least 3 others. Then, by Theorem A, Γ must be



The signs of (r_i, r_j) for adjacent roots are known except for the ones involving r_k. One of these signs involving r_k may be chosen arbitrarily and the other two are then determined by Lemma 19, using suitable cycles in Γ. This completes the proof.

THEOREM B. *Let w be an element of W_0 with admissible diagram Γ. Then the characteristic polynomial of w on V is determined by Γ.*

PROOF. Let $w = w_1 w_2$ where

$$w_1 = w_{r_1} w_{r_2} \cdots w_{r_k} \qquad w_2 = w_{r_{k+1}} w_{r_{k+2}} \cdots w_{r_{k+h}}$$

as in § 3, the roots $r_1, r_2 \cdots r_k$ being mutually orthogonal and also the roots $r_{k+1}, \cdots r_{k+h}$. Let U be the subspace of V spanned by these roots. Then $r_1, r_2, \cdots r_{k+h}$ form a basis for U. Let A be the $(k+h)\times(k+h)$ matrix with coefficients $(2(r_i, r_j))/(r_i, r_i)$. Then A admits a decomposition as a block matrix

$$A = \begin{pmatrix} 2I_k & B \\ C & 2I_h \end{pmatrix}$$

We now consider the matrices representing w_1, w_2 and w with respect to the above basis of U. These can be expressed in terms of B and C. In fact

$$w_1 = \begin{pmatrix} -I_k & -B \\ 0 & I_h \end{pmatrix} \qquad w_2 = \begin{pmatrix} I_k & 0 \\ -C & -I_h \end{pmatrix}$$

and it follows that

$$w = w_1 w_2 = \begin{pmatrix} BC-I_k & B \\ -C & -I_h \end{pmatrix}.$$

Suppose we are given the graph Γ, and know also where relevant which nodes of Γ correspond to long and short roots. Then A is determined to within alterations produced by sign changes of certain roots, by Proposition 20. Given the matrix A the matrix representing w is determined as above, and so the characteristic polynomial of w on U is determined. If we change certain r_i into their negatives we obtain a matrix for w similar to the one above, and so the characteristic polynomial is unchanged. Thus the characteristic polynomial of w on U is determined by the graph Γ and the designation of the long and short roots. However, the latter is clearly irrelevant, as an interchange of long and short roots does not affect the corresponding reflections. Hence the characteristic polynomial is determined by the graph Γ. Finally the characteristic polynomial on V is obtained by multiplying the characteristic polynomial on U by $(t-1)^{\dim V - \dim U}$.

We now calculate these characteristic polynomials explicitly.

PROPOSITION 21. *The characteristic polynomials determined by the connected admissible diagrams are those shown in Table* 3.

(In Table 3 a graph labelled A_l, B_l etc. is the Dynkin diagram of this type. The remaining graphs are those listed in Table 2).

If an admissible diagram is disconnected its characteristic polynomial is obtained by multiplying together the characteristic polynomials of the connected components. If an admissible diagram has fewer than l nodes, one must also multiply by the appropriate power of $t-1$ to obtain the characteristic polynomial on V.

There is a particularly simple formula for the trace of w in terms of its admissible diagram.

PROPOSITION 22. *Let w be an element of W_0 with admissible diagram Γ. Let V_1 be the subspace of V of elements fixed by w and $V_1^\perp$ the orthogonal complement. Then the trace of w on $V_1^\perp$ is given by:*

$tr_{(V_1^\perp)} w$ = *Number of bonds in Γ − Number of nodes in Γ.*

PROOF. With the notation of Theorem B we have

TABLE 3

Admissible diagram	Characteristic polynomial
A_l	$t^l+t^{l-1}+\cdots+t+1$
B_l, C_l	t^l+1
D_l	$(t^{l-1}+1)(t+1)$
$D_l(a_1)$	$(t^{l-2}+1)(t^2+1)$
$D_l(a_2)$	$(t^{l-3}+1)(t^3+1)$
$\vdots$	
$D_l(a_{\frac{1}{2}l-1})$, $D_l(b_{\frac{1}{2}l-1})$	$(t^{\frac{1}{2}l}+1)^2$ (if l even)
G_2	t^2-t+1
F_4	t^4-t^2+1
$F_4(a_1)$	$(t^2-t+1)^2$
E_6	$(t^4-t^2+1)(t^2+t+1)$
$E_6(a_1)$	t^6+t^3+1
$E_6(a_2)$	$(t^2-t+1)^2(t^2+t+1)$
E_7	$(t^6-t^3+1)(t+1)$
$E_7(a_1)$	$(t^6-t^5+t^4-t^3+t^2-t+1)(t+1)$
$E_7(a_2)$, $E_7(b_2)$	$(t^4-t^2+1)(t^2-t+1)(t+1)$
$E_7(a_3)$	$(t^4-t^3+t^2-t+1)(t^2-t+1)(t+1)$
$E_7(a_4)$	$(t^2-t+1)^3(t+1)$
E_8	$t^8+t^7-t^5-t^4-t^3+t+1$
$E_8(a_1)$	t^8-t^4+1
$E_8(a_2)$	$t^8-t^6+t^4-t^2+1$
$E_8(a_3)$, $E_8(b_3)$	$(t^4-t^2+1)^2$
$E_8(a_4)$	$(t^6-t^3+1)(t^2-t+1)$
$E_8(a_5)$, $E_8(b_5)$	$t^8-t^7+t^5-t^4+t^3-t+1$
$E_8(a_6)$	$(t^4-t^3+t^2-t+1)^2$
$E_8(a_7)$	$(t^4-t^2+1)(t^2-t+1)^2$
$E_8(a_8)$	$(t^2-t+1)^4$

$$w = \begin{pmatrix} BC-I_k & B \\ -C & -I_h \end{pmatrix}$$

on $V_1^\perp$. Thus

$$\operatorname{tr} w = \operatorname{tr}(BC)-(k+h).$$

Now $k+h$ is the number of nodes in Γ. Also

$$\begin{aligned}\operatorname{tr}(BC) &= \sum_{i,j} B_{ij}C_{ji} = \sum_{i=1}^{k}\sum_{j=k+1}^{k+h} \frac{2(r_i,r_j)}{(r_i,r_i)}\cdot\frac{2(r_j,r_i)}{(r_j,r_j)} \\ &= \sum_{i=1}^{k}\sum_{j=k+1}^{k+h} \qquad \text{(Number of bonds joining } r_i, r_j)\end{aligned}$$

Thus $\operatorname{tr}(BC)$ is the number of bonds in Γ.

NOTE. The trace of w on V is obtained by adding $\dim V-\dim V_1^\perp$ to the above value.

Thus

$$\operatorname{tr}_V w = \dim V + (\textit{Number of bonds}) - 2.\ (\textit{Number of nodes}).$$

With the information obtained so far we can calculate the characteristic polynomials of all elements of W_0.

7. Weyl groups of classical types

Although every conjugacy class of W in W_0 determines an admissible diagram, the correspondence between such conjugacy classes and admissible diagrams is not in general a one-one correspondence. As we shall see, it can happen that a given conjugacy class can be described by more than one admissible diagram, or that two or more classes can have the same admissible diagram. We shall describe the situation first in the groups of classical type A_l, B_l, C_l, D_l. Although the results for these groups could be derived entirely in terms of root systems, it is more convenient to use the language of permutation groups to describe the conjugacy classes.

PROPOSITION 23. *Let W be the Weyl group of type A_l. There is a one-one correspondence between conjugacy classes in W and admissible diagrams of form*

$$A_{i_1}+A_{i_2}+\cdots+A_{i_k}$$

where $$\Sigma(i_r+1) = l+1.$$

(*Note. A graph $A_{i_1}+A_{i_2}+\cdots+A_{i_k}$ is one whose connected components are Dynkin diagrams of type $A_{i_1}, A_{i_2}\cdots A_{i_k}$. The case in which some of the i_r are zero is included, A_0 being interpreted as the empty set*).

PROOF. Since $W(A_l)$ is isomorphic to the symmetric group S_{l+1}, the conjugacy classes are in one-one correspondence with partitions of $l+1$; the partition corresponding to the permutation w being the cycle-type of w. Consider a permutation which is itself a cycle. Multiplying from right to left we have

$$(123\cdots i) = (12)(23)\cdots(i-1\ i).$$

Now the transpositions (12), (23), $\cdots$ $(i-1\ i)$ form a complete set of fundamental reflections for the Weyl subgroup S_i of all permutations of $1, 2, \cdots i$. Thus $(123\cdots i)$ is a Coxeter element of S_i. Similarly any i-cycle in W is a Coxeter element of some Weyl subgroup of type A_{i-1}. Thus the conjugacy class of i-cycles has admissible diagram A_{i-1}. Now consider an arbitrary element of W. Since disjoint cycles operate on mutually orthogonal subspaces of V, a permutation with cycle-type $(\lambda_1, \lambda_2, \cdots)$ is represented by an admissible diagram $A_{\lambda_1-1}+A_{\lambda_2-1}+\cdots$

We now turn to the group $W(B_l)\cong W(C_l)$. In describing the conjugacy classes it is more convenient to think of this as a group of type C_l.

PROPOSITION 24. *Let W be the Weyl group of type C_l. There is a one-one correspondence between conjugacy classes in W and admissible diagrams of the form*

$$A_{i_1}+A_{i_2}+\cdots+C_{j_1}+C_{j_2}+\cdots$$

where $\Sigma(i_r+1)+\Sigma j_r = l.$

PROOF. The elements of $W(C_l)$ operate on the orthonormal basis $e_1, e_2, \cdots e_l$ of V by permuting the basis vectors and changing the sign of an arbitrary subset of them. By ignoring the sign-changes, each element w of W determines a permutation of $1, 2, \cdots l$ and this permutation can be expressed in the usual way as a product of disjoint cycles. Let $(k_1 k_2 \cdots k_r)$ be such a cycle. Then

$$e_{k_1} \to \pm e_{k_2} \to \pm e_{k_3} \to \cdots \to \pm e_{k_r} \to \pm e_{k_1}$$

The cycle $(k_1 k_2 \cdots k_r)$ is said to be positive if $w^r(e_{k_1}) = e_{k_1}$ and negative if $w^r(e_{k_1}) = -e_{k_1}$. Then the lengths of the cycles together with their signs give a set of positive or negative integers called the signed cycle-type of w.

Now it is easy to see that two elements of W are conjugate if and only if they have the same signed cycle-type. Thus there is a one-one correspondence between conjugacy classes and signed cycle-types. We consider the special cases of a single positive cycle and a single negative cycle. A positive i-cycle, denoted by $[i]$, is (as in type A_l) a Coxeter element of a Weyl subgroup of type A_{i-1}, and w is represented by an admissible diagram A_{i-1}. Consider the negative i-cycle

$$e_1 \to e_2 \to \cdots \to e_{i-1} \to e_i \to -e_1.$$

This can be expressed as the product of elements

$$(12)\cdot(23)\cdots(i-1\ i)\cdot w_i$$

where w_i changes e_i into $-e_i$ and fixes e_j for all $j \neq i$. These factors form a complete set of fundamental reflections of the Weyl subgroup of type C_i operating on $e_1, e_2, \cdots e_i$, and so the given element is a Coxeter element of this Weyl subgroup. Similarly any negative i-cycle, denoted by $[\bar{i}]$, is a Coxeter element of some Weyl subgroup of type C_i, and so is represented by an admissible diagram C_i.

Now consider an arbitrary element of W, expressed as a product of disjoint positive or negative cycles. Since disjoint cycles operate on orthogonal subspaces of V, the admissible diagram splits into connected components corresponding to the cycle decomposition, and so has form

$$A_{\lambda_1-1}+A_{\lambda_2-1}+\cdots+C_{\mu_1}+C_{\mu_2}+\cdots$$

where $\Sigma\lambda_i+\Sigma\mu_i = l.$

If we define two partitions λ, μ by $\lambda = (\lambda_1, \lambda_2, \cdots)$ $\mu = (\mu_1, \mu_2, \cdots)$ we recover Young's classical result that there is a one-one correspondence between conjugacy classes in $W(C_l)$ and pairs of partitions (λ, μ) with $|\lambda|+|\mu| = l$.

We observe that the admissible diagrams used in Proposition 24 are not the only ones which could have been taken. For example $W(C_l)$ contains a Weyl subgroup $W(D_l)$ generated by the reflections corresponding to all the short roots, and so D_l is an admissible diagram for $W(C_l)$. However since the admissible diagrams of Proposition 24 are in one-one correspondence with the conjugacy classes, there is no need to consider the remainder, which duplicate some of the graphs already used.

We now turn to $W(D_l)$, where the situation is rather more complicated. $W(D_l)$ is a subgroup of $W(C_l)$ of index 2. An element of $W(C_l)$ lies in $W(D_l)$ if and only if it changes the sign of an even number of basis vectors e_i.

Proposition 25. *An element of $W(C_l)$ lies in $W(D_l)$ if and only if it has an even number of negative terms in its signed cycle-type. Two elements of $W(D_l)$ are conjugate if and only if they have the same signed cycle-type, except that if all the cycles are even and positive there are two conjugacy classes. A positive i-cycle $[i]$ is represented by the admissible diagram A_{i-1} and the pair of negative cycles $[\bar{i}\bar{j}]$ with $i \geqq j$ is represented by the admissible diagram D_{i+1} if $j = 1$ and $D_{i+j}(a_{j-1})$ if $j > 1$. The admissible diagram representing any other class is obtained by splitting the signed cycle-type into positive cycles and pairs of negative cycles, and then taking the union of the admissible diagrams corresponding to these.*

Example. The conjugacy class with signed cycle-type $[31\bar{3}\bar{3}2\bar{1}]$ may be represented by the graph $A_2+D_3+D_6(a_2)$ or alternatively by the graph $A_2+D_4+D_5(a_1)$.

Proof. Since a positive cycle changes the sign of an even number of e_i and a negative cycle changes the sign of an odd number, it follows that the elements of $W(C_l)$ in $W(D_l)$ are the ones with an even number of negative cycles. We consider when two such elements are conjugate in $W(D_l)$. Conjugate elements certainly have the same signed cycle-type, and we investigate whether the converse is true. Now two elements with the same signed cycle-type are conjugate in $W(C_l)$, and we must determine whether a transforming element can be found in $W(D_l)$.

Let w_1, w_2 be elements of $W(D_l)$ with the same signed cycle-type and w be an element of $W(C_l)$ such that $ww_1w^{-1} = w_2$. Let

$$\begin{aligned} w_1(e_i) &= \varepsilon_i e_{\sigma(i)} \\ w_2(e_i) &= \eta_i e_{\tau(i)} \\ w(e_i) &= \zeta_i e_{\rho(i)} \end{aligned}$$

where ε_i, η_i, ζ_i are ± 1 and σ, τ, ρ are permutations of $1, 2 \cdots l$. Now $ww_1(e_i) = w_2 w(e_i)$ gives

$$\varepsilon_i \zeta_{\sigma(i)} e_{\rho\sigma(i)} = \zeta_i \eta_{\rho(i)} e_{\tau\rho(i)}$$

Thus we have $\rho\sigma = \tau\rho$ and

$$\frac{\zeta_{\sigma(i)}}{\zeta_i} = \frac{\eta_{\rho(i)}}{\varepsilon_i}.$$

Now the ε's and η's are given, and we are solving these equations for the ζ's. Let

$$C = (i, \sigma(i), \sigma^2(i), \cdots \sigma^{k-1}(i))$$

be a cycle of σ. Then if ζ is chosen arbitrarily (viz 1 or -1) on one element of this cycle it is determined on the whole cycle. Moreover this can only be done provided

$$\varepsilon_i \varepsilon_{\sigma(i)} \cdots \varepsilon_{\sigma^{k-1}(i)} = \eta_{\rho(i)} \eta_{\rho\sigma(i)} \cdots \eta_{\rho\sigma^{k-1}(i)}$$

i.e. provided sign C = sign $\rho(C)$.

We therefore choose ρ to satisfy $\rho\sigma\rho^{-1} = \tau$ such that sign $\rho(C)$ = sign C for all such cycles C. (This can be done since w_1, w_2 have the same signed cycle-type). We then consider the product of the ζ's on the cycle C.

$$\zeta_i \zeta_{\sigma(i)} \cdots \zeta_{\sigma^{k-1}(i)} = \zeta_i^k \left(\frac{\eta_{\rho(i)}}{\varepsilon_i}\right)^{k-1} \left(\frac{\eta_{\rho\sigma(i)}}{\varepsilon_{\sigma(i)}}\right)^{k-2} \cdots \frac{\eta_{\rho\sigma^{k-2}(i)}}{\varepsilon_{\sigma^{k-2}(i)}}.$$

If k is odd this product is ζ_i multiplied by a term involving ε's and η's. It can therefore be made 1 or -1 according to the choice of ζ_i. Thus w can be chosen as an element of $W(D_l)$ provided w_1 has at least one odd cycle.

Suppose now that all the cycles of w_1 are even. Then

$$\zeta_i \zeta_{\sigma(i)} \zeta_{\sigma^2(i)} \cdots \zeta_{\sigma^{k-1}(i)} = \frac{\eta_{\rho(i)} \eta_{\rho\sigma^2(i)} \cdots \eta_{\rho\sigma^{k-2}(i)}}{\varepsilon_i \varepsilon_{\sigma^2(i)} \cdots \varepsilon_{\sigma^{k-2}(i)}}$$

Suppose C is a negative cycle. Then so is $\rho(C)$ and therefore

$$\eta_{\rho(i)} \eta_{\rho\sigma(i)} \eta_{\rho\sigma^2(i)} \cdots \eta_{\rho\sigma^{k-1}(i)} = -1$$

so that

$$\eta_{\rho(i)} \eta_{\rho\sigma^2(i)} \cdots \eta_{\rho\sigma^{k-2}(i)} = -\eta_{\rho\sigma(i)} \eta_{\rho\sigma^3(i)} \cdots \eta_{\rho\sigma^{k-1}(i)}.$$

Then by replacing ρ on C by $\rho\sigma$ we change the sign of

$$\zeta_i \zeta_{\sigma(i)} \zeta_{\sigma^2(i)} \cdots \zeta_{\sigma^{k-1}(i)}.$$

Thus by choosing ρ suitably on the negative cycle C we can ensure that w is an element of $W(D_l)$. Thus w may be chosen inside $W(D_l)$ provided w_1 has some negative cycle.

Now suppose that all the cycles of w_1 are even and positive. Then

$$\zeta_i \zeta_{\sigma(i)} \zeta_{\sigma^2(i)} \cdots \zeta_{\sigma^{k-1}(i)}$$

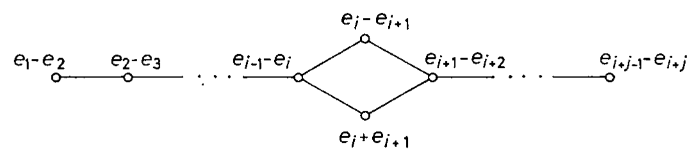
is determined for each cycle irrespective of the choice of ρ. Thus $\zeta_1 \zeta_2 \cdots \zeta_l$ is determined by the choice of w_1, w_2 and can be either 1 or -1. This means there are two conjugacy classes in $W(D_l)$ with this signed cycle-type.

We now consider admissible diagrams representing the conjugacy classes. A positive i-cycle $[i]$ is, as before, a Coxeter element of a Weyl subgroup of type A_{i-1}, so is represented by a graph A_{i-1}. Consider the element

$$(12) \cdot (23) \cdots (i-1\ i)(i\ i+1) w_{i+1}$$

where w_{i+1} changes e_i into $-e_{i+1}$, e_{i+1} into $-e_i$, and fixes e_j for all $j \neq i$, $i+1$. This element is exhibited as the product of a complete set of fundamental reflections of a Weyl subgroup of type D_{i+1}, so is a Coxeter element of this Weyl subgroup. It is therefore represented by a graph D_{i+1}. Its signed cycle-type is $[\bar{i}\ \bar{1}]$, and so the conjugacy class with this signed cycle-type is represented by the graph D_{i+1}.

Finally consider an element with signed cycle-type $[\bar{i}\ \bar{j}]$ where $i \geqq j \geqq 2$. Such an element may be obtained by considering the set of roots shown below.



Dividing these roots in the only way possible into two subsets S_1, S_2 each mutually orthogonal, and taking the product of the corresponding reflections

$$\prod_{r \in S_1} w_r \cdot \prod_{r \in S_2} w_r$$

we obtain an element with signed cycle-type $[\bar{i}\ \bar{j}]$. Thus the conjugacy class with this signed cycle-type can be represented by the graph $D_{i+j}(a_{j-1})$. This completes the proof.

Observe that in type D_l we may have more than one graph representing a conjugacy class and more than one conjugacy class represented by a given graph. One obtains more than one graph representing a given class

TABLE 4

The group $W(D_4)$

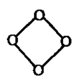
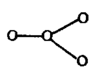
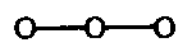
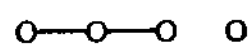
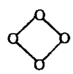
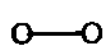
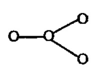
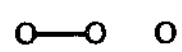
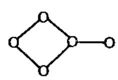
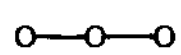
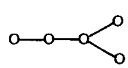
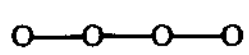
Signed cycle-type	Conjugacy class	Graph
$[1111]$	ϕ	ϕ
$[11\bar{1}\bar{1}]$	D_2	o o
$[\bar{1}\bar{1}\bar{1}\bar{1}]$	D_2+D_2	o o o o
$[211]$	A_1	o
$[2\bar{1}\bar{1}]$	A_1+D_2	o o o
$[\bar{2}1\bar{1}]$	D_3	o—o—o
$[22]$	$(A_1+A_1)'$	o o
$[22]$	$(A_1+A_1)''$	o o
$[\bar{2}\bar{2}]$	$D_4(a_1)$	
$[31]$	A_2	o—o
$[\bar{3}\bar{1}]$	D_4	
$[4]$	$(A_3)'$	o—o—o
$[4]$	$(A_3)''$	o—o—o

TABLE 5

The group $W(D_5)$

Signed cycle-type	Conjugacy class	Graph
[11111]	ϕ	ϕ
$[111\bar{1}\bar{1}]$	D_2	
$[1\bar{1}\bar{1}\bar{1}\bar{1}]$	D_2+D_2	
[2111]	A_1	
$[21\bar{1}\bar{1}]$	A_1+D_2	
$[\bar{2}11\bar{1}]$	D_3	
$[\bar{2}\bar{1}\bar{1}\bar{1}]$	D_3+D_2	
[221]	A_1+A_1	
$[2\bar{2}\bar{1}]$	A_1+D_3	
$[\bar{2}\bar{2}1]$	$D_4(a_1)$	
[311]	A_2	
$[3\bar{1}\bar{1}]$	A_2+D_2	
$[\bar{3}1\bar{1}]$	D_4	
[32]	A_2+A_1	
$[\bar{3}\bar{2}]$	$D_5(a_1)$	
[41]	A_3	
$[\bar{4}\bar{1}]$	D_5	
[5]	A_4	

because a signed cycle-type may be decomposable in several ways into positive cycles and pairs of negative cycles. On the other hand one obtains two conjugacy classes with each graph of form

$$A_{i_1}+A_{i_2}+\cdots+A_{i_k} \qquad \sum(i_r+1)=l$$

where all the i_r are odd. There is also another reason why one sometimes obtains more than one class with a given graph. This occurs because of the exceptional isomorphisms $D_2 = A_1+A_1$ and $D_3 = A_3$. Thus classes with signed cycle-types [22] and $[\bar{1}\bar{1}]$ are represented by the same graph, and so are the classes with signed cycle-types [4] and $[\bar{2}\bar{1}]$. The same thing happens with more complicated signed cycle-types which contain the above types as components.

EXAMPLES. We give in Tables 4, 5 graphs representing the conjugacy classes in $W(D_4)$ and $W(D_5)$.

Notice the way in which the graphs in Table 4 emphasise the 'triality' which exists in the group D_4, but which is not clearly apparent from the signed cycle-types.

We observe that in the groups of type A_l, B_l, C_l, D_l discussed so far every conjugacy class can be represented by an admissible diagram, and so $W_0 = W$.

8. The exceptional Weyl groups

In dealing with the conjugacy classes of the Weyl groups of type G_2, F_4, E_6, E_7, E_8 we do not have available the description in terms of permutation groups which simplifies the discussion in the classical types. We must therefore work entirely in terms of roots and reflections. We have in § 5 determined the admissible diagrams in these cases, and in § 6 determined the characteristic polynomials corresponding to these admissible diagrams. We must now decide whether distinct admissible diagrams with the same characteristic polynomial give rise to one or more conjugacy classes, and also when there can be more than one conjugacy class with the same admissible diagram.

We shall now distinguish between admissible diagrams corresponding to long and short roots in the types containing roots of different lengths. The admissible diagram of an element of type A_i expressible as a product of reflections corresponding to long roots will be denoted by A_i, and for an element expressible as a product of reflections corresponding to short roots the diagram will be denoted by $\tilde{A}_i$.

We first consider the case of certain pairs of admissible diagrams which give the same characteristic polynomial.

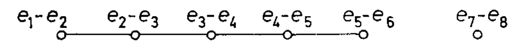
LEMMA 26. *For each of the following pairs of graphs Γ_1, Γ_2 there exist elements $w_1, w_2 \in W_0$ such that w_1 has admissible diagram Γ_1, w_2 has admissible diagram Γ_2 and w_1, w_2 are not conjugate in W.*

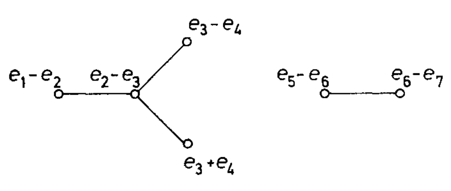
Type	Γ_1	Γ_2	*Characteristic polynomial*
G_2	A_1	$\tilde{A}_1$	$t+1$
F_4	A_1	$\tilde{A}_1$	$t+1$
	$2A_1$	$A_1+\tilde{A}_1$	$(t+1)^2$
	A_2	$\tilde{A}_2$	t^2+t+1
	$3A_1$	$2A_1+\tilde{A}_1$	$(t+1)^3$
	A_3	B_2+A_1	$(t^2+1)(t+1)$
	C_3	B_3	t^3+1
	$\tilde{A}_2+A_1$	$A_2+\tilde{A}_1$	$(t^2+t+1)(t+1)$
	D_4	C_3+A_1	$(t^3+1)(t+1)$
E_8	A_5+A_1	D_4+A_2	$(t^3+1)(t^2+t+1)(t+1)$

PROOF. We choose w_1, w_2 in each case to be Coxeter elements of suitable Weyl subgroups, and show that w_1, w_2 either fix or invert different numbers of long or short roots. This is sufficient, since conjugate elements of W must fix the same number of long roots and the same number of short roots, and invert the same numbers.

Elements of type A_1 and $\tilde{A}_1$ in G_2 invert two long roots and two short roots respectively, so cannot be conjugate.

The following sets of roots in E_8 are systems of type A_5+A_1 and D_4+A_2.





The roots fixed by the products w_1, w_2 of the corresponding reflections are the roots orthogonal to all the roots shown. There are roots orthogonal to all the roots in the first diagram, but no such roots for the second diagram. Thus w_1, w_2 are not conjugate.

The various cases in type F_4 are dealt with similarly, but we omit the details.

We now consider certain pairs of conjugacy classes with the same admissible diagram. It was shown in Proposition 25 that there are two distinct conjugacy classes in $W(D_l)$ with graph

$$A_{i_1}+A_{i_2}+\cdots+A_{i_k}$$

where i_r is odd and $\sum_{r=1}^{k}(i_r+1) = l$. In particular $W(D_8)$ contains two classes associated with each of the graphs $4A_1$, A_3+2A_1, $2A_3$, A_5+A_1, A_7. Now $W(D_8)$ is a Weyl subgroup of $W(E_8)$ and we shall show that these two classes remain non-conjugate in $W(E_8)$. Similarly $W(D_6)$ contains two classes associated with each of the graphs $3A_1$, A_3+A_1, A_5. Hence $W(D_6+A_1) = W(D_6)\times W(A_1)$ contains two classes associated with each of the graphs $3A_1$, A_3+A_1, A_5, $4A_1$, A_3+2A_1, A_5+A_1. Now $W(D_6+A_1)$ is a Weyl subgroup of $W(E_7)$, as may be seen by inspecting the extended Dynkin diagram of E_7. We shall show that these two classes remain non-conjugate in $W(E_7)$.

LEMMA 27. *For each of the following graphs there exist w_1, $w_2 \in W_0$ corresponding to the graph such that w_1, w_2 are not conjugate in W.*

Type	*Graph*
E_7	$3A_1$
	A_3+A_1
	A_5
	$4A_1$
	A_3+2A_1
	A_5+A_1
E_8	$4A_1$
	A_3+2A_1
	$2A_3$
	A_5+A_1
	A_7

PROOF. In each case in E_8 we take for w_1 a product of reflections corresponding to roots of form e_i-e_j. For w_2 we take a product of reflections corresponding to roots of form e_i-e_j with one exception, which has form e_i+e_j. Then we show that w_1, w_2 fix different numbers of roots. In E_7 the idea is similar, except that e_1+e_8 appears sometimes in w_1, w_2 because of the description of the root system of E_7 given in Table 1. Defining w_2 from w_1 by changing one sign only, we verify that w_1, w_2 fix different numbers of roots. The reason for this is that $\frac{1}{2}(\sum_{i=1}^{8}\varepsilon_i e_i)$, $\varepsilon_i = \pm 1$, is a root of E_8 if $\prod_{i=1}^{8}\varepsilon_i = 1$ but not if $\prod_{i=1}^{8}\varepsilon_i = -1$.

For example, considering elements of type $4A_1$ in E_8 we take the two sets of roots

$$\underset{\circ}{e_1-e_2}\quad \underset{\circ}{e_3-e_4}\quad \underset{\circ}{e_5-e_6}\quad \underset{\circ}{e_7-e_8}$$

$$\underset{\circ}{e_1-e_2}\quad \underset{\circ}{e_3-e_4}\quad \underset{\circ}{e_5-e_6}\quad \underset{\circ}{e_7+e_8}$$

thus defining elements w_1, w_2. The roots fixed by w_1 are those orthogonal to the first set of roots, which are

$$\pm(e_1+e_2),\ \pm(e_3+e_4),\ \pm(e_5+e_6),\ \pm(e_7+e_8)$$
$$\tfrac{1}{2}(\pm(e_1+e_2)\pm(e_3+e_4)\pm(e_5+e_6)+(e_7+e_8)).$$

The roots fixed by w_2 are those orthogonal to the second set of roots, which are

$$\pm(e_1+e_2),\ \pm(e_3+e_4),\ \pm(e_5+e_6),\ \pm(e_7-e_8).$$

Thus w_1, w_2 fix different numbers of roots so cannot be conjugate. The idea is the same in the other cases.

The set $\mathscr{C}$ of conjugacy classes. Let W be a Weyl group of one of the exceptional types. We determine all the admissible diagrams associated with W by the methods described earlier, and for each admissible diagram we calculate the corresponding characteristic polynomial. Different admissible diagrams may give rise to the same characteristic polynomial. For each characteristic polynomial we choose one admissible diagram giving rise to it, except that in the cases discussed in Lemma 26 we choose both graphs. Let $\mathscr{A}$ be the set of admissible diagrams obtained in this way. For each Γ in $\mathscr{A}$ we now choose one conjugacy class corresponding to Γ, except that in the cases discussed in Lemma 27 we choose two classes. We denote by $\mathscr{C}$ the set of conjugacy classes obtained in this way. There appears to be some arbitrariness in the choice of $\mathscr{C}$. This is not actually so, however, for we shall show by a counting argument that $\mathscr{C}$ is the full set of conjugacy classes of W.

9. The number of elements in a conjugacy class

In order to be able to use such a counting argument we need a way of determining the number of elements in a given conjugacy class. The following method seems the most convenient.

Let Γ be a graph in $\mathscr{A}$ and S be a set of roots corresponding to the nodes of Γ. Let Φ_1 be the smallest root subsystem of Φ containing S, i.e. the intersection of all the root systems in Φ containing S. Let V_1 be the subspace of V spanned by Φ_1 and W_1 be the Weyl group of Φ_1. Thus W_1 is a Weyl subgroup of W. The roots in Φ orthogonal to V_1 form a subsystem Φ_2 with Weyl group W_2. Then $\Phi_1 \cup \Phi_2$ is also a subsystem of Φ and has Weyl group $W_1 \times W_2$.

PROPOSITION 28. *$W_1 \times W_2$ is a normal subgroup of $N_W(W_1)$, the normalizer of W_1. The factor group $N_W(W_1)/W_1 \times W_2$ is isomorphic to the group of symmetries of the Dynkin diagram of W_1 induced by transformations by elements of W.*

PROOF. Let $w \in N_W(W_1)$. For each root $r \in \Phi_1$ we have

$$ww_r w^{-1} = w_{w(r)} \in W_1 .$$

Now $w_{w(r)}$ is an involution in W_1, so is a product of reflections corresponding to a set of orthogonal roots in Φ_1, by Lemma 5. However $w_{w(r)}$ is itself a reflection. Thus $w(r) \in \Phi_1$. It follows that $w(\Phi_1) = \Phi_1$.

Let Π_1 be a fundamental system of roots in Φ_1. Then $w(\Pi_1)$ is also a fundamental system in Φ_1. However any two fundamental systems are equivalent under the Weyl group, and so $w(\Pi_1) = w_1(\Pi_1)$ for some $w_1 \in W_1$. Let $N_W(\Pi_1)$ be the set of elements of W such that $w(\Pi_1) = \Pi_1$. Then $w_1^{-1} w \in N_W(\Pi_1)$ and so

$$N_W(W_1) = W_1 \cdot N_W(\Pi_1).$$

Also we have

$$W_1 \cap N_W(\Pi_1) = N_{W_1}(\Pi_1) = 1$$

since the only element of a Weyl group leaving invariant a fundamental system of roots is the identity. Let $C_W(\Pi_1)$ be the set of elements of W such that $w(r) = r$ for each $r \in \Pi_1$. By Lemma 1, $C_W(\Pi_1) = W_2$. Thus

$$N_W(W_1)/W_1 \times W_2 \cong N_W(\Pi_1)/C_W(\Pi_1)$$

and this is the group of symmetries of the Dynkin diagram of W_1 induced by transformations by elements of W.

If the group of induced symmetries of the Dynkin diagram of W_1 is known, then the number of conjugates $|W : N_W(W_1)|$ of W_1 in W may be calculated by Proposition 28.

Let w be an element of W_0 obtained as a product of reflections corresponding to the roots in S. We shall calculate the number of conjugates of w in W using the following lemma.

LEMMA 29. *The number of conjugates of w in W is equal to abc/d, where:*

a is the number of elements in the conjugacy class of W_1 containing w.

b is the number of conjugacy classes of W_1 contained in the conjugacy class of W containing w.

c is the number of conjugates of W_1 in W.

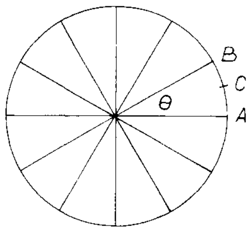
d is the number of subgroups conjugate to W_1 in W containing w.

PROOF. This is clear.

We now discuss the calculation of the numbers a, b, c, d. We denote by $\mathrm{ccl}_{W_1}(w)$ the conjugacy class of W_1 containing w, and consider $a = |\mathrm{ccl}_{W_1}(w)|$. If Γ is connected and has no cycles, w is a Coxeter element of W_1. The following result enables us to find $|\mathrm{ccl}_{W_1}(w)|$ in this case.

PROPOSITION 30. *A Coxeter element of a Weyl group W commutes only with its powers.*

PROOF. This follows from the results in Steinberg's paper [11]. Let w be a Coxeter element of W and h be the order of w. Then it is shown in [11] that $\zeta = e^{2\pi i/h}$ occurs as an eigenvalue of w with multiplicity 1, and that the corresponding 1-dimensional eigenspace is not orthogonal to any root. Let $w = w_1 w_2$ be an expression of w as a product of two involutions as in § 3. Steinberg shows how to construct a 2-dimensional subspace P of V invariant under the dihedral group $\langle w_1, w_2 \rangle$ generated by w_1, w_2. The reflecting hyperplanes of V intersect P in h lines through the origin such that adjacent lines are inclined at an angle π/h. w_1 operates on P as the reflection in one of these lines, w_2 is the reflection in one of the adjacent lines, and $w = w_1 w_2$ operates as a rotation through an angle $2\pi/h$. Let U be the unit circle in P and A, B be the intersections of U with the reflecting lines of w_1, w_2 respectively.



Let C be a point of U lying strictly between A and B. Then C does not lie in any reflecting hyperplane of V, so is in the interior of one of the Weyl chambers of V.

Now $h = 2N/l$, the total number of roots divided by the number of fundamental roots [11]. Excluding the trivial case A_1 we have $h > 2$ and so $\zeta = e^{2\pi i/h}$ is not real. Let v be an eigenvector in P with eigenvalue ζ and let x be any element of W commuting with w. Then

$$wx(v) = xw(v) = \zeta x(v)$$

and so $x(v)$ is also an eigenvector of w with eigenvalue ζ. Since ζ occurs as eigenvalue with multiplicity 1 we have $x(v) = \mu v$ for some μ. Thus $x(\bar{v}) = \bar{\mu}\bar{v}$. Hence x leaves invariant the subspace spanned by v, $\bar{v}$ and this is the complexification of P since v is not real. Thus $x(P) = P$.

Now $x(C)$ lies in the interior of one of the sectors of the circle U described above. Considering the way in which the dihedral group $\langle w_1, w_2 \rangle$ operates on U it is evident that there is some element $w' \in \langle w_1, w_2 \rangle$ such that $x(C)$ lies in the same sector of U as $w'(C)$. Thus $w'^{-1}x(C)$ lies in the same sector as C, viz between A and B. Now C and $w'^{-1}x(C)$ lie in the interior of the same Weyl chamber of W in V, and this can only happen if $w'^{-1}x = 1$. Thus $x \in \langle w_1, w_2 \rangle$. Now

$\langle w_1, w_2 \rangle$ is a dihedral group of order $2h$ containing the cyclic subgroup $\langle w \rangle$ of order h. Thus every element x of $\langle w_1, w_2 \rangle$ which commutes with w must be a power of w.

COROLLARY. *If w is a Coxeter element of W then*

$$|\mathrm{ccl}_W(w)| = \frac{|W| \cdot l}{2N},$$

where $2N$ is the total number of roots.

If the graph Γ of w is disconnected then $|\mathrm{ccl}_{W_1}(w)|$ is the product of analogous factors for the connected components. Thus the only case which remains is that in which Γ is connected but contains cycles. If W_1 is of classical type D_l, $|\mathrm{ccl}_{W_1}(w)|$ is readily calculated using § 7. If W_1 is of exceptional type one can often make use of the fact that

$$|\mathrm{ccl}_W(w)| = |\mathrm{ccl}_W(zw)|$$

where z is an element of the centre of W. If $z \neq 1$, $|\mathrm{ccl}_W(zw)|$ will often be known already. The only cases which cannot be dealt with in this way are $\Gamma = F_4(a_1), E_6(a_1), E_6(a_2), E_8(a_1), E_8(a_2), E_8(a_3)$, and in these cases the order of the centralizer must be calculated directly. One obtains the following results:

W	Γ	Order of w	$\lvert C_W(w)\rvert$
F_4	$F_4(a_1)$	6	72
E_6	$E_6(a_1)$	9	9
	$E_6(a_2)$	6	72
E_8	$E_8(a_1)$	24	24
	$E_8(a_2)$	20	20
	$E_8(a_3)$	12	288

NOTE. These are the elements whose centralizers $C_W(w)$ cannot be calculated by using properties of Coxeter elements or reducing the calculation to a simpler case. It is remarkable that a formula can be given for $|C_W(w)|$ which is valid for all such elements w. Let $d_1, d_2, \cdots d_l$ be the degrees of the basic polynomial invariants of W. Then $|W| = d_1 d_2 \cdots d_l$ and, for suitable elements w including the ones above, $|C_W(w)|$ is the product of those d_i which are divisible by the order of w. T. A. Springer has an unpublished proof of this fact, using methods of algebraic geometry.

We now turn to the calculation of the number b in Lemma 29; the number of W_1-conjugacy classes into which $\mathrm{ccl}_W(w) \cap W_1$ splits. We require two preliminary results.

LEMMA 31. *Let Γ be an admissible diagram associated with Φ but no subsystem of Φ. Then there is only one conjugacy class in the set $\mathscr{C}$ whose elements have the characteristic polynomial determined by Γ.*

PROOF. The distinct conjugacy classes in $\mathscr{C}$ with the same characteristic polynomial are those arising from the cases discussed in Lemmas 26 and 27. However, all the admissible diagrams in these cases are associated with proper subsystems of Φ.

PROPOSITION 32. *Any two isomorphic subsystems of Φ of maximal rank are equivalent under the action of W.*

PROOF. This result is stated by Dynkin [3] P. 146, in terms of the corresponding subalgebras of a simple Lie algebra.

We can now show that $b = 1$ in all the cases which need be considered in the exceptional groups. In proving this we shall assume inductively that $\mathscr{C}$ is the full set of conjugacy classes in Weyl groups of smaller rank than W.

PROPOSITION 33. *Let W be a Weyl group of rank at most 8. Let w be an element of W_0 with graph Γ, let Φ_1 be the smallest subsystem containing the roots corresponding to the nodes of Γ, and let W_1 be the Weyl group of Φ_1. Then*

$$\mathrm{ccl}_W(w) \cap W_1 = \mathrm{ccl}_{W_1}(w)$$

except when $W = W(E_8)$ and $\Gamma = D_4+D_4(a_1)$.

NOTE. If $\Gamma = D_4+D_4(a_1)$, we have $W_1 \cong W(D_4)\times W(D_4)$ and $\mathrm{ccl}_W(w) \cap W_1$ splits into two classes in W_1, as there is an element of $W(E_8)$ permuting the two orthogonal subsystems of type D_4. However this case need not be considered at all, since the graph $D_4+D_4(a_1)$ has the same characteristic polynomial as $D_5(a_1)+A_3$, which is also an admissible diagram for E_8. Thus we may choose $D_5(a_1)+A_3$ to be in $\mathscr{A}$ instead of $D_4+D_4(a_1)$.

PROOF. Suppose first that Φ_1 has maximal rank, i.e. rank $\Phi_1 = $ rank Φ. Let x be an element of W such that $xwx^{-1} \in W_1$. W_1 is either a Weyl group of classical type, or is decomposable into Weyl groups of smaller rank. In either case we know that $W_1 = (W_1)_0$ and so xwx^{-1} determines a certain admissible diagram in W_1. We show this is Γ. Now xwx^{-1} operates on the indecomposable components of Φ_1, which correspond to the connected components of Γ. For each component of Γ of type A, xwx^{-1} must operate as a Coxeter element of the corresponding Weyl group. For the Coxeter elements are the only elements of $W(A_i)$ with $\bar{l}(w) = i$. Thus if all components of Γ have type A, Γ is the graph of

xwx^{-1}. It follows that xwx^{-1} and w are conjugate in W_1. If there is just one component of Γ not of type A, the eigenvalues of xwx^{-1} on this component are the same as the eigenvalues of w. The graph of w on this component is not associated with any proper Weyl subgroup. Thus, using § 7 if this component is a classical group, or Lemma 31 if it is an exceptional group, we deduce that xwx^{-1} is conjugate to w in W_1. Finally suppose Γ has at least two components not of type A. The only possibilities are

$$\Phi = F_4 \qquad \Phi_1 = B_2+B_2$$

or

$$\Phi = E_8 \qquad \Phi_1 = D_4+D_4.$$

In the former case w must be a Coxeter element of W_1, and so must xwx^{-1} since it has the same eigenvalues. Thus xwx^{-1} is conjugate to w in W_1. In the latter case Γ must be D_4+D_4, $D_4+D_4(a_1)$, or $D_4(a_1)+D_4(a_1)$. The graph of xwx^{-1} must be Γ also, since xwx^{-1} has the same eigenvalues as w. xwx^{-1} and w are conjugate in W_1 if Γ is D_4+D_4 or $D_4(a_1)+D_4(a_1)$, but need not be if Γ is $D_4+D_4(a_1)$ since xwx^{-1}, w may act in opposite ways on the two components, which are interchanged by an element of W.

Now suppose that Φ_1 is not necessarily of maximal rank. Let $x \in W$ and $xwx^{-1} \in W_1$. Let V_1 be the subspace of V spanned by Φ_1. Then w operates on V_1 without fixing a vector, but fixes every vector in the orthogonal subspace $V_1^{\perp}$. Thus V_1 is the subspace orthogonal to the fixed space of w. Similarly V_1 is the subspace orthogonal to the fixed space of xwx^{-1}, since $xwx^{-1} \in W_1$. Thus $x(V_1) = V_1$. Now we have $W_1 = W(\Phi_1)$ and so $xW_1x^{-1} = W(x(\Phi_1))$. Let $\bar{\Phi}_1 = \Phi \cap V_1$. Then Φ_1 and $x(\Phi_1)$ are isomorphic subsystems of $\bar{\Phi}_1$ of maximal rank. Thus $x(\Phi_1) = w'(\Phi_1)$ for some $w' \in \bar{W}_1 = W(\bar{\Phi}_1)$, by Proposition 32. Thus $w'^{-1}x(\Phi_1) = \Phi_1$ and so $w'^{-1}x \in N(W_1)$. Hence $x = w'w''$ where $w' \in \bar{W}_1$ and $w'' \in N(W_1)$. Now $w''ww''^{-1}$ and $w'w''ww''^{-1}w'^{-1}$ are both in W_1, which is of maximal rank in $\bar{W}_1$. Thus these two elements are conjugate in W_1 as above. Now w and $w''ww''^{-1}$ are elements of W_1 with the same characteristic polynomial, and the graph Γ of w is not associated with any proper Weyl subgroup of W_1. Thus, using § 7 if W_1 is a classical group and Lemma 31 if W_1 is an exceptional group, we see that w and $w''ww''^{-1}$ are conjugate in W_1. Hence w and xwx^{-1} are conjugate in W_1.

Now consider the determination of the number c of Lemma 29. We have

$$c = |W : N_W(W_1)|$$

and, by Proposition 28,

$$|N_W(W_1)| = |W_1| \cdot |W_2| \operatorname{Aut}_W(\Delta_1)$$

where $\operatorname{Aut}_W(\Delta_1)$ is the group of symmetries of the Dynkin diagram of W_1 induced by elements of W. Given a conjugacy class in $\mathscr{C}$, the type of W_1 is determined by the graph Γ of the class, by Lemma 10. The type of W_2 is then the type of the largest subsystem of Φ orthogonal to Φ_1. This can be read off from the list of all subsystems of Φ (c.f. Dynkin [3], p. 149), except in the few cases where there are two isomorphic subsystems which are not conjugate. These cases are described by Dynkin in [3], p. 147, 148 and coincide with the cases described in Lemma 27 of non-conjugate elements of W with the same graph. In these cases the orthogonal subsystem Φ_2 is obtained directly by inspection of the root system, and one usually obtains different types for Φ_2 for isomorphic non-conjugate subsystems Φ_1. Thus $|W_1|$ and $|W_2|$ are now known, and to calculate $|W : N_W(W_1)|$ we require in addition a knowledge of $\operatorname{Aut}_W(\Delta_1)$. If W_1 is indecomposable, its Dynkin diagram Δ_1 is connected. The order of the induced group of symmetries is then very restricted – in fact it is either 1 or 2 except when W_1 has type D_4, when it could also be 3 or 6. It is easy to find $\operatorname{Aut}_W(\Delta_1)$ in all such cases by inspection of the root system.

If W_1 is decomposable it may be much more difficult to calculate $\operatorname{Aut}_W(\Delta_1)$, but in this case $|W : N_W(W_1)|$ may be calculated directly from a knowledge of what happens in the indecomposable case. This follows from the next lemma.

Lemma 34. *$|W : N_W(W_1)|$ is the number of subsystems of Φ equivalent to Φ_1 under W.*

Proof. Let $x \in W$. We show $x(\Phi_1) = \Phi_1$ if and only if $x \in N(W_1)$. Suppose $x(\Phi_1) = \Phi_1$ and let $r \in \Phi_1$. Then $xw_rx^{-1} = w_{x(r)}$, and $w_{x(r)}$ is in W_1 since $x(r)$ is in Φ_1. Thus $x \in N(W_1)$ since the elements w_r generate W_1.

Now suppose conversely that $x \in N(W_1)$, and let $r \in \Phi_1$. Then $xw_rx^{-1} = w_{x(r)} \in W_1$. However the only reflections in W_1 are reflections corresponding to roots in Φ_1. Thus $x(r) \in \Phi_1$. It follows that the number of conjugates of W_1 in W is equal to the number of subsystems equivalent to Φ_1 under W.

Let the Dynkin diagram Δ_1 be decomposable and write $\Delta_1 = \Delta_1' + \Delta_1''$ where Δ_1' is an indecomposable component and Δ_1'' is the union of the other components. Assume first that isomorphic subsystems of the types being considered are equivalent under W. Then the number of subsystems isomorphic to Δ_1' in Φ is assumed known, since Δ_1' is indecomposable. Let the systems in Φ orthogonal to systems of type Δ_1' have type Δ_2'. Then

Δ_1'' is the type of a subsystem in a system of type Δ_2'', and the number of subsystems isomorphic to Δ_1'' may be assumed known by induction. Then by dividing by the number of components of Δ_1 isomorphic to Δ_1' we obtain the number of systems isomorphic to Δ_1 in Φ.

This procedure needs slight modification when we are dealing with a subsystem not equivalent to all its isomorphic copies in Φ. These are the subsystems described in Lemma 27. We omit the details.

In the manner described above we can calculate $|W : N_W(W_1)|$ and $|\mathrm{Aut}_W(\Delta_1)|$ for all classes in $\mathscr{C}$.

Finally we discuss the calculation of the number d of Lemma 29. d is the number of subgroups conjugate to W_1 in W containing w. Let Φ_1, V_1 be as before, let $\bar{\Phi}_1 = \Phi \cap V_1$ and $\bar{W}_1$ be the Weyl group of $\bar{\Phi}_1$. We show first that

$$d \leqq |N_W(\bar{W}_1) : N_W(W_1)|$$

LEMMA 35. (i) *$|N_W(\bar{W}_1) : N_W(W_1)|$ is the number of subsystems of $\bar{\Phi}_1$ equivalent to Φ_1 under an element of W.*

(ii) *d is the number of subsystems of $\bar{\Phi}_1$ equivalent to Φ_1 under an element of W and containing w in the Weyl group.*

PROOF. Let x be an element of W. Now $x(\Phi_1)$ is contained in $\bar{\Phi}_1$ if and only if it is contained in the subspace V_1. The elements x transforming V_1 into itself are just the elements such that $x(\bar{\Phi}_1) = \bar{\Phi}_1$. The argument of Lemma 34 now shows that $x(\bar{\Phi}_1) = \bar{\Phi}_1$ if and only if $x \in N_W(\bar{W}_1)$. Thus $x(\Phi_1)$ is in $\bar{\Phi}_1$ if and only if $x \in N_W(\bar{W}_1)$. The elements of $N_W(\bar{W}_1)$ operate transitively on the subsystems of form $x(\Phi_1)$ in $\bar{\Phi}_1$. The stabilizer of Φ_1 is $N_W(W_1)$, again by the argument of Lemma 34. Thus the number of distinct subsystems $x(\Phi_1)$ of $\bar{\Phi}_1$ is $|N_W(\bar{W}_1) : N_W(W_1)|$.

Now the Weyl group of the root system $x(\Phi_1)$ is xW_1x^{-1}. Thus the number of subgroups xW_1x^{-1} containing w is equal to the number of subsystems $x(\Phi_1)$ containing w in the Weyl group.

COROLLARY. *If $\Phi_1 = \Phi \cap V_1$ then $d = 1$.*

If $\Phi_1 \neq \bar{\Phi}_1$ the number $|N_W(\bar{W}_1) : N_W(W_1)|$ can be calculated by the methods described earlier, thus giving an upper bound for d. We now show that the calculation of d can be reduced to the case in which $V_1 = V$, i.e. Φ_1 has maximal rank in Φ.

LEMMA 36. *d is the number of subsystems of $\bar{\Phi}_1$ equivalent to Φ_1 under an element of $\bar{W}_1$ and containing w in the Weyl group.*

PROOF. Let x be an element of W such that $x(\Phi_1)$ is contained in $\bar{\Phi}_1$. Now Φ_1 and $x(\Phi_1)$ are isomorphic subsystems of $\bar{\Phi}_1$ of maximal rank. Thus $x(\Phi_1) = y(\Phi_1)$ for some $y \in \bar{W}_1$, by Proposition 32. The result now follows from Lemma 35.

Thus it is sufficient to consider the root system $\bar{\Phi}_1$ in the calculation of d, and Φ_1 has maximal rank in $\bar{\Phi}_1$.

Suppose Φ_1 is decomposable, and let

$$\Phi_1 = \Psi_1 + \Psi_1'$$

where Ψ_1, Ψ_1' are orthogonal subsystems of Φ_1. Let U_1, U_1' be the subspaces spanned by Ψ_1, Ψ_1' and let $\bar{\Psi}_1 = \Phi \cap U_1$, $\bar{\Psi}_1' = \Phi \cap U_1'$.

Now let $\Phi_2 = x(\Phi_1)$ be any subsystem containing w in its Weyl group, where $x \in W$. Then

$$\Phi_2 = \Psi_2 + \Psi_2'$$

where $\Psi_2 = x(\Psi_1)$, $\Psi_2' = x(\Psi_1')$ and we define U_2, U_2', $\bar{\Psi}_2$, $\bar{\Psi}_2'$ in the analogous way.

We consider the possible ways of choosing Φ_2, given w. Consider first the choice of U_2. This is a w-invariant subspace of V spanned by roots on which w operates with given eigenvalues, i.e. the eigenvalues of w on U_1. Now every w-invariant subspace of V is the direct sum of its intersections with the eigenspaces of w. Using this information, and the matrix form for w given in the proof of Theorem B it is possible to determine the subspaces U_2 which can occur. Given U_2, the orthogonal subspace U_2' and the subsystems $\bar{\Psi}_2$, $\bar{\Psi}_2'$ are determined. w operates on $\bar{\Psi}_2$, $\bar{\Psi}_2'$ either as an element of the Weyl group or as an element of the Weyl group combined with a symmetry of the Dynkin diagram, by Proposition 28. We require w to operate as an element of the Weyl group in both cases, so must exclude the subspaces U_2 (if any) giving rise to non-trivial symmetries. Now w operates on $\bar{\Psi}_2$, $\bar{\Psi}_2'$ with the same characteristic polynomials as for its operation on $\bar{\Psi}_1$, $\bar{\Psi}_1'$. We may assume by induction on the rank that information is already available about the conjugacy classes of the Weyl groups of $\bar{\Psi}_2$, $\bar{\Psi}_2'$. Thus we may assume that w operates on $\bar{\Psi}_2$, $\bar{\Psi}_2'$ as the same type of element as on $\bar{\Psi}_1$, $\bar{\Psi}_1'$.

Now we may choose Ψ_1 so that any subsystem of Φ isomorphic to $\bar{\Psi}_1$ is equivalent to $\bar{\Psi}_1$ under W. For example we may certainly choose $\bar{\Psi}_1$ to be indecomposable of rank at most $\frac{1}{2}l$, and then the results of Dynkin [3] mentioned earlier show that this is the case, since the only counter-examples are the cases described in Lemma 27. Thus there exists $x \in W$ such that $x(\bar{\Psi}_1) = \bar{\Psi}_2$. Hence $x(\bar{\Psi}_1') = \bar{\Psi}_2'$ and we have

$$x(\Psi_1) \subseteq \bar{\Psi}_2 \quad x(\Psi_1') \subseteq \bar{\Psi}_2'$$

Now the number of subsystems Ψ_2 of $\bar{\Psi}_2$ equivalent to $x(\Psi_1)$ under an element of $W(\bar{\Psi}_2)$ and containing $w|\bar{\Psi}_2$ in the Weyl group may be assumed known by induction. The same applies to the number of subsystems Ψ_2' of $\bar{\Psi}_2'$ equivalent to $x(\Psi_1')$ under $W(\bar{\Psi}_2')$ containing $w|\bar{\Psi}_2'$ in the Weyl group. Thus the number of subsystems $\Phi_2 = \Psi_2 + \Psi_2'$ of the required type can be obtained. One must divide by the number of components of Φ_2 isomorphic to Ψ_2 as the same system may have been obtained several times over.

In the inductive procedure outlined above we have assumed that the number d is known for systems of smaller rank. In carrying out the calculations in E_7 and E_8 we need to know these numbers in some ot the classical Weyl groups $W(A_l)$ and $W(D_l)$.

LEMMA 37. *Let w be an element of* W_0 *with graph* Γ. *Then*:

(i) *If* $W = W(A_l)$ *then* $\Gamma = A_{i_1-1}+A_{i_2-1}+\cdots+A_{i_r-1}$ *and* $d = 1$.

(ii) *Let* $W = W(D_l)$ *and*

$$\Gamma = A_{i_1-1}+A_{i_2-1}+\cdots+A_{i_r-1}+D_{j_1+k_1}(a_{k_1-1})+\cdots+D_{j_s+k_s}(a_{k_s-1})$$

Then the signed cycle-type of w is

$$[i_1 i_2 \cdots i_r j_1 \bar{k}_1 j_2 \bar{k}_2 \cdots j_s \bar{k}_s]$$

where

$$\sum_{t=1}^{r} i_t + \sum_{t=1}^{s} (j_t+k_t) = l.$$

Let the multiplicity of the negative i-cycle $\bar{i}$ *in the signed cycle-type be* m_i *and the multiplicity of the graph* $D_{j+k}(a_{k-1})$ *as component of* Γ *be* $e_{j,k}$. *Let u be the number of components* $D_{j+k}(a_{k-1})$ *of* Γ *with* $j = k$. *Then*

$$d = \frac{\prod_i (m_i!)}{2^u \prod_{j,k} (e_{j,k}!)}$$

PROOF. (i) If Φ is a root system of type A_l and Ψ is a subsystem of Φ, every proper subsystem of Ψ has smaller rank than Ψ. Thus if Φ_1 is the smallest subsystem of Φ containing all the roots corresponding to nodes of Γ we have $\Phi_1 = \bar{\Phi}_1$. Hence $d = 1$, by Lemma 35.

(ii) Now let $W = W(D_l)$. Then

$$\Gamma = A_{i_1-1}+\cdots+A_{i_r-1}+D_{j_1+k_1}(a_{k_1-1})+\cdots+D_{j_s+k_s}(a_{k_s-1})$$

by Proposition 25, with the convention that $D_{j+1}(a_0) = D_{j+1}$. Then Φ_1 has type

$$A_{i_1-1}+\cdots+A_{i_r-1}+D_{j_1+k_1}+\cdots+D_{j_s+k_s}$$

and $\bar{\Phi}_1$ has type D_n where

$$n = \sum_{t=1}^{r} (i_t-1)+\sum_{t=1}^{s} (j_t+k_t).$$

Consider the subsystems of $\bar{\Phi}_1$ isomorphic to Φ_1 containing w in the Weyl group. Given the expression of w as a product of disjoint positive or negative cycles, each positive cycle determines uniquely the corresponding subsystem A_{i-1}. However, the negative cycles do not determine uniquely the components of type D. Each pair of negative cycles determines a component of type D, and the negative cycles can be paired in various ways to give components of the given dimensions $j_1+k_1, j_2+k_2, \cdots j_s+k_s$. The negative cycles of w can be inserted in the brackets

$$[\bar{j}_1\bar{k}_1][\bar{j}_2\bar{k}_2]\cdots[\bar{j}_s\bar{k}_s]$$

in $\Pi_i(m_i!)$ ways. When this is done the same subsystem is obtained with multiplicity $2^u \cdot \Pi_{j,k}(e_{j,k}!)$ due to permutations of equivalent pairs $[\bar{j}\bar{k}]$ and repetitions of pairs $[\bar{j}\bar{k}]$ with $j = k$. Thus the total number d of subsystems of the required type is

$$\frac{\prod_i (m_i!)}{2^u \prod_{j,k} (e_{j,k}!)}.$$

Finally we must calculate d in the cases in which Φ_1 is indecomposable and of maximal rank. It is often possible in these cases to replace w by zw where z is a non-unit element of the centre of W, and use the fact that $|\mathrm{ccl}(w)| = |\mathrm{ccl}(zw)|$. If $|\mathrm{ccl}(zw)|$ is already known $|\mathrm{ccl}(w)|$ can be obtained. This method can in fact be used to by-pass the calculation of d in other cases also.

Suppose Φ_1 is indecomposable and of maximal rank, that

$$|N_W(\overline{W}_1) : N_W(W_1)| \neq 1,$$

and that no information can be obtained by multiplying by a non-unit element of the centre. There are very few such cases, which are listed in Table 6, and in which it is probably simplest to calculate $|\mathrm{ccl}(w)|$ directly and deduce the value of d by Lemma 29.

TABLE 6

W	Γ	d
F_4	B_4	1
E_8	A_8	3
	$D_8(a_1)$	3
	$D_8(a_3)$	3

The methods outlined in this section enable us to calculate the integers a, b, c, d for each conjugacy class in $\mathscr{C}$ and hence, by Lemma 29, the number of elements in $\mathscr{C}$. The details for the exceptional Weyl groups of types G_2, F_4, E_6, E_7, E_8 are shown in Tables 7, 8, 9, 10, 11, at the end of the paper.

Having calculated the number of elements in each conjugacy class in $\mathscr{C}$, we now sum these integers, and establish the following proposition.

PROPOSITION 38. $\Sigma_{c \in \mathscr{C}}|C| = |W|$.

COROLLARIES. (i) *$\mathscr{C}$ is the complete set of conjugacy classes of W.*

(ii) $W = W_0$.

(iii) *Every element of W is associated with some admissible diagram in the set $\mathscr{A}$.*

(iv) *For each graph in the set $\mathscr{A}$ there is just one corresponding conjugacy class of W; except for the graphs in Lemma 27, for which there are two conjugacy classes.*

We have now proved the following equivalent statements about the group W.

THEOREM C. (i) *Every element of a Weyl group W is expressible as the product of two involutions.*

(ii) *Every element of W is contained in some dihedral subgroup.*

(iii) *For each element $w \in W$ there is an involution $i \in W$ such that $iwi = w^{-1}$.*

COROLLARY. *Every element of W is conjugate to its inverse.*

NOTE. It seems very desirable to have a proof of Theorem C which does not make use of the elaborate calculations necessary in the present discussion. In particular one would like an algebraic proof based on a definition of W by generators and relations, or a geometric proof using the operation of W as a Euclidean reflection group.

However, even if such a proof were available, it is not clear that this would remove the necessity of the counting argument of Proposition 38. One would have to show in some alternative way that the set of graphs needed to parametrise the conjugacy classes is not larger than $\mathscr{A}$, and that the only cases where two or more classes are parametrised by the same graph are those of Lemma 27.

10. Applications

The results we have obtained about conjugacy classes in the Weyl groups give useful information about the Chevalley groups over finite fields. Let W be a Weyl group and $G(q)$ a corresponding Chevalley group over the field $k = GF(q)$. $G(q)$ may be considered as the group of k-rational points of a simple algebraic group G with Weyl group W. A subgroup of G is called a torus if it is a closed subgroup isomorphic to the direct product of a number of groups isomorphic to the multiplicative group (of the base field), and a subgroup of $G(q)$ is called a torus if it is the group of k-rational points of some torus in G. A maximal torus of G is a torus which is not contained in any larger torus of G, and a maximal torus of $G(q)$ is, by definition, the group of k-rational points of some maximal torus of G. Now it is known that there is a one-one correspondence between the conjugacy classes of maximal tori in $G(q)$ and conjugacy classes of W. (See (18) E). The proof uses a theorem of Lang [8]. Let $T_w(q)$ be a maximal torus corresponding to the element w of W. Then the order of $T_w(q)$ is $f(q)$, where $f(t)$ is the characteristic polynomial of w on V. Thus our results on the Weyl groups determine the orders of all the maximal tori in $G(q)$. Let T_w be the torus in G corresponding to $T_w(q)$ and let N_w be the normalizer of T_w in G. Let $N_w(q)$ be the group of k-rational points of N_w. Then it follows from the result of Lang mentioned above that

$$|N_w(q) : T_w(q)| = |C_W(w)|$$

where $C_W(w)$ is the centralizer of w in W. Thus our results on the Weyl group determine this index for all maximal tori. $N_w(q)$ is usually, although not always, the normalizer of $T_w(q)$ in $G(q)$. It is likely that the subgroups $N_w(q)$ will have a significant rôle to play in the representation theory of $G(q)$.

A second application of our results concerns the classification of nilpotent elements in a simple Lie algebra over an algebraically closed field of characteristic 0. Let $\mathscr{L}$ be such a Lie algebra and G be its adjoint group. Thus G is a Chevalley group over the same field as $\mathscr{L}$ and operates on $\mathscr{L}$ as a group of automorphisms. It appears that the admissible graphs may be used to parametrise the classes of nilpotent elements of $\mathscr{L}$ under the action of G, and also the conjugacy classes of unipotent elements in G. By results of Dynkin [3] every nilpotent element of $\mathscr{L}$ is conjugate under the action of G to a regular or semi-regular nilpotent element of some regular semi-simple subalgebra of $\mathscr{L}$. Every regular nilpotent element of a semi-simple algebra is conjugate to the element $\sum_{r\in\Pi} e_r$, where Π is a system of fundamental roots of the algebra and the e_r are corresponding root-vectors. It is not difficult to show, using the results of Dynkin, that every semi-regular nilpotent element is conjugate to an element $\sum_{r\in\Omega} e_r$ where Ω is some set of positive roots associated with an admissible graph Γ. Thus we may associate at least one admissible graph with each class of nilpotent elements. If n is a regular nilpotent element of a regular semi-simple subalgebra of $\mathscr{L}$, the admissible graph representing n may be taken to be the Dynkin diagram of this subalgebra. If n is a semi-regular nilpotent element which is not regular, n is in one of the classes $D_l(a_1)$, $D_l(a_2)$, $\cdots$ $E_6(a_1)$, $E_7(a_1)$, $E_7(a_2)$, $E_8(a_1)$, $E_8(a_2)$ in the notation of Dynkin [3]. We have chosen our notation so that an admissible graph Γ corresponding to n may be denoted in the same way.

However, as with the Weyl group, the correspondence between classes of nilpotent elements and admissible graphs is not bijective. This is clear from Dynkin's work. Moreover it is quite possible for two admissible graphs to represent the same class in the Weyl group but not the same class of nilpotent elements, and vice-versa. There is, however, a natural injection from the set of semi-regular nilpotent classes into the set of classes of the Weyl group [17]. It would be desirable to gain a better understanding of the relationship between the classes in the Weyl group and the classes of nilpotent elements in the Lie algebra.

It is a pleasure to acknowledge many helpful discussions with T.A. Springer about the work in this paper, and R. Steinberg also made a number of useful suggestions.

TABLE 7

Conjugacy classes in the Weyl group $W(G_2)$

Γ	W_1	$W_1 \times W_2$	$\bar{W}_1$	$\lvert \mathrm{Aut}_W(\Pi_1) \rvert$	$\lvert W : N_W(W_1) \rvert$	$\lvert \mathrm{ccl}_{W_1}(w) \rvert$	$\lvert N_W(\bar{W}_1) : N_W(W_1) \rvert$	d	$\lvert \mathrm{ccl}_W(w) \rvert$
ϕ	1	G_2	1	1	1	1	1	1	1
A_1	A_1	$A_1 \times \tilde{A}_1$	A_1	1	3	1	1	1	3
$\tilde{A}_1$	$\tilde{A}_1$	$A_1 \times \tilde{A}_1$	$\tilde{A}_1$	1	3	1	1	1	3
$A_1 \times \tilde{A}_1$	$A_1 \times \tilde{A}_1$	$A_1 \times \tilde{A}_1$	G_2	1	3	1	3	3	1
A_2	A_2	A_2	G_2	2	1	2	1	1	2
G_2	G_2	G_2	G_2	1	1	2	1	1	2

TABLE 8

Conjugacy classes in the Weyl group $W(F_4)$

Γ	W_1	$W_1 \times W_2$	$\bar{W}_1$	$\lvert \mathrm{Aut}_W(\Pi_1)\rvert$	$\lvert W : N_W(W_1)\rvert$	$\lvert \mathrm{ccl}_{W_1}(w)\rvert$	$\lvert N_W(\bar{W}_1) : N_W(W_1)\rvert$	d	$\lvert \mathrm{ccl}_W(w)\rvert$
ϕ	1	F_4	1	1	1	1	1	1	1
A_1	A_1	$C_3 \times A_1$	A_1	1	12	1	1	1	12
$\tilde{A}_1$	$\tilde{A}_1$	$B_3 \times \tilde{A}_1$	A_1	1	12	1	1	1	12
A_1^2	A_1^2	$B_2 \times A_1^2$	B_2	2	18	1	1	1	18
$A_1 \times \tilde{A}_1$	$A_1 \times \tilde{A}_1$	$A_1^2 \times \tilde{A}_1^2$	$A_1 \times \tilde{A}_1$	1	72	1	1	1	72
A_2	A_2	$A_2 \times \tilde{A}_2$	A_2	2	16	2	1	1	32
$\tilde{A}_2$	$\tilde{A}_2$	$A_2 \times \tilde{A}_2$	$\tilde{A}_2$	2	16	2	1	1	32
B_2	B_2	B_2^2	B_2	1	18	2	1	1	36
A_1^3	A_1^3	A_1^4	C_3	6	12	1	1	1	12
$A_1^2 \times \tilde{A}_1$	$A_1^2 \times \tilde{A}_1$	$A_1^2 \times \tilde{A}_1^2$	B_3	2	36	1	3	3	12
A_3	A_3	$A_3 \times \tilde{A}_1$	B_3	2	12	6	1	1	72
$B_2 \times A_1$	$B_2 \times A_1$	$B_2 \times A_1^2$	C_3	1	36	2	3	1	72
C_3	C_3	$C_3 \times A_1$	C_3	1	12	8	1	1	96
B_3	B_3	$B_3 \times \tilde{A}_1$	B_3	1	12	8	1	1	96
$\tilde{A}_2 \times A_1$	$\tilde{A}_2 \times A_1$	$\tilde{A}_2 \times A_1$	$\tilde{A}_2 \times A_1$	2	48	2	1	1	96
$A_2 \times \tilde{A}_1$	$A_2 \times \tilde{A}_1$	$A_2 \times \tilde{A}_1$	$A_2 \times \tilde{A}_1$	2	48	2	1	1	96
A_1^4	A_1^4	A_1^4	F_4	24	3	1	3	3	1
$A_2 \times \tilde{A}_2$	$A_2 \times \tilde{A}_2$	$A_2 \times \tilde{A}_2$	F_4	2	16	4	16	4	16
$A_3 \times \tilde{A}_1$	$A_3 \times \tilde{A}_1$	$A_3 \times \tilde{A}_1$	F_4	2	12	6	12	2	36
$C_3 \times A_1$	$C_3 \times A_1$	$C_3 \times A_1$	F_4	1	12	8	12	3	32
D_4	D_4	D_4	F_4	6	1	32	1	1	32
$D_4(a_1)$	D_4	D_4	F_4	6	1	12	1	1	12
B_4	B_4	B_4	F_4	1	3	48	3	1	144
F_4	F_4	F_4	F_4	1	1	96	1	1	96
$F_4(a_1)$	F_4	F_4	F_4	1	1	16	1	1	16

TABLE 9

Conjugacy classes in the Weyl group $W(E_6)$

Γ	W_1	$W_1\times W_2$	$\bar{W}_1$	$\lvert\mathrm{Aut}_W(\Pi_1)\rvert$	$\lvert W:N_W(W_1)\rvert$	$\lvert\mathrm{ccl}_{W_1}(w)\rvert$	$\lvert N_W(\bar{W}_1):N_W(W_1)\rvert$	d	$\lvert\mathrm{ccl}_W(w)\rvert$
ϕ	1	E_6	1	1	1	1	1	1	1
A_1	A_1	$A_5\times A_1$	A_1	1	36	1	1	1	36
A_1^2	A_1^2	$A_3\times A_1^2$	A_1^2	2	270	1	1	1	270
A_2	A_2	A_2^3	A_2	2	120	2	1	1	240
A_1^3	A_1^3	A_1^4	A_1^3	6	540	1	1	1	540
$A_2\times A_1$	$A_2\times A_1$	$A_2^2\times A_1$	$A_2\times A_1$	1	720	2	1	1	1440
A_3	A_3	$A_3\times A_1^2$	A_3	2	270	6	1	1	1620
A_1^4	A_1^4	A_1^4	D_4	24	135	1	3	3	45
$A_2\times A_1^2$	$A_2\times A_1^2$	$A_2\times A_1^2$	$A_2\times A_1^2$	2	1080	2	1	1	2160
A_2^2	A_2^2	A_2^3	A_2^2	2	120	4	1	1	480
$A_3\times A_1$	$A_3\times A_1$	$A_3\times A_1^2$	$A_3\times A_1$	1	540	6	1	1	3240
A_4	A_4	$A_4\times A_1$	A_4	1	216	24	1	1	5184
D_4	D_4	D_4	D_4	6	45	32	1	1	1440
$D_4(a_1)$	D_4	D_4	D_4	6	45	12	1	1	540
$A_2^2\times A_1$	$A_2^2\times A_1$	$A_2^2\times A_1$	$A_2^2\times A_1$	2	360	4	1	1	1440
$A_3\times A_1^2$	$A_3\times A_1^2$	$A_3\times A_1^2$	D_5	2	270	6	10	3	540
$A_4\times A_1$	$A_4\times A_1$	$A_4\times A_1$	$A_4\times A_1$	1	216	24	1	1	5184
A_5	A_5	$A_5\times A_1$	A_5	1	36	120	1	1	4320
D_5	D_5	D_5	D_5	1	27	240	1	1	6480
$D_5(a_1)$	D_5	D_5	D_5	1	27	160	1	1	4320
A_2^3	A_2^3	A_2^3	E_6	6	40	8	40	4	80
$A_5\times A_1$	$A_5\times A_1$	$A_5\times A_1$	E_6	1	36	120	36	3	1440
E_6	E_6	E_6	E_6	1	1	4320	1	1	4320
$E_6(a_1)$	E_6	E_6	E_6	1	1	5760	1	1	5760
$E_6(a_2)$	E_6	E_6	E_6	1	1	720	1	1	720

TABLE 10

Conjugacy classes in the Weyl group $W(E_7)$

Γ	W_1	$W_1 \times W_2$	$\bar{W}_1$	$\lvert \mathrm{Aut}_W(\Pi_1) \rvert$	$\lvert W : N_W(W_1) \rvert$	$\lvert \mathrm{ccl}_{W_1}(w) \rvert$	$\lvert N_W(\bar{W}_1) : N_W(W_1) \rvert$	d	$\lvert \mathrm{ccl}_W(w) \rvert$
ϕ	1	E_7	1	1	1	1	1	1	1
A_1	A_1	$D_6 \times A_1$	A_1	1	$3^2 \cdot 7$	1	1	1	$3^2 \cdot 7$
$A_1{}^2$	$A_1{}^2$	$D_4 \times A_1{}^3$	$A_1{}^2$	2	$3^3 \cdot 5 \cdot 7$	1	1	1	$3^3 \cdot 5 \cdot 7$
A_2	A_2	$A_5 \times A_2$	A_2	2	$2^4 \cdot 3 \cdot 7$	2	1	1	$2^5 \cdot 3 \cdot 7$
$(A_1{}^3)'$	$A_1{}^3$	$D_4 \times A_1{}^3$	$A_1{}^3$	6	$3^2 \cdot 5 \cdot 7$	1	1	1	$3^2 \cdot 5 \cdot 7$
$(A_1{}^3)''$	$A_1{}^3$	$A_1{}^7$	$A_1{}^3$	6	$2^2 \cdot 3^3 \cdot 5 \cdot 7$	1	1	1	$2^2 \cdot 3^3 \cdot 5 \cdot 7$
$A_2 \times A_1$	$A_2 \times A_1$	$A_3 \times A_2 \times A_1$	$A_2 \times A_1$	2	$2^4 \cdot 3^2 \cdot 5 \cdot 7$	2	1	1	$2^5 \cdot 3^2 \cdot 5 \cdot 7$
A_3	A_3	$A_3{}^2 \times A_1$	A_3	2	$2^2 \cdot 3^2 \cdot 5 \cdot 7$	6	1	1	$2^3 \cdot 3^3 \cdot 5 \cdot 7$
$(A_1{}^4)'$	$A_1{}^4$	$A_1{}^7$	$A_1{}^4$	6	$2^2 \cdot 3^3 \cdot 5 \cdot 7$	1	1	1	$2^2 \cdot 3^3 \cdot 5 \cdot 7$
$(A_1{}^4)''$	$A_1{}^4$	$A_1{}^7$	D_4	24	$3^3 \cdot 5 \cdot 7$	1	3	3	$3^2 \cdot 5 \cdot 7$
$A_2 \times A_1{}^2$	$A_2 \times A_1{}^2$	$A_2 \times A_1{}^3$	$A_2 \times A_1{}^2$	4	$2^4 \cdot 3^3 \cdot 5 \cdot 7$	2	1	1	$2^5 \cdot 3^3 \cdot 5 \cdot 7$
$A_2{}^2$	$A_2{}^2$	$A_2{}^3$	$A_2{}^2$	4	$2^5 \cdot 3 \cdot 5 \cdot 7$	4	1	1	$2^7 \cdot 3 \cdot 5 \cdot 7$
$(A_3 \times A_1)'$	$A_3 \times A_1$	$A_3{}^2 \times A_1$	$A_3 \times A_1$	2	$2^2 \cdot 3^2 \cdot 5 \cdot 7$	6	1	1	$2^3 \cdot 3^3 \cdot 5 \cdot 7$
$(A_3 \times A_1)''$	$A_3 \times A_1$	$A_3 \times A_1{}^3$	$A_3 \times A_1$	2	$2^3 \cdot 3^3 \cdot 5 \cdot 7$	6	1	1	$2^4 \cdot 3^4 \cdot 5 \cdot 7$
A_4	A_4	$A_4 \times A_2$	A_4	2	$2^5 \cdot 3^2 \cdot 7$	24	1	1	$2^8 \cdot 3^3 \cdot 7$
D_4	D_4	$D_4 \times A_1{}^3$	D_4	6	$3^2 \cdot 5 \cdot 7$	32	1	1	$2^5 \cdot 3^2 \cdot 5 \cdot 7$
$D_4(a_1)$	D_4	$D_4 \times A_1{}^3$	D_4	6	$3^2 \cdot 5 \cdot 7$	12	1	1	$2^2 \cdot 3^3 \cdot 5 \cdot 7$
$A_1{}^5$	$A_1{}^5$	$A_1{}^7$	$D_4 \times A_1$	8	$3^4 \cdot 5 \cdot 7$	1	3	3	$3^3 \cdot 5 \cdot 7$
$A_2 \times A_1{}^3$	$A_2 \times A_1{}^3$	$A_2 \times A_1{}^3$	$A_2 \times A_1{}^3$	12	$2^4 \cdot 3^2 \cdot 5 \cdot 7$	2	1	1	$2^5 \cdot 3^2 \cdot 5 \cdot 7$
$A_2{}^2 \times A_1$	$A_2{}^2 \times A_1$	$A_2{}^2 \times A_1$	$A_2{}^2 \times A_1$	4	$2^5 \cdot 3^2 \cdot 5 \cdot 7$	4	1	1	$2^7 \cdot 3^2 \cdot 5 \cdot 7$
$(A_3 \times A_1{}^2)'$	$A_3 \times A_1{}^2$	$A_3 \times A_1{}^3$	$A_3 \times A_1{}^2$	2	$2^3 \cdot 3^3 \cdot 5 \cdot 7$	6	1	1	$2^4 \cdot 3^4 \cdot 5 \cdot 7$
$(A_3 \times A_1{}^2)''$	$A_3 \times A_1{}^2$	$A_3 \times A_1{}^3$	D_5	4	$2^2 \cdot 3^3 \cdot 5 \cdot 7$	6	10	3	$2^3 \cdot 3^3 \cdot 5 \cdot 7$
$A_3 \times A_2$	$A_3 \times A_2$	$A_3 \times A_2 \times A_1$	$A_3 \times A_2$	2	$2^4 \cdot 3^2 \cdot 5 \cdot 7$	12	1	1	$2^6 \cdot 3^3 \cdot 5 \cdot 7$

TABLE 10 (*continued*)

Γ	W_1	$W_1 \times W_2$	$\bar{W}_1$	$\lvert \mathrm{Aut}_W(\Pi_1)\rvert$	$\lvert W : N_W(W_1)\rvert$	$\lvert \mathrm{ccl}_{W_1}(w)\rvert$	$\lvert N_W(\bar{W}_1) : N_W(W_1)\rvert$	d	$\lvert \mathrm{ccl}_W(w)\rvert$
$A_4 \times A_1$	$A_4 \times A_1$	$A_4 \times A_1$	$A_4 \times A_1$	2	$2^5 \cdot 3^3 \cdot 7$	24	1	1	$2^8 \cdot 3^4 \cdot 7$
$(A_5)'$	A_5	$A_5 \times A_2$	A_5	2	$2^4 \cdot 3 \cdot 7$	120	1	1	$2^7 \cdot 3^2 \cdot 5 \cdot 7$
$(A_5)''$	A_5	$A_5 \times A_1$	A_5	2	$2^4 \cdot 3^2 \cdot 7$	120	1	1	$2^7 \cdot 3^3 \cdot 5 \cdot 7$
$D_4 \times A_1$	$D_4 \times A_1$	$D_4 \times A_1^3$	$D_4 \times A_1$	2	$3^3 \cdot 5 \cdot 7$	32	1	1	$2^5 \cdot 3^3 \cdot 5 \cdot 7$
$D_4(a_1) \times A_1$	$D_4 \times A_1$	$D_4 \times A_1^3$	$D_4 \times A_1$	2	$3^3 \cdot 5 \cdot 7$	12	1	1	$2^2 \cdot 3^4 \cdot 5 \cdot 7$
D_5	D_5	$D_5 \times A_1$	D_5	2	$2 \cdot 3^3 \cdot 7$	240	1	1	$2^5 \cdot 3^4 \cdot 5 \cdot 7$
$D_5(a_1)$	D_5	$D_5 \times A_1$	D_5	2	$2 \cdot 3^3 \cdot 7$	160	1	1	$2^6 \cdot 3^3 \cdot 5 \cdot 7$
A_1^6	A_1^6	A_1^7	D_6	24	$3^3 \cdot 5 \cdot 7$	1	15	15	$3^2 \cdot 7$
A_2^3	A_2^3	A_2^3	E_6	12	$2^5 \cdot 5 \cdot 7$	2^3	40	4	$2^6 \cdot 5 \cdot 7$
$A_3 \times A_1^3$	$A_3 \times A_1^3$	$A_3 \times A_1^3$	$D_5 \times A_1$	4	$2^2 \cdot 3^3 \cdot 5 \cdot 7$	$2 \cdot 3$	10	3	$2^3 \cdot 3^3 \cdot 5 \cdot 7$
$A_3 \times A_2 \times A_1$	$A_3 \times A_2 \times A_1$	$A_3 \times A_2 \times A_1$	$A_3 \times A_2 \times A_1$	2	$2^4 \cdot 3^2 \cdot 5 \cdot 7$	$2^2 \cdot 3$	1	1	$2^6 \cdot 3^3 \cdot 5 \cdot 7$
A_3^2	A_3^2	$A_3^2 \times A_1$	D_6	4	$2 \cdot 3^2 \cdot 5 \cdot 7$	$2^2 \cdot 3^2$	10	2	$2^2 \cdot 3^4 \cdot 5 \cdot 7$
$A_4 \times A_2$	$A_4 \times A_2$	$A_4 \times A_2$	$A_4 \times A_2$	2	$2^5 \cdot 3^2 \cdot 7$	$2^4 \cdot 3$	1	1	$2^9 \cdot 3^3 \cdot 7$
$(A_5 \times A_1)'$	$A_5 \times A_1$	$A_5 \times A_1$	$A_5 \times A_1$	2	$2^4 \cdot 3^2 \cdot 7$	$2^3 \cdot 3 \cdot 5$	1	1	$2^7 \cdot 3^3 \cdot 5 \cdot 7$
$(A_5 \times A_1)''$	$A_5 \times A_1$	$A_5 \times A_1$	E_6	2	$2^4 \cdot 3^2 \cdot 7$	$2^3 \cdot 3 \cdot 5$	36	3	$2^7 \cdot 3^2 \cdot 5 \cdot 7$
A_6	A_6	A_6	A_6	2	$2^5 \cdot 3^2$	$2^4 \cdot 3^2 \cdot 5$	1	1	$2^9 \cdot 3^4 \cdot 5$
$D_4 \times A_1^2$	$D_4 \times A_1^2$	$D_4 \times A_1^3$	D_6	2	$3^3 \cdot 5 \cdot 7$	2^5	15	3	$2^5 \cdot 3^2 \cdot 5 \cdot 7$
$D_5 \times A_1$	$D_5 \times A_1$	$D_5 \times A_1$	$D_5 \times A_1$	2	$2 \cdot 3^3 \cdot 7$	$2^4 \cdot 3 \cdot 5$	1	1	$2^5 \cdot 3^4 \cdot 5 \cdot 7$
$D_5(a_1) \times A_1$	$D_5 \times A_1$	$D_5 \times A_1$	$D_5 \times A_1$	2	$2 \cdot 3^3 \cdot 7$	$2^5 \cdot 5$	1	1	$2^6 \cdot 3^3 \cdot 5 \cdot 7$
D_6	D_6	$D_6 \times A_1$	D_6	2	$3^2 \cdot 7$	$2^8 \cdot 3^2$	1	1	$2^8 \cdot 3^4 \cdot 7$
$D_6(a_1)$	D_6	$D_6 \times A_1$	D_6	2	$3^2 \cdot 7$	$2^5 \cdot 3^2 \cdot 5$	1	1	$2^5 \cdot 3^4 \cdot 5 \cdot 7$
$D_6(a_2)$	D_6	$D_6 \times A_1$	D_6	2	$3^2 \cdot 7$	$2^7 \cdot 5$	1	1	$2^7 \cdot 3^2 \cdot 5 \cdot 7$
E_6	E_6	E_6	E_6	2	$2^2 \cdot 7$	$2^5 \cdot 3^3 \cdot 5$	1	1	$2^7 \cdot 3^3 \cdot 5 \cdot 7$
$E_6(a_1)$	E_6	E_6	E_6	2	$2^2 \cdot 7$	$2^7 \cdot 3^2 \cdot 5$	1	1	$2^9 \cdot 3^2 \cdot 5 \cdot 7$
$E_6(a_2)$	E_6	E_6	E_6	2	$2^2 \cdot 7$	$2^4 \cdot 3^2 \cdot 5$	1	1	$2^6 \cdot 3^2 \cdot 5 \cdot 7$
A_1^7	A_1^7	A_1^7	E_7	168	$3^3 \cdot 5$	1	135	135	1

TABLE 10 (*continued*)

Γ	W_1	$W_1 \times W_2$	$\bar{W}_1$	$\lvert \mathrm{Aut}_W(\Pi_1) \rvert$	$\lvert W : N_W(W_1) \rvert$	$\lvert \mathrm{ccl}_{W_1}(w) \rvert$	$\lvert N_W(\bar{W}_1) : N_W(W_1) \rvert$	d	$\lvert \mathrm{ccl}_W(w) \rvert$
$A_3{}^2 \times A_1$	$A_3{}^2 \times A_1$	$A_3{}^2 \times A_1$	E_7	4	$2 \cdot 3^2 \cdot 5 \cdot 7$	$2^2 \cdot 3^2$	630	6	$2^2 \cdot 3^3 \cdot 5 \cdot 7$
$A_5 \times A_2$	$A_5 \times A_2$	$A_5 \times A_2$	E_7	2	$2^4 \cdot 3 \cdot 7$	$2^4 \cdot 3 \cdot 5$	336	4	$2^6 \cdot 3^2 \cdot 5 \cdot 7$
A_7	A_7	A_7	E_7	2	$2^2 \cdot 3^2$	$2^4 \cdot 3^2 \cdot 5 \cdot 7$	36	2	$2^5 \cdot 3^4 \cdot 5 \cdot 7$
$D_4 \times A_1{}^3$	$D_4 \times A_1{}^3$	$D_4 \times A_1{}^3$	E_7	6	$3^2 \cdot 5 \cdot 7$	2^5	315	15	$2^5 \cdot 3 \cdot 7$
$D_6 \times A_1$	$D_6 \times A_1$	$D_6 \times A_1$	E_7	1	$3^2 \cdot 7$	$2^8 \cdot 3^2$	63	3	$2^8 \cdot 3^3 \cdot 7$
$D_6(a_2) \times A_1$	$D_6 \times A_1$	$D_6 \times A_1$	E_7	1	$3^2 \cdot 7$	$2^7 \cdot 5$	63	3	$2^7 \cdot 3 \cdot 5 \cdot 7$
E_7	E_7	E_7	E_7	1	1	$2^9 \cdot 3^2 \cdot 5 \cdot 7$	1	1	$2^9 \cdot 3^2 \cdot 5 \cdot 7$
$E_7(a_1)$	E_7	E_7	E_7	1	1	$2^9 \cdot 3^4 \cdot 5$	1	1	$2^9 \cdot 3^4 \cdot 5$
$E_7(a_2)$	E_7	E_7	E_7	1	1	$2^7 \cdot 3^3 \cdot 5 \cdot 7$	1	1	$2^7 \cdot 3^3 \cdot 5 \cdot 7$
$E_7(a_3)$	E_7	E_7	E_7	1	1	$2^9 \cdot 3^3 \cdot 7$	1	1	$2^9 \cdot 3^3 \cdot 7$
$E_7(a_4)$	E_7	E_7	E_7	1	1	$2^6 \cdot 5 \cdot 7$	1	1	$2^6 \cdot 5 \cdot 7$

TABLE 11

Conjugacy classes in the Weyl group $W(E_8)$

Γ	W_1	$W_1 \times W_2$	$\bar{W}_1$	$\lvert \mathrm{Aut}_W(\Pi_1)\rvert$	$\lvert W : N_W(W_1)\rvert$	$\lvert \mathrm{ccl}_{W_1}(w)\rvert$	$\lvert N_W(\bar{W}_1) : N_W(W_1)\rvert$	d	$\lvert \mathrm{ccl}_W(w)\rvert$
ϕ	1	E_8	1	1	1	1	1	1	1
A_1	A_1	$E_7 \times A_1$	A_1	1	$2^3 \cdot 3 \cdot 5$	1	1	1	$2^3 \cdot 3 \cdot 5$
A_1^2	A_1^2	$D_6 \times A_1^2$	A_1^2	2	$2^2 \cdot 3^3 \cdot 5 \cdot 7$	1	1	1	$2^2 \cdot 3^3 \cdot 5 \cdot 7$
A_2	A_2	$E_6 \times A_2$	A_2	2	$2^5 \cdot 5 \cdot 7$	2	1	1	$2^6 \cdot 5 \cdot 7$
A_1^3	A_1^3	$D_4 \times A_1^4$	A_1^3	6	$2^3 \cdot 3^3 \cdot 5^2 \cdot 7$	1	1	1	$2^3 \cdot 3^3 \cdot 5^2 \cdot 7$
$A_2 \times A_1$	$A_2 \times A_1$	$A_5 \times A_2 \times A_1$	$A_2 \times A_1$	2	$2^7 \cdot 3^2 \cdot 5 \cdot 7$	2	1	1	$2^8 \cdot 3^2 \cdot 5 \cdot 7$
A_3	A_3	$D_5 \times A_3$	A_3	2	$2^3 \cdot 3^3 \cdot 5 \cdot 7$	$2 \cdot 3$	1	1	$2^4 \cdot 3^4 \cdot 5 \cdot 7$
$(A_1^4)'$	A_1^4	$D_4 \times A_1^4$	D_4	24	$2 \cdot 3^3 \cdot 5^2 \cdot 7$	1	3	3	$2 \cdot 3^2 \cdot 5^2 \cdot 7$
$(A_1^4)''$	A_1^4	A_1^8	A_1^4	24	$2^3 \cdot 3^4 \cdot 5^2 \cdot 7$	1	1	1	$2^3 \cdot 3^4 \cdot 5^2 \cdot 7$
$A_2 \times A_1^2$	$A_2 \times A_1^2$	$A_3 \times A_2 \times A_1^2$	$A_2 \times A_1^2$	4	$2^6 \cdot 3^3 \cdot 5^2 \cdot 7$	2	1	1	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$
A_2^2	A_2^2	A_2^4	A_2^2	8	$2^7 \cdot 3 \cdot 5^2 \cdot 7$	2^2	1	1	$2^9 \cdot 3 \cdot 5^2 \cdot 7$
$A_3 \times A_1$	$A_3 \times A_1$	$A_3^2 \times A_1^2$	$A_3 \times A_1$	2	$2^5 \cdot 3^3 \cdot 5^2 \cdot 7$	$2 \cdot 3$	1	1	$2^6 \cdot 3^4 \cdot 5^2 \cdot 7$
A_4	A_4	A_4^2	A_4	2	$2^7 \cdot 3^3 \cdot 7$	$2^3 \cdot 3$	1	1	$2^{10} \cdot 3^4 \cdot 7$
D_4	D_4	D_4^2	D_4	6	$2 \cdot 3^2 \cdot 5^2 \cdot 7$	2^5	1	1	$2^6 \cdot 3^2 \cdot 5^2 \cdot 7$
$D_4(a_1)$	D_4	D_4^2	D_4	6	$2 \cdot 3^2 \cdot 5^2 \cdot 7$	$2^2 \cdot 3$	1	1	$2^3 \cdot 3^3 \cdot 5^2 \cdot 7$
A_1^5	A_1^5	A_1^8	$D_4 \times A_1$	24	$2^3 \cdot 3^4 \cdot 5^2 \cdot 7$	1	3	3	$2^3 \cdot 3^3 \cdot 5^2 \cdot 7$
$A_2 \times A_1^3$	$A_2 \times A_1^3$	$A_2 \times A_1^4$	$A_2 \times A_1^3$	12	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	2	1	1	$2^8 \cdot 3^3 \cdot 5^2 \cdot 7$
$A_2^2 \times A_1$	$A_2^2 \times A_1$	$A_2^3 \times A_1$	$A_2^2 \times A_1$	4	$2^8 \cdot 3^2 \cdot 5^2 \cdot 7$	2^2	1	1	$2^{10} \cdot 3^2 \cdot 5^2 \cdot 7$
$(A_3 \times A_1^2)'$	$A_3 \times A_1^2$	$A_3^2 \times A_1^2$	D_5	4	$2^4 \cdot 3^3 \cdot 5^2 \cdot 7$	$2 \cdot 3$	10	3	$2^5 \cdot 3^3 \cdot 5^2 \cdot 7$
$(A_3 \times A_1^2)''$	$A_3 \times A_1^2$	$A_3 \times A_1^4$	$A_3 \times A_1^2$	4	$2^5 \cdot 3^4 \cdot 5^2 \cdot 7$	$2 \cdot 3$	1	1	$2^6 \cdot 3^5 \cdot 5^2 \cdot 7$
$A_3 \times A_2$	$A_3 \times A_2$	$A_3 \times A_2 \times A_1^2$	$A_3 \times A_2$	4	$2^6 \cdot 3^3 \cdot 5^2 \cdot 7$	$2^2 \cdot 3$	1	1	$2^8 \cdot 3^4 \cdot 5^2 \cdot 7$
$A_4 \times A_1$	$A_4 \times A_1$	$A_4 \times A_2 \times A_1$	$A_4 \times A_1$	2	$2^8 \cdot 3^3 \cdot 5 \cdot 7$	$2^3 \cdot 3$	1	1	$2^{11} \cdot 3^4 \cdot 5 \cdot 7$
A_5	A_5	$A_5 \times A_2 \times A_1$	A_5	2	$2^7 \cdot 3^2 \cdot 5 \cdot 7$	$2^3 \cdot 3 \cdot 5$	1	1	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7$
$D_4 \times A_1$	$D_4 \times A_1$	$D_4 \times A_1^4$	$D_4 \times A_1$	6	$2^3 \cdot 3^3 \cdot 5^2 \cdot 7$	2^5	1	1	$2^8 \cdot 3^3 \cdot 5^2 \cdot 7$
$D_4(a_1) \times A_1$	$D_4 \times A_1$	$D_4 \times A_1^4$	$D_4 \times A_1$	6	$2^3 \cdot 3^3 \cdot 5^2 \cdot 7$	$2^2 \cdot 3$	1	1	$2^5 \cdot 3^4 \cdot 5^2 \cdot 7$

TABLE 11 (*continued*)

Γ	W_1	$W_1\times W_2$	$\bar{W}_1$	$\lvert\mathrm{Aut}_W(\Pi_1)\rvert$	$\lvert W:N_W(W_1)\rvert$	$\lvert\mathrm{ccl}_{W_1}(w)\rvert$	$\lvert N_W(\bar{W}_1):N_W(W_1)\rvert$	d	$\lvert\mathrm{ccl}_W(w)\rvert$
D_5	D_5	$D_5\times A_3$	D_5	2	$2^3\cdot 3^3\cdot 5\cdot 7$	$2^4\cdot 3\cdot 5$	1	1	$2^7\cdot 3^4\cdot 5^2\cdot 7$
$D_5(a_1)$	D_5	$D_5\times A_3$	D_5	2	$2^3\cdot 3^3\cdot 5\cdot 7$	$2^5\cdot 5$	1	1	$2^8\cdot 3^3\cdot 5^2\cdot 7$
$A_1{}^6$	$A_1{}^6$	$A_1{}^8$	D_6	48	$2^2\cdot 3^4\cdot 5^2\cdot 7$	1	15	15	$2^2\cdot 3^3\cdot 5\cdot 7$
$A_2\times A_1{}^4$	$A_2\times A_1{}^4$	$A_2\times A_1{}^4$	$D_4\times A_2$	48	$2^5\cdot 3^3\cdot 5^2\cdot 7$	2	3	3	$2^6\cdot 3^2\cdot 5^2\cdot 7$
$A_2{}^2\times A_1{}^2$	$A_2{}^2\times A_1{}^2$	$A_2{}^2\times A_1{}^2$	$A_2{}^2\times A_1{}^2$	8	$2^7\cdot 3^3\cdot 5^2\cdot 7$	2^2	1	1	$2^9\cdot 3^3\cdot 5^2\cdot 7$
$A_2{}^3$	$A_2{}^3$	$A_2{}^4$	E_6	12	$2^8\cdot 5^2\cdot 7$	2^3	40	4	$2^9\cdot 5^2\cdot 7$
$A_3\times A_1{}^3$	$A_3\times A_1{}^3$	$A_3\times A_1{}^4$	$D_5\times A_1$	4	$2^5\cdot 3^4\cdot 5^2\cdot 7$	$2\cdot 3$	10	3	$2^6\cdot 3^4\cdot 5^2\cdot 7$
$A_3\times A_2\times A_1$	$A_3\times A_2\times A_1$	$A_3\times A_2\times A_1{}^2$	$A_3\times A_2\times A_1$	2	$2^7\cdot 3^3\cdot 5^2\cdot 7$	$2^2\cdot 3$	1	1	$2^9\cdot 3^4\cdot 5^2\cdot 7$
$(A_3{}^2)'$	$A_3{}^2$	$A_3{}^2\times A_1{}^2$	D_6	8	$2^3\cdot 3^3\cdot 5^2\cdot 7$	$2^2\cdot 3^2$	10	2	$2^4\cdot 3^5\cdot 5^2\cdot 7$
$(A_3{}^2)''$	$A_3{}^2$	$A_3{}^2$	$A_3{}^2$	8	$2^5\cdot 3^3\cdot 5^2\cdot 7$	$2^2\cdot 3^2$	1	1	$2^7\cdot 3^5\cdot 5^2\cdot 7$
$A_4\times A_1{}^2$	$A_4\times A_1{}^2$	$A_4\times A_1{}^2$	$A_4\times A_1{}^2$	4	$2^7\cdot 3^4\cdot 5\cdot 7$	$2^3\cdot 3$	1	1	$2^{10}\cdot 3^5\cdot 5\cdot 7$
$A_4\times A_2$	$A_4\times A_2$	$A_4\times A_2\times A_1$	$A_4\times A_2$	2	$2^8\cdot 3^3\cdot 5\cdot 7$	$2^4\cdot 3$	1	1	$2^{12}\cdot 3^4\cdot 5\cdot 7$
$(A_5\times A_1)'$	$A_5\times A_1$	$A_5\times A_2\times A_1$	E_6	2	$2^7\cdot 3^2\cdot 5\cdot 7$	$2^3\cdot 3\cdot 5$	36	3	$2^{10}\cdot 3^2\cdot 5^2\cdot 7$
$(A_5\times A_1)''$	$A_5\times A_1$	$A_5\times A_1{}^2$	$A_5\times A_1$	2	$2^7\cdot 3^3\cdot 5\cdot 7$	$2^3\cdot 3\cdot 5$	1	1	$2^{10}\cdot 3^4\cdot 5^2\cdot 7$
A_6	A_6	$A_6\times A_1$	A_6	2	$2^8\cdot 3^3\cdot 5$	$2^4\cdot 3^2\cdot 5$	1	1	$2^{12}\cdot 3^5\cdot 5^2$
$D_4\times A_1{}^2$	$D_4\times A_1{}^2$	$D_4\times A_1{}^4$	D_6	4	$2^2\cdot 3^4\cdot 5^2\cdot 7$	2^5	15	3	$2^7\cdot 3^3\cdot 5^2\cdot 7$
$D_4\times A_2$	$D_4\times A_2$	$D_4\times A_2$	$D_4\times A_2$	12	$2^5\cdot 3^2\cdot 5^2\cdot 7$	2^6	1	1	$2^{11}\cdot 3^2\cdot 5^2\cdot 7$
$D_4(a_1)\times A_2$	$D_4\times A_2$	$D_4\times A_2$	$D_4\times A_2$	12	$2^5\cdot 3^2\cdot 5^2\cdot 7$	$2^3\cdot 3$	1	1	$2^8\cdot 3^3\cdot 5^2\cdot 7$
$D_5\times A_1$	$D_5\times A_1$	$D_5\times A_1{}^2$	$D_5\times A_1$	2	$2^4\cdot 3^4\cdot 5\cdot 7$	$2^4\cdot 3\cdot 5$	1	1	$2^8\cdot 3^5\cdot 5^2\cdot 7$
$D_5(a_1)\times A_1$	$D_5\times A_1$	$D_5\times A_1{}^2$	$D_5\times A_1$	2	$2^4\cdot 3^4\cdot 5\cdot 7$	$2^5\cdot 5$	1	1	$2^9\cdot 3^4\cdot 5^2\cdot 7$
D_6	D_6	$D_6\times A_1{}^2$	D_6	2	$2^2\cdot 3^3\cdot 5\cdot 7$	$2^8\cdot 3^2$	1	1	$2^{10}\cdot 3^5\cdot 5\cdot 7$
$D_6(a_1)$	D_6	$D_6\times A_1{}^2$	D_6	2	$2^2\cdot 3^3\cdot 5\cdot 7$	$2^5\cdot 3^2\cdot 5$	1	1	$2^7\cdot 3^5\cdot 5^2\cdot 7$
$D_6(a_2)$	D_6	$D_6\times A_1{}^2$	D_6	2	$2^2\cdot 3^3\cdot 5\cdot 7$	$2^7\cdot 5$	1	1	$2^9\cdot 3^3\cdot 5^2\cdot 7$
E_6	E_6	$E_6\times A_2$	E_6	2	$2^5\cdot 5\cdot 7$	$2^5\cdot 3^3\cdot 5$	1	1	$2^{10}\cdot 3^3\cdot 5^2\cdot 7$
$E_6(a_1)$	E_6	$E_6\times A_2$	E_6	2	$2^5\cdot 5\cdot 7$	$2^7\cdot 3^2\cdot 5$	1	1	$2^{12}\cdot 3^2\cdot 5^2\cdot 7$
$E_6(a_2)$	E_6	$E_6\times A_2$	E_6	2	$2^5\cdot 5\cdot 7$	$2^4\cdot 3^2\cdot 5$	1	1	$2^9\cdot 3^2\cdot 5^2\cdot 7$

TABLE 11 (*continued*)

Γ	W_1	$W_1\times W_2$	$\bar{W}_1$	$\lvert\mathrm{Aut}_W(\Pi_1)\rvert$	$\lvert W:N_W(W_1)\rvert$	$\lvert\mathrm{ccl}_{W_1}(w)\rvert$	$\lvert N_W(\bar{W}_1):N_W(W_1)\rvert$	d	$\lvert\mathrm{ccl}_W(w)\rvert$
$A_1{}^7$	$A_1{}^7$	$A_1{}^8$	E_7	168	$2^3\cdot3^4\cdot5^2$	1	135	135	$2^3\cdot3\cdot5$
$A_2{}^3\times A_1$	$A_2{}^3\times A_1$	$A_2{}^3\times A_1$	$E_6\times A_1$	12	$2^8\cdot3\cdot5^2\cdot7$	2^3	40	4	$2^9\cdot3\cdot5^2\cdot7$
$A_3\times A_1{}^4$	$A_3\times A_1{}^4$	$A_3\times A_1{}^4$	D_7	16	$2^3\cdot3^4\cdot5^2\cdot7$	$2\cdot3$	105	15	$2^4\cdot3^4\cdot5\cdot7$
$A_3\times A_2\times A_1{}^2$	$A_3\times A_2\times A_1{}^2$	$A_3\times A_2\times A_1{}^2$	$D_5\times A_2$	4	$2^6\cdot3^3\cdot5^2\cdot7$	$2^2\cdot3$	10	3	$2^8\cdot3^3\cdot5^2\cdot7$
$A_3{}^2\times A_1$	$A_3{}^2\times A_1$	$A_3{}^2\times A_1{}^2$	E_7	4	$2^4\cdot3^3\cdot5^2\cdot7$	$2^2\cdot3^2$	630	6	$2^5\cdot3^4\cdot5^2\cdot7$
$A_4\times A_2\times A_1$	$A_4\times A_2\times A_1$	$A_4\times A_2\times A_1$	$A_4\times A_2\times A_1$	2	$2^8\cdot3^3\cdot5\cdot7$	$2^4\cdot3$	1	1	$2^{12}\cdot3^4\cdot5\cdot7$
$A_4\times A_3$	$A_4\times A_3$	$A_4\times A_3$	$A_4\times A_3$	2	$2^7\cdot3^3\cdot5\cdot7$	$2^4\cdot3^2$	1	1	$2^{11}\cdot3^5\cdot5\cdot7$
$A_5\times A_1{}^2$	$A_5\times A_1{}^2$	$A_5\times A_1{}^2$	$E_6\times A_1$	2	$2^7\cdot3^3\cdot5\cdot7$	$2^3\cdot3\cdot5$	36	3	$2^{10}\cdot3^3\cdot5^2\cdot7$
$A_5\times A_2$	$A_5\times A_2$	$A_5\times A_2\times A_1$	E_7	2	$2^7\cdot3^2\cdot5\cdot7$	$2^4\cdot3\cdot5$	336	4	$2^9\cdot3^3\cdot5^2\cdot7$
$A_6\times A_1$	$A_6\times A_1$	$A_6\times A_1$	$A_6\times A_1$	2	$2^8\cdot3^3\cdot5$	$2^4\cdot3^2\cdot5$	1	1	$2^{12}\cdot3^5\cdot5^2$
$(A_7)'$	A_7	$A_7\times A_1$	E_7	2	$2^5\cdot3^3\cdot5$	$2^4\cdot3^2\cdot5\cdot7$	36	2	$2^8\cdot3^5\cdot5^2\cdot7$
$(A_7)''$	A_7	A_7	A_7	2	$2^6\cdot3^3\cdot5$	$2^4\cdot3^2\cdot5\cdot7$	1	1	$2^{10}\cdot3^5\cdot5^2\cdot7$
$D_4\times A_1{}^3$	$D_4\times A_1{}^3$	$D_4\times A_1{}^4$	E_7	6	$2^3\cdot3^3\cdot5^2\cdot7$	2^5	315	15	$2^8\cdot3^2\cdot5\cdot7$
$D_4\times A_3$	$D_4\times A_3$	$D_4\times A_3$	D_7	4	$2^3\cdot3^3\cdot5^2\cdot7$	$2^6\cdot3$	35	2	$2^8\cdot3^4\cdot5^2\cdot7$
$D_4(a_1)\times A_3$	$D_4\times A_3$	$D_4\times A_3$	D_7	4	$2^3\cdot3^3\cdot5^2\cdot7$	$2^3\cdot3^2$	35	3	$2^6\cdot3^4\cdot5^2\cdot7$
$D_5\times A_1{}^2$	$D_5\times A_1{}^2$	$D_5\times A_1{}^2$	D_7	4	$2^3\cdot3^4\cdot5\cdot7$	$2^4\cdot3\cdot5$	21	3	$2^7\cdot3^4\cdot5^2\cdot7$
$D_5\times A_2$	$D_5\times A_2$	$D_5\times A_2$	$D_5\times A_2$	2	$2^5\cdot3^3\cdot5\cdot7$	$2^5\cdot3\cdot5$	1	1	$2^{10}\cdot3^4\cdot5^2\cdot7$
$D_5(a_1)\times A_2$	$D_5\times A_2$	$D_5\times A_2$	$D_5\times A_2$	2	$2^5\cdot3^3\cdot5\cdot7$	$2^6\cdot5$	1	1	$2^{11}\cdot3^3\cdot5^2\cdot7$
$D_6\times A_1$	$D_6\times A_1$	$D_6\times A_1{}^2$	E_7	2	$2^3\cdot3^3\cdot5\cdot7$	$2^8\cdot3^2$	63	3	$2^{11}\cdot3^4\cdot5\cdot7$
$D_6(a_2)\times A_1$	$D_6\times A_1$	$D_6\times A_1{}^2$	E_7	2	$2^3\cdot3^3\cdot5\cdot7$	$2^7\cdot5$	63	3	$2^{10}\cdot3^2\cdot5^2\cdot7$
$E_6\times A_1$	$E_6\times A_1$	$E_6\times A_1$	$E_6\times A_1$	2	$2^5\cdot3\cdot5\cdot7$	$2^5\cdot3^3\cdot5$	1	1	$2^{10}\cdot3^4\cdot5^2\cdot7$
$E_6(a_1)\times A_1$	$E_6\times A_1$	$E_6\times A_1$	$E_6\times A_1$	2	$2^5\cdot3\cdot5\cdot7$	$2^7\cdot3^2\cdot5$	1	1	$2^{12}\cdot3^3\cdot5^2\cdot7$
$E_6(a_2)\times A_1$	$E_6\times A_1$	$E_6\times A_1$	$E_6\times A_1$	2	$2^5\cdot3\cdot5\cdot7$	$2^4\cdot3^2\cdot5$	1	1	$2^9\cdot3^3\cdot5^2\cdot7$
D_7	D_7	D_7	D_7	2	$2^3\cdot3^3\cdot5$	$2^8\cdot3\cdot5\cdot7$	1	1	$2^{11}\cdot3^4\cdot5^2\cdot7$
$D_7(a_1)$	D_7	D_7	D_7	2	$2^3\cdot3^3\cdot5$	$2^8\cdot3^2\cdot7$	1	1	$2^{11}\cdot3^5\cdot5\cdot7$
$D_7(a_2)$	D_7	D_7	D_7	2	$2^3\cdot3^3\cdot5$	$2^7\cdot3\cdot5\cdot7$	1	1	$2^{10}\cdot3^4\cdot5^2\cdot7$

TABLE 11 (*continued*)

Γ	W_1	$W_1 \times W_2$	$\bar{W}_1$	$\lvert \mathrm{Aut}_W(\Pi_1) \rvert$	$\lvert W : N_W(W_1) \rvert$	$\lvert \mathrm{ccl}_{W_1}(w) \rvert$	$\lvert N_W(\bar{W}_1) : N_W(W_1) \rvert$	d	$\lvert \mathrm{ccl}_W(w) \rvert$
E_7	E_7	$E_7 \times A_1$	E_7	1	$2^3 \cdot 3 \cdot 5$	$2^9 \cdot 3^2 \cdot 5 \cdot 7$	1	1	$2^{12} \cdot 3^3 \cdot 5^2 \cdot 7$
$E_7(a_1)$	E_7	$E_7 \times A_1$	E_7	1	$2^3 \cdot 3 \cdot 5$	$2^9 \cdot 3^4 \cdot 5$	1	1	$2^{12} \cdot 3^5 \cdot 5^2$
$E_7(a_2)$	E_7	$E_7 \times A_1$	E_7	1	$2^3 \cdot 3 \cdot 5$	$2^7 \cdot 3^3 \cdot 5 \cdot 7$	1	1	$2^{10} \cdot 3^4 \cdot 5^2 \cdot 7$
$E_7(a_3)$	E_7	$E_7 \times A_1$	E_7	1	$2^3 \cdot 3 \cdot 5$	$2^9 \cdot 3^3 \cdot 7$	1	1	$2^{12} \cdot 3^4 \cdot 5 \cdot 7$
$E_7(a_4)$	E_7	$E_7 \times A_1$	E_7	1	$2^3 \cdot 3 \cdot 5$	$2^6 \cdot 5 \cdot 7$	1	1	$2^9 \cdot 3 \cdot 5^2 \cdot 7$
${A_1}^8$	${A_1}^8$	${A_1}^8$	E_8	1344	$3^4 \cdot 5^2$	1	$3^4 \cdot 5^2$	2025	1
${A_2}^4$	${A_2}^4$	${A_2}^4$	E_8	48	$2^6 \cdot 5^2 \cdot 7$	2^4	$2^6 \cdot 5^2 \cdot 7$	40	$2^7 \cdot 5 \cdot 7$
${A_3}^2 \times {A_1}^2$	${A_3}^2 \times {A_1}^2$	${A_3}^2 \times {A_1}^2$	E_8	4	$2^3 \cdot 3^3 \cdot 5^2 \cdot 7$	$2^2 \cdot 3^2$	$2^3 \cdot 3^3 \cdot 5^2 \cdot 7$	36	$2^3 \cdot 3^3 \cdot 5^2 \cdot 7$
${A_4}^2$	${A_4}^2$	${A_4}^2$	E_8	4	$2^6 \cdot 3^3 \cdot 7$	$2^6 \cdot 3^2$	$2^6 \cdot 3^3 \cdot 7$	6	$2^{11} \cdot 3^4 \cdot 7$
$A_5 \times A_2 \times A_1$	$A_5 \times A_2 \times A_1$	$A_5 \times A_2 \times A_1$	E_8	2	$2^7 \cdot 3^2 \cdot 5 \cdot 7$	$2^4 \cdot 3 \cdot 5$	$2^7 \cdot 3^2 \cdot 5 \cdot 7$	12	$2^9 \cdot 3^2 \cdot 5^2 \cdot 7$
$A_7 \times A_1$	$A_7 \times A_1$	$A_7 \times A_1$	E_8	2	$2^5 \cdot 3^3 \cdot 5$	$2^4 \cdot 3^2 \cdot 5 \cdot 7$	$2^5 \cdot 3^3 \cdot 5$	4	$2^7 \cdot 3^5 \cdot 5^2 \cdot 7$
A_8	A_8	A_8	E_8	2	$2^6 \cdot 3 \cdot 5$	$2^7 \cdot 3^2 \cdot 5 \cdot 7$	$2^6 \cdot 3 \cdot 5$	3	$2^{13} \cdot 3^2 \cdot 5^2 \cdot 7$
$D_4 \times {A_1}^4$	$D_4 \times {A_1}^4$	$D_4 \times {A_1}^4$	E_8	24	$2 \cdot 3^3 \cdot 5^2 \cdot 7$	2^5	$2 \cdot 3^3 \cdot 5^2 \cdot 7$	135	$2^6 \cdot 5 \cdot 7$
${D_4}^2$	${D_4}^2$	${D_4}^2$	E_8	12	$3^2 \cdot 5^2 \cdot 7$	2^{10}	$3^2 \cdot 5^2 \cdot 7$	6	$2^9 \cdot 3 \cdot 5^2 \cdot 7$
$D_4(a_1)^2$	${D_4}^2$	${D_4}^2$	E_8	12	$3^2 \cdot 5^2 \cdot 7$	$2^4 \cdot 3^2$	$3^2 \cdot 5^2 \cdot 7$	15	$2^4 \cdot 3^3 \cdot 5 \cdot 7$
$D_5(a_1) \times A_3$	$D_5 \times A_3$	$D_5 \times A_3$	E_8	2	$2^3 \cdot 3^3 \cdot 5 \cdot 7$	$2^6 \cdot 3 \cdot 5$	$2^3 \cdot 3^3 \cdot 5 \cdot 7$	6	$2^8 \cdot 3^3 \cdot 5^2 \cdot 7$
$D_6 \times {A_1}^2$	$D_6 \times {A_1}^2$	$D_6 \times {A_1}^2$	E_8	2	$2^2 \cdot 3^3 \cdot 5 \cdot 7$	$2^8 \cdot 3^2$	$2^2 \cdot 3^3 \cdot 5 \cdot 7$	15	$2^{10} \cdot 3^4 \cdot 7$
D_8	D_8	D_8	E_8	1	$3^3 \cdot 5$	$2^{13} \cdot 3^2 \cdot 5$	$3^3 \cdot 5$	2	$2^{12} \cdot 3^5 \cdot 5^2$
$D_8(a_1)$	D_8	D_8	E_8	1	$3^3 \cdot 5$	$2^{11} \cdot 3 \cdot 5 \cdot 7$	$3^3 \cdot 5$	3	$2^{11} \cdot 3^3 \cdot 5^2 \cdot 7$
$D_8(a_2)$	D_8	D_8	E_8	1	$3^3 \cdot 5$	$2^{13} \cdot 3 \cdot 7$	$3^3 \cdot 5$	2	$2^{12} \cdot 3^4 \cdot 5 \cdot 7$
$D_8(a_3)$	D_8	D_8	E_8	1	$3^3 \cdot 5$	$2^8 \cdot 3^2 \cdot 5 \cdot 7$	$3^3 \cdot 5$	3	$2^8 \cdot 3^4 \cdot 5^2 \cdot 7$
$E_6 \times A_2$	$E_6 \times A_2$	$E_6 \times A_2$	E_8	2	$2^5 \cdot 5 \cdot 7$	$2^6 \cdot 3^3 \cdot 5$	$2^5 \cdot 5 \cdot 7$	4	$2^9 \cdot 3^3 \cdot 5^2 \cdot 7$
$E_6(a_2) \times A_2$	$E_6 \times A_2$	$E_6 \times A_2$	E_8	2	$2^5 \cdot 5 \cdot 7$	$2^5 \cdot 3^2 \cdot 5$	$2^5 \cdot 5 \cdot 7$	4	$2^8 \cdot 3^2 \cdot 5^2 \cdot 7$
$E_7 \times A_1$	$E_7 \times A_1$	$E_7 \times A_1$	E_8	1	$2^3 \cdot 3 \cdot 5$	$2^9 \cdot 3^2 \cdot 5 \cdot 7$	$2^3 \cdot 3 \cdot 5$	3	$2^{12} \cdot 3^2 \cdot 5^2 \cdot 7$
$E_7(a_2) \times A_1$	$E_7 \times A_1$	$E_7 \times A_1$	E_8	1	$2^3 \cdot 3 \cdot 5$	$2^7 \cdot 3^3 \cdot 5 \cdot 7$	$2^3 \cdot 3 \cdot 5$	3	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7$
$E_7(a_4) \times A_1$	$E_7 \times A_1$	$E_7 \times A_1$	E_8	1	$2^3 \cdot 3 \cdot 5$	$2^6 \cdot 5 \cdot 7$	$2^3 \cdot 3 \cdot 5$	3	$2^9 \cdot 5^2 \cdot 7$

TABLE 11 (*continued*)

Γ	W_1	$W_1 \times W_2$	$\bar{W}_1$	$\lvert \mathrm{Aut}_W(\Pi_1) \rvert$	$\lvert W : N_W(W_1) \rvert$	$\lvert \mathrm{ccl}_{W_1}(w) \rvert$	$\lvert N_W(\bar{W}_1) : N_W(W_1) \rvert$	d	$\lvert \mathrm{ccl}_W(w) \rvert$
E_8	E_8	E_8	E_8	1	1	$2^{13} \cdot 3^4 \cdot 5 \cdot 7$	1	1	$2^{13} \cdot 3^4 \cdot 5 \cdot 7$
$E_8(a_1)$	E_8	E_8	E_8	1	1	$2^{11} \cdot 3^4 \cdot 5^2 \cdot 7$	1	1	$2^{11} \cdot 3^4 \cdot 5^2 \cdot 7$
$E_8(a_2)$	E_8	E_8	E_8	1	1	$2^{12} \cdot 3^5 \cdot 5 \cdot 7$	1	1	$2^{12} \cdot 3^5 \cdot 5 \cdot 7$
$E_8(a_3)$	E_8	E_8	E_8	1	1	$2^9 \cdot 3^3 \cdot 5^2 \cdot 7$	1	1	$2^9 \cdot 3^3 \cdot 5^2 \cdot 7$
$E_8(a_4)$	E_8	E_8	E_8	1	1	$2^{13} \cdot 3^2 \cdot 5^2 \cdot 7$	1	1	$2^{13} \cdot 3^2 \cdot 5^2 \cdot 7$
$E_8(a_5)$	E_8	E_8	E_8	1	1	$2^{13} \cdot 3^4 \cdot 5 \cdot 7$	1	1	$2^{13} \cdot 3^4 \cdot 5 \cdot 7$
$E_8(a_6)$	E_8	E_8	E_8	1	1	$2^{11} \cdot 3^4 \cdot 7$	1	1	$2^{11} \cdot 3^4 \cdot 7$
$E_8(a_7)$	E_8	E_8	E_8	1	1	$2^9 \cdot 3^3 \cdot 5^2 \cdot 7$	1	1	$2^9 \cdot 3^3 \cdot 5^2 \cdot 7$
$E_8(a_8)$	E_8	E_8	E_8	1	1	$2^7 \cdot 5 \cdot 7$	1	1	$2^7 \cdot 5 \cdot 7$

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