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CONJUGACY CLASSES OF HYPERBOLIC MATRICES IN Sl(n, $\mathbf{Z}$) AND IDEAL CLASSES IN AN ORDER

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ABSTRACT. A bijection is proved between Sl($n,\mathbf{Z}$)-conjugacy classes of hyperbolic matrices with eigenvalues $\{\lambda_1,\ldots,\lambda_n\}$ which are units in an n-degree number field, and narrow ideal classes of the ring $R_k = \mathbf{Z}[\lambda_i]$. A bijection between Gl($n,\mathbf{Z}$)-conjugacy classes and the wide ideal classes, which had been known, is repeated with a different proof.

In 1980, Peter Sarnak was able to obtain an estimate of the growth of the class number of real quadratic number fields using the Selberg Trace Formula and a bijection between hyperbolic elements of Sl(2, $\mathbf{Z}$) and quadratic forms. The "class number" counted was the number of congruence classes of quadratic forms as studied by Gauss [**1**]. In this paper we will translate this bijection into modern number-theoretic terms by counting ideal classes in a ring of integers associated to a given field. In this way a bijection is proved between conjugacy classes of hyperbolic matrices in Sl(2, $\mathbf{Z}$) with a given set of eigenvalues and ideal classes in a certain order (i.e. subring of dimension n over $\mathbf{Z}$) associated to the ring of integers O_K in a real nth degree number field K. This more direct method is necessary for generalizing the bijection to higher dimensional cases because Sarnak's result depends upon quadratic forms, Pell's equation and other things which are well understood only in the case of Sl(2, $\mathbf{Z}$).

We must mention the work of Latimer and MacDuffee [**3**] who first proved Theorem 2 in a slightly different fashion. Important also is the extensive work of Taussky [**7–9**], who simplified the results of Latimer and MacDuffee and extended them in certain directions, as well as doing much work on the Sl(2, $\mathbf{Z}$) case.

It follows from a brief examination of the characteristic polynomial for a matrix A in Sl($n,\mathbf{Z}$) that the eigenvalues of A are conjugate units in an extension of $\mathbf{Q}$. We shall insist in the remainder of this paper that A be "hyperbolic" with irreducible characteristic polynomial, that is, A will have distinct real eigenvalues $\lambda^{(i)}$, each of which is of degree n over $\mathbf{Q}$.

PROPOSITION 1. *If λ is an eigenvalue for a matrix $A \in \mathrm{SL}(n,\mathbf{Z})$, then for any field K containing λ there exists an eigenvector ω, ${}^T\omega = (\omega_1,\ldots,\omega_n)$, with $\omega_i \in O_K$.*

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PROOF. We can construct an eigenvector for A by solving $A \circ x = \lambda x$ for $x = (1, x_2, \ldots, x_n)^T$. This matrix identity yields n simultaneous equations with coefficients in $Z \oplus Z\lambda$. Therefore the solution constructed lies in $\mathbf{Q}(\lambda)$. If we multiply the solution vector by a constant it will still be an eigenvector, so this allows us to clear the denominators of $x_2, \ldots, x_n$. The resulting vector ω has entries which all lie in O_K.

PROPOSITION 2. *For a given ideal I in O_K and ordered set of elements ${}^T\omega = (\omega_1, \ldots, \omega_n)$ which are a $\mathbf{Z}$-basis of I, and for λ a unit with norm equal to* 1, *there exists a matrix A in* $\mathrm{Sl}(n, \mathbf{Z})$ *such that $A \circ \omega = \lambda\omega$.*

PROOF. Because λ is a unit, $\lambda^T\omega = (\lambda\omega_1, \ldots, \lambda\omega_n)$ is also a $\mathbf{Z}$-basis of I. Therefore, some matrix A in $\mathrm{Gl}(n, \mathbf{Z})$ takes ω to $\lambda\omega$. If we inspect the action of A on $K^{(j)}$, the jth conjugate field of K, we see that $I^{(j)}$ has a $\mathbf{Z}$-basis ${}^T\omega^{(j)} = (\omega_1^{(j)}, \ldots, \omega_n^{(j)})$. Furthermore, $A\omega^{(j)} = \lambda^{(j)}\omega^{(j)}$. Thus the eigenvalues of A are $\lambda^{(1)}, \ldots, \lambda^{(n)}$, conjugate units in a field. Therefore $\det A = \prod_j \lambda^{(j)} = 1$ and A is in $\mathrm{Sl}(n, \mathbf{Z})$.

PROPOSITION 3. *Let I, ω, A, λ be as in Proposition* 2. *Let ${}^T\alpha$ be another vector whose elements are a basis of I. Then there exists B in* $\mathrm{Sl}(n, \mathbf{Z})$ *with eigenvalues $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(n)}$ and eigenvectors $\alpha^{(1)}, \ldots, \alpha^{(n)}$, and B is conjugate to A by an element of* $\mathrm{Gl}(n, \mathbf{Z})$.

PROOF. Because $\{\omega_i\}$ and $\{\alpha_i\}$ are bases of I there exists C in $\mathrm{Gl}(n, \mathbf{Z})$ with $C \circ \alpha = \omega$. Let $B = C^{-1}AC$. Thus

$$B \circ \alpha = C^{-1}AC\alpha = C^{-1}A \circ \omega = C^{-1} \circ \lambda\omega = \lambda C^{-1} \circ \omega = \lambda\alpha.$$

COROLLARY TO PROPOSITION 3. *If n is odd then we can choose the C of Proposition* 3 *to be in* $\mathrm{Sl}(n, \mathbf{Z})$.

PROOF. If $\det C = -1$ then replace C by $-1 \cdot C$.

PROPOSITION 4. *An ideal class in $K = \mathbf{Q}(\lambda)$ uniquely determines a conjugacy class in* $\mathrm{Gl}(n, \mathbf{Z})$ *of matrices with eigenvalue λ (and determinant $+1$).*

PROOF. If ω is a vector whose entries are a basis for an ideal I and if J is another ideal in the same ideal class as I, then there is a constant k such that $k\omega$ is a basis of J. By Proposition 2 there is a matrix A with eigenvalue λ and eigenvector ω, thus $k\omega$ is also on eigenvector of A. We can therefore take ω to represent an entire ideal class, of which each ideal has a basis $k\omega$ which is an eigenvector of A. It suffices then to show the converse of Proposition 3; this is, if B is conjugate to A then it corresponds to the same ideal class. But the eigenvector of $B = C^{-1}AC$ with eigenvalue λ is exactly $C^{-1} \circ \omega$, and $C^{-1} \circ \omega$ has entries which are also a basis of I.

Gauss [**1**] showed several examples of symmetric 2×2-matrices for which the number of equivalence classes under $\mathrm{Sl}(2, \mathbf{Z})$ is twice the number of classes under $\mathrm{Gl}(2, \mathbf{Z})$. The import of these examples for our purposes lies in calculating conjugacy classes of a determinant 1 hyperbolic. Sarnak [**5**] shows that a correspondence exists between congruence classes of symmetric matrices of a given determinant and $\mathrm{Sl}(2, \mathbf{Z})$-conjugacy classes of associated hyperbolic matrices in $\mathrm{Sl}(2, \mathbf{Z})$ with certain eigenvalues. Consequently, we do not expect the number of $\mathrm{Gl}(n, \mathbf{Z})$-conjugacy classes of a hyperbolic to agree with the number of $\mathrm{Sl}(n, \mathbf{Z})$-conjugacy classes when n is even. However, we already have a weak result which follows directly from Proposition 4, which we can now state.

THEOREM 1. *Let K be an nth degree extension of $\mathbf{Q}$ and let $\lambda \in K^+$ be a unit. If $K = \mathbf{Q}(\lambda)$ then the number of $\mathrm{Gl}(n, \mathbf{Z})$-conjugacy classes of matrices in $\mathrm{Sl}(n, \mathbf{Z})$, which are hyperbolic with eigenvalues $\{\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(n)}\}$, is greater than the class number of K.*

PROOF. The proof follows immediately from Proposition 4.

The application Sarnak makes of his version of this information is in a computation involving the Selberg trace formula. One term in the trace formula contains a sum over the $\mathrm{Sl}(2, \mathbf{Z})$-conjugacy classes of a hyperbolic matrix with multiplier λ^2. It is clear that the number of $\mathrm{Sl}(2, \mathbf{Z})$-conjugacy classes of such matrices is greater than or equal to the number of $\mathrm{Gl}(2, \mathbf{Z})$-conjugacy classes. But it would be useful to have some bound on how many more of these classes there are under $\mathrm{Sl}(2, \mathbf{Z})$ than under $\mathrm{Gl}(2, \mathbf{Z})$. This estimate is quite easy to make. In fact, the bound is the same for any $\mathrm{Gl}(n, \mathbf{Z})$ when n is even.

PROPOSITION 5. *Let $\{\lambda^{(1)}, \ldots, \lambda^{(n)}\}$ be n distinct conjugate units. Then the number of conjugacy classes in $\mathrm{Sl}(n, \mathbf{Z})$ of matrices with eigenvalues $\{\lambda^{(1)}, \ldots, \lambda^{(n)}\}$ is less than or equal to twice the number of $\mathrm{Gl}(n, \mathbf{Z})$-conjugacy classes of the same matrices.*

PROOF. Let A and B be conjugates in $\mathrm{Gl}(n, \mathbf{Z})$. Then B is conjugate to either A or DAD in $\mathrm{Sl}(n, \mathbf{Z})$, where $D = D^{-1}$ and $\det D = -1$, e.g.

$$D = \begin{pmatrix} -1 & & 0 & & \\ & +1 & & & \\ & & +1 & & \\ & 0 & & \ddots & \\ & & & & +1 \end{pmatrix}.$$

Suppose $C^{-1}BC = A$ and $\det C = -1$. Then $DC^{-1}D \cdot DBD \cdot DCD = DAD$, so $DC^{-1} \cdot B \cdot CD = DAD$ because $D^2 = I$ and $(CD)^{-1}B(CD) = DAD$ because $D = D^{-1}$. But $\det CD = 1$. Therefore a conjugacy class in $\mathrm{Gl}(n, \mathbf{Z})$ divides into at most two classes in $\mathrm{Sl}(n, \mathbf{Z})$ by sending an element B to either the class whose representative is A or the class represented by DAD.

To fully generalize Sarnak's result to the $\mathrm{Sl}(n, \mathbf{Z})$ case, it is necessary to establish a bijection between conjugacy classes of a family of matrices and ideal classes in O_K, where K is some field. It is possible to make this bijection easily in a special case, namely, when O_K is generated as a $\mathbf{Z}$-module over $\{1, \lambda, \lambda^2, \ldots, \lambda^{n-1}\}$, where λ is the unit corresponding to the given family of matrices in $\mathrm{Sl}(n, \mathbf{Z})$. In other cases, the bijection will still be between classes of matrices and ideal classes in $\mathbf{Z}[\lambda]$, a subring of O_K.

DEFINITION 1. Let $R = \mathbf{Z}[\lambda]$. Two ideals I and J in R are said to be in the same *ideal class* if there is some constant r in the quotient field of R such that $rI = J$.

Proposition 6. *Let $\{I\}$ be the ideal class containing I. Then $\{I\}$ is well defined; that is, membership to an ideal class is an equivalence relation on ideals.*

Proof. Suppose $J \in \{I\}$. Then there is $c \in$ (quotient field of R) such that $cJ = I$. Therefore $c^{-1}I = J$ and $I \in \{J\}$. Also if $J \in \{I\}$ and $H \in \{J\}$, then there are c and d in the quotient field of R with $cJ = I$ and $dH = J$. Thus $cdH = I$ and $H \in \{I\}$. Obviously $J \in \{J\}$. Therefore membership to an ideal class in R is an equivalence relation on ideals. Thus $\{I\}$ is well defined.

It is worthwhile to note here the need for Proposition 6. $R = \mathbf{Z}[\lambda]$ is a ring consisting of integers but is not in general the complete ring of integers for a field K, although it is contained in O_K; $K = \mathbf{Q}(\lambda)$. R is not necessarily Dedekind, although principal ideals in R are invertible. Thus, if we wished to make ideals in the same class differ by a principal ideal or principal inverse ideal, we could do so without changing or invalidating Definition 1. However, because R is not necessarily Dedekind we do not have an ideal class *group*, as in the case of O_K.

Proposition 7. *Let A be in $\mathrm{Sl}(n,\mathbf{Z})$ with eigenvalues $\{\lambda^{(1)},\dots,\lambda^{(n)}\}$ distinct conjugate units. Let $R = \mathbf{Z}[\lambda]$. Suppose $A \cdot \omega = \lambda\omega$ for some vector ${}^T\omega = (\omega_1,\dots,\omega_n)$. Then $\{\omega_1,\dots,\omega_n\}$ are a $\mathbf{Z}$-basis for an ideal in R.*

Proof. It suffices to show that for $r \in R$, $r\omega_i \in \oplus_i \mathbf{Z}\omega_i$. But $r = \sum_{j=0}^{n-1}\alpha_j\lambda^j$, $\alpha_j \in \mathbf{Z}$. Thus it suffices to show that $\lambda^j \cdot \omega_i$ is in $\oplus_i \mathbf{Z}\omega_i$. But ${}^T\omega = (\omega_1,\dots,\omega_n)$ is an eigenvector of A^j with eigenvalue λ^j. Thus $\lambda^j\omega_i$ is a linear combination of the ω_i. Therefore $\oplus_i \mathbf{Z}\omega_i$ is an ideal in R.

Theorem 2. *Let U_λ be the set of $\mathrm{Gl}(n,\mathbf{Z})$-conjugacy classes of matrices in $\mathrm{Sl}(n,\mathbf{Z})$ whose elements have eigenvalues $\{\lambda^{(1)},\lambda^{(2)},\dots,\lambda^{(n)}\}$. Let $R = \mathbf{Z}[\lambda]$. Let $\{I\}$ be the set of ideal classes of R. Then there is a well-defined map $\Phi_\lambda\colon U_\lambda \to \{I_i\}$ which is one-to-one and onto.*

Proof. We choose $\Phi_\lambda(\{A\})$ to be the ideal generated by the entries of the λ-eigenvector of any representative of $\{A\}$. $\Phi_\lambda(\{A\})$ exists and is an ideal generated as a $\mathbf{Z}$-module by $\{\omega_i\}$, (the entries of the eigenvector) by Proposition 7. Φ_λ is thus well defined by Proposition 3 and onto by Proposition 2. By Proposition 4, Φ_λ^{-1} is also well defined and onto. Therefore Φ_λ is one-to-one and onto.

For convenience, let $h(K)$ denote the class number of K and let $H(\lambda)$ denote the number of $\mathrm{Gl}(n,\mathbf{Z})$-conjugacy classes of matrices with eigenvalues $\{\lambda^{(1)},\lambda^{(2)},\dots,\lambda^{(n)}\}$. Then from Theorem 2 we have

Corollary to Theorem 2. *If $R = O_K$ then $h(K) = H(\lambda)$.*

Proof. The proof follows immediately from Theorem 2.

The following proposition and theorem give yet another criterion for the equality of $h(K)$ and $H(\lambda)$. They will yield a large class of discriminants for which the number field has the desired property.

Proposition 8. *Let λ be a unit with minimal polynomial of degree n and let $A \circ \omega = \lambda\omega$ for $A \in \mathrm{Sl}(n,\mathbf{Z})$ and $\omega \in \mathbf{R}^n$. Let ${}^T\omega = (\omega_1,\omega_2,\dots,\omega_n)$. Then $\{\omega_i\}$ is a linearly independent set over $\mathbf{Q}$.*

PROOF. Suppose not. Then $\lambda\omega_i = \sum_{i=1}^{n-1} b_j \omega_s$ for all i. Thus $B \circ \omega = \lambda\omega$, where B has rational entries and the nth column is zero. Therefore λ is an eigenvector of the $((n-1) \times (n-1))$-matrix in the upper left-hand corner of B. Therefore the minimal polynomial of λ has degree less than n.

THEOREM 3. *Suppose $K = \mathbf{Q}(\lambda)$ is a number field of degree n and λ is a unit. If the discriminant of Λ is square-free then $h(K) = H(\lambda)$, where $\Lambda = (1, \lambda, \ldots, \lambda^{n-1})$.*

PROOF. $\Lambda = (1, \lambda, \lambda^2, \ldots, \lambda^{n-1})$ is a $\mathbf{Z}$-basis for R. It is also a $\mathbf{Q}$-basis for K. Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ be a $\mathbf{Z}$-basis of O_K. By a theorem in Samuel [**4**], if $D_K(1, \lambda, \ldots, \lambda^{n-1})$ is square-free then $(1, \lambda, \ldots)$ is a $\mathbf{Z}$-basis of O_K. But $D_K(1, \lambda, \ldots, \lambda^{n-1}) = D(K)$, which is assumed to be square-free. Therefore $R = O_K$ and by Corollary to Theorem 2, $h(K) = H(\lambda)$.

We should note that in the Selberg trace formula the conjugacy classes of hyperbolics are always taken to be conjugates by elements of $\mathrm{Sl}(2, \mathbf{Z})$. The bijection obtained in Theorem 2 of this paper, however, is between (wide) ideal classes in a ring R_K associated to a field K and Gl(n, $\mathbf{Z}$)-conjugacy classes of hyperbolic matrices. We can convert the problem of study of $\mathrm{Sl}(n, \mathbf{Z})$-conjugacy classes if we count the narrow ideal classes of R_K instead of the usual (wide) ideal classes.

DEFINITION 2. Two ideals I and J of a ring O_K of integers of a number field K are said to be *narrowly equivalent* if $I = rJ$, $r \in O_K$, or $J = rI$, $r \in K$, such that r is a number of positive norm.

It is well known that the above equivalence relation divides the set of ideals of O_K into classes which are called the *narrow ideal classes*. The number of these is the *narrow class number*. It is easy to see that if R_K is the order described previously which is associated to the number field K, the ideals of R_K can be grouped into narrow ideal classes by the same definition. For the remainder of this paper we will be referring (unless otherwise stated) to the narrow ideal classes and narrow class number of R_K rather than O_K.

Let λ be a unit of degree n and let $\mathbf{Q}(\lambda) = K$. Let R_K and O_K be the rings associated with K as before, namely, O_K being the ring of integers and $R_K = \mathbf{Z}[\lambda]$. Let $R = R_K$ or O_K in general. We now begin a process of associating to every narrow ideal class a basis vector and two sets of basis vectors on which we will study the action of $\mathrm{Sl}(n, \mathbf{Z})$.

Let $\{I_1\}, \ldots, \{I_k\}$ be the wide ideal classes of ring R. Let $I_1, \ldots, I_k$ be arbitrary choices of representatives of each class. Let $\omega^1, \ldots, \omega^k$ be arbitrary choices of vectors consisting of ordered bases for I_j, respectively. To each ideal J_j in the same wide ideal class as I_j associate the basis vector $r \cdot \omega^j$, $r \in K$, choosing r such that $n(r) > 0$ if I_j and J_j are in the same narrow ideal class. Thus to every ideal J_j in R we have associated a basis $r\omega^j$ which, although ω^j is chosen arbitrarily, we shall consider fixed.

We know that all bases of I_j can be obtained by the action of $\mathrm{Gl}(n, \mathbf{Z})$ on ω^j. We can then split the set of bases elements of an ideal I_j into two sets, namely those which arise from the action of elements of $\mathrm{Sl}(n, \mathbf{Z})$ on ω^j and the rest. Define $\Omega^+_{I_j}$ to be the set of all $\nu = A \cdot \omega^j$, $A \in \mathrm{Sl}(n, \mathbf{Z})$, and define $\Omega^-_{I_j}$ to be all the other basis vectors. Note that if $\nu \in \Omega_{I_j}^-$ then $B \cdot \nu = \omega^j$, det $B = -1$, $B \in \mathrm{Gl}(n, \mathbf{Z})$, although the converse is not necessarily true.

It is easy to see that the following holds:

1. The elements $\Omega_{I_j}^-$ are equivalent under the action of $\mathrm{Sl}(n, \mathbf{Z})$.

2. We can define Ω_J^+ and Ω_J^- analogously for all J widely equivalent to I_j.

3. Any $\nu \in \Omega_j^-$ is related to any $\nu' \in \Omega_J^+$ *only* by a matrix B of negative determinant.

PROPOSITION 9. *To each element $\nu \in \Omega_{I_j}^+$ we can associate a unique matrix $A_\nu \in$ $\mathrm{Sl}(n, \mathbf{Z})$ such that $A_\nu \cdot \nu^j = \lambda \nu$.*

PROPOSITION 10. *To $\Omega_{I_j}^+$ we can associate a conjugacy class under the action of $\mathrm{Sl}(n, \mathbf{Z})$ of matrices equivalent to A_ν.*

The proofs of these propositions are exactly the same as those given for Propositions 2 and 3, hence will not be repeated.

Needless to say, we are attempting to construct a bijection between ($\mathrm{Sl}(n, \mathbf{Z})$ -conjugacy classes of hyperbolic matrices and narrow ideal classes of the ring R_K, $K = \mathbf{Q}(\lambda)$. This bijection has been partially supplied by Theorem 2, which reduces the problem to the case of two ideals in the same wide ideal class but different narrow ideal classes. We can prove the desired result after the next lemma.

LEMMA 1. *If $J = rI_j$, $r \in K$, and ω^j is the basis associated to I_j, then there exists a matrix C of positive determinant and having rational entries, and a basis ν^j of I_j such that $C\nu^j = r\omega^j$, the canonical basis of J. Furthermore, $\nu^j \in \Omega_{I_j}^+$ if and only if $n(r) > 0$.*

PROOF. Because J and I_j are in the same wide ideal class, there is a matrix B with rational entries such that $B \cdot \omega^j = r\omega^j$. By inspecting the action of B on the conjugate fields of K we see that

$$B \cdot \left(\omega_{(i)}^j\right) = r_{(i)} \cdot \omega_{(i)}^j,$$

hence det $B = n(r)$. If $n(r) > 0$, we are done. Otherwise let $D = D^{-1}$ and det $D = -1$, e.g.

$$D = \begin{pmatrix} -1 & & 0 & & \\ & +1 & & & \\ & & +1 & & \\ & 0 & & \ddots & \\ & & & & +1 \end{pmatrix}.$$

Then $D\omega^j = \nu^j$ and det $BD = -n(r) > 0$. If $C = BD$ then $C \cdot \nu^j = r\omega^j$ and det $C > 0$ and $\nu^j \in \Omega_{I_j}^-$.

THEOREM 4. *Let U_λ be the set of $\mathrm{Sl}(n, \mathbf{Z})$-conjugacy classes of hyperbolic matrices with eigenvalues $\{\lambda^{(1)}, \ldots, \lambda^{(n)}\}$. Let $R_K = \mathbf{Z}[\lambda]$ and let $\{I_j\}$ be the set of narrow ideal classes of R. Then there is a well-defined map $\Phi_\lambda \colon U_\lambda \to \{I_j\}$ which is one-to-one and onto.*

PROOF. By Theorem 2 it suffices to show that if I_j and I_K are two widely equivalent ideal classes ($rI_j = I_K$) with associated bases ω^j and ω^k such that $r\omega^j = \omega^k$, and if $A\omega^j = \lambda\omega^j$ and $B(\omega^k) = \lambda\omega^k$, we have the following:

(1) If $CAC^{-1} = B$ and if $\det C = +1$ then I_j and I_k are narrowly equivalent.

(2) If I_j and I_K are narrowly equivalent then there exists C with $\det C = +1$ such that $CAC^{-1} = B$.

PROOF OF (1). Suppose $CAC^{-1} = B$ and $\det C = +1$. Note that $\omega^j \in \Omega^+_{I_j}$ and $r\omega^j = \omega^k \in \Omega^+_{I_k}$. Let $\hat{\omega}^k = C^{-1}\omega^k$. Then $\hat{\omega}^k \in \Omega^+_{I_k}$ by the definition of $\Omega^+_{I_k}$. Also we have that $A\omega^j = \lambda\omega^j$ and

$$A\hat{\omega}^k = C^{-1}BC \cdot C^{-1}\omega^k = C^{-1}B \cdot \omega^k = C^{-1} \cdot \lambda\omega^k = \lambda \cdot \hat{\omega}^k.$$

Therefore $\hat{\omega}^k = \hat{r} \cdot \omega^j$ and by Lemma 1, $n(\hat{r}) > 0$. Therefore I_j and I_k are narrowly equivalent.

PROOF OF (2). Suppose I_j and I_k are narrowly equivalent and $A\omega^j = \lambda\omega^j$, $B\omega^k = \lambda\omega^k$. Then there exists r ($n(r) > 0$) such that $\hat{\omega}^k = r\omega^j$ is a basis of I_K. By Lemma 1, $\hat{\omega}^k \in \Omega^+_{I_K}$. By the definition of $\Omega^+_{I_K}$ there exists a matrix $C \in \mathrm{Sl}(n, \mathbf{Z})$ such that $C\hat{\omega}^k = \omega^k$. We then have $A\hat{\omega}^k = \lambda\hat{\omega}^k$ because $\hat{\omega}^k$ is a multiple of ω^j and

$$CAC^{-1}\omega^k = CA\hat{\omega}^k = C \cdot \lambda\hat{\omega}^k = \lambda \cdot \omega^k.$$

By noting the action of CAC^{-1} on conjugate fields of K, we have the CAC^{-1} has the same eigenvalues and eigenvectors as B and hence $CAC^{-1} = B$.

By comparing the result of this theorem with the Corollary to Proposition 3, Proposition 5, and Theorem 1, we can conclude the following corollaries. Let K be a field with irreducible minimal polynomial $P(\lambda)$ for λ a unit.

COROLLARY 1. *If the degree n of K is odd, then the wide class number of R_K equals the narrow class number.*

COROLLARY 2. *The narrow class number of R_K is less than or equal to twice the wide class number of R_K.*

COROLLARY 3. *The narrow class number of O_K is less than or equal to the narrow class number of R_K.*

Thus we see that for a hyperbolic element of $\mathrm{Sl}(n, \mathbf{Z})$ with nth degree irreducible characteristic polynomial $P(\lambda)$, the coefficient associated to that term in the n-dimensional version of the trace formula will, in part, be the narrow class number of the ring $R_K = \mathbf{Z}[\lambda]$. When n is odd it will also be the wide class number of R_K. In either case there will be number-theoretic information associated to that particular term which may possibly be retrieved by estimates involving the trace formula. For further information on the Selberg trace formula and number-theoretic applications thereof, please see [**2**, **6**, **10**, **12**].

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