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CONJUGACY CLASSES OF n-TUPLES IN LIE ALGEBRAS AND ALGEBRAIC GROUPS

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Let G be a reductive algebraic group over an algebraically closed field F of characteristic zero and let $L(G)$ be the Lie algebra of G. Then G acts on G by inner automorphisms and it acts on $L(G)$ by the adjoint representation. By taking the diagonal actions, we get actions of G on the spaces of n-tuples G^n and $L(G)^n$. In this paper we will prove a number of geometric properties of the orbits of G on these spaces of n-tuples.

For $n = 1$ the situation has been studied in great detail (see [12, 30]). For example, if $x \in G$, then the orbit $G \cdot x$ is closed (resp. unstable) if and only if x is semisimple (resp. unipotent). Let x have Jordan decomposition $x = su$, with s semisimple and u unipotent. Then the stabilizer G_x is the intersection of G_s and G_u and $G \cdot s$ is the unique closed orbit in the closure of $G \cdot x$. Let V be the affine variety corresponding to the algebra $F[G]^G$ of regular class functions on G and let $\pi\colon G \to V$ be the morphism of affine varieties corresponding to the inclusion homomorphism $F[G]^G \to F[G]$. Then each fibre $\pi^{-1}(v)$, $v \in V$, has codimension equal to the rank of G.

All of these results, plus a number of others, can be generalized to the action of G on n-tuples. For now, we will restrict our discussion to the action of G on G^n. Let $\mathbf{x} = (x_1, \ldots, x_n) \in G^n$ and let $A(\mathbf{x})$ be the algebraic subgroup of G generated by $\{x_1, \ldots, x_n\}$. We say that $\mathbf{x}$ is a *semisimple n-tuple* (resp. *unipotent n-tuple*) if $A(\mathbf{x})$ is a linearly reductive (resp. unipotent) algebraic group. We show that the orbit $G \cdot \mathbf{x}$ is closed (resp. unstable) if and only if $\mathbf{x}$ is a semisimple (resp. unipotent) n-tuple. Let L be a Levi subroup of $A(\mathbf{x})$. Then each x_i can be written uniquely in the form $x_i = y_i z_i$, with $y_i \in L$ and $z_i \in R_u(A(\mathbf{x}))$. Let $\mathbf{y} = (y_1, \ldots, y_n)$ and let $\mathbf{z} = (z_1, \ldots, z_n)$. The decomposition $\mathbf{x} = \mathbf{y} \cdot \mathbf{z}$ is called a Levi decomposition of $\mathbf{x}$. (Such a decomposition of n-tuples is not unique.) We show that the stabilizer $G_\mathbf{x}$ is the intersection of $G_\mathbf{y}$ and $G_\mathbf{z}$ and that $G \cdot \mathbf{y}$ is the unique closed orbit in the closure of $G \cdot \mathbf{x}$. Let G^n/G be the affine variety corresponding to the algebra $F[G^n]^G$ of invariants and let $\pi\colon G^n \to G^n/G$ be the morphism corresponding to the inclusion homomorphism $F[G^n]^G \to F[G^n]$. Let $\mathbf{x} \in G^n$ be a semisimple n-tuple and let A be a maximal torus of the stabilizer $G_\mathbf{x}$. Then the dimension of the fibre $\pi^{-1}(\pi(\mathbf{x}))$ depends only on the G-conjugacy class of A and we give a precise formula for this dimension. (For $n = 1$, A is a maximal torus of G.) We also characterize the stable points of G^n and the smooth points of the quotient variety G^n/G.

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Most of our results can be generalized to the following situation: Let S be a linearly reductive group which acts on G by automorphisms and let $K = G^S$ be the fixed point subgroup. Then we consider the actions of K on G^n and on $L(G)^n$. In this case our results seem to be new even for the case of $n = 1$. As an example, we prove the following: Let $\mathbf{x} = (x_1, \ldots, x_n) \in G^n$ and let $A_S(\mathbf{x})$ be the smallest S-stable algebraic subgroup of G containing $\{x_1, \ldots, x_n\}$. Then the orbit $K \cdot \mathbf{x}$ is closed if and only if $A_S(\mathbf{x})$ is a linearly reductive algebraic group.

A number of our arguments carry over to reductive real algebraic groups or, more generally, to real reductive Lie groups. The results on real groups are given in §11 and §15.

In our proofs, a crucial role is played by the Hilbert–Mumford theorem in the strengthened form due to Kempf [11]. In fact, the whole paper is, to some extent, an exercise in the Hilbert–Mumford theorem. In most cases, the proofs for n-tuples in the Lie algebra $L(G)$ are exact parallels of the proofs for n-tuples in the group G. In order to avoid constant repetition of similar proofs, we have usually given the proofs for G^n (which tend to be slightly more complicated) and omitted the proofs for $L(G)^n$.

Let $G = GL(V)$ and let K be either the orthogonal group $O(V)$ or the symplectic group $\mathrm{Sp}(V)$. For the actions of G and K on $L(G)^n$, many of our results were obtained by Procesi in [22]. In these cases, one can use more elementary methods and the Hilbert–Mumford theorem is not needed.

The hypothesis of characteristic zero is necessary for most of our theorems. In §16 we discuss a few results along the same line which hold in characteristic p.

An announcement and brief discussion of some of these results (without proofs) was given in [27].

§1. Preliminaries. We let $\mathbf{Z}$ denote the ring of rational integers, $\mathbf{R}$ the field of real numbers, and $\mathbf{C}$ the field of complex numbers.

1.1. Our basic reference for algebraic groups and algebraic geometry is the book of Borel [3] and in general we follow the conventions and notation therein. All algebraic groups and algebraic varieties are taken over an algebraically closed field F. Except where explicitly indicated otherwise, we assume that F is of characteristic zero. We always let k denote a subfield of F. By an algebraic group, we always mean an affine algebraic group. If X is a k-variety (a variety defined over k), then $X(k)$ denotes the set of k-rational points of X. If the algebraic group G acts morphically on the algebraic variety X, then we say that X is a G-variety; we say that X is a G-variety defined over k if X, G, and the action of G on X are all defined over k.

Let G be a grop and let X be a G-set. If $g \in G$ and $x \in X$, then $g \cdot x$ denotes the action of g on x, G_x is the stabilizer, or isotropy subgroup, of G at x and $G \cdot x$ is the G-orbit of x. We let X^G denote the set of fixed points of G on X.

If G is an algebraic group, we consider (the algebraic variety) G as an affine G-variety, with G acting by inner automorphisms. Thus, if S is a closed subgroup of G, then G^S is equal to $Z_G(S)$, the centralizer of S in G.

Since we are in characteristic zero, every unipotent algebraic group is connected. If G is a (not necessarily connected) algebraic group, then $R_u(G)$ is the unipotent radical of G (the maximal closed, connected, normal unipotent subgroup of G, or of G^0).

We let $\mathbf{X}(G)$ (resp. $\mathbf{Y}(G)$) denote the set of characters (resp. one-parameter multiplicative subgroups) of the algebraic group G. If G is a k-group, then $\mathbf{X}(G)_k$ (resp. $\mathbf{Y}(G)_k$) is the set of characters (resp. one-parameter subgroups) defined over k. If $\lambda \in \mathbf{Y}(G)$ and $\mu \in \mathbf{X}(G)$, then $\langle \lambda, \mu \rangle \in \mathbf{Z}$ is defined by $\mu(\lambda(t)) = t^{\langle \lambda, \mu \rangle}$ ($t \in F^*$). If X is an affine G-variety and if $\lambda \in \mathbf{Y}(G)$, then we frequently write X^λ for $X^{\text{image}(\lambda)}$.

1.2. *Linearly reductive groups and Levi subgroups.* (See [10], chap. 8.) An algebraic group G is *reductive* if G is connected and if $R_u(G) = \{1\}$. The algebraic group G is *linearly reductive* if every rational representation of G is semisimple. Since we are in characteristic zero, this is equivalent to the condition that the identity component G^0 is reductive. If G is a linearly reductive group, then $Z(G)$, the center of G, is a diagonalizable group.

The following two results are well known:

1.2.1. *Let G be a closed subgroup of $GL(V)$. Then G is linearly reductive if and only if V is a semisimple G-module.*

1.2.2. *Let η: $G \to H$ be a surjective homomorphism of algebraic groups and let K be the kernel of η. Then G is linearly reductive if and only if both K and H are linearly reductive.*

As an easy consequence of 1.2.1 and 1.2.2 we have

1.2.3. *Let G be linearly reductive and let H be a closed subgroup of G. Then H is linearly reductive if and only if the adjoint representation of H on the Lie algebra $L(G)$ is semisimple.*

We omit the proof.

Let G be a k-group. A k-subgroup L of G is a *Levi k-subgroup* of G if G is the semidirect product (in the sense of algebraic groups) of L and $R_u(G)$. It follows easily that L is linearly reductive. The following result is due to Mostow (see [10], chap. 8):

1.2.4. *Let G be a k-group.* (a) *There exist Levi k-subgroups of G.* (b) *If L_1 and L_2 are Levi k-subgroups of G, then there exists $u \in R_u(G)(k)$ such that $uL_1u^{-1} = L_2$.* (c) *Let M be a linearly reductive k-subgroup of G. Then M is contained in a Levi k-subgroup of G.*

1.3. *The quotient of X by G.* (See [14, 15].) Let G be a linearly reductive group and let X be an affine G-variety. Then the algebra $F(X)^G$ of G-invariant regular functions on X is a finitely generated F-algebra. Let X/G be the affine variety corresponding to $F[X]^G$ and let π_X: $X \to X/G$ be the morphism of affine varieties corresponding to the inclusion homomorphism $F[X]^G \to F[X]$. We say that X/G is the *quotient of X by G* and that π_X is the *quotient morphism.*

Remark 1.3.1. Our terminology differs from that of Borel [3], pages 172–174. In general the variety X/G is not a "quotient of X by G" and the morphism π_X is not a "quotient morphism" in the sense of [3].

The following properties of the quotient morphism π_X are known:

1.3.2. (a) *π_X is a surjective map.* (b) *For each $\xi \in X/G$, the fibre $\pi_X^{-1}(\xi)$ contains a unique closed orbit, which we denote by $T(\xi)$. For $x \in X$, we have $\pi_X(x) = \xi$ if and only if the closure of the orbit $G \cdot x$ meets $T(\xi)$.*

We shall frequently need the following property of closed orbits [25]:

1.3.3. *Let $x \in X$ be such that the orbit $G \cdot x$ is closed. Then the stabilizer G_x is linearly reductive.*

1.4. *Stable and unstable points.* (See [18, 20].) Let G be a linearly reductive group and let X be an affine G-variety. Let $Z = \bigcap_{x \in X} G_x$ denote the kernel of the action of G on X. A point x of X is a *stable point* (or a G-stable point, when reference to G is necessary) if the orbit $G \cdot x$ is closed and if G_x/Z is finite. Let $X^{(s)}$ be the set of stable points of X. Then $X^{(s)}$ is an open (possibly empty) G-stable subset of X, the image $\pi_X(X^{(s)})$ is an open subset of X/G, and $X^{(s)} = \pi_X^{-1}(\pi_X(X^{(s)}))$. For each stable point x of X the fibre $\pi_X^{-1}(\pi_X(x))$ is equal to the orbit $G \cdot x$. The image $\pi_X(X^{(s)})$ is an open subset of X/G and is a "geometric quotient of X by G" in the sense of [18].

Remark 1.4.1. Our definition of stable points is a slight generalization of the definitions in [18, 20]. A point x of X is stable in our sense if it is stable in the sense of [18] (see p. 147) or [20] for the action of G/Z on X.

Let G be linearly reductive and let (X, x_0) be a pointed affine G-variety with G-invariant base point x_0. Then a point x of X is *unstable* if x_0 is in the closure of the orbit $G \cdot x$. If x is unstable, we say that the orbit $G \cdot x$ is unstable. If E is a rational G-module, then we consider E as a pointed G-variety with base point 0. We consider G^n as a pointed G-variety with base point $\mathbf{1} = (1, \ldots, 1)$.

1.5. *Algebraic Lie subalgebras.* (See [10].) Let G be an algebraic group. Then a subalgebra $\mathfrak{a}$ of the Lie algebra $L(G)$ is an *algebraic Lie subalgebra* of $L(G)$ if there exists a connected closed subgroup A of G such that $L(A) = \mathfrak{a}$. In this case the subgroup A is uniquely determined by $\mathfrak{a}$. We say that $\mathfrak{a}$ is a *reductive algebraic Lie subalgebra* if the corresponding subgroup A is a reductive algebraic group. Let G be connected. Then the algebraic Lie subalgebra $\mathfrak{a}$ of $L(G)$ is a *Levi subalgebra* of $L(G)$ if the corresponding algebraic group A is a Levi subgroup of G.

§2. Parabolic subgroups and one-parameter subgroups. See [14, 32]. Let ϕ: $F^* \to X$ be a morphism of algebraic varieties. We say that $\lim_{t \to 0} \phi(t)$ exists if there exists a morphism ψ: $F \to X$ whose restriction to F^* is ϕ; in this case we write $\lim_{t \to 0} \phi(t)$ for $\psi(0)$. In particular, let G be an algebraic group, let X be a

G-variety, and let $\lambda \in \mathbf{Y}(G)$. If $x \in X$ and if $\lim_{t \to 0} \lambda(t) \cdot x$ exists, we often denote $\lim_{t \to 0} \lambda(t) \cdot x$ by $\lambda(0) \cdot x$. Clearly, $\lambda(0) \cdot x \in X^{\lambda}$.

We shall frequently use the following strengthened form of the Hilbert–Mumford theorem (henceforth denoted by HMT), due to Kempf [11]:

2.1 (HMT). *Let G be a linearly reductive k-group and let X be an affine G-variety defined over k. Let $x \in X(k)$ and let $\mathcal{O}$ be the unique closed orbit in the closure of $G \cdot x$. Then $\mathcal{O}$ is a k-variety and there exists $\lambda \in \mathbf{Y}(G)_k$ and $y \in \mathcal{O}(k)$ such that* $\lim_{t \to 0} \lambda(t) \cdot x = y$.

The proof of 2.1 is difficult. For the case when $F = \mathbf{C}$ and $k = \mathbf{R}$, an elementary proof is given in [2].

Let G be a linearly reductive group and let $\lambda \in \mathbf{Y}(G)$. We define subsets $P(\lambda)$ and $U(\lambda)$ of G and subsets $\mathfrak{p}(\lambda)$ and $\mathfrak{u}(\lambda)$ of $L(G)$ as follows:

(a) $P(\lambda) = \{g \in G \mid \lim_{t \to 0} \lambda(t) \cdot g \text{ exists}\}$.
(b) $U(\lambda) = \{g \in G \mid \lim_{t \to 0} \lambda(t) \cdot g = 1\}$.
(c) $p(\lambda) = \{x \in L(G) \mid \lim_{t \to 0} \lambda(t) \cdot x \text{ exists}\}$.
(d) $u(\lambda) = \{x \in L(G) \mid \lim_{t \to 0} \lambda(t) \cdot x = 0\}$.

When reference to G is necessary, we write $P_G(\lambda)$, $U_G(\lambda)$, $\mathfrak{p}_{L(G)}(\lambda)$, and $\mathfrak{u}_{L(G)}(\lambda)$ instead of $P(\lambda)$, $U(\lambda)$, $\mathfrak{p}(\lambda)$, and $\mathfrak{u}(\lambda)$.

2.2. *Let G be a linearly reductive k-group and let $\lambda \in \mathbf{Y}(G)_k$.* (a) *$P(\lambda)$ is a closed k-subgroup of G and $U(\lambda)$ is the unipotent radical of $P(\lambda)$.* (b) *If G is connected, then $P(\lambda)$ is a parabolic subgroup of G.* (c) $\mathfrak{p}(\lambda) = L(P(\lambda))$ *and* $\mathfrak{u}(\lambda) = L(U(\lambda))$. (d) *Define a morphism h_λ: $P(\lambda) \to G^{\lambda}$ by $h_\lambda(g) = \lambda(0) \cdot g$. Then h_λ is a surjective k-homomorphism of algebraic groups and $h_\lambda(g) = g$ for $g \in G^{\lambda}$.*

Proof. Parts (a) and (b) are proved in [14, 18] and the proofs of (c) and (d) follow easily from the proofs there.

2.3. *Let G be a reductive k-group, let P be a parabolic k-subgroup of G, and let L be a Levi k-subgroup of P. Let A be the unique maximal k-split torus of $Z(L)$. Then there exists $\lambda \in \mathbf{Y}(A) = \mathbf{Y}(A)_k$ such that $P(\lambda) = P$, $L = G^{\lambda} = G^{A}$, and $U(\lambda) = R_u(P)$.*

Proof. This follows easily from [4], Theorem 4.15.

We need to extend 2.3 to the case of linearly reductive groups.

PROPOSITION 2.4. *Let G be a linearly reductive k-group, let P be a parabolic k-subgroup of G^0, and let L be a Levi k-subgroup of $N_G(P)$. Then there exists $\lambda \in \mathbf{Y}(G)_k$ such that $N_G(P) = P(\lambda)$, $L = G^{\lambda}$, and $R_u(P) = U(\lambda)$.*

Proof. Let $Z = Z(L^0)^0$ Then Z is a k-torus and $L^0 = (G^0)^Z$. The finite group L/L^0 acts on Z by conjugation. Let Γ_1 denote the image of L/L^0 in $\mathrm{Aut}(Z)$. Let $\bar{k}$ denote the algebraic closure of k in F. Since Z is defined over k, the Galois group $\mathrm{Gal}(\bar{k}, k)$ acts on Z by automorphisms. Let Γ_2 denote the image of $\mathrm{Gal}(\bar{k}, k)$ in $\mathrm{Aut}(Z)$ and let Γ denote the subgroup of $\mathrm{Aut}(Z)$ generated by Γ_1 and Γ_2. By [3], page 213, the group Γ_2 is finite and it is clear that Γ_2 normalizes Γ_1, so that Γ is a finite subgroup of $\mathrm{Aut}(Z)$. Let E denote the real vector space $\mathbf{Y}(Z) \otimes_{\mathbf{Z}} \mathbf{R}$. We consider $\mathbf{X}(Z)$ as a subspace of the dual space E^* in the obvious way. Let $\Psi \subset \mathbf{X}(Z)$ be the set of nonzero weights of Z on the Lie algebra $L(G)$ and let

$$E_0 = \{ v \in E \mid \langle \alpha, v \rangle \neq 0 \, (\alpha \in \Psi) \}.$$

Then E_0 is an open dense subset of E. The finite group Γ acts on E and E_0 is Γ-stable.

We need the following lemma:

LEMMA 2.5. *Let $\mu_1, \mu_2 \in (E_0 \cap \mathbf{Y}(Z))$. Then $P(\mu_1)^0 = P(\mu_2)^0$ if and only if μ_1 and μ_2 belong to the same connected component of E_0.*

Proof. Let $P_i = P(\mu_i)^0$ and let $\Psi_i = \{\alpha \in \Psi \mid \langle \mu_i, \alpha \rangle > 0\}$, $i = 1, 2$. Then the Lie algebra $L(P_i)$ is spanned by $L(G)^Z$ and the weight spaces $L(G)_\alpha$ with $\alpha \in \Psi_i$. Thus we see that $P_1 = P_2$ if and only if $\Psi_1 = \Psi_2$. But it is clear that $\Psi_1 = \Psi_2$ if and only if μ_1 and μ_2 belong to the same connected component of E_0. This proves the lemma.

Now we can finish the proof of Proposition 2.4. Let A be the unique maximal k-split torus of Z. By 2.3, there exists $\mu \in \mathbf{Y}(A)$ such that $P = P(\mu)^0$, $L^0 = (G^0)^\mu = (G^0)^A$ and $R_u(P) = U(\mu)$. Since $(G^0)^Z = L^0 = (G^0)^\mu$, we see that $\mu \in E_0$. Let C be the connected component of E^0 containing μ. Since P is a k-subgroup and $L \subset P$, we see from Lemma 2.5 that C is Γ-stable. Let $\lambda \in \mathbf{Y}(Z)$ be defined by $\lambda = \sum_{\gamma \in \Gamma} \gamma \cdot \mu$. Since C is an open convex cone, $\lambda \in C$ and hence $P(\lambda)^0 = P(\mu)^0 = P$. It is immediate that λ is a fixed point of Γ. Thus λ is defined over k and $L \subset G^\lambda$. Clearly $P(\lambda) \subset N_G(P)$. Since G^λ is linearly reductive and L is a Levi subgroup of $N_G(P)$, we see that $G^\lambda = L$. By Lemma 2.2, $U(\lambda) = R_u(P) = R_u(N_G(P))$. This proves Proposition 2.4.

PROPOSITION 2.6. *Let G be a linearly reductive k-group, let H be a k-subgroup of G, and let L be a Levi k-subgroup of H. Then there exists $\lambda \in \mathbf{Y}(G)_k$ such that $H \subset P(\lambda)$, $R_u(H) \subset U(\lambda)$, and $L \subset G^\lambda$.*

Proof. We use a construction of Borel–Tits [5]. We define inductively two increasing sequences (U_i) and (N_i) of k-subgroups of G as follows:

$$U_1 = R_u(H),\ N_1 = N_G(U_1), \ldots,\ U_i = R_u(N_{i-1}),\ N_i = N_G(U_i), \ldots .$$

By [5], there exists $m > 0$ such that $U_m = U_{m+1}$ and N_m^0 is a parabolic subgroup of G^0. It follows easily that N_m is the normalizer of N_m^0. Let $U = U_m$ and let $P = N_m^0$. Clearly $H \subset N_m = N_G(P)$ and $R_u(H) \subset U$. By 1.2.4, there exists a Levi

k-subgroup M of $N_G(P)$ which contains L. Proposition 2.6 now follows from Proposition 2.4.

The following elementary lemma will be needed in a later section.

LEMMA 2.7. *Let G be a linearly reductive algebraic group and let $\lambda \in \mathbf{Y}(Z(G^0))$. Let $x \in G$ and assume that $\lim_{t\to 0}\lambda(t)\cdot x$ exists. Then $\lim_{t\to 0}\lambda(t)\cdot x = x$.*

Proof. We define one-parameter subgroups μ and γ of the algebraic torus $Z(G^0)^0$ by $\mu(t) = x^{-1}\lambda(t)x$ and $\gamma(t) = \mu(t)\lambda(t)^{-1}$. Then it is clear that

$$\lim_{t\to 0}\gamma(t) = \lim_{t\to 0}x^{-1}\lambda(t)x\lambda(t)^{-1} = x^{-1}\lim_{t\to 0}\lambda(t)x\lambda(t)^{-1}$$

exists. But it is immediate that this implies that γ is the trivial one-parameter subgroup, hence that $\lim_{t\to 0}\lambda(t)x\lambda(t)^{-1} = x$.

§3. Closed orbits and unstable orbits. In this section we consider the diagonal action of an algebraic group G on n-tuples $\mathbf{x} = (x_1,\ldots,x_n)$ in G^n and $L(G)^n$. If $\mathbf{x} = (x_1,\ldots,x_n) \in G^n$, we let $\Gamma(\mathbf{x})$ denote the (abstract) subgroup of G generated by $\{x_1,\ldots,x_n\}$ and we let $A(\mathbf{x})$ be the Zariski closure of $\Gamma(\mathbf{x})$ in G; $A(\mathbf{x})$ is the algebraic subgroup of G generated by $\{x_1,\ldots,x_n\}$. If $\mathbf{x} = (x_1,\ldots,x_n) \in L(G)^n$, then $\mathfrak{c}(\mathbf{x})$ denotes the subalgebra of $L(G)$ generated by $\{x_1,\ldots,x_n\}$ and $\mathfrak{a}(\mathbf{x})$ denotes the algebraic hull of $\{x_1,\ldots,x_n\}$. (The algebraic hull of a subset X of $L(G)$ is the smallest algebraic Lie subalgebra of $L(G)$ containing X.) In this case we let $A(\mathbf{x})$ be the unique closed connected subgroup of G with $L(A(\mathbf{x})) = \mathfrak{a}(\mathbf{x})$. If $\mathbf{x}$ is an n-tuple in either G^n or in $L(G)^n$, then it is clear that $G_{\mathbf{x}} = G^{A(\mathbf{x})}$ and that $L(G_{\mathbf{x}}) = L(G)^{A(\mathbf{x})}$.

LEMMA 3.1. *Let $\phi\colon G \to H$ be a homomorphism of algebraic groups and let $\mathbf{x} = (x_1,\ldots,x_n)$ be an n-tuple in G^n (resp. in $L(G)^n$). Write $\phi(\mathbf{x}) = (\phi(x_1),\ldots,\phi(x_n))$ (resp. $\phi(\mathbf{x}) = (d\phi(x_1),\ldots,d\phi(x_n))$). Then $\phi(A(\mathbf{x})) = A(\phi(\mathbf{x}))$.*

The proof is immediate.

LEMMA 3.2. *Let G be a semisimple algebraic group and let $\mathbf{x} \in L(G)^n$.* (a) *If $\mathfrak{a}(\mathbf{x})$ is a semisimple Lie algebra, then $\mathfrak{a}(\mathbf{x}) = \mathfrak{c}(x)$.* (b) *If $L(G)$ is a semisimple $\mathfrak{a}(\mathbf{x})$-module and if $L(G)^{\mathfrak{a}(\mathbf{x})} = \{0\}$, then $\mathfrak{a}(\mathbf{x})$ is a semisimple Lie algebra.*

Proof. (a) It follows from standard properties of algebraic Lie subalgebras that $L(G)$ is a semisimple $\mathfrak{a}(\mathbf{x})$-module if and only if it is a semisimple $\mathfrak{c}(\mathbf{x})$-module. Assume that $\mathfrak{a}(\mathbf{x})$ is a semisimple Lie algebra. Then $L(G)$ is a semisimple $\mathfrak{a}(\mathbf{x})$-module, hence a semisimple $\mathfrak{c}(\mathbf{x})$-module. This implies that $\mathfrak{c}(\mathbf{x})$ is a reductive Lie algebra. If $\mathfrak{z}$ denotes the center of $\mathfrak{c}(\mathbf{x})$, then $\mathfrak{z}$ is contained in the center of $\mathfrak{a}(\mathbf{x})$, hence $\mathfrak{z} = \{0\}$. Thus $\mathfrak{c}(\mathbf{x})$ is semisimple. But this implies that $\mathfrak{c}(\mathbf{x}) = \mathfrak{a}(\mathbf{x})$. (b) If $L(G)$ is a semisimple $\mathfrak{a}(\mathbf{x})$-module, then $\mathfrak{a}(\mathbf{x})$ is a reductive

Lie algebra. If, in addition, $L(G)^{\mathfrak{a}(\mathbf{x})} = \{0\}$, then $\mathfrak{a}(\mathbf{x})$ has trivial center, and hence is semisimple.

LEMMA 3.3. *Let G be a reductive group.* (a) *There exists $\mathbf{x} \in G^2$ such that $A(\mathbf{x}) = G$.* (b) *There exists $\mathbf{x} \in L(G)^2$ such that $\mathfrak{a}(\mathbf{x}) = L(G)$.*

Proof. (a) If G is semisimple, then (a) follows from [31], Theorem 3. Let G be reductive and let $D(G) = [G, G]$. Let $S = Z(G)^0$. Let $\mathbf{y} = (y_1, y_2) \in D(G)^2$ be such that $A(\mathbf{y}) = D(G)$. Since S is a torus, there exists $s \in S$ which generates a Zariski dense subgroup of S. Let $\mathbf{x} = (sy_1, y_2)$. Then $A(\mathbf{x}) = G$. (b) Let $\mathfrak{t}$ be a Cartan subalgebra of $L(G)$. Then there exists $a \in \mathfrak{t}$ such that the algebraic hull of $\{a\}$ is $\mathfrak{t}$. Let $L(G) = \mathfrak{t} + \Sigma_\alpha L(G)_\alpha$ be the root space decomposition of $L(G)$ with respect to $\mathfrak{t}$. Let $v = \Sigma v_\alpha \in \Sigma L(G)_\alpha$ be such that each $v_\alpha \neq 0$. If $\mathbf{x} = (v, a)$, it is clear that $\mathfrak{a}(\mathbf{x}) = L(G)$.

Definition 3.4. Let G be an algebraic group and let $\mathbf{x} \in G^n$ (resp. $\mathbf{x} \in L(G)^n$). If $A(\mathbf{x})$ is a linearly reductive group, then we say that $\mathbf{x}$ is a *semisimple n-tuple*. If $A(\mathbf{x})$ is a unipotent group, then we say that $\mathbf{x}$ is a *unipotent* (resp. *nilpotent*) *n-tuple*.

Remarks. (a) For $n = 1$, these definitions agree with the usual definitions of semisimple, unipotent, and nilpotent elements in G and in $L(G)$. (b) Let ϕ: $G \to H$ be a homomorphism of algebraic groups, let $\mathbf{x} \in G^n$, and let $\mathbf{y} = \phi(\mathbf{x}) \in H^n$. If $\mathbf{x}$ is a semisimple (resp. unipotent) n-tuple, then $\mathbf{y}$ is a semisimple (resp. unipotent) n-tuple. A similar remark holds for n-tuples in $L(G)$.

Example. Let G be a semisimple algebraic group and let (x, h, y) be an $\mathfrak{sl}_2$-triple in $L(G)$ (see [7], chap. 8, §11, for definition). Then (x, y) is a semisimple 2-tuple in $L(G)$, although both x and y are nilpotent elements of $L(G)$.

LEMMA 3.5. *Let G be a linearly reductive and let $\mathbf{x} \in G^n$ (resp. $\mathbf{x} \in L(G)^n$). Then $\mathbf{x}$ is a semisimple n-tuple if and only if $L(G)$ is a semisimple $A(\mathbf{x})$-module.*

The proof follows from Lemma 1.2.3.

After all of these preliminaries, we can now give a quick proof of our first main theorem.

THEOREM 3.6. *Let G be a linearly reductive group and let $\mathbf{x} = (x_1, \ldots, x_n) \in G^n$ (resp. $\mathbf{x} \in L(G)^n$). Then the following conditions are equivalent*: (i) $\mathbf{x}$ *is a semisimple n-tuple*; *and* (ii) *the orbit $G \cdot \mathbf{x}$ is closed.*

Proof. We shall only give the proof for $\mathbf{x} \in G^n$. The proof for $\mathbf{x} \in L(G)^n$ is similar. (ii) $\Rightarrow$ (i). Assume that $A(\mathbf{x})$ is not linearly reductive and let L be a Levi subgroup of $A(\mathbf{x})$. By Proposition 2.6, there exists $\lambda \in Y(G)$ such that $A(\mathbf{x}) \subset P(\lambda)$, $L \subset G^\lambda$, and $R_u(A(\mathbf{x})) \subset U(\lambda)$. Let h_λ: $P(\lambda) \to G^\lambda$ be defined by $h_\lambda(g) = \lambda(0) \cdot g$. Let $y_i = h_\lambda(x_i)$, $i = 1, \ldots, n$, and let $\mathbf{y} = (y_1, \ldots, y_n)$. Then $\mathbf{y} = \lim_{\lambda(t) \to 0} \lambda(t) \cdot \mathbf{x}$. It follows from Lemmas 2.3 and 3.1 that $h_\lambda(A(\mathbf{x})) =$

$A(\mathbf{y}) = L$. Since $A(\mathbf{x})$ is not linearly reductive, the kernel of the restriction of h_λ to $A(\mathbf{x})$ is nontrivial and consequently $A(\mathbf{x})$ is not conjugate to $A(\mathbf{y})$. Thus $\mathbf{y} \notin G \cdot \mathbf{x}$ and therefore $G \cdot \mathbf{x}$ is not closed.

(i) $\Rightarrow$ (ii). Assume that $A(\mathbf{x})$ is linearly reductive. Let $\mathcal{O}$ be the unique closed orbit in the closure of $G \cdot \mathbf{x}$. By HMT, there exists $\lambda \in Y(G)$ and $\mathbf{y} = (y_1, \ldots, y_n) \in \mathcal{O}$ such that $\lambda(0) \cdot \mathbf{x} = \mathbf{y}$. Thus $A(\mathbf{x}) \subset P(\lambda)$. By 1.2.4 and 2.2, there exists $u \in U(\lambda)$ such that $uA(\mathbf{x})u^{-1} \subset G^\lambda$. Let h_λ: $P(\lambda) \to G^\lambda$ be as above. Then, for each $i = 1, \ldots, n$, we see that $ux_iu^{-1} \in G^\lambda$, hence $h_\lambda(ux_iu^{-1}) = ux_iu^{-1}$. On the other hand, $u \in U(\lambda) = \text{kernel}(h_\lambda)$ and consequently $h_\lambda(ux_iu^{-1}) = h_\lambda(x_i) = y_i$. Thus $\mathbf{y} = u \cdot \mathbf{x}$ and $\mathbf{x} \in \mathcal{O}$.

THEOREM 3.7. *Let G be a linearly reductive group and let $\mathbf{x} \in G^n$ (resp. $\mathbf{x} \in L(G)^n$). Then the following conditions are equivalent*: (i) $\mathbf{x}$ *is an unstable point of G^n (resp. of $L(G)^n$); and* (ii) $\mathbf{x}$ *is a unipotent (resp. nilpotent) n-tuple.*

Proof. We shall only give the proof for $\mathbf{x} \in G^n$. Assume that $\mathbf{x}$ is an unstable point. Then by HMT there exists $\lambda \in Y(G)$ such that $\lim_{t\to 0}\lambda(t) \cdot \mathbf{x} = \mathbf{1}$. It follows that $A(\mathbf{x})$ is contained in the unipotent group $U(\lambda)$. Thus (i) implies (ii). Assume now that $\mathbf{x}$ is a unipotent n-tuple. Then there exists a Borel subgroup B of G^0 such that $A(\mathbf{x}) \subset R_u(B)$. By Proposition 2.4, there exists $\lambda \in Y(G)$ such that $R_u(B) = U(\lambda)$. Thus $\lim_{t\to 0}\lambda(t) \cdot \mathbf{x} = \mathbf{1}$ and $\mathbf{x}$ is an unstable point.

§4. Stable n-tuples.

THEOREM 4.1. *Let G be a reductive group and let $\mathbf{x} \in G^n$ (resp. $\mathbf{x} \in L(G)^n$). Then the following two conditions are equivalent*: (i) $A(\mathbf{x})$ *is not contained in any proper parabolic subgroup of G; and* (ii) $\mathbf{x}$ *is a stable point of G^n (resp. of $L(G)^n$).*

Remarks. (a) For $\mathbf{x} \in L(G)^n$, condition (i) is equivalent to the condition that $\mathfrak{a}(\mathbf{x})$ is not contained in any proper parabolic subalgebra of $L(G)$. (b) If G is nonabelian of positive dimension and if $n = 1$, then there are no stable points, since the centralizer of every semisimple element contains a maximal torus of G.

Proof of Theorem 4.1. We only give the proof for $\mathbf{x} \in G^n$. (i) $\Rightarrow$ (ii). Assume that $\mathbf{x}$ is not a stable point. First consider the case in which $G \cdot \mathbf{x}$ is not closed. Then by HMT there exists $\lambda \in Y(G)$ and $\mathbf{y} \in G^n$ such that $G \cdot \mathbf{y}$ is closed and $\lim_{t\to 0}\lambda(t) \cdot \mathbf{x} = \mathbf{y}$. Thus $A(\mathbf{x}) \subset P(\lambda)$. Since $\mathbf{x} \neq \mathbf{y}$, we see that $\lambda \notin Y(Z(G))$ and hence $P(\lambda)$ is a proper parabolic subgroup of G. Now assume that $G \cdot \mathbf{x}$ is closed. Then the stabilizer $G_\mathbf{x}$ is linearly reductive by 1.3.3. Since $\mathbf{x}$ is not stable, $G_\mathbf{x}^0 \neq Z(G)^0$. As $G_\mathbf{x}^0$ is reductive, this implies that there exists $\lambda \in Y(G_\mathbf{x})$ such that $\lambda \notin Y(Z(G))$. Hence $P(\lambda)$ is a proper parabolic subgroup of G containing $A(\mathbf{x})$.

(ii) $\Rightarrow$ (i). Assume that $A(\mathbf{x})$ is contained in a proper parabolic subgroup P of G. By 2.3, there exists $\lambda \in Y(G)$ such that $P = P(\lambda)$. Let $\mathbf{y} = \lambda(0) \cdot \mathbf{x}$. Then $A(\mathbf{y}) \subset G^\lambda$, hence image($\lambda$) $\subset G_\mathbf{y}$ and $\mathbf{y}$ is not a stable point. If $G \cdot \mathbf{x}$ is closed, then $\mathbf{y} \in G \cdot \mathbf{x}$, which shows that $\mathbf{x}$ is not a stable point. If $G \cdot \mathbf{x}$ is not closed, then clearly $\mathbf{x}$ is not a stable point.

Examples. (a) Let $G = GL(V)$. If $E \neq \{0\}$ is a proper vector subspace of V, let $P_E = \{g \in G | g \cdot E = E\}$. Then P_E is a maximal proper parabolic subgroup of G and every maximal proper parabolic subgroup of G is of the form P_E for some subspace E. Thus $\mathbf{x} \in G^n$ is stable if and only if V is an irreducible $A(\mathbf{x})$-module. (b) Let $G = \mathrm{SO}(V)$ or $\mathrm{Sp}(V)$. Then the maximal proper parabolic subgroups of G are the stabilizers in G of nontrivial totally isotropic subspaces of V. Thus $\mathbf{x} \in G^n$ is stable if and only if there are no nontrivial totally isotropic subspaces of V which are $A(\mathbf{x})$-stable.

The examples above are given in [22].

The following result gives a different characterization of stable n-tuples for the case when G is a semisimple group:

PROPOSITION 4.2. *Let G be a semisimple algebraic group and let $n > 1$.* (a) *An n-tuple $\mathbf{x} \in G^n$ is stable if and only if $L(G)$ is a semisimple $A(\mathbf{x})$-module and $L(G)^{A(\mathbf{x})} = \{0\}$.* (b) *An n-tuple $\mathbf{x} \in L(G)^n$ is stable if and only if $\mathfrak{c}(\mathbf{x})$ is a semisimple Lie algebra and $L(G)^{\mathfrak{c}(\mathbf{x})} = \{0\}$.*

The proof follows easily from 1.2.3, Lemma 3.2, and the remark preceding Lemma 3.1.

Remark. If G is reductive, then it follows easily from Lemma 3.3 and Theorem 4.1 that the set of stable points of G^n (resp. $L(G)^n$) is nonempty if $n > 1$.

§5. An analogue of the Jordan decomposition for *n*-tuples. Let G be an algebraic group. If $\mathbf{y} = (y_1, \ldots, y_n)$ and $\mathbf{z} = (z_1, \ldots, z_n)$ are in G^n, then $\mathbf{y} \cdot \mathbf{z}$ denotes the n-tuple $(y_1 z_1, \ldots, y_n z_n)$. Thus $\mathbf{y} \cdot \mathbf{z}$ is the product of $\mathbf{y}$ and $\mathbf{z}$ in the product group G^n.

Definition 5.1. (a) Let G be a k-group. If $\mathbf{x} \in G(k)^n$, then a decomposition $\mathbf{x} = \mathbf{y} \cdot \mathbf{z}$, with $\mathbf{y}$ and $\mathbf{z}$ in $G(k)^n$, is a *Levi k-decomposition* of $\mathbf{x}$ if $A(\mathbf{y})$ is a Levi k-subgroup of $A(\mathbf{x})$ and if $A(\mathbf{z})$ is contained in the unipotent radical of $A(\mathbf{x})$. (b) If $\mathbf{x} \in L(G)(k)^n$, then a decomposition $\mathbf{x} = \mathbf{y} + \mathbf{z}$, with $\mathbf{y}$ and $\mathbf{z}$ in $L(G)(k)^n$, is a *Levi k-decomposition* of $\mathbf{x}$ if $\mathfrak{a}(\mathbf{y})$ is a Levi subalgebra of $\mathfrak{a}(\mathbf{x})$ and if $A(\mathbf{z}) \subset R_u(A(\mathbf{x}))$.

Remarks. (a) If $n = 1$, then the Levi decomposition of $x \in G$ (resp. $x \in L(G)$) agrees with the usual Jordan decomposition. In this case the Levi decomposition is unique. (b) For $n > 1$, the Levi decomposition of $\mathbf{x} \in G^n$ (resp. $\mathbf{x} \in L(G)^n$) is not necessarily unique. In fact, there is a bijective correspondence between Levi decompositions of $\mathbf{x}$ and Levi subgroups of $A(\mathbf{x})$ (resp. Levi subalgebras of $\mathfrak{a}(\mathbf{x})$).

Let $\mathbf{x} = (x_1, \ldots, x_n) \in G(k)^n$. Then Levi k-decompositions of $\mathbf{x}$ exist and every Levi k-decomposition of $\mathbf{x}$ can be obtained as follows: Let L be a Levi k-subgroup of $A(\mathbf{x})$. For each index i we may write x_i uniquely in the form $x_i = y_i z_i$ with $y_i \in L(k)$ and $z_i \in R_u(A(\mathbf{x}))(k)$. Let $\mathbf{y} = (y_1, \ldots, y_n)$ and $\mathbf{z} = (z_1, \ldots, z_n)$. Then I claim that $\mathbf{x} = \mathbf{y} \cdot \mathbf{x}$ is a Levi decomposition of $\mathbf{x}$. To prove this, it is necessary to show that $A(\mathbf{y}) = L$. Let M denote the quotient group $A(\mathbf{x})/R_u(A(\mathbf{x}))$ and let ϕ: $A(\mathbf{x}) \to M$ be the quotient map. Then $\phi(x_i) = \phi(y_i)$ for each index i and it follows from Lemma 3.1 that $M = \phi(A(\mathbf{x}))$. Moreover,

$$\phi(A(\mathbf{x})) = A(\phi(\mathbf{x})) = A(\phi(\mathbf{y})) = \phi(A(\mathbf{y})).$$

Since ϕ maps L isomorphically onto M, we see that $A(\mathbf{y}) = L$.

An analogous construction gives all Levi k-decompositions of $\mathbf{x} \in L(G)(k)^n$.

THEOREM 5.2. *Let G be a linearly reductive k-group and let $\mathbf{x} \in G(k)^n$ (resp. $\mathbf{x} \in L(G)(k)^n$). Let $\mathbf{x} = \mathbf{y} \cdot \mathbf{z}$ (resp. $\mathbf{x} = \mathbf{y} + \mathbf{z}$) be a Levi k-decomposition of $\mathbf{x}$. Then $G_{\mathbf{x}} = G_{\mathbf{y}} \cap G_{\mathbf{z}}$ and $G \cdot \mathbf{y}$ is the unique closed orbit in the closure of $G \cdot \mathbf{x}$. Moreover, there exists $\lambda \in \mathrm{Y}(G)_k$ such that $\lim_{t \to 0} \lambda(t) \cdot \mathbf{x} = \mathbf{y}$.*

Proof. We give the proof for $\mathbf{x} \in G^n$. The Levi k-subgroup $A(\mathbf{y})$ of $A(\mathbf{x})$ is linearly reductive and hence the orbit $G \cdot \mathbf{y}$ is closed. By Proposition 2.6 there exists $\lambda \in \mathrm{Y}(G)_k$ such that $A(\mathbf{x}) \subset P(\lambda)$, $A(\mathbf{y}) \subset G^{\lambda}$ and $R_u(A(\mathbf{x})) \subset U(\lambda)$. It follows that $\lim_{t \to 0} \lambda(t) \cdot \mathbf{x} = \mathbf{y}$. Thus $G \cdot \mathbf{y}$ is the unique closed orbit in the closure of $G \cdot \mathbf{x}$. Since $A(\mathbf{y})$ and $A(\mathbf{z})$ are subgroups of $A(\mathbf{x})$, we have $G_{\mathbf{x}} \subset G_{\mathbf{y}} \cap G_{\mathbf{z}}$. Since $\mathbf{x} = \mathbf{y} \cdot \mathbf{z}$, $G_{\mathbf{x}} \supset G_{\mathbf{y}} \cap G_{\mathbf{z}}$. This proves the theorem.

§6. Closed orbits for n-tuples in arbitrary algebraic groups. In this section we will prove the following theorem:

THEOREM 6.1. *Let G be an algebraic group and let $\mathbf{x} \in G^n$ (resp. $\mathbf{x} \in L(G)^n$) be a semisimple n-tuple. Then the orbit $G \cdot \mathbf{x}$ is closed.*

The proof of Theorem 6.1 is based on the proof for $n = 1$ given in [3]. The proof will be given in a series of lemmas. We will only give the proof for $\mathbf{x} \in G^n$.

LEMMA 6.2. *Let E be a finite-dimensional vector space. For every $\mathbf{a} = (a_1, \ldots, a_n) \in \mathrm{End}(E)^n$ let*

$$E(\mathbf{a}) = \{v \in E \mid a_i \cdot v = v \ (i = 1, \ldots, n)\}.$$

Then for every positive integer r, the set

$$A(r, n) = \{\mathbf{a} \in \mathrm{End}(E)^n \mid \dim E(\mathbf{a}) \leqslant r\}$$

is an open subset of $\mathrm{End}(E)^n$.

Proof. If V and W are finite-dimensional vector spaces and s is an integer, then it is clear that the set

$$H_s(V,W) = \{A \in \mathrm{Hom}(V,W) | \mathrm{rank}(A) \geqslant s)\}$$

is open in Hom(V, W). Define a morphism ϕ: $\mathrm{End}(E)^n \to \mathrm{Hom}(E, E^n)$ by

$$\phi(a_1,\dots,a_n)\cdot v = (a_1\cdot v - v,\dots,a_n\cdot v - v),$$

for $v \in E$. It is clear that $A(r,n) = \phi^{-1}(H_s(E, E^n)$, where $s = \dim E - r$. Thus $A(r,n)$ is open. This proves the lemma.

Now let G be an algebraic group and let $\mathbf{x} \in G^n$ be a semisimple n-tuple. We may assume that G is a closed subgroup of $SL(E)$ for some finite-dimensional vector space E. We consider G, $L(G)$, and $SL(E)$ as closed G-stable subvarieties of End(E); here G acts on End(E) by conjugation. Since $A(\mathbf{x})$ is linearly reductive, the orbit $SL(E)\cdot\mathbf{x}$ is closed in $SL(E)^n$, hence closed in $\mathrm{End}(E)^n$. Thus $X = (SL(E)\cdot\mathbf{x}) \cap G^n$ is closed in $\mathrm{End}(E)^n$. For $\mathbf{y} = (y_1,\dots,y_n) \in \mathrm{End}(E)^n$, let

$$C(\mathbf{y}) = \{a \in \mathrm{End}(E) | ay_i = y_i a \ (i = 1,\dots,n)\}.$$

It is clear that the stabilizer $GL(E)_{\mathbf{y}}$ is equal to $GL(E) \cap C(\mathbf{y})$, so that $\dim GL(E)_{\mathbf{y}} = \dim C(\mathbf{y})$. Set $d(\mathbf{y}) = \dim C(\mathbf{y})$.

The Lie algebra $\mathfrak{g} = L(G)$ is a G-stable subspace of End(E). Thus we get representations ϕ: $G \to GL(\mathfrak{g})$ and η: $G \to GL(\mathrm{End}(E)/\mathfrak{g})$. The representation ϕ is the adjoint representation of G; it is a subrepresentation of the adjoint representation of G on $\mathrm{End}(E) = L(GL(E))$ and η is the corresponding quotient representation.

LEMMA 6.3. *If r and s are integers, then the sets*

$$\{\mathbf{y} \in G^n | \dim \mathfrak{g}^{A(\mathbf{y})} \leqslant r\} \quad \text{and} \quad \{\mathbf{y} \in G^n | \dim(\mathrm{End}(E)/\mathfrak{g})^{A(\mathbf{y})} \leqslant s\}$$

are open subsets of G^n.

Proof. This follows from Lemma 6.2.

LEMMA 6.4. *Let $\mathbf{y} \in X$. Then*

$$\dim \mathfrak{g}^{A(\mathbf{y})} + \dim(\mathrm{End}(E)/\mathfrak{g})^{A(\mathbf{y})} = d(\mathbf{x}).$$

Proof. Since $\mathbf{y} \in GL(E)\cdot\mathbf{x}$, we have $d(\mathbf{x}) = \dim GL(E)_{\mathbf{x}} = \dim GL(E)_{\mathbf{y}}$. Since $A(\mathbf{x})$ is linearly reductive, $A(\mathbf{y})$ is linearly reductive. Thus

$$d(\mathbf{x}) = \dim GL(E)_{\mathbf{y}} = \dim \mathrm{End}(E)^{A(\mathbf{y})} = \dim \mathfrak{g}^{A(\mathbf{y})} + \dim(\mathrm{End}(E)/\mathfrak{g})^{A(\mathbf{y})}.$$

Now we can prove Theorem 6.1. It follows from Lemmas 6.3 and 6.4 that the set $\{y \in X | \dim \mathfrak{g}^{A(y)} = r\}$ is open and closed in X. Consequently the integer-valued function $y \to \dim \mathfrak{g}^{A(y)}$ is constant on each component of X. Let X_1 be the connected component of X containing x. Since $\dim \mathfrak{g}^{A(y)} = \dim G_y$ for all $y \in X$, we see that all G-orbits on X_1 have the same dimension. Hence all G-orbits on X_1 are closed. Thus $G \cdot \mathbf{x}$ is closed in X_1, hence closed in $\mathrm{End}(E)^n$. This proves Theorem 6.1.

§7. Dimensions of fibres. In §7, G will denote a reductive algebraic group.

Definition 7.1. A torus S of G is a *standard torus* of G if $S = Z(G^S)^0$.

Standard tori can be obtained in the following manner. Let T be a maximal torus of G and let Δ be a basis for the set of roots $\Phi(T, G)$. If J is a subset of Δ, let $T_J = (\bigcap_{\alpha \in J} \ker(\alpha))^0$. Then T_J is a standard torus of G and every standard torus of G is conjugate to some T_J.

LEMMA 7.1. *Let $\mathbf{x} \in G^n$ (resp. $\mathbf{x} \in L(G)^n$) and let S be a maximal torus of the stabilizer $G_{\mathbf{x}}$. Then S is a standard torus of G. If $n > 1$, then for every standard torus S of G there exists a semisimple n-tuple $\mathbf{x} \in G^n$ (resp. $\mathbf{x} \in L(G)^n$) such that $S = G_{\mathbf{x}}^0$.*

Proof. We give the proof for $\mathbf{x} \in G^n$. Let S be a maximal torus of $G_{\mathbf{x}}$. Then G^S is a reductive group and $G^S \supset A(\mathbf{x})$. Let $A = Z(G^S)^0 = Z_G(G^S)^0$. Then $A \supset S$ and A is a torus of $G_{\mathbf{x}}$. Thus $A = S$. Therefore S is a standard torus of G. Now let $n > 1$ and let S be a standard torus of G. Then G^S is a reductive group. By Lemma 3.3, there exists $\mathbf{x} \in (G^S)^n$ such that $A(\mathbf{x}) = G^S$. Thus $\mathbf{x}$ is a semisimple n-tuple and $S = G_{\mathbf{x}}^0$.

Let π: $G^n \to G^n/G$ (resp. π: $L(G) \to L(G)^n/G$) denote the quotient morphism and, for each $\mathbf{x} \in G^n$ (resp. $\mathbf{x} \in L(G)^n$), let $\mathcal{F}_{\mathbf{x}}$ denote the fibre $\pi^{-1}(\pi(\mathbf{x}))$. The following result is a special case of more general theorems proved in [28]:

THEOREM 7.3. *Let $\mathbf{x}$ be a semisimple n-tuple in G^n (resp. $L(G)^n$) and let S be a maximal torus of the stabilizer $G_{\mathbf{x}}$. Let $2q(\mathbf{x}) = \dim G - \dim G^S$ and let $r(\mathbf{x}) = \dim G^S - \dim S$. If C is an irreducible component of the fibre $\mathcal{F}_{\mathbf{x}}$, then $\dim C = r(\mathbf{x}) + (n+1)q(\mathbf{x})$.*

Remark. In [28], there is also an easy combinatorial rule for computing the number of irreducible components of the fibre $F_{\mathbf{x}}$.

Example. Let $\mathbf{x}$ be a semisimple n-tuple such that $G_{\mathbf{x}}$ contains a maximal torus T of G. If C is an irreducible component of $\mathcal{F}_{\mathbf{x}}$, then $2\dim C = (n + 1)(\dim G - \dim T)$.

§8. Smooth points of the quotient variety.

THEOREM 8.1. *Let $\mathfrak{g}$ be a semisimple Lie algebra such that each simple factor of $\mathfrak{g}$ has rank at least two and let G be the adjoint group of $\mathfrak{g}$. Let $n > 1$ and let π:*

$\mathfrak{g}^n \to \mathfrak{g}^n/G$ *be the quotient morphism. Let* $\mathbf{x} \in \mathfrak{g}^n$ *be a semisimple n-tuple. Then the following conditions are equivalent*: (i) $G_{\mathbf{x}} = \{1\}$; *and* (ii) $\pi(\mathbf{x})$ *is a smooth point of* $\mathfrak{g}^n/G$.

Remark. Let $\mathfrak{g} = \mathfrak{sl}_2(F)$ and let a be a regular semisimple element of $\mathfrak{g}$. Let $\mathbf{x} = (a, 0) \in \mathfrak{g}^2$. Then $G_{\mathbf{x}}$ is a maximal torus of the adjoint group G of $\mathfrak{g}$. In this case an easy argument using 8.2 below shows that $\pi(\mathbf{x})$ is a smooth point of the quotient variety $\mathfrak{g}^2/G$. This shows that the restriction that each simple factor of $\mathfrak{g}$ has rank at least two is necessary.

The proof of Theorem 8.1 will be given in a series of lemmas. First we recall some known results. Let K be a linearly reductive group and let E be a (rational) K-module. Following G. Schwarz [29], we say that (E, K) is *coregular* if the quotient variety E/K is a smooth variety. This is equivalent to the condition that the algebra of invariants $F[E]^K$ is a graded polynomial algebra.

8.2. *Let K be linearly reductive and let E be a K-module.* (a) *Let* $v \in E$ *be such that the orbit* $K \cdot v$ *is closed. Then* $\pi_E(v)$ *is a smooth point of* E/K *if and only if the slice representation of the stabilizer* K_v *at v is coregular.* (b) *Assume that* (E, K) *is coregular. If* E_1 *is a K-submodule of E, then* (E_1, K) *is coregular.*

See [29] for the proof.

8.3. *Let A be a finite subgroup of* $GL(E)$. *Then* (E, A) *is coregular if and only if A is generated by pseudoreflections.*

For the proof, see [6], chapter 5, §5.
The following result is well known:

8.4. *Let A be a graded polynomial algebra. Then any minimal set* $(f_l, \dots, f_m)$ *of homogeneous generators of A is algebraically independent.*

Lemma 8.5. *Let* $\mathfrak{g}$ *be a semisimple Lie algebra and let A be a nontrivial finite subgroup of the adjoint group G of* $\mathfrak{g}$. *Then* $(\mathfrak{g}, A)$ *is not coregular.*

Proof. Each element of A is contained in a maximal torus T of G. But it is easy to see from the root space decomposition of $\mathfrak{g}$ that no element of a maximal torus of G can be a pseudoreflection.

Lemma 8.6. *Let G and* $\mathfrak{g}$ *be as in Theorem* 8.1 *and let T be a maximal torus of G. Then* $(\mathfrak{g}, T)$ *is not coregular.*

Proof. Because of the assumptions on $\mathfrak{g}$, there exist roots $\alpha, \beta \in \Phi(T, G)$ such that $\alpha + \beta$ is a root. Let v_1 (resp. v_2, v_3, v_4, v_5, v_6) be a nonzero element of the root space $\mathfrak{g}_\alpha$ (resp. $\mathfrak{g}_\beta, \mathfrak{g}_{\alpha+\beta}, \mathfrak{g}_{-\alpha}, \mathfrak{g}_{-\beta}, \mathfrak{g}_{-\alpha-\beta}$). Let E be the subspace of G spanned by $\{v_1, \dots, v_6\}$ and let $(x_1, \dots, x_6)$ be the basis of the dual space E^* dual to the basis $(v_1, \dots, v_6)$. It suffices to show that (E, T) is not coregular. It is clear that

$$x_1x_4,\ x_2x_5,\ x_3x_6,\ x_1x_2x_6, \text{ and } x_3x_4x_5$$

are in the invariant algebra $F[E]^T$, and it is also clear that these five homogeneous polynomials are contained in a minimal set of homogeneous generators of $F[E]^T$. However, these invariants satisfy the relation

$$(x_1x_4)(x_2x_5)(x_3x_6) = (x_1x_2x_6)(x_3x_4x_5).$$

It follows from 8.4 that $F[E]^T$ is not a graded polynomial algeba, hence that (E, T) is not coregular.

LEMMA 8.7. *Let G and $\mathfrak{g}$ be as above and let S be a nontrivial standard torus of G. Then $S = Z(G^S)$. Moreover, $(\mathfrak{g}, S)$ is not coregular.*

Proof. The result is clear if S is a maximal torus. We assume that S is not a maximal torus. Then there exists a maximal torus T containing S, a basis Δ of the root system $\Phi(T, G)$, and a proper subset J of Δ such that $S = T_J$. Since $G^T = T$, we have $T \supset Z(G^S)$. Since the roots of T on $L(G^S)$ are integral combinations of the elements of J, we see that $Z(G^S) = \bigcap_{\alpha \in J}\ker(\alpha)$. Let $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$. Since G is adjoint, the homomorphism $T \to (F^*)^\ell$ given by $t \to (\alpha_1(t), \dots, \alpha_\ell(t))$ is an isomorphism. Consequently $\bigcap_{\alpha \in J}\ker(\alpha)$ is connected and $S = Z(G^S)$.

Now to prove that $(\mathfrak{g}, S)$ is not coregular. By 8.2, it suffices to prove the result for the case of simple $\mathfrak{g}$. Since $\operatorname{rank}(\mathfrak{g}) > 1$, there exists $\alpha \in J$ and $\beta \in (\Delta - J)$ such that $\alpha + \beta$ is a root. Let v_1 (resp. v_2, v_3, v_4) be a basis of the root space $\mathfrak{g}_\beta$ (resp. $\mathfrak{g}_{\alpha+\beta}, g_{-\beta}, \mathfrak{g}_{-\alpha-\beta}$). Let E be the subspace of $\mathfrak{g}$ with basis $(v_1, \dots, v_4)$ and let $(x_1, \dots, x_4)$ be the dual basis of E^*. It suffices to show that (E, S) is not coregular. There are no homogeneous elements of degree 1 in $F[E]^S$ and x_1x_3, x_2x_3, x_1x_4, and x_2x_4 are homogeneous invariants of degree two which satisfy the relation $(x_1x_3)(x_2x_4) = (x_1x_4)(x_2x_3)$. It follows from 8.4 that (E, S) is not coregular.

We can now prove Theorem 8.1. If $G_{\mathbf{x}} = \{1\}$, then it follows from 8.2 that $\pi(\mathbf{x})$ is a smooth point of the quotient. Thus (i) implies (ii). Let $\mathbf{x} \in G^n$ be such that $G \cdot \mathbf{x}$ is closed and let $H = G_{\mathbf{x}}$. Let E denote the slice representation of H at $\mathbf{x}$. Since H is linearly reductive, E is isomorphic as an H-module to $L(H) \oplus \mathfrak{g}^{n-1}$. Thus it will suffice to prove the following lemma:

LEMMA 8.8. *Let G and $\mathfrak{g}$ be as in Theorem 8.1, let $H \neq \{1\}$ be a linearly reductive subgroup of G, and let S be a maximal torus of H. Assume that S is a standard torous of G. Then $(L(H) \oplus \mathfrak{g}, H)$ is not coregular.*

Proof. We need to consider three separate cases.

Case 1: $S = \{1\}$. Since H is linearly reductive, this implies that H is finite. It follows from Lemma 8.5 that $(\mathfrak{g}, H)$ is not coregular. The lemma now follows from 8.2.

Case 2: *S is a maximal torus of G*. Let $y \in L(S)$ be such that $G_y = S$. Consider the point $(y, 0)$ of $L(H) \oplus \mathfrak{g}$. Then $G_{(y,0)} = S$. It will suffice to show that the slice representation of S at $(y, 0)$ is not coregular. But the adjoint representation of S on $\mathfrak{g}$ is a subrepresentation of this representation and, by Lemma 8.6, $(\mathfrak{g}, S)$ is not coregular.

Case 3: *$S \neq \{1\}$ and S is not a maximal torus*. Let $y \in L(S)$ be such that $H_y^0 = S$ and $G_y = G^S$. Let $K = H_y$. Then $G^S \supset K$ and $S = K^0$. Clearly $K = H_{(y,0)}$. Since the stabilizer $H_{(y,0)}$ contains the maximal torus S of H, the orbit $H \cdot (y, 0)$ is closed (see [14], p. 183). Moreover, the adjoint representation of K on $\mathfrak{g}$ is a subrepresentation of the slice representation of K at $(y, 0)$. It will suffice to show that $(\mathfrak{g}, K)$ is not regular. If $K = S$, this follows from Lemma 8.7. Assume $K \neq S$. Let D denote the commutator subgroup of G^S; D is a semisimple group and $D \neq \{1\}$, since S is not a maximal torus. Since $Z(G^S) = S$, the finite group K/S acts nontrivially on $L(D)$. More precisely, the image of K in the adjoint group $M = D/Z(D)$ of $L(D)$ is a nontrivial finite subgroup of this adjoint group. By Lemma 8.6, $(L(D), K/S)$ is not coregular. Hence $(L(D), K)$ is not coregular and consequently $(\mathfrak{g}, K)$ is not regular. This completes the proof of Theorem 8.1.

A similar result holds for the action of G on G^n.

THEOREM 8.9. *Let G be a semisimple group and assume that each simple factor of $L(G)$ has rank at least two. Let $n > 1$ and let π: $G^n \to G^n/G$ denote the quotient morphism. Let $\mathbf{x} \in G^n$ be such that the orbit $G \cdot \mathbf{x}$ is closed. Then the following two conditions are equivalent*: (i) $G_{\mathbf{x}} = Z(G)$; *and* (ii) $\pi(\mathbf{x})$ *is a smooth point of G^n/G.*

Proof. The action of G on G^n induces an action of the adjoint group $\mathrm{Ad}(G) = G/Z(G)$ on G^n and $G^n/G = G^n/\mathrm{Ad}(G)$. If $\mathrm{Ad}(G)_{\mathbf{x}} = \{1\}$, then it follows from 8.2 that $\pi(\mathbf{x})$ is a smooth point. Thus (i) implies (ii). Let $H = \mathrm{Ad}(G)_{\mathbf{x}}$ and assume $H \neq \{1\}$. Then H is linearly reductive, since the orbit $\mathrm{Ad}(G) \cdot \mathbf{x}$ is closed. The slice representation of H at $\mathbf{x}$ is equivalent to the representation of H on $L(H) \oplus L(G)^{n-1}$. We see from Lemmas 8.8 and 7.2 that $(L(H) \oplus L(G), H)$ is not coregular. Thus (ii) implies (i).

§9. *n*-tuples in $L(G)$ and conjugacy classes of subalgebras. In this section we will prove several technical results concerning n-tuples in a Lie algebra which will be used in §10. Throughout this section $\mathfrak{g}$ denotes a finite-dimensional Lie algebra over F and G is a closed subgroup of the algebraic group $\mathrm{Aut}(\mathfrak{g})$ of all Lie algebra automorphisms of $\mathfrak{g}$. Recall that if $\mathbf{x} = (x_1, \ldots, x_n) \in \mathfrak{g}^n$, then $\mathfrak{c}(\mathbf{x})$ denotes the Lie subalgebra of $\mathfrak{g}$ generated by $\{x_1, \ldots, x_n\}$. Two subalgebras $\mathfrak{a}$ and $\mathfrak{b}$ of $\mathfrak{g}$ are G-conjugate if there exists $g \in G$ such that $g \cdot \mathfrak{a} = \mathfrak{b}$. We let $L_n = L_n(X_1, \ldots, X_n)$ denote the free Lie algebra over F on the indeterminates $X_1, \ldots, X_n$. Each n-tuple $\mathbf{x} = (x_1, \ldots, x_n) \in \mathfrak{g}^n$ determines a Lie algebra homomorphism $\eta_{\mathbf{x}}$: $L_n \to \mathfrak{g}$ given by $\eta_{\mathbf{x}}(X_i) = x_i$ $(i = 1, \ldots, n)$. The image of $\eta_{\mathbf{x}}$ is

$\mathfrak{c}(\mathbf{x})$. For each $a \in L_n$, the map $\mathbf{x} \to \eta_{\mathbf{x}}(a)$ is a morphism from $\mathfrak{g}^n$ to $\mathfrak{g}$. Let δ denote the integer-valued function of $\mathfrak{g}^n$ given by $\delta(\mathbf{x}) = \dim \mathfrak{c}(\mathbf{x})$.

PROPOSITION 9.1. *δ is a lower semicontinuous function.*

Proof. Let $\mathbf{x} \in \mathfrak{g}^n$ and let $d = \delta(\mathbf{x})$. Then there exist elements $a_1, \dots, a_d$ of L_n such that $(\eta_{\mathbf{x}}(a_1), \dots, \eta_{\mathbf{x}}(a_d))$ is an F-basis of $\mathfrak{c}(\mathbf{x})$. The map $\tau\colon \mathfrak{g}^n \to \mathfrak{g}^d$ given by $\tau(\mathbf{y}) = (\eta_{\mathbf{y}}(a_1), \dots, \eta_{\mathbf{y}}(a_d))$ is a morphism of algebraic varieties. Thus the set U of all $\mathbf{y} \in \mathfrak{g}^n$ such that the vectors $\eta_{\mathbf{y}}(a_1), \dots, \eta_{\mathbf{y}}(a_d)$ are linearly independent is an open neighbourhood of $\mathbf{x}$ in $\mathfrak{g}^n$. Clearly, if $\mathbf{y} \in U$, then $\delta(\mathbf{y}) \geqslant \delta(\mathbf{x})$. This proves that δ is lower semicontinuous.

COROLLARY 9.2. *For each $d \geqslant 0$, let $V(d) = \{\mathbf{x} \in \mathfrak{g}^n | \dim \mathfrak{c}(\mathbf{x}) \leqslant d\}$ and let $V(d)' = \{\mathbf{x} \in \mathfrak{g}^n | \dim \mathfrak{c}(\mathbf{x}) = d\}$. Then $V(d)$ is a closed subset of $\mathfrak{g}^n$ and $V(d)'$ is relatively open in $V(d)$.*

Let $\mathrm{Gr}_d(\mathfrak{g})$ be the Grassmann variety of all d-dimensional subspaces of $\mathfrak{g}$ and let $\mathbf{A}_d = \mathbf{A}_d(\mathfrak{g})$ be the closed subvariety of $\mathrm{Gr}_d(\mathfrak{g})$ consisting of all d-dimensional subalgebras of $\mathfrak{g}$. Let $X = \{(\mathfrak{a}, \mathbf{x}) \in \mathbf{A}_d \times \mathfrak{g}^n | \mathbf{x} \in \mathfrak{a}^n\}$; X is a closed subvariety of $\mathbf{A}_d \times \mathfrak{g}^n$. Let $p_1\colon X \to \mathbf{A}_d$ and $p_2\colon X \to \mathfrak{g}^n$ denote the restrictions to X of the projections $\mathrm{pr}_1\colon \mathbf{A}_d \times \mathfrak{g}^n \to \mathbf{A}_d$ and $\mathrm{pr}_2\colon \mathbf{A}_d \times \mathfrak{g}^n \to \mathfrak{g}^n$. Then p_1 defines X as a vector bundle over $\mathbf{A}_d$; the fibre over $\mathfrak{a} \in \mathbf{A}_d$ is isomorphic to $\mathfrak{a}^n$. It is clear that $V(d) \supset p_2(X)$. Let $X' = p_2^{-1}(V(d)')$. Then X' is an open subset of X. Let $\pi_2\colon X' \to V(d)'$ denote the restiction of p_2. Then π_2 is a bijection. The inverse bijection $\psi\colon V(d)' \to X'$ is given by $\psi(\mathbf{x}) = (\mathfrak{c}(\mathbf{x}), \mathbf{x})$.

LEMMA 9.3. *ψ is a morphism of algebraic varieties. Thus π_2 is an isomorphism of algebraic varieties.*

Proof. Let $\mathbf{x} \in V(d)'$ and let $a_1, \dots, a_d$ be elements of L_n such that $(\eta_{\mathbf{x}}(a_1), \dots, \eta_{\mathbf{x}}(a_d))$ is a basis of $\mathfrak{c}(\mathbf{x})$. Let

$$E = \{\mathbf{y} \in V(d)' | \eta_{\mathbf{y}}(a_1) \wedge \cdots \wedge \eta_{\mathbf{y}}(a_d) \neq 0\};$$

then E is an open subset of $V(d)'$ and $\mathbf{x} \in E$. It suffices to show that the restriction of ψ to E is a morphism. Let D_d be the set of decomposable vectors in $\wedge^d(\mathfrak{g})$ and let $\phi\colon D_d \to \mathrm{Gr}_d(\mathfrak{g})$ be the morphism which assigns to each decomposable d-vector the corresponding d-dimensional subspace of $\mathfrak{g}$. Define $\beta\colon E \to D_d$ by $\beta(\mathbf{y}) = \eta_{\mathbf{y}}(a_1) \wedge \cdots \wedge \eta_{\mathbf{y}}(a_d)$. Then it follows immediately that $\mathfrak{c}(\mathbf{y}) = \phi(\beta(\mathbf{y}))$ and hence that $\psi(\mathbf{y}) = (\phi(\beta(\mathbf{y})), \mathbf{y})$. Thus the restriction of ψ to E is a morphism.

Now let $\mathfrak{a} \in \mathbf{A}_d$ and let $\mathcal{O}(\mathfrak{a})$ denote the G-orbit of $\mathfrak{a}$ in $\mathbf{A}_d$. Then $\mathcal{O}(\mathfrak{a})$ is a smooth locally closed subvariety of $\mathbf{A}_d$. Let $Y = p_1^{-1}(\mathcal{O}(\mathfrak{a}))$. Since $p_1\colon X \to \mathbf{A}_d$ is a vector bundle over $\mathbf{A}_d$, it follows that Y is a vector bundle over $\mathcal{O}(\mathfrak{a})$. In particular, Y is a smooth variety. Let

$$Y' = Y \cap X' = \{(\mathfrak{b}, \mathbf{x}) \in X | \mathfrak{b} \in \mathcal{O}(\mathfrak{a}) \text{ and } \mathfrak{c}(\mathbf{x}) = \mathfrak{b}\}.$$

Then Y' is an open G-stable subvariety of Y. Hence Y' is a smooth subvariety of X'.

PROPOSITION 9.4. *Let* $\mathfrak{a}$ *be a d-dimensional subalgebra of* $\mathfrak{g}$. *Then the set* $C(n, \mathfrak{a}) = \{\mathbf{x} \in \mathfrak{g}^n | \mathfrak{c}(\mathbf{x})$ *is G-conjugate to* $\mathfrak{a}\}$ *is a G-stable smooth connected subvariety of* $\mathfrak{g}^n$.

Proof. Let the notation be as above. Then it is clear that $\pi_2(Y') = C(n, \mathfrak{a})$. Since π_2 is an isomorphism of varieties, the result follows.

Remark. Let G be a connected algebraic group and let $\mathfrak{g} = L(G)$. If $\mathbf{x} \in \mathfrak{g}^n$, it is not necessarily the case that

$$\{\mathbf{y} \in \mathfrak{g}^n | \mathfrak{a}(\mathbf{y}) \text{ is } G\text{-conjugate to } \mathfrak{a}(\mathbf{x})\}$$

is a locally closed subset of $\mathfrak{g}^n$. Similarly, if $\mathbf{x} \in G^n$, the set

$$\{\mathbf{y} \in G^n | A(\mathbf{y}) \text{ is } G\text{-conjugate to } A(\mathbf{x})\}$$

is not necessarily locally closed in G^n. For example, if G is a nontrivial algebraic torus, then $\{x \in G | A(x) = G\}$ is not locally closed in G.

§10. Partition of the set of stable n-tuples. In this section G denotes a semisimple algebraic group. We always assume that $n > 1$.

It is known that there are only a finite number of conjugacy classes of semisimple subalgebras of $L(G)$ [8, 23]. Let $\mathfrak{s}_1, \ldots, \mathfrak{s}_r$ be a set of representatives for the conjugacy classes of semisimple subalgebras of $L(G)$. Each $\mathfrak{s}_i$ is an algebraic Lie subalgebra of $L(G)$. We let S_i be the unique closed connected subgroup of G such that $L(S_i) = \mathfrak{s}_i$. Let $d_i = \dim S_i$. Let $\mathcal{S}_i = \{\mathbf{x} \in L(G)^n | \mathfrak{c}(\mathbf{x})$ is conjugate to $\mathfrak{s}_i\}$. Then it follows from Proposition 9.4 that $\mathcal{S}_i$ is a connected, smooth G-stable subvariety of $L(G)^n$. It follows from Lemma 3.3 that each $\mathcal{S}_i$ is nonempty.

For each $d \geqslant 0$, let $V(d)$ and $V(d)'$ be defined as in §9. We have shown that $V(d)'$ is locally closed in $L(G)^n$.

LEMMA 10.1. $\mathcal{S}_i$ *is an open subset of* $V(d_i)'$.

Proof. Let $d = d_i$. It was shown in the proof of Lemma 9.3 that the map θ: $V(d)' \to \mathbf{A}_d$ given by $\theta(\mathbf{x}) = \mathfrak{c}(\mathbf{x})$ is a morphism of algebraic varieties. Let $\mathcal{O}(\mathfrak{s}_i)$ denote the G-orbit of $\mathfrak{s}_i$ in $\mathbf{A}_d$. It is shown in [20] that $\mathcal{O}(\mathfrak{s}_i)$ is open in $\mathbf{A}_d$ (semisimple subalgebras are "rigid"). Thus $\mathcal{S}_i = \theta^{-1}(\mathcal{O}(\mathfrak{s}_i))$ is open in $V(d)'$.

We define a partial order on the set $\{\mathfrak{s}_1, \ldots, \mathfrak{s}_r\}$ by "$\mathfrak{s}_i \leqslant \mathfrak{s}_j$ *if and only if* $\mathfrak{s}_i$ *is conjugate to a subalgebra of* $\mathfrak{s}_j$."

LEMMA 10.2. *For each* $i, j = 1, \ldots, r$ *the following three conditions are equivalent*: (i) $\mathcal{S}_i$ *meets the closure of* $\mathcal{S}_j$; (ii) $\mathcal{S}_i$ *is contained in the closure of* $\mathcal{S}_j$; *and* (iii) $\mathfrak{s}_i \leqslant \mathfrak{s}_j$.

Proof. Clearly (ii) implies (i). Assume that $\mathfrak{s}_i \leqslant \mathfrak{s}_j$ and let $\mathbf{x} \in \mathscr{S}_i$. Then there exists $g \in G$ such that $g \cdot \mathfrak{s}_j \supset \mathfrak{c}(\mathbf{x})$. Let $\mathfrak{m} = g \cdot \mathfrak{s}_j$. Then $\mathbf{x} \in \mathfrak{m}^n$. By Lemma 2.3, Lemma 3.3, and Corollary 9.2, the set $C = \{\mathbf{y} \in \mathfrak{m}^n | \mathfrak{c}(\mathbf{y}) = \mathfrak{m}\}$ is a nonempty open subset of $\mathfrak{m}^n$. Thus $\mathbf{x}$ belongs to the closure of C and clearly $\mathscr{S}_j \supset C$. Therefore (iii) implies (ii).

It remains to prove that (i) implies (iii). Assume that $\mathbf{x} \in \mathscr{S}_i$ is in the closure of $\mathscr{S}_j$. Since the closure of $\mathscr{S}_j$ is G-stable, we may assume that $\mathfrak{c}(\mathbf{x}) = \mathfrak{s}_i$. We will need the following result:

10.2.1. *There exists a d_j-dimensional subalgebra $\mathfrak{a}$ of $L(G)$ which contains $\mathfrak{s}_i$ and is in the closure of $\mathscr{O}(\mathfrak{s}_j)$.*

Proof. To simplify notation, for this proof we let $E = L(G)$, $V = \mathfrak{s}_j$, and $d = d_j$. Let $P = \{g \in GL(E) | g \cdot V = V\}$. Thus P is a parabolic subgroup of $\mathrm{GL}(E)$ and the homogeneous space $\mathrm{GL}(E)/P$ can be indentified with $\mathrm{Gr}_d(L(G))$. Let $\mathscr{E}$ denote the homogeneous vector bundle $GL(E) \times^P V^n$, where P acts diagonally on V^n. (If we let P act on $GL(E) \times V^n$ by $p \cdot (g, \mathbf{v}) = (gp^{-1}, p \cdot \mathbf{v})$, then the points of $\mathscr{E}$ can be identified with the P-orbits on $GL(E) \times V^n$.) The morphism $GL(E) \times V^n \to E^n$ given by $(g, \mathbf{v}) \to g \cdot \mathbf{v}$ is constant on P-orbits and determines a morphism β: $\mathscr{E} \to E^n$. Moreover, β is a proper morphism. Let D denote the closure of $\mathscr{O}(V)$, the G-orbit of $V = \mathfrak{s}_j$ in $\mathbf{A}_d$. Let π: $\mathscr{E} \to \mathrm{Gr}_d(E) = GL(E)/P$ be the bundle map. Then $\pi^{-1}(D)$ is closed in $\mathscr{E}$ and hence $\beta(\pi^{-1}(D))$ is closed in E^n. It follows from Proposition 9.1 that $\beta(\pi^{-1}(D))$ is the closure of $\mathscr{S}_j$. Thus $\mathbf{x} \in \beta(\pi^{-1}(D))$. Consequently there exists a subalgebra $\mathfrak{a} \in D$ such that $\mathfrak{c}(\mathbf{x}) = \mathfrak{s}_i$ is contained in $\mathfrak{a}$. This proves 10.2.1.

Let $d = d_j$. We see from 10.2.1 that $\mathfrak{s}_i$ is contained in the d-dimensional subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ and that $\mathfrak{a}$ is contained in the closure of the orbit $\mathscr{O}(\mathfrak{s}_j)$. It follows from [24], Theorem 9.11, that there exists an open neighbourhood $N(\mathfrak{a})$ of $\mathfrak{a}$ in $\mathbf{A}_d$ such that if $\mathfrak{b} \in N(\mathfrak{a})$, then $\mathfrak{b}$ contains a subalgebra $\mathfrak{c}$ such that $\mathfrak{c}$ is conjugate in $L(G)$ to $\mathfrak{s}_i$. Clearly $\mathscr{O}(\mathfrak{s}_j)$ meets $N(\mathfrak{a})$. Consequently $\mathfrak{s}_i \leqslant \mathfrak{s}_j$. This completes the proof of Lemma 10.2.

We index the semisimple subalgebras $\mathfrak{s}_1, \ldots, \mathfrak{s}_r$ such that $\mathfrak{z}_{L(G)}(\mathfrak{s}_i) = \{0\}$ if and only if $i \leqslant q$, where $q \leqslant r$. It follows from Proposition 4.3 that an n-tuple $\mathbf{x} \in L(G)^n$ is stable if and only if $\mathfrak{c}(\mathbf{x})$ is conjugate to some $\mathfrak{s}_j$ with $j \leqslant q$. Thus we have proved

THEOREM 10.3. *Let G be a semisimple algebraic group and assume that $n > 1$. Let $\mathfrak{s}_1, \ldots, \mathfrak{s}_q$ be a set of representatives for the conjugacy classes of semisimple subalgebras $\mathfrak{s}$ of $L(G)$ such that $\mathfrak{z}_{L(G)}(\mathfrak{s}) = \{0\}$. For each $j = 1, \ldots, q$, let*

$$\mathscr{S}_j = \left\{\mathbf{x} \in L(G)^n | \mathfrak{c}(\mathbf{x}) \text{ is conjugate to } \mathfrak{s}_j\right\}.$$

Let $(L(G)^n)^{(s)}$ be the set of G-stable points of $L(G)^n$. Then each $\mathscr{S}_j$ is a smooth, G-stable connected subvariety of $(L(G)^n)^{(s)}$ and $(L(G)^n)^{(s)}$ is the disjoint union of

$\mathscr{S}_1, \ldots, \mathscr{S}_q$. *For each* $j = 1, \ldots, q$, *let* $A_j = \{i = 1, \ldots, q \mid \mathfrak{s}_i \leqslant \mathfrak{s}_j\}$. *Then the closure of* $\mathscr{S}_j$ *in* $(L(G)^n)^{(s)}$ *is the union of the* $\mathscr{S}_i$, $i \in A_j$.

Theorem 10.3 gives us a partition of the set $(L(G)^n)^{(s)}$ of stable n-tuples by the smooth, connected subvarieties $\mathscr{S}_i$, $i = 1, \ldots, q$. Moreover, we have an explicit description of the closure relations between the $\mathscr{S}_i$'s.

§11. *n*-tuples in real algebraic groups and Lie algebras. In §11 we will carry over many of our results on n-tuples to the case of real reductive algebraic groups and, more generally, real reductive Lie groups. Throughout §11, the base field F will be the field $\mathbf{C}$ of complex numbers and we will always denote complex algebraic varieties by boldface letters $\mathbf{X}$, $\mathbf{E}$, $\mathbf{G}$, $\mathbf{L(G)}$, etc. If $\mathbf{X}$ is a complex algebraic variety, we need to consider two distinct topologies on $\mathbf{X}$, the Zariski topology and the classical (Hausdorff) topology induced by the usual topology of $\mathbf{C}$. In §11, all references to topological terms which refer to the Zariski topology will be given the prefix *Zariski*. Thus a subset Y of $\mathbf{X}$ is *closed* (resp. *Zariski-closed*) if it is closed in the classical topology (resp. the Zariski topology). If $\mathbf{X}$ is a complex $\mathbf{R}$-variety (a complex variety defined over the field $\mathbf{R}$ of real numbers), then $\mathbf{X(R)}$ will be given the classical topology (the topology induced by the classical topology on $\mathbf{X}$). If $\mathbf{G}$ is an algebraic $\mathbf{R}$-group, then $\mathbf{G(R)}$ is considered as a real Lie group and $\mathbf{G(R)}^0$ denotes the identity component of $\mathbf{G(R)}$. It is known that $\mathbf{G(R)}/\mathbf{G(R)}^0$ is finite. It is not necessarily true that $\mathbf{G(R)}^0 = \mathbf{G}^0\mathbf{(R)}$.

Let H be a real Lie group and let $\mathbf{x} = (x_1, \ldots, x_n) \in H^n$. Then $\Gamma(\mathbf{x})$ denotes the (abstract) subgroup of H generated by $\{x_1, \ldots, x_n\}$. If $\mathbf{x} = (x_1, \ldots, x_n) \in L(H)^n$, then $\mathfrak{c}(\mathbf{x})$ denotes the subalgebra of $L(H)$ generated by $\{x_1, \ldots, x_n\}$. Assume that $\mathbf{H}$ is a complex algebraic group and that H is a closed subgroup of $\mathbf{H(R)}$ containing $\mathbf{H(R)}^0$. If $\mathbf{x} \in H^n$, then $\mathbf{A(x)}$ denotes the Zariski closure of $\Gamma(\mathbf{x})$ in $\mathbf{H}$. If $\mathbf{x} \in L(H)^n$, then $\mathfrak{a}(\mathbf{x})$ denotes the algebraic hull of $\mathfrak{c}(\mathbf{x})$ in $\mathbf{L(H)}$ and $\mathbf{A(x)}$ is the connected algebraic subgroup of $\mathbf{H}$ such that $\mathbf{L(A(x))} = \mathfrak{a}(\mathbf{x})$.

For the rest of §11, $\mathbf{G}$ will denote a linearly reductive complex algebraic $\mathbf{R}$-group and G will denote a closed subgroup of $\mathbf{G(R)}$ containing $\mathbf{G(R)}^0$.

Definition 11.1. Let $\mathbf{x} \in G^n$ (resp. $\mathbf{x} \in L(G)^n$). Then $\mathbf{x}$ is a *semisimple n-tuple* in G^n (resp. in $L(G)^n$) if $L(G)$ is a semisimple $\Gamma(\mathbf{x})$-module (resp. a semisimple $\mathfrak{c}(\mathbf{x})$-module). We note that this definition depends only on the Lie group structure of G.

LEMMA 11.2. *Let* $\mathbf{x} \in G^n$ (*resp.* $\mathbf{x} \in L(G)^n$). *Then* $\mathbf{x}$ *is a semisimple n-tuple in* G^n (*resp. in* $L(G)^n$) *if and only if* $\mathbf{x}$ *is a semisimple n-tuple in* $\mathbf{G}^n$ (*resp. in* $\mathbf{L(G)}^n$).

Proof. We give the proof for $\mathbf{x} \in G^n$. Since $\mathbf{A(x)}$ is the Zariski closure of $\Gamma(\mathbf{x})$, the following three conditions are equivalent: (i) $L(G)$ is a semisimple $\Gamma(\mathbf{x})$-module; (ii) $\mathbf{L(G)}$ is a semisimple $\Gamma(\mathbf{x})$-module; and (iii) $\mathbf{L(G)}$ is a semisimple $\mathbf{A(x)}$-module. The proof now follows from Lemma 3.5.

The following result is due to Birkes [2]:

11.3. *Let* η: $\mathbf{G} \to \mathbf{GL}(\mathbf{E})$ *be a rational representation defined over* $\mathbf{R}$ *and let* $x \in \mathbf{E}(\mathbf{R})$. *Then the orbit* $G \cdot x$ *is closed in* $\mathbf{E}(\mathbf{R})$ *if and only if* $\mathbf{G} \cdot x$ *is Zariski-closed in* $\mathbf{E}$.

THEOREM 11.4. *Let* $\mathbf{x} \in G^n$ (*resp.* $\mathbf{x} \in L(G)^n$). *Then the following conditions are equivalent*: (i) *the orbit* $G \cdot \mathbf{x}$ *is closed*; *and* (ii) $\mathbf{x}$ *is a semisimple n-tuple*.

Proof. This follows from 11.3, Theorem 3.7, and Lemma 11.2.

Definition 11.5. (a) Let $\mathbf{x} \in G^n$. Then $\mathbf{x}$ is a *unipotent n-tuple* if $\Gamma(\mathbf{x})$ is a unipotent subgroup of $\mathbf{G}$. (b) Let $\mathbf{x} \in L(G)^n$. Then $\mathbf{x}$ is a *nilpotent n-tuple* if each $y \in \mathfrak{c}(\mathbf{x})$ is a nilpotent element of the real algebraic Lie algebra $L(\mathbf{G}(\mathbf{R}))$.

Definition 11.6. Let H be a real Lie group and let $\mathbf{x} \in H^n$ (resp. $\mathbf{x} \in L(H)^n$). Then $\mathbf{x}$ is an *unstable point* of H^n (resp. $L(H)^n$) if $\mathbf{1} = (1, \dots, 1)$ belongs to the closure of the orbit $H \cdot \mathbf{x}$ (resp. if $\mathbf{0} = (0, \dots, 0)$ belongs to the closure of the orbit $H \cdot \mathbf{x}$). If $\mathbf{x}$ is an unstable point, we say that the orbit $H \cdot \mathbf{x}$ is unstable.

THEOREM 11.7. *Let* $\mathbf{x} \in G^n$ (*resp.* $\mathbf{x} \in L(G)^n$). *Then the orbit* $G \cdot \mathbf{x}$ *is unstable if and only if* $\mathbf{x}$ *is a unipotent* (*resp. nilpotent*) *n-tuple*.

The proof follows from HMT and Theorem 3.7.

11.8. *Stable points*. We assume that $\mathbf{G}$ is a Zariski-closed subgroup of $\mathbf{GL}(\mathbf{E})$, where $\mathbf{E}$ is a finite-dimensional complex vector space with real structure E. Let $\mathbf{E}^{(s)}$ be the set of $\mathbf{G}$-stable points of $\mathbf{E}$. Then $\mathbf{E}^{(s)}$ is an $\mathbf{R}$-subvariety of $\mathbf{E}$. It follows from [18], page 41, that the action of $\mathbf{G}$ on $\mathbf{E}^{(s)}$ is proper (in the sense of algebraic geometry). If we now consider $\mathbf{G}$ and $\mathbf{E}^{(s)}$ with their classical topology, it follows that the action of $\mathbf{G}$ on $\mathbf{E}^{(s)}$ is proper in the sense of proper actions of locally compact groups acting on locally compact spaces [13]. (It is known that a proper mapping of complex algebraic varieties is also a proper mapping of the corresponding complex spaces.) Let $E^{(s)} = \mathbf{E}^{(s)} \cap E$. Thus $E^{(s)}$ is the set of all $x \in E$ such that the orbit $G \cdot x$ is closed and such that the stabilizer G_x is finite. The group G is a closed subgroup of (the Lie group) $\mathbf{G}$ and $E^{(s)}$ is a closed subset of (the locally compact space) $\mathbf{E}^{(s)}$. It follows that G acts properly on $E^{(s)}$.

PROPOSITION 11.9. *Let the notation be as above, let* X *be a closed G-stable differentiable submanifold of* E, *and let* $X^{(s)} = E^{(s)} \cap X$. *Let* $X^{(s)}/G$ *be the set of orbits on* G *on* $X^{(s)}$, *supplied with the quotient topology. Then* G *acts properly on* $X^{(s)}$ *and the orbit space* $X^{(s)}/G$ *is a Hausdorff space. Moreover,* $X^{(s)}/G$ *has the structure of a V-manifold.*

See [1] for the definitions regarding V-manifolds.

Proof. Since $X^{(s)}$ is closed in $E^{(s)}$, the action of G on $X^{(s)}$ is proper. Hence the orbit space is Hausdorff [13]. By the results of Luna [16] or Palais [21], there exists a differentiable slice at each point $x \in X^{(s)}$. Since each stabilizer G_x is finite, the orbit space has a natural structure of V-manifold.

11.10. *Stable n-tuples for G.* We may consider $\mathbf{G}$ as a Zariski-closed $\mathbf{R}$-subgroup of $\mathbf{SL}(\mathbf{E})$, where $\mathbf{E}$ has an $\mathbf{R}$-structure E. Thus $\mathbf{G}$ and $\mathbf{L}(\mathbf{G})$ are Zariski-closed subvarieties of $\mathrm{End}_{\mathbf{C}}(\mathbf{E})$. Now $\mathbf{G}$ acts on $\mathrm{End}_{\mathbf{C}}(\mathbf{E})$ by $g \cdot a = gag - 1$. We let $\mathbf{W} = \mathrm{End}_{\mathbf{C}}(\mathbf{E})^{\mathbf{Z}(\mathbf{G})}$. Then $\mathbf{W}$ is a $\mathbf{G}$-stable linear subspace of $\mathrm{End}_{\mathbf{C}}(E)$ with $\mathbf{R}$-structure $W = \mathrm{End}_{\mathbf{R}}(E) \cap \mathbf{W}$. Moreover, $\mathbf{G}$ and $\mathbf{L}(\mathbf{G})$ are Zariski-closed $\mathbf{G}$-stable subvarieites of $\mathbf{W}$. We define a linear representation η of $\mathbf{G}$ on $\mathbf{W}^n$ by $\eta(g) \cdot (a_1, \ldots, a_n) = (g \cdot a_1, \ldots, g \cdot a_n)$. Let $\mathbf{H} = \eta(\mathbf{G})$ and let $H = \eta(G)$. Then $\mathbf{H}$ is a Zariski-closed linearly reductive $\mathbf{R}$-subgroup of $\mathbf{SL}(\mathbf{W}^n)$ and $\mathbf{H}(\mathbf{R})^0 \subset H \subset \mathbf{H}(\mathbf{R})$. Let $(\mathbf{W}^n)^{(s)}$ be the set of $\mathbf{H}$-stable points of $\mathbf{W}^n$ and let $(W^n)^{(s)} = (\mathbf{W}^n)^{(s)} \cap W^n$. Then an n-tuple $\mathbf{x} \in W^n$ is in $(W^n)^{(s)}$ if and only if the orbit $H \cdot \mathbf{x}$ is closed in W^n and the stabilizer $H_{\mathbf{x}}$ is finite.

Definition 11.10.1. Let $\mathbf{x} \in G^n$ (resp. $\mathbf{x} \in L(G)^n$). Then $\mathbf{x}$ is a G-stable point of G^n (resp. $L(G)^n$) if the orbit $G \cdot \mathbf{x}$ is closed and if $G_{\mathbf{x}}/Z(G)$ is finite.

Let $(G^n)^{(s)}$ (resp. $(L(G)^n)^{(s)}$) be the set of G-stable points of G^n (resp. $L(G)^n$). Then it is clear that $(G^n)^{(s)} = G^n \cap (W^n)^{(s)}$ (resp. $(L(G)^n)^{(s)} = L(G)^n \cap (W^n)^{(s)}$).

PROPOSITION 11.11. *The action of $G/Z(G)$ on $(G^n)^{(s)}$ (resp. $(L(G)^n)^{(s)}$) is proper and the orbit space $(G^n)^{(s)}/G$ (resp. $(L(G)^n)^{(s)}/G$) has the structure of a V-manifold.*

The proof of Proposition 11.11 follows immediately from Proposition 11.9.

Remark. Let the notation be as above and let $\mathbf{x} \in G^n$ (resp. $\mathbf{x} \in L(G)^n$). Then it follows easily from the definitions that $\mathbf{x}$ is a G-stable point of G^n (resp. $L(G)^n$) if and only if it is a $\mathbf{G}$-stable point of $\mathbf{G}^n$ (resp. $\mathbf{L}(\mathbf{G})^n$). Assume that $\mathbf{G}$ is connected, hence reductive. It is then a consequence of Theorem 4.1 that $\mathbf{x}$ is a G-stable n-tuple if and only if $\mathbf{A}(\mathbf{x})$ is not contained in any proper parabolic subgroup of $\mathbf{G}$. However, this condition is not, in general, equivalent to the condition that $\Gamma(\mathbf{x})$ is not contained in any proper $\mathbf{R}$-parabolic subgroup of $\mathbf{G}$, as one can see from elementary examples.

PROPOSITION 11.12. *Assume that $\mathbf{G}$ is semisimple. Let $\mathbf{x} \in G^n$. Then $\mathbf{x}$ is a G-stable n-tuple if and only if $L(G)$ is a semisimple $\Gamma(\mathbf{x})$-module and $L(G)^{\Gamma(\mathbf{x})} = \{0\}$. Let $\mathbf{x} \in L(G)^n$. Then $\mathbf{x}$ is a stable n-tuple if and only if $L(G)$ is a semisimple $\mathfrak{c}(\mathbf{x})$-module and $L(G)^{\mathfrak{c}(\mathbf{x})} = \{0\}$.*

Proof. This follows from Proposition 4.3.

As a special case of Proposition 11.12 we have

COROLLARY 11.13. *Let E be an odd-dimensional real vector space and let $G = GL(E)$. Let $\mathbf{x} \in G^n$ (resp. $\mathbf{x} \in L(G)^n$). Then $\mathbf{x}$ is a G-stable n-tuple if and only if E is a simple $\Gamma(\mathbf{x})$-module (resp. E is a simple $\mathfrak{c}(\mathbf{x})$-module).*

Proof. We give the proof for $x \in G^n$. Let $D = \mathrm{End}_{\mathbf{R}}(E)^{\Gamma(\mathbf{x})}$. It is an easy consequence of Proposition 11.12, Theorem 11.4, and Example 4.2 that $\mathbf{x}$ is G-stable if and only if E is a simple $\Gamma(\mathbf{x})$-module and $\dim_{\mathbf{R}} D = 1$. On the other hand, it follows from Schur's lemma that if E is a simple $\Gamma(\mathbf{x})$-module, then D is a division algebra over $\mathbf{R}$. There are three possibilities: (i) $D = \mathbf{R}$; (ii) $D = \mathbf{C}$; and (iii) $D = \mathbf{H}$ (the quaternions). Since E is odd-dimensional, possibilities (ii) and (iii) are ruled out. This proves the corollary.

Let $\mathbf{x} \in G^n$ and let $\mathbf{x} = \mathbf{y} \cdot \mathbf{z}$ be a Levi $\mathbf{R}$-decomposition of $\mathbf{x}$ (considered as a point of $\mathbf{G}^n$). Since a unipotent real algebraic group is connected (as a real Lie group), it follows easily that $\mathbf{z} \in G^n$, and hence that $\mathbf{y} \in G^n$.

PROPOSITION 11.14. *Let* $\mathbf{x} \in G^n$ *(resp.* $\mathbf{x} \in L(G)^n$*) and let* $\mathbf{x} = \mathbf{y} \cdot \mathbf{z}$ *(resp.* $\mathbf{x} = \mathbf{y} + \mathbf{z}$*) be a Levi* $\mathbf{R}$*-decomposition of* $\mathbf{x}$*. Then* $G_{\mathbf{x}} = G_{\mathbf{y}} \cap G_{\mathbf{z}}$ *and* $G \cdot \mathbf{y}$ *is the unique closed orbit in the closure of* $G \cdot \mathbf{x}$*. Furthermore, there exists* $\lambda \in \mathbf{Y}(G)_{\mathbf{R}}$ *such that* $\lim_{t \to 0} \lambda(t) \cdot \mathbf{x} = \mathbf{y}$.

Proof. It follows from a result of Luna [16] that there exists a unique closed G-orbit in the closure of $G \cdot \mathbf{x}$. The other conclusions follow easily from Theorem 5.2.

Definition 11.15. A real Lie group H is a *real reductive Lie group* if there exists a reductive complex algebraic $\mathbf{R}$-group $\mathbf{G}$ and a Lie group homomorphism η: $H \to \mathbf{G}(\mathbf{R})$ with finite kernel such that $\eta(H)$ contains $\mathbf{G}(\mathbf{R})^0$. Let H be a real reductive Lie group. Then an n-tuple $\mathbf{x} \in H^n$ is *H-stable* if the orbit $H \cdot \mathbf{x}$ is closed in H^n and if $H_{\mathbf{x}}/Z(H)$ is finite.

LEMMA 11.16. *Let H be a real reductive Lie group, let* $\eta, \mathbf{G}$ *be as above, and let* $G = \eta(H)$*. Let* $\mathbf{x} = (x_1, \dots, x_n) \in H^n$ *and let* $\mathbf{y} = \eta(\mathbf{x}) = (\eta(x_1), \dots, \eta(x_n))$*. Then* $H \cdot \mathbf{x}$ *is closed in* H^n *if and only if* $G \cdot \mathbf{y}$ *is closed in* G^n*. Moreover,* $\mathbf{x}$ *is an H-stable point of* H^n *if and only if* $\mathbf{y}$ *is a G-stable point of* G^n.

We omit the proof, which is an easy exercise.

THEOREM 11.17. *Let H be a real reductive Lie group and let* $\mathbf{x} \in H^n$*. Then the orbit* $H \cdot \mathbf{x}$ *is closed if and only if* $L(H)$ *is a semisimple* $\Gamma(\mathbf{x})$*-module. Assume further that H is a semisimple Lie group. Then* $\mathbf{x}$ *is a stable point of* H^n *if and only if* $L(H)$ *is a semisimple* $\Gamma(\mathbf{x})$*-module and* $L(H)^{\Gamma(\mathbf{x})} = \{0\}$.

The proof follows easily from Lemma 11.16 and the corresponding results for G.

§12. S-groups. Let S and G be algebraic k-groups. We say that G is an S-group, defined over k, if S acts k-morphically on G such that for every $s \in S$, the morphism $G \to G$ given by $g \to s \cdot g$ is an automorphism of algebraic groups.

Remark. Let S and G be k-subgroups of an algebraic k-group H, with G normal in H. Then the action of S on G by inner automorphisms determines an S-group structure on G defined over k. By taking semidirect products, any S-group structure on G, defined over k, can be obtained in this way.

Let G be an S-group, defined over k, and assume that both S and G are linearly reductive groups. Let $K = G^S$ be the fixed point subgroup. Then K is a k-subgroup of G and is linearly reductive ([26], Prop. 10.1.5). In §13–§15 we will extend many of the results of earlier sections to the actions of K on G^n and $L(G)^n$. In this section, we extend some of the foundational results on parabolic subgroups and Levi subgroups given in §1 and §2 to the framework of S-groups.

For the remainder of §12, S denotes a linearly reductive k-group.

PROPOSITION 12.1. *Let H be an S-group defined over k. Then there exists an S-stable Levi k-subgroup of H.*

Proof. Let C denote the semidirect product $H \ltimes S$; then C is a k-group. We consider H and S as subgroups of C in the usual way. Let $U = R_u(H)$. Then U is a normal unipotent subgroup of C and it is clear that C/U is isomorphic to $(H/U) \ltimes S$. By 1.2.2, $(H/U) \ltimes S$ is linearly reductive, hence C/U is linearly reductive. This implies that $U = R_u(C)$. Since S is a linearly reductive k-subgroup of C, there exists a Levi k-subgroup M of C which contains S. Let $L = M \cap H$. Then L is a k-subgroup of both H and M and is normalized by M. It follows from 1.2.2 that L is linearly reductive. Thus L is contained in a Levi k-subgroup L_1 of H. Let $x \in L_1$. We may write $x = au$, with $a \in M$ and $u \in U$. Thus $a = xu^{-1} \in M \cap H = L$. Consequently $u = a^{-1}x \in L_1 \cap U = \{1\}$. Therefore $x = a \in L$ and $L = L_1$. Since M contains S, L is an S-stable Levi k-subgroup of H.

PROPOSITION 12.2. *Let H be an S-group defined over k and let M be an S-stable, linearly reductive k-subgroup of H. Then M is contained in an S-stable Levi k-subgroup of H.*

Proof. Let R denote the semidirect product $M \ltimes S$; then R is linearly reductive. The actions of M (by inner automorphisms) and S on H are compatible and hence determine a k-morphic action of R on H (see [26], §2). Thus H is an R-group defined over k. By Proposition 12.1, there exists an R-stable Levi k-subgroup L of H. Thus L is S-stable and $N_H(L) \supset M$. It is clear that L is a Levi subgroup of $N_H(L)$. Since L is normal in $N_H(L)$ and all Levi subgroups of $N_H(L)$ are conjugate in $N_H(L)$, we see that L is the unique Levi subgroup of $N_H(L)$. Since M is linearly reductive, it follows from 1.2.4 that $L \supset M$.

PROPOSITION 12.3. *Let G be a linearly reductive S-group defined over k and let $K = G^S$. Let P be an S-stable parabolic k-subgroup of G^0. If L is an S-stable Levi k-subgroup of $N_G(P)$, then there exists $\lambda \in Y(K)_k$ such that $N_G(P) = P(\lambda)$, $L = G^\lambda$, and $R_u(P) = U(\lambda)$.*

Proof. The actions of L (by inner automorphisms) and S on G are compatible and determine a k-morphic action of the semidirect product $M = L \ltimes S$ on G. The group M is a linearly reductive k-group and P, L, and $U = R_u(P)$ are M-stable k-subgroups of G. Let $Z = Z(L^0)^0$. Then Z is the unique maximal central torus of L^0. The torus Z is an M-stable k-subgroup of G. Let A be the unique maximal k-split torus of Z (see [4], §1). It follows easily from 2.3 that there exists $\mu \in Y(A) = Y(A)_k$ such that $P(\mu)^0 = P$, $(G^0)^\mu = (G^0)^A = L^0 = (G^0)^Z$, and $U(\mu) = U$. Let Ψ be the set of nonzero weights of Z on $L(G)$. If $\alpha \in \Psi$, then an easy argument shows that $\langle \mu, \alpha \rangle \neq 0$. An argument similar to the argument given in the proof of Proposition 2.4 now shows that there exists $\lambda \in Y(K)_k$ such that $N_G(P) = P(\lambda)$, $L = G^\lambda$, and $R_u(P) = U(\lambda)$.

PROPOSITION 12.4. *Let G be a linearly reductive S-group defined over k and let $K = G^S$. Let H be an S-stable k-subgroup of G and let L be an S-stable Levi k-subgroup of H. Then there exists $\lambda \in Y(K)_k$ such that $P(\lambda) \supset H$, $G^\lambda \supset L$, and $U(\lambda) \supset R_u(H)$.*

The proof is almost exactly the same as the proof of Proposition 2.6. We omit the details.

§13. K-orbits on G^n and $L(G)^n$. In §13 S denotes a linearly reductive k-group, G is a linearly reductive S-group defined over k, and $K = G^S$.

In the next few sections, we will study the actions of K on G^n and on $L(G)^n$.

Let $\mathbf{x} = (x_1, \ldots, x_n) \in G^n$. We let $A_S(\mathbf{x})$ be the intersection of all closed S-stable subgroups of G containing $\{x_1, \ldots, x_n\}$. Similarly, if $\mathbf{x} = (x_1, \ldots, x_n) \in L(G)^n$, we let $\mathfrak{a}_S(\mathbf{x})$ be the algebraic hull of $\{s \cdot x_i | s \in S,\ i = 1, \ldots, n\}$ and we let $A_S(\mathbf{x})$ be the unique closed connected subgroup of G such that $L(A_S(\mathbf{x})) = \mathfrak{a}_S(\mathbf{x})$.

Definition 13.1. Let $\mathbf{x} \in G^n$ (resp. $\mathbf{x} \in L(G)^n$). Then $\mathbf{x}$ is an *S-semisimple n-tuple* if $A_S(\mathbf{x})$ is a linearly reductive group. The n-tuple $\mathbf{x}$ is *S-unipotent* (resp. *S-nilpotent*) if $A_S(\mathbf{x})$ is a unipotent group.

THEOREM 13.2. *Let $\mathbf{x} \in G^n$ (resp. $\mathbf{x} \in L(G)^n$). Then the following conditions are equivalent*: (i) *the orbit $K \cdot \mathbf{x}$ is closed*; *and* (ii) $\mathbf{x}$ *is an S-semisimple n-tuple.*

Proof. We only give the proof for $\mathbf{x} \in G^n$.

(i) $\Rightarrow$ (ii). Assume that $A_S(\mathbf{x})$ is not linearly reductive and let $U = R_u(A_S(\mathbf{x}))$. By Proposition 12.4, there exists $\lambda \in Y(K)$ such that $P(\lambda) \supset A_S(\mathbf{x})$ and $U(\lambda) \supset U$. Let $\mathbf{y} = \lambda(0) \cdot \mathbf{x}$. Define the homomorphism h_λ: $P(\lambda) \to G^\lambda$ by $h_\lambda(g) = \lambda(0) \cdot g$ ($g \in P(\lambda)$). Since $\lambda \in Y(K)$, the subgroup $P(\lambda)$ is S-stable and $h_\lambda(s \cdot g) = s \cdot h_\lambda(g)$ ($s \in S$, $g \in G$). It is an easy consequence of this that $h_\lambda(A_S(\mathbf{x})) = A_S(\mathbf{y})$. Since $\dim U > 0$ and U is contained in the kernel of h_λ, we see that $\dim A_S(\mathbf{y}) < \dim A_S(\mathbf{x})$. Thus $\mathbf{y} \notin K \cdot \mathbf{x}$ and the orbit $K \cdot \mathbf{x}$ is not closed.

(ii) $\Rightarrow$ (i). Assume that $A_S(\mathbf{x})$ is linearly reductive. Let T be a maximal torus of the stabilizer $K_{\mathbf{x}}$. It follows from a result of Luna [17] that the orbit $K \cdot \mathbf{x}$ is closed in G^n if and only if $K^T \cdot \mathbf{x}$ is closed in $(G^T)^n$. It is clear that G^T is S-stable and that $A_S(\mathbf{x}) \subset G^T$. Thus we can reduce to the case $G = G^T$ and $K = K^T$. So we may assume that T is a central torus in G and in K. Assume now that $K \cdot \mathbf{x}$ is not closed. By HMT there exists $\lambda \in \mathrm{Y}(K)$ and $\mathbf{y} \in G^n$ such that $\lim_{t \to 0} \lambda(t) \cdot \mathbf{x} = \mathbf{y}$ and such that the orbit $K \cdot \mathbf{y}$ is closed. Since $P(\lambda)$ is S-stable, we see that $A_S(\mathbf{x}) \subset P(\lambda)$. It follows from Lemma 2.7 that $\lambda \notin \mathrm{Y}(Z(G^0))$, hence that $P = P(\lambda)^0$ is a proper parabolic subgroup of G^0. Let $Q = N_G(P)$. Then $P(\lambda) \subset Q$ and Q is S-stable. By Propositions 12.2 and 12.3, there exists $\mu \in \mathrm{Y}(K)$ such that $P(\mu) = Q$ and $A_S(\mathbf{x}) \subset G^{\mu}$. Let T_1 be the torus generated by T and image(μ). Then $T_1 \subset K_{\mathbf{x}}$. Since $P(\mu)^0 = Q^0 = P \neq G$, image($\mu$) is not contained in T. Thus $\dim T_1 > \dim T$, which contradicts the choice of T.

Example. Let G be a semisimple group and let θ be an automorphism of G of period two. Let $K = G^{\theta} = \{g \in G | \theta(g) = g\}$. Let $x \in G$. Then the orbit $K \cdot x$ is closed in G if and only if $A(\{x, \theta(x)\})$ is a linearly reductive group. More generally, let θ be an automorphism of G of finite order r and let $K = G^{\theta}$. Let $x \in G$. Then $K \cdot x$ is closed if and only if $A(\{x, \theta(x), \dots, \theta^{r-1}(x)\})$ is a linearly reductive group.

THEOREM 13.3. *Let* $\mathbf{x} \in G^n$ (*resp.* $\mathbf{x} \in L(G)^n$). *Then the following conditions are equivalent*: (i) *the orbit* $K \cdot \mathbf{x}$ *is unstable*; *and* (ii) $\mathbf{x}$ *is an S-unipotent* (*resp. S-nilpotent*) *n-tuple.*

Proof. We give the proof for $\mathbf{x} \in G^n$. (ii) $\Rightarrow$ (i). Assume that $A_S(\mathbf{x})$ is unipotent. Then, by Proposition 12.4, there exists $\lambda \in \mathrm{Y}(K)$ such that $U(\lambda)$ contains $A_S(\mathbf{x})$. This shows that $K \cdot \mathbf{x}$ is unstable. (i) $\Rightarrow$ (ii). Assume that $\mathbf{x} = (x_1, \dots, x_n)$ is K-unstable. By HMT, there exists $\lambda \in \mathrm{Y}(K)$ such that $\lim_{t \to 0} \lambda(t) \cdot \mathbf{x} = (1, \dots, 1)$ Thus each $x_i \in U(\lambda)$. Since $U(\lambda)$ is S-stable, $U(\lambda) \supset A_S(\mathbf{x})$, which shows that $A_S(\mathbf{x})$ is a unipotent group.

13.4. *The Levi S-decomposition.*

Definition 13.4.1. (a) Let $\mathbf{x} \in G(k)^n$. Then a decomposition $\mathbf{x} = \mathbf{y} \cdot \mathbf{z}$, with $\mathbf{y}, \mathbf{z} \in G(k)^n$, is a *Levi S-decomposition of* $\mathbf{x}$ *defined over* k if the following conditions hold: (i) $A_S(\mathbf{y})$ is an S-stable Levi k-subgroup of $A_S(\mathbf{x})$; and (ii) $R_u(A_S(\mathbf{x}) \supset A_S(\mathbf{z})$. (b) Let $\mathbf{x} \in L(G)(k)^n$. Then a decomposition $\mathbf{x} = \mathbf{y} + \mathbf{z}$, with $\mathbf{y}, \mathbf{z} \in L(G)(k)^n$, is a *Levi S-decomposition of* $\mathbf{x}$ *defined over* k if the following conditions hold: (i) $A_S(\mathbf{y})$ is a Levi k-subgroup of $A_S(\mathbf{x})$; and (ii) $R_u(A_S(\mathbf{x})) \supset A_S(\mathbf{z})$.

Let $\mathbf{x} = (x_1, \dots, x_n) \in G(k)^n$ and let L be an S-stable Levi k-subgroup of $A_S(\mathbf{x})$. For each index i, we may write x_i uniquely in the form $x_i = y_i z_i$, with $y_i \in L(k)$ and $z_i \in R_u(A_S(\mathbf{x}))(k)$. An argument similar to the argument preceding the statement of Theorem 5.2 shows that $A_S(\mathbf{y}) = L$. Hence $\mathbf{x} = \mathbf{y} \cdot \mathbf{z}$ is a Levi S-decomposition of $\mathbf{x}$ defined over k. Conversely, a straightforward argument shows that every Levi S-decomposition of $\mathbf{x}$ defined over k can be obtained in this way. Thus there is a bijective correspondence between Levi S-decompositions of $\mathbf{x}$ defined over k and S-stable Levi k-subgroups of $A_S(\mathbf{x})$. One can give a similar construction for Levi S-decompositions defined over k for an n-tuple $\mathbf{x} \in L(G)(k)^n$.

THEOREM 13.5. *Let $\mathbf{x} \in G(k)^n$ (resp. $\mathbf{x} \in L(G)(k)^n$) and let $\mathbf{x} = \mathbf{y} \cdot \mathbf{z}$ (resp. $\mathbf{x} = \mathbf{y} + \mathbf{z}$) be a Levi S-decomposition of $\mathbf{x}$ defined over k. Then $K_{\mathbf{x}} = K_{\mathbf{y}} \cap K_{\mathbf{z}}$ and $K \cdot \mathbf{y}$ is the unique closed orbit in the closure of $K \cdot \mathbf{x}$. Moreover, there exists $\lambda \in \mathrm{Y}(K)_k$ such that $\lim_{t \to 0} \lambda(t) \cdot \mathbf{x} = \mathbf{y}$.*

The proof is parallel to the proof of Theorem 5.2 and follows easily from Proposition 12.4 and Theorem 13.2. We omit details.

We recall that an algebraic k-group H is *k-anisotropic* if $\mathrm{Y}(H)_k = \{0\}$, where 0 denotes the trivial one-parameter subgroup of H. The following result is an immediate consequence of HMT.

13.6. *Let H be a linearly reductive k-group which is k-anisotropic and let H act k-morphically on an affine k-variety X. If $\mathbf{x} \in X(k)$, the the orbit $H \cdot \mathbf{x}$ is closed.*

PROPOSITION 13.7. *Assume that K is k-anisotropic. Let M be an S-stable k-subgroup of G. Then M is linearly reductive.*

Proof. Choose $x_1, \dots, x_n \in L(M)(k)$ which generate $L(M)$. Let $\mathbf{x} = (x_1, \dots, x_n)$. Then clearly $\mathfrak{a}_S(\mathbf{x}) = L(M)$. By 13.6, the orbit $K \cdot \mathbf{x}$ is closed. Thus $A_S(\mathbf{x}) = M^0$ is a reductive algebraic group by Theorem 13.2. Therefore M is linearly reductive.

Proposition 13.7 generalizes a well-known result used in representation theory ([33], Cor. 1.1.5.4, p. 42).

13.8. *Fibres of the quotient morphism.* Assume now that G is reductive. Let π: $G^n \to G^n/K$ (resp. π: $L(G)^n \to L(G)^n/K$) denote the quotient morphism. For each n-tuple $\mathbf{x} \in G^n$ (resp. $\mathbf{x} \in L(G)^n$), let $\mathscr{F}_{\mathbf{x}}$ denote the fibre $\pi^{-1}(\pi(\mathbf{x}))$. We let $m(K)$ be the dimension of the flag manifold of K; thus $\dim K = 2m(K) + \operatorname{rank}(K)$. If T is a subtorus of K, we define $d(T) = m(K) + m(K^T) + \operatorname{rank}(K) - \dim T$.

The following result is proved in [28], Propositions 10.3 and 10.4.

PROPOSITION 13.8.1. *Let $\mathbf{x} \in G^n$ (resp. $\mathbf{x} \in L(G)^n$) be such that the orbit $K \cdot \mathbf{x}$ is closed and let T be a maximal torus of the stabilizer $K_{\mathbf{x}}$. Then the fibre $\mathscr{F}_{\mathbf{x}}$ is equidimensional and*

$$\dim \mathscr{F}_{\mathbf{x}} = d(T) + n\big(\dim L(G) - \dim L(G)^T\big).$$

In [28], there is also a combinatorial rule giving the number of irreducible components of the fibre $\mathscr{F}_{\mathbf{x}}$.

§14. K-stable n-tuples. In §14, S denotes a linearly reductive group, G is a reductive S-group, and $K = G^S$.

An n-tuple in G^n (resp. $L(G)^n$) is *K-stable* if it is stable for the action of K on G^n (resp. $L(G)^n$).

THEOREM 14.1. *Let* $\mathbf{x} \in G^n$ (*resp.* $\mathbf{x} \in L(G)^n$). *Then the following two conditions are equivalent*: (i) $\mathbf{x}$ *is K-stable*; *and* (*ii*) $A(\mathbf{x})$ *is not contained in any S-stable proper parabolic subgroup of* G.

Remarks. (a) If Q is an S-stable parabolic subgroup of G, then $A(\mathbf{x})$ is contained in Q if and only if $A_S(\mathbf{x})$ is contained in Q. Thus, in condition (i) above, $A(\mathbf{x})$ can be replaced by $A_S(\mathbf{x})$. (b) We see from Theorem 14.1 and Theorem 4.1 that a G-stable n-tuple is also K-stable. It is not necessarily true that a K-stable n-tuple is G-stable. (c) If G is nonabelian and of positive dimension, then there are no G-stable elements in G or $L(G)$. However, we will show that, with a few obvious exceptions, there are always K-stable elements in G and $L(G)$.

Proof of Theorem 14.1. We only give the proof for $\mathbf{x} \in G^n$. (i) $\Rightarrow$ (ii). Assume that there exists a proper S-stable parabolic subgroup P of G which contains $A(\mathbf{x})$. By Proposition 12.3 there exist $\lambda \in Y(K)$ such that $P = P_G(\lambda)$. Since $P \neq G$, $\lambda \notin Y(Z(G))$. Let $\mathbf{y} = \lim_{t\to 0}\lambda(t) \cdot \mathbf{x}$. There are two possible cases. (1) $\mathbf{y} \notin K \cdot \mathbf{x}$. In this case, $K \cdot \mathbf{x}$ is not closed and hence $\mathbf{x}$ is not K-stable. (2) $\mathbf{y} \in K \cdot \mathbf{x}$. Then image($\lambda$) is contained in $K_{\mathbf{y}}$ and thus $\mathbf{y}$ is not K-stable. Consequently $\mathbf{x}$ is not K-stable.

(ii) $\Rightarrow$ (i). Assume that $\mathbf{x}$ is not K-stable. Again we have two possible cases. (1) $K \cdot \mathbf{x}$ *is not closed*. Then by HMT there exists $\mathbf{y} \in G^n$ and $\lambda \in Y(K)$, $\lambda \notin Y(Z(G))$, such that $\lim_{t\to 0}\lambda(t) \cdot \mathbf{x} = \mathbf{y}$ and $K \cdot \mathbf{y}$ is closed. Thus $P_G(\lambda)$ contains $A(\mathbf{x})$ and $P_G(\lambda)$ is a proper S-stable parabolic subgroup of G. (2) $K \cdot \mathbf{x}$ *is closed*. By 1.3.3, $K_{\mathbf{x}}$ is linearly reductive. Since $\mathbf{x}$ is not K-stable, $\dim K_{\mathbf{x}} > \dim(K \cap Z(G))$. It follows from this that there exists $\lambda \in Y(K_{\mathbf{x}})$, $\lambda \notin Y(Z(G))$. Thus $P_G(\lambda)$ is a proper S-stable parabolic subgroup of G which contains $A(\mathbf{x})$.

14.2. *Existence of K-stable points.* We wish to give a necessary and sufficient condition for the existence of K-stable points in G^n and in $L(G)^n$. First we need a few results concerning parabolic subgroups. Let P be an S-stable parabolic subgroup of G and let $\mathscr{P}$ be the set of all G-conjugates of P. We identify $\mathscr{P}$ with the coset space G/P in the usual way. If $Q \in \mathscr{P}$, then $Q = {}^gP$ for some $g \in G$. Thus, if $s \in S$, we have $s \cdot Q = {}^{s(g)}P$. (Here we have written $s(g)$ for $s \cdot g$ to avoid confusing notation.) Hence S acts on $\mathscr{P}$. Under the identification of $\mathscr{P}$ with G/P, this corresponds to the action of S on G/P given by $s \cdot (gP) = s(g)P$.

Let P act on itself by conjugation. Let $\mathscr{E} = \mathscr{E}(P)$ denote the homogeneous fibre bundle $G \times^P P$ with base space $G/P = \mathscr{P}$ and standard fibre P. Define a morphism ϕ_0: $G \times P \to G$ by $\phi_0(g, x) = gxg^{-1}$. Then ϕ_0 determines a morphism ϕ: $\mathscr{E} \to G$. The morphism ϕ is a proper surjective morhpism and $\dim \mathscr{E} = \dim G$. Let ζ: $\mathscr{E} \to \mathscr{P}$ be the bundle map and let $\mathscr{E}' = \mathscr{E}'(P) = \zeta^{-1}(\mathscr{P}^S)$. Then

$\mathscr{E}'$ is a closed subset of $\mathscr{E}$ and $\phi(\mathscr{E}')$ is a closed subset of G. The set $\phi(\mathscr{E}')$ is the set of all $g \in G$ which are contained in an S-stable parabolic subgroup of G which is conjugate to P. Since $\dim \mathscr{E} = \dim G$, it is clear that $\phi(\mathscr{E}') = G$ if and only if $\mathscr{P}^S = \mathscr{P}$.

THEOREM 14.3. *Assume that $G \neq K$. Then the following four conditions are equivalent*: (i) *there exist K-stable points of G*; (i′) *there exist K-stable points of $L(G)$*; (ii) *if P is a proper S-stable parabolic subgroup of G, then there exists $g \in G$ such that ${}^gP$ is not S-stable*; *and* (iii) *$L(K)$ does not contain any nonabelian ideal of $L(G)$*.

Proof. We shall only prove the equivalence of (i), (ii), and (iii).

(i) $\Rightarrow$ (ii). Let $x \in G$ be a K-stable point and let P be a proper parabolic S-stable subgroup of G. Then $x \in {}^gP$ for some $g \in G$ and we see from Theorem 14.1 that ${}^gP$ is not S-stable.

(ii) $\Rightarrow$ (i). Let $P_1, \dots, P_r$ be proper S-stable parabolic subgroups of G such that every proper S-stable parabolic subgroup of G is conjugate in G to exactly one of the P_i's. (Note that is there are no proper S-stable parabolic subgroups of G, then (i) holds by Theorem 14.1.) For each $i = 1, \dots, r$, let $\mathscr{E}_i = \mathscr{E}(P_i)$, $\mathscr{E}_i' = \mathscr{E}'(P_i)$, and ϕ_i: $\mathscr{E}_i \to G$ be defined as in 14.2. If (ii) holds, then each $\phi_i(\mathscr{E}_i')$ is a proper closed subset of G. Thus $D = \{g \in G | g \notin \phi_i(\mathscr{E}_i') \ (i = 1, \dots, r)\}$ is a nonempty open subset of G. It follows from Theorem 14.1 that D is precisely the set of K-stable points of G.

It suffices to prove the equivalence of (ii) and (iii) in the case when G is semisimple.

(ii) $\Rightarrow$ (iii). Assume that $L(K)$ contains a nontrivial ideal $\mathfrak{h}$ of $L(G)$. Then $L(G)$ is the direct sum of ideals $\mathfrak{h}$ and $\mathfrak{m}$, and $\mathfrak{h}$ and $\mathfrak{m}$ are semisimple Lie algebras. There exist closed, normal semisimple subgroups H and M of G with $L(H) = \mathfrak{h}$ and $L(M) = \mathfrak{m}$. It is clear that K contains H and that the product map $H \times M \to G$, $(h, m) \to hm$, is an isogeny. Furthermore, the ideal $\mathfrak{m}$ is S-stable and hence M is S-stable. Let P_1 be a proper parabolic subgroup of H. Then $P = P_1M$ is an S-stable proper parabolic subgroup of G. If $g \in G$ and $s \in S$, then

$$s \cdot ({}^gP) = s \cdot \left({}^gP_1{}^gM\right) = s \cdot ({}^gP_1M) = ({}^gP_1)M = {}^gP.$$

Thus $(G/P)^S = G/P$ and (ii) does not hold.

(iii) $\Rightarrow$ (ii). Assume that there exists a proper S-stable parabolic subgroup of G such that $(G/P)^S = G/P$. Let $\mathscr{P}$ denote the set of all G-conjugates of P. For each $s \in S$, define a morphism η_s: $G \to G$ by $\eta_s(g) = g^{-1}s \cdot g$. If $Q \in \mathscr{P}$, $s \in S$, and $g \in G$, then ${}^{s(g)}Q = {}^gQ$, since Q and ${}^gQ$ are S-stable. Therefore $\eta_s(g) \in N_G(Q) = Q$. Set $M = \bigcap_{Q \in \mathscr{P}} Q$. Then $\eta_s(G) \subset M$ and hence $\eta_s(G) \subset M^0$, since $\eta_s(G)$ is irreducible. The group M^0 is a closed, connected normal subgroup of G. Since $K \neq G$, there exists $s \in S$ such that $\eta_s(G) \neq \{1\}$. Thus M^0 is a proper, nontrivial, normal S-stable subgroup of G. Therefore, there exists a nontrivial S-stable ideal $\mathfrak{h}$ of $L(G)$ such that $L(G) = L(M) \oplus \mathfrak{h}$. The Lie algebra $\mathfrak{h}$ is a semisimple Lie algebra. Let H be the unique closed connected normal subgroup of G such that $L(H) = \mathfrak{h}$. Then H is an S-stable normal subgroup of G. For $s \in S$, we have $\eta_s(H) \subset \eta_s(G) \subset M^0$ and, since H is S-stable, $\eta_s(H) \subset H$. Thus $\eta_s(H)$ is contained in $(H \cap M)^0 = \{1\}$. But this implies that $H \subset K$, hence that $L(H) \subset L(K)$.

§15. Real semisimple S-groups. In this section $F = \mathbf{C}$ and we use the conventions of §11. In particular, complex linear algebraic groups will be denoted by boldface letters and we distinguish between the classical topology and the Zariski topology

We let G be a semisimple real Lie group with finite center and let S be an (abstract) subgroup of Aut(G), the group of all Lie group automorphisms of G. Then S acts on $L(G)$ by Lie algebra automorphisms. In §15, we will always assume that $L(G)$ is a semisimple S-module. Let $K = G^S$. We consider the action of K on G^n and $L(G)^n$. For $\mathbf{x} = (x_1, \ldots, x_n) \in G^n$, we let $\Gamma_S(x)$ denote the (abstract) subgroup of G generated by $\{s \cdot x_i | s \in S,\ i = 1, \ldots, n\}$. If $\mathbf{x} \in L(G)^n$, then $\mathfrak{c}_S(\mathbf{x})$ denotes the Lie subalgebra of $L(G)$ generated by $\{s \cdot x_i | s \in S,\ i = 1, \ldots, n\}$.

THEOREM 15.1. *Let the notation be as above Let* $\mathbf{x} \in G^n$ *(resp.* $\mathbf{x} \in L(G)^n$*). Then the following two conditions are equivalent*: (i) $K \cdot \mathbf{x}$ *is closed*; *and* (ii) $L(G)$ *is a semisimple* $\Gamma_S(\mathbf{x})$*-module (resp. a semisimple* $\mathfrak{c}_S(\mathbf{x})$*-module).*

The proof of Theorem 15.1 will occupy most of the rest of §15. First we reduce to the case of real algebraic groups. Let G_1 denote $\mathrm{Aut}(L(G))^0$, the identity component of the real Lie group Aut($L(G)$) of all Lie algebra automorphisms of $L(G)$. It is known that the adjoint representation Ad_G: $G \to \mathrm{Aut}(L(G))$ maps G onto G_1 and has kernel $Z(G)$. By assumption, $Z(G)$ is finite. Let $y_i = \mathrm{Ad}_G(x_i)$, $i = 1, \ldots, n$, and let $\mathbf{y} = (y_1, \ldots, y_n)$. The group S acts on Aut($L(G)$) by inner automorphisms and hence acts on G_1. Let $K_1 = G_1^S$. It is clear that $K_1^0 \subset \mathrm{Ad}_G(K) \subset K_1$. By [34], $\mathrm{Aut}(L(G))^S$ has only a finite number of components. Thus $K_1/\mathrm{Ad}_G(K)$ is finite.

LEMMA 15.2. *The orbit* $K \cdot \mathbf{x}$ *is closed in* G^n *if and only if* $K_1 \cdot \mathbf{y}$ *is closed in* G_1^n.

We omit the proof, which is a straightforward exercise.

It is clear that $\mathrm{Ad}_G(\Gamma_S(\mathbf{x}))$ is generated by $\{s \cdot y_i | s \in S,\ i = 1, \ldots, n\}$. Thus Lemma 15.2 reduces the proof of Theorem 15.1 to the case in which $G = G_1$. From now on, we assume that $G = G_1$.

Let $\mathbf{G}$ denote the adjoint group of the complex semisimple Lie algebra $L(G) \otimes_{\mathbf{R}} \mathbf{C}$. Then $\mathbf{G}$ is a semisimple complex algebraic group and $\mathbf{L}(\mathbf{G})$ can be identified with $L(G) \otimes_{\mathbf{R}} \mathbf{C}$. Thus we have $G = \mathbf{G}(\mathbf{R})^0$ and $L(G) = \mathbf{L}(\mathbf{G})(\mathbf{R})$. Each element of S gives an automorphism of $\mathbf{L}(\mathbf{G})$. Hence we may consider S as a subgroup of $\mathrm{Aut}(\mathbf{L}(\mathbf{G}))$ and S acts on $\mathbf{G} = \mathrm{Aut}(\mathbf{L}(\mathbf{G}))^0$ by conjugation. Let $\mathbf{M}$ denote the Zariski closure of S in $\mathrm{Aut}(\mathbf{L}(\mathbf{G}))$. Then $\mathbf{M}$ is an $\mathbf{R}$-subgroup of

$\mathrm{Aut}(\mathbf{L}(\mathbf{G}))$ and S is contained in $\mathbf{M}(\mathbf{R})$. Clearly $\mathbf{G}^S = \mathbf{G}^{\mathbf{M}}$. Since $L(G) = \mathbf{L}(\mathbf{G})(\mathbf{R})$ is a semisimple S-module, it follows that $\mathbf{L}(\mathbf{G})$ is a semisimple $\mathbf{M}$-module and hence $\mathbf{M}$ is a linearly reductive algebraic group. Let $\mathbf{K} = \mathbf{G}^{\mathbf{M}} = \mathbf{G}^S$. Then $\mathbf{K}$ is a linearly reductive group and it is clear from the definitions that $K = G^S$ satisfies $\mathbf{K}(\mathbf{R})^0 \subset K \subset \mathbf{K}(\mathbf{R})$.

Consider the following two conditions on an n-tuple $\mathbf{x} \in G^n$ (resp. $\mathbf{x} \in L(G)^n$): (iii) $\mathbf{K} \cdot \mathbf{x}$ is Zariski-closed in $\mathbf{G}^n$ (resp. in $\mathbf{L}(\mathbf{G})^n$); and (iv) $\mathbf{L}(\mathbf{G})$ is a semisimple $\mathbf{A}_{\mathbf{M}}(\mathbf{x})$-module. Then it follows from 11.3 that condition (i) of Theorem 15.1 and condition (iii) are equivalent. It follows from Theorem 13.2 that (iii) and (iv) are equivalent. Since $\Gamma_S(\mathbf{x})$ is Zariski-dense in $\mathbf{A}_{\mathbf{M}}(\mathbf{x})$, we see that (iv) is equivalent to (ii) of Theorem 15.1 for $\mathbf{x} \in G^n$. For the case $\mathbf{x} \in L(G)^n$, an easy argument shows that $\mathfrak{a}_{\mathbf{M}}(\mathbf{x})$ is the algebraic hull of $\mathfrak{c}_S(\mathbf{x})$ in $\mathbf{L}(\mathbf{G})$. Thus (iv) and (ii) are equivalent for $\mathbf{x} \in L(G)^n$. This proves Theorem 15.1.

THEOREM 15.3. *Let* $\mathbf{x} \in L(G)^n$. *Then the closure of* $K \cdot \mathbf{x}$ *contains* $\mathbf{0} = (0, \ldots, 0)$ *if and only if* $\mathrm{ad}_{L(G)}(y)$ *is nilpotent for every* $y \in \mathfrak{c}_S(\mathbf{x})$.

We omit the proof, which follows in a straightforward manner from Theorem 13.1 and HMT.

Example. Let G be as above and let θ be an involutive automorphism of G. Let K be the fixed point subgroup of θ. For $x \in G$, let $\Gamma_\theta(x)$ be the subgroup of G generated by $\{x, \theta(x)\}$. Then it follows easily from Theorem 15.1 that the following three conditions are equivalent: (i) $K \cdot x$ is closed in G; (ii) $G \cdot (x, \theta(x))$ is closed in G^2; and (iii) $L(G)$ is a semisimple $\Gamma_\theta(x)$-module.

§16. The case of characteristic p. In this section, the base field F is of arbitrary characteristic and G denotes a reductive group.

The proof of most of our results in earlier sections depended on the characteristic zero hypothesis and, in many cases, the corresponding results in positive characteristic p are not true. In this section, we will give a characterization of those n-tuples $\mathbf{x} \in G^n$ such that the orbit $G \cdot \mathbf{x}$ is closed which holds for arbitrary characteristic.

Definition 16.1. Let H be a closed subgroup of G and let S be a maximal torus of G^H. Then H is *strongly reductive* in G if H is not contained in any proper parabolic subgroup of G^S.

It is clear that the definition does not depend on the choice of the maximal torus S of G^H. In characteristic zero, it is not difficult to show that H is strongly reductive in G if and only if H is linearly reductive. However, in characteristic p the situation is more complicated.

LEMMA 16.2. *Let H be a closed subgroup of $GL(E)$. Then H is strongly reductive in $GL(E)$ if and only if E is a semisimple H-module.*

Proof. Let S be a maximal torus of $GL(E)^H$ and let Ψ be the set of weights of S on E. The condition that H is not contained in any proper parabolic subgroup of $GL(E)^S$ is equivalent to the condition that each weight space E_α, ($\alpha \in \Psi$), is a simple H-module. This proves the lemma.

LEMMA 16.3. *Let H be a closed subgroup of G which is strongly reductive in G. Then H^0 is reductive.*

Proof. Let S be a maximal torus of G^H. Then G^S is a reductive group. Assume that $R_u(H) \neq \{1\}$. Then by a result of Borel–Tits [5] there exists a (proper) parabolic subgroup P of G^S such that $H \subset P$ and $R_u(H) \subset R_u(P)$. This contradicts the hypothesis that H is strongly reductive in G.

If $\mathbf{x} \in G^n$, then $A(\mathbf{x})$ is defined as in §3. The following theorem characterizes closed orbits in G^n.

THEOREM 16.4. *Let $\mathbf{x} \in G^n$. Then the orbit $G \cdot \mathbf{x}$ is closed if and only if $A(\mathbf{x})$ is strongly reductive in G.*

Before giving the proof of Theorem 16.4, we need some preliminary results.

16.5. *Let X be an affine G-variety and let S be a linearly reductive subgroup of G. Let $x \in X^S$. Then $G \cdot x$ is closed in X if and only if $G^S \cdot x$ is closed in X^S.*

See [26], Theorem C.

The result below gives a nice characterization of closed orbits in an affine G-variety.

LEMMA 16.6. *Let X be an affine G-variety, let $x \in X$, and let S be a maximal torus of the stabilizer G_x. Then the following two conditions are equivalent:* (i) $G \cdot x$ *is closed; and (ii) x is a stable point for the action of G^S on X^S.*

Proof. (ii) → (i). If x is stable for (X^S, G^S), then $G^S \cdot x$ is closed in X^S. It follows from 16.5 that $G \cdot x$ is closed. (i) ⇒ (ii). If $G \cdot x$ is closed, then $G^S \cdot x$ is closed. Hence, by [25], the identity component of the stabilizer $H = (G^S)_x$ is reductive. Since S is central in H and is a maximal torus of H^0, we see that $H^0 = S$. Therefore x is stable for the action of G^S on X^S.

We note that the results of 2.1, 2.2, and 2.3 hold for a reductive k-group G if k is perfect.

Proposition 16.7 below shows that the characterization of stable points of G^n given in Theorem 4.1 carries over to characteristic p.

PROPOSITION 16.7. *Let $\mathbf{x} \in G^n$. Then $\mathbf{x}$ is a stable point of G^n if and only if $A(\mathbf{x})$ is not contained in any proper parabolic subgroup of G.*

Proof. Assume that $A(\mathbf{x})$ is not contained in any proper parabolic subgroup of G. Then it is an easy consequence of HMT that the orbit $G \cdot \mathbf{x}$ is closed. Hence, by [25], $G_{\mathbf{x}}^0$ is a reductive group. Let S be a maximal torus of $G_{\mathbf{x}}$. If $S \neq Z(G)^0$, then there exists $\lambda \in \mathbf{Y}(S)$, $\lambda \notin \mathbf{Y}(Z(G))$. Clearly $A(\mathbf{x}) \subset G^\lambda \subset$

$P_G(\lambda)$, which gives a contradiction. Thus $S = Z(G)^0$. Since $G_{\mathbf{x}}^0$ is reductive, this implies that $G_{\mathbf{x}}^0 = Z(G)^0$, and thus $\mathbf{x}$ is a stable point.

For the converse, assume that $A(\mathbf{x}) \subset P(\lambda)$, where λ is a noncentral one-parameter subgroup of G. Let $\mathbf{y} = \lambda(0) \cdot \mathbf{x}$. Then $\lambda \in \mathrm{Y}(G_{\mathbf{y}})$, and thus $\dim G_{\mathbf{y}} > \dim Z(G)$, so that $\mathbf{y}$ is not a stable point. If $\mathbf{y} \in G \cdot \mathbf{x}$, then $\mathbf{x}$ is not stable. On the other hand, if $\mathbf{y} \notin G \cdot \mathbf{x}$, then the orbit $G \cdot \mathbf{x}$ is not closed and hence $\mathbf{x}$ is not a stable point.

We can now prove Theorem 16.4. Let S be a maximal torus of $G_{\mathbf{x}}$. Assume that $A(\mathbf{x})$ is strongly reductive in G. Then by Proposition 16.7, as applied to G^S, $\mathbf{x}$ is a stable point for the action of G^S on $(G^S)^n$. In particular, the orbit $G^S \cdot \mathbf{x}$ is closed and hence, by 16.5, the orbit $G \cdot \mathbf{x}$ is closed. Assume now that $G \cdot \mathbf{x}$ is closed. By Lemma 16.6 and Proposition 16.7, $A(\mathbf{x})$ is not contained in any proper parabolic subgroup of G^S. Thus $A(\mathbf{x})$ is strongly reductive in G. This completes the proof.

In characteristic p, the condition that a closed subgroup H of G be strongly reductive is hard to pin down concretely. If H is a linearly reductive group, then it follows from [26], Proposition 6.1, that H is strongly reductive in G for *any* reductive group G which contains H as a closed subgroup. However, if H is reductive but not a torus, then the property that H be strongly reductive in G depends not only on H but on the embedding of H in G. For example, let H be a reductive group which is not a torus. Then, by [19], there exists a faithful linear representation η: $H \to GL(E)$ which is not semisimple. It follows from Lemma 16.2 that $\eta(H)$ is not strongly reductive in $GL(E)$.

In order to get a few more examples of strongly reductive subgroups, we will use the following result ([26], Theorem C):

16.8. *Let G be a closed connected normal subgroup of the algebraic group H and assume that G is a reductive group. Let S be a linearly reductive subgroup of H and let $K = G^S$. Let H act morphically on the affine variety X and let $x \in X^S$. Let $\mathcal{O}$ denote the unique closed G-orbit in the closure of the orbit $G \cdot x$. Then there exists $\lambda \in \mathrm{Y}(K)$ and $y \in \mathcal{O}$ such that* $\lim_{t \to 0} \lambda(t) \cdot x = y$. *In particular, $G \cdot x$ is closed if and only if $K \cdot x$ is closed.*

As an application of 16.8, we prove the following result:

PROPOSITION 16.9. *Let S be a linearly reductive group, let G be a reductive S-group, and let $K = G^S$. Let H be a closed subgroup of K^0. Then H is a strongly reductive subgroup of K^0 if and only if it is a strongly reductive subgroup of G.*

Proof. We may assume that the base field F is of prime characteristic p. It follows from [26], Proposition 10.1.5, that K^0 is a reductive group. It is clear that the property of being a strongly reductive subgroup is preserved if the base field is extended to an algebraically closed field which is transcendental over the prime field $\mathbf{F}_p$. Thus we may assume that F is transcendental over $\mathbf{F}_p$. By Lemma 16.3, we may assume that H^0 is reductive. Under the assumptions on F, a straightforward argument shows that there exists an integer $n > 0$ and $\mathbf{x} \in K^n$ such that

$A(\mathbf{x}) = H$. (The essential point is that every one-dimensional group over F is generated, as an algebraic group, by one element.) It follows from 16.8 that $K^0 \cdot \mathbf{x}$ is closed in $(K^0)^n$ if and only if $G \cdot \mathbf{x}$ is closed in G^n. It follows from Theorem 16.4 that $K^0 \cdot \mathbf{x}$ (resp. $G \cdot \mathbf{x}$) is closed in $(K^0)^n$ (resp. G^n) if and only if $H = A(\mathbf{x})$ is strongly reductive in K^0 (resp. G^n). This proves Proposition 16.9.

COROLLARY 16.10. (char$(F) \neq 2$). *Let V be a finite-dimensional vector space with a nondegenerate symmetric (resp. alternating) bilinear form and let $O(V)$ (resp.* $\mathrm{Sp}(V)$*) be the corresponding orthogonal (resp. symplectic) group. Let H be a closed subgroup of* $\mathrm{SO}(V)$ *(resp.* $\mathrm{Sp}(V)$*). Then H is a strongly reductive subgroup of* $\mathrm{SO}(V)$ *(resp.* $\mathrm{Sp}(V)$*) if and only if V is a semisimple H-module.*

REFERENCES

1. W. BAILEY, *The decomposition theorem for V-manifolds*, Amer. J. Math. **78** (1956), 862–888.
2. D. BIRKES, *Orbits of linear algebraic groups*, Ann. of Math. (2) **93** (1971), 459–479.
3. A. BOREL, *Linear Algebraic Groups*, W. A. Benjamin, New York, 1969.
4. A. BOREL AND J. TITS, *Groupes réductifs*, Inst. Hautes Etudes Sci. Publ. Math. **27** (1965), 55–150.
5. ______, *Eléments unipotents et sous-groupes paraboliques de groupes réductifs*, Invent. Math. **12** (1971), 95–104.
6. N. BOURBAKI, *Groupes et Algèbres de Lie*, chaps. 4–6, Act. Sci. Ind. **1337**, Hermann, Paris, 1968.
7. ______, *Groupes et Algèbres de Lie*, chaps. 7–8, Act. Sci. Ind. **1364**, Hermann, Paris, 1974.
8. E. DYNKIN, *Semisimple subalgebras of semisimple Lie algebras*, Amer. Math. Soc. Russian Translations **6** (1957), 111–244.
9. S. HELGASON, *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, New York–San Francisco–London, 1978.
10. G. HOCHSCHILD, *Basic Theory of Algebraic Groups and Lie Algebras*, Springer-Verlag, Berlin–Heidelberg–New York, 1981.
11. G. KEMPF, *Instability in invariant theory*, Ann. of Math. (2) **108** (1978), 299–316.
12. B. KOSTANT, *Lie group representations on polynomial rings*, Amer. J. Math. **85** (1963), 327–404.
13. J. KOSZUL, *Lectures on Groups of Transformations*, Tata Institute of Fundamental Research, Bombay, 1965.
14. H. KRAFT, *Geometrische Methoden in der Invariantentheorie*, Vieweg, Braunschweig, 1985.
15. D. LUNA, *Slices étales*, Bull. Soc. Math. France, Mémoire **33** (1973), 81–105.
16. ______, *Sur certaines opérations différentiables de groupes de Lie*, Amer. J. Math. **97** (1975), 172–181.
17. ______, *Adhérences d'orbite et invariants*, Invent. Math. **29** (1975), 231–238.
18. D. MUMFORD AND J. FOGARTY, *Geometric Invariant Theory* (2nd ed.), Springer-Verlag, Berlin–Heidelberg–New York, 1982.
19. M. NAGATA, *Complete reducibility of representations of a matric group*, J. Math. Kyoto Univ. **1** (1961–1962), 87–99.
20. P. NEWSTEAD, *Introduction to Moduli Problems and Orbit Spaces*, Tata Institute Lecture Notes, Springer-Verlag, Berlin–Heidelberg–New York, 1978.
21. R. PALAIS, *On the existence of slices for actions of non-compact Lie groups*, Ann. of Math. (2) **73** (1961), 295–323.
22. C. PROCESI, *The invariant theory of $n \times n$ matrices*, Adv. in Math. **19** (1976), 306–381.
23. R. RICHARDSON, *A rigidity theorem for subalgebras of Lie and associative algebras*, Illinois J. Math. **11** (1967), 92–110.
24. ______, *Deformations of subalgebras of Lie algebras*, J. of Differential Geom. **3** (1969), 289–308.
25. ______, *Affine coset spaces of reductive algebraic groups*, Bull. London Math. Soc. **9** (1977), 38–41.

26. ______, *On orbits of algebraic groups and Lie groups*, Bull. Australian Math. Soc. **25** (1982), 1–29.
27. ______, *Simultaneous conjugacy of n-tuples in Lie algebras and algebraic groups*, Canad. Math. Soc. Conference Proceedings **6** (1986), 377–387.
28. ______, *Irreducible components of the nullcone*, to appear.
29. G. SCHWARZ, *Representations of a simple Lie group with a regular ring of invariants*, Invent. Math. **49** (1978), 167–191.
30. R. STEINBERG, *Regular elements of semisimple algebraic groups*, Inst. Hautes Etudes Sci. Publ. Math. **25** (1965), 49–80.
31. J. TITS, *Free subgroups in linear groups*, J. Algebra **20** (1972), 250–270.
32. T. VUST, *Opération de groupes réductifs dans un type de cones presque homogènes*, Bull. Math. Soc. France **102** (1974), 317–334.
33. G. WARNER, *Harmonic analysis on semisimple Lie groups*, vol. 1, Springer-Verlag, Berlin–Heidelberg–New York, 1972.
34. H. WHITNEY, *Elementary structure of real algebraic varieties*, Ann. of Math. (2) **66** (1957), 545–556.

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