CONJUGATIONS ON COMPLEX MANIFOLDS AND EQUIVARIANT HOMOTOPY OF MU[1]

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1. **Introduction.** Let $\rho: \Omega_*^U \to \mathfrak{N}_*$ denote the natural homomorphism from the stably complex bordism ring into the unoriented bordism ring. Milnor showed in [8] that the image of ρ consists of all squares $([M]_2)^2$ in $\mathfrak{N}_*$. Since $\mathfrak{N}_*$ is a polynomial algebra over Z_2, an epimorphism $R: \Omega_{2n}^U \to \mathfrak{N}_n$ is defined by the condition that $R^2 = \rho$. Milnor made use of the following result of Conner and Floyd [3, p. 64]: if τ is a conjugation on a closed almost complex $2n$-manifold M, then the fixed point set $F(M)$ is an n-manifold and $[M]_2 = ([F(M)]_2)^2$ in $\mathfrak{N}_{2n}$, i.e. $R([M]) = [F(M)]_2$. Hence, if a conjugation is present we may regard R as "passage to the fixed point set." We shall develop a bordism theory in which such a "fixed point homomorphism" is a natural feature.

From the homotopy point of view, Ω_*^U coincides with the (stable) homotopy $\pi_*(MU)$ of the Milnor spectrum MU [7]. In fact, the Thom spaces $MU(n)$ carry involutions making it possible to define equivariant homotopy groups $\Omega_{p,q}^U = \pi_{p,q}(MU)$. The details follow.

Give C^m the involution $(z_1, \cdots, z_m) \mapsto (\bar{z}_1, \cdots, \bar{z}_m)$. Then the Grassmannian $G_n(C^m)$ of n-planes in C^m inherits an involution, as does the classifying space $BU(n) = G_n(C^\infty)$. Moreover, the universal complex n-plane bundle $E^n \to BU(n)$ inherits an involution which makes E^n a real vector bundle over the real space $BU(n)$ in the sense of Atiyah [1]. Thus $MU(n) = B(E^n)/S(E^n)$ is endowed with an involution fixing the base point. Notice that the corresponding fixed point sets are R^m, $G_n(R^m)$, $BO(n)$ and $MO(n)$.

Following Atiyah [1] let $B^{p,q}$ and $S^{p,q}$ denote the unit ball and unit sphere in a Euclidean space $R^{p,q}$ of dimension $p+q$ carrying an orthogonal involution with fixed point set R^q. If X is a space with involution and fixed base point $*$, let $\pi_{p,q}(X)$ denote the set of equivariant homotopy classes of maps $(B^{p,q}, S^{p,q}) \to (X, *)$. For $q \geqq 2$, $\pi_{p,q}(X)$ is an abelian group.

There are equivariant suspension maps $i_n: MU(n) \wedge (B^{1,1}/S^{1,1}) \to MU(n+1)$, and so homomorphisms

$$\pi_{p+k,q+k}(MU(k)) \to \pi_{p+k+1,q+k+1}(MU(k+1)).$$

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Hence we may define

$$(1.1)\qquad \overset{U}{\Omega}_{p,q} = \pi_{p,q}(MU) = \lim_{k\to\infty} \pi_{p+k,q+k}(MU(k))$$

for integers p, q. There is a forgetful homomorphism ψ, and a fixed point homomorphism ϕ obtained by restriction to the fixed point sets:

$$\overset{U}{\Omega}_{p+q} \overset{\psi}{\leftarrow} \overset{U}{\Omega}_{p,q} \overset{\phi}{\rightarrow} \mathfrak{N}_q.$$

We shall state a number of results about the groups $\Omega^U_{p,q}$ and the homomorphisms ψ and ϕ. The results of [5], on fixed point free conjugations and the existence of equivariant maps are a by-product of this study. A similar investigation of equivariant stable stems has been made by Bredon [2].

2. **The exact sequence.** The inclusions $R^{p+k,q+k}\rightarrow R^{p+k+1,q+k}$ give rise to a homomorphism χ so that the diagram

$$\begin{array}{ccc} \overset{U}{\Omega}_{p+1,q} & \overset{\chi}{\rightarrow} & \overset{U}{\Omega}_{p\,q} \\ \phi\searrow & & \swarrow\phi \\ & \mathfrak{N}_q & \end{array}$$

is commutative. The image of χ consists of elements of order 2. As in [6] there is an exact sequence

$$(2.1)\qquad \cdots \rightarrow \overset{U}{\Omega}_{p+1,q} \overset{\chi}{\rightarrow} \overset{U}{\Omega}_{p,q} \overset{\psi}{\rightarrow} \overset{U}{\Omega}_{p+q} \overset{\omega}{\rightarrow} \overset{U}{\Omega}_{p+1,q-1} \rightarrow \cdots .$$

It follows from the exact sequence of [6] that ϕ: $\Omega^U_{p,q}\rightarrow\mathfrak{N}_q$ is an isomorphism for $p+q<0$; this gives a basis for induction on $p+q$.

THEOREM 2.2. *$\Omega^U_{p,q}$ is a finitely generated abelian group in which all torsion is of order 2. The torsion subgroup is the kernel of ψ: $\Omega^U_{p,q}\rightarrow\Omega^U_{p+q}$.*

3. **Transversality.** Given an equivariant map f from $(B^{p+k,q+k}, S^{p+k,q+k})$ into $(MU(k), *)$, is f equivariantly homotopic to a map g which is transversal to $BU(k)\subset MU(k)$? (As is customary, we approximate $BU(k)$ and $MU(k)-\{*\}$ by smooth manifolds.) That this is not generally true follows from the fact that ϕ: $\Omega^U_{p,q}\rightarrow\mathfrak{N}_q$ is an isomorphism for $p+q<0$.

THEOREM 3.1. *If $p\geqq q$, each element of $\Omega^U_{p,q}$ is represented by a map f: $(B^{p+k,q+k}, S^{p+k,q+k})\rightarrow(MU(k), *)$ which is transversal to $BU(k)\subset MU(k)$.*

This follows by examination of a more general situation, in the category of smooth manifolds with involution and smooth equivariant maps. Let $f\colon M\to W$ be given, and let V be a closed invariant submanifold of W. We assume that each fixed point set $F(M)$, $F(V)$, $F(W)$ is of uniform dimension. Put $m=\dim M$, $m'=\dim F(M)$, etc.

LEMMA 3.2. *If $(m-2m')+(v-2v')\geqq(w-2w')$, f is equivariantly homotopic to a map g which is transversal to V.*

COROLLARY 3.3. *The diagram*

$$\begin{array}{ccc} \Omega^U_{n,n} & \xrightarrow{\psi} & \Omega^U_{2n} \\ & \phi\searrow \quad \swarrow R & \\ & \mathfrak{N}_n & \end{array}$$

is commutative.

COROLLARY 3.4. *The homomorphism $\phi\colon \Omega^U_{p,q}\to\mathfrak{N}_q$ is onto if $p\leqq q$ and is zero if $p>q$.*

The sequence

$$0\to\Omega^U_{n+1,n}\xrightarrow{\chi}\Omega^U_{n,n}\xrightarrow{\psi}\Omega^U_{2n}\to 0 \tag{3.5}$$

is exact. I conjecture that $\Omega^U_{n+1,n}=0$ for all n, and have verified this for $n\leqq 4$.

4. The spectral sequence. We do not have a complete description of the groups $\Omega^U_{p,q}$. In particular, the extent of the torsion and the image of ψ are not known in general. The difficulties are measured by the spectral sequence of the bigraded exact couple (2.1), which we now write as

$$\cdots\to\Omega^U_{p+1,q}\xrightarrow{\chi}\Omega^U_{p,q}\xrightarrow{\psi}E^1_{p,q}\xrightarrow{\omega}\Omega^U_{p+1,q-1}\to\cdots$$

where $E^1_{p,q}=\Omega^U_{p+q}$. The differential $d^r\colon E^r_{p-r,q+1}\to E^r_{p,q}$ of the spectral sequence $\{E^r_{p,q}\}$ $(r>0)$ arises from the diagram

$$\begin{array}{ccc} & & \Omega^U_{p,q}\xrightarrow{\psi}E^1_{p,q} \\ & & \downarrow\chi^{r-1} \\ E^1_{p-r,q+1} & \xrightarrow{\omega} & \Omega^U_{p-r+1,q}. \end{array}$$

We are able to determine d^1 and d^3 $(d^2=0)$, and so reach the following conclusions.

THEOREM 4.1. (a) *If* $p \not\equiv q$ mod 4, $\Omega^U_{p,q}$ *is finite*; (b) *if* $p-q \equiv 4$ mod 8, $\psi: \Omega^U_{p,q} \to \Omega^U_{p+q}$ *has image* $2\Omega^U_{p+q}$; (c) *if* $p \equiv q$ mod 8, *the image of* ψ *contains* $2\Omega^U_{p+q} + [CP(1)]\Omega^U_{p+q-2}$.

COROLLARY 4.2. *If* $p \equiv q$ mod 4, $\Omega^U_{p,q}$ *has the same rank as* Ω^U_{p+q}.

The differential d^1: $E^1_{p-1,q+1} \to E^1_{p,q}$ is zero if $p \not\equiv q$ mod 4, and is multiplication by 2 otherwise. This is proved with K-theory and KR-theory characteristic numbers [**4**], [**9**]; notice that the composition $\tilde{K}(S^n)^r \to [KO]^{\sim}(S^n) \xrightarrow{c} \tilde{K}(S^n)$ is zero if $n \not\equiv 0$ mod 4, and is multiplication by 2 otherwise. Thus $E^2_{p,q} \cong \Omega^U_{p+q} \otimes Z_2$ if $p \equiv q$ mod 4, otherwise $E^2_{p,q} = 0$. Moreover, $E^3 = E^2$. With the help of characteristic numbers, we show that d^3: $E^3_{p-3,q+1} \to E^3_{p,q}$ is multiplication by $[CP(1)]$ if $p \equiv q$ mod 8, otherwise $d^3 = 0$. Then $E^7 \cong \cdots \cong E^4$; I conjecture that d^7: $E^7_{p-7,q+1} \to E^7_{p,q}$ is multiplication by the class of the quadric Q^6 if $p \equiv q$ mod 16, otherwise d^7 is zero.

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