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Connected domination game played on Cartesian products

Abstract: The connected domination game on a graph G is played by Dominator and Staller according to the rules of the standard domination game with the additional requirement that at each stage of the game the selected vertices induce a connected subgraph of G. If Dominator starts the game and both players play optimally, then the number of vertices selected during the game is the connected game domination number of G. Here this invariant is studied on Cartesian product graphs. A general upper bound is proved and demonstrated to be sharp on Cartesian products of stars with paths or cycles. The connected game domination number is determined for Cartesian products of P_3 with arbitrary paths or cycles, as well as for Cartesian products of an arbitrary graph with K_k for the cases when k is relatively large. A monotonicity theorem is proved for products with one complete factor. A sharp general lower bound on the connected game domination number of Cartesian products is also established.

Keywords: domination game; connected domination game; Cartesian product of graphs; paths and cycles

MSC: 05C57, 05C69, 91A43

1 Introduction

Connected domination game on a graph G is played by two players, usually named Dominator and Staller. They play in turns, at each move selecting a single vertex of G such that it dominates at least one vertex that is not yet dominated with the previously played vertices and such that at each stage of the game the selected vertices induce a connected subgraph of G. If Dominator has the first move, then we speak of a *D-game*, otherwise they play an *S-game*. When the game is finished, that is, when there is no legal move available, the players have determined a connected dominating set D of G. The goal of Dominator is to finish with $|D|$ being as small as possible, the goal of Staller is just the opposite. If both players play optimally, then $|D|$ is unique. In the D-game it is called the *connected game domination number* $\gamma_{\mathrm{cg}}(G)$ of G, while when S-game is played, the corresponding invariant is denoted by $\gamma'_{\mathrm{cg}}(G)$. The connected domination game is thus defined as the standard domination game [1] with the additional requirement that the players maintain connectedness of the selected vertices at all times. Other games closely related to the domination game are the total domination game [2–5], the disjoint domination game [6], the transversal game on hypergraphs [7, 8], Maker-Breaker domination

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game [9, 10], Maker-Breaker total domination game [11], as well as the whole variety of domination games as described in [12].

The connected domination game was introduced by Borowiecki, Fiedorowicz, and Sidorowicz in [13]. Among other results they proved a sharp upper bound on γ_{cg} of 2-trees, studied the game on the Cartesian product, and considered the S-game and Staller-pass games. The game does not admit the so-called Continuation Principle [14], which is an utmost useful tool for the standard game, cf. [15–19]. Instead, [13] brings the so-called game with Chooser (to be described below) which plays a similar role for the connected game. The game was further investigated in [20] where a question from [13] on a relation between the D-game and the S-game is answered, the relation of the game with the diameter investigated, the game on the lexicographic product completely explained, and the effect of a vertex predomination studied.

In this paper we focus on the connected domination game played on Cartesian products. In Section 2 we give a general upper bound on it that in particular extends the upper bound on $\gamma_{\mathrm{cg}}(G \square K_m)$ from [13]. The sharpness of the bound is demonstrated by several earlier exact results. In the subsequent section we determine the connected game domination number of the Cartesian product of stars with paths or with cycles. This yields additional families for which the upper bound from the earlier section is sharp. We continue by determining $\gamma_{\mathrm{cg}}(P_3 \square P_n)$ and $\gamma_{\mathrm{cg}}(P_3 \square C_n)$ in Section 4. In the subsequent section we consider products in which one factor is complete. We first prove that $\gamma_{\mathrm{cg}}(G \square K_k) = 2n(G) - 1$, provided that k is not too small. Then we obtain a monotonicity theorem asserting that $\gamma_{\mathrm{cg}}(K_{n+1} \square G) \geq \gamma_{\mathrm{cg}}(K_n \square G)$ holds for $n \geq 1$. In Section 6 we prove a general lower bound on the connected game domination number of Cartesian products and establish its sharpness on several infinite families. We conclude with some general thoughts about the connected domination game and pose several open problems.

In the rest of the introduction we present the connected domination game with Chooser and give further definitions needed later on.

The *connected domination game with Chooser* [13] has similar rules as the normal game, except that there is another player, Chooser, who can make zero, one, or more moves after any move of Dominator or Staller. The conditions for his move to be legal are the same as for Dominator and Staller. Chooser has no specific goal, he can help either Dominator or Staller. We recall the Chooser Lemma from [13] to be used later on.

Lemma 1.1 (Chooser Lemma)**.** *Consider the connected domination game with Chooser on a graph G. Suppose that in the game Chooser picks k vertices, and that both Dominator and Staller play optimally. Then at the end of the game the number of played vertices is at most $\gamma_{\mathrm{cg}}(G) + k$ and at least $\gamma_{\mathrm{cg}}(G) - k$.*

The *Cartesian product $G \square H$* of graphs G and H has the vertex set $V(G) \times V(H)$, vertices (g, h) and (g', h') being adjacent if either $gg' \in E(G)$ and $h = h'$, or $g = g'$ and $hh' \in E(H)$. If $h \in V(H)$, then the subgraph of $G \square H$ induced by the vertex set $\{(g, h) : g \in V(G)\}$ is isomorphic to G, called a *G-layer*, and denoted with G^h. Analogously the H-layers are defined and denoted with ^gH for a fixed vertex $g \in V(G)$. For more on the Cartesian product of graphs see [21].

The *closed neighborhood* of a vertex v in a graph G is denoted by $N[v]$. The closed neighborhood of a set of vertices $S \subseteq V(G)$ is $N[S] = \bigcup_{v \in S} N[v]$. Similarly, the *open neighborhood* of v is $N(v) = N[v] \setminus \{v\}$. A set $S \subseteq V(G)$ is a *dominating set* of the graph G if $N[S] = V(G)$. The minimal cardinality of a dominating set is the *domination number $\gamma(G)$* of G. A *connected dominating set* is a set S that is both connected and dominating. The smallest cardinality of such set is the *connected domination number $\gamma_{\mathrm{c}}(G)$* of G. The largest size of an independent set of vertices in G is the *independence number $\alpha(G)$* of G. Additionally, we denote $[n] = \{1, \ldots, n\}$, $\Delta(G)$ as the maximum degree of vertices of G, and $n(G)$ as the number of vertices of G.

2 A general upper bound

In this section we prove a general upper bound on the connected game domination number of Cartesian products and demonstrate its sharpness by several earlier results. A key for the bound is the following theorem from the seminal paper [13]. We reprove the result here in more detail, to give additional insight.

Theorem 2.1. [13, Theorem 1] *If G is a connected graph, then $\gamma_{cg}(G) \leq 2\gamma_c(G) - 1$.*

Proof. Let $\ell = \gamma_c(G)$ and let $D = \{x_1, \ldots, x_\ell\}$ be a connected dominating set in G, where for every $i \in [\ell]$, the set $\{x_1, \ldots, x_i\}$ induces a connected subgraph of G. As D is a connected dominating set, such an ordering of the vertices of D is clearly possible.

The strategy of Dominator is to consecutively play the vertices $x_1, \ldots, x_\ell$, in particular, his first move is on the vertex x_1. Suppose now that x_i, $i \geq 2$, is the next vertex to be played by Dominator according to his strategy and that he is not able to play it.

It is possible that Dominator cannot play x_i because x_i was already played by Staller in the course of the game. If this is the case, Dominator simply plays x_{i+1}, provided this is a legal move. If it is not and also x_{i+1} was already played by Staller, we continue this process until we reach a vertex x_j, $j \geq i$, that was neither played earlier by Staller nor it is playable by Dominator. Because of the latter fact, all the vertices from $N[x_j]$ are already dominated at this stage of the game. If each of the vertices $x_{j+1}, \ldots, x_\ell$ was either already played by Staller or is unplayable by Dominator, then the set of vertices selected so far constitutes a connected dominating set of G. Indeed, all the vertices from each of $N[x_1], \ldots, N[x_\ell]$ are dominated, hence G is dominated because $\{x_1, \ldots, x_\ell\}$ is a dominating set. Moreover, it is a connected dominating set because the strategy of Dominator preserves the connectedness (and Staller must follow it anyway).

Assume therefore that there exits a vertex x_k, $j + 1 \leq k \leq \ell$, that is undominated and let k be the smallest possible index of such a vertex. If x_k is adjacent to some vertex already played, then Dominator plays x_k. Otherwise, consider an arbitrary neighbor x_s of x_k from the sequence $x_1, \ldots, x_{k-1}$. If x_s was played in the game played so far, then Dominator can extend the game by playing x_k. So suppose that x_s was not played before. This means that x_s was (and is) an unplayable vertex. In particular, its neighbor x_k must have been dominated by some vertex already played. But this means again that Dominator can play x_k because it is adjacent to an already played vertex.

By the above strategy of Dominator, the game is finished no later than at the moment when Dominator plays x_ℓ. If this happens and if Dominator played all the vertices $x_1, \ldots, x_\ell$, then the number of vertices played is exactly $2\ell - 1 = 2\gamma_c(G) - 1$. If, however, some of the vertices x_i were played by Staller or became unplayable during the game, the number of vertices played is even smaller. □

We can now quickly derive the announced upper bound.

Theorem 2.2. *If G and H are connected graphs, then*

$$\gamma_{cg}(G \,\square\, H) \leq \min\{2\gamma_c(G)n(H), 2\gamma_c(H)n(G)\} - 1 \,.$$

Proof. Let X_G be a connected dominating set of G with $|X_G| = \gamma_c(G)$. Since X_G induces a connected subgraph of G, the set of vertices $X_G \times V(H)$ induces a connected subgraph of $G \,\square\, H$ and hence it is a connected dominating set of $G \,\square\, H$. Therefore, $\gamma_c(G \,\square\, H) \leq 2\gamma_c(G)n(H)$ and thus $\gamma_{cg}(G \,\square\, H) \leq 2\gamma_c(G)n(H) - 1$ by Theorem 2.1. By a parallel argument (or by referring to the commutativity of the Cartesian product) we also have $\gamma_{cg}(G \,\square\, H) \leq 2\gamma_c(H)n(G) - 1$. □

Before listing known results that demonstrate the sharpness of the bound of Theorem 2.2, we note that the following result is a direct consequence of the theorem.

Corollary 2.3. [13, Theorem 5] *If G is a connected graph, then $\gamma_{cg}(G \,\square\, K_m) \leq \min\{2m\gamma_c(G)) - 1, 2n(G) - 1\}$.*

In [13, Theorem 6, Theorem 7] it was proved that if $2 \leq m \neq n \geq 4$, then

$$\gamma_{cg}(K_{1,n-1} \,\square\, K_m) = \min\{2n - 1, 2m - 1\} \,,$$

and if $m, n \geq 4$, then

$$\gamma_{\mathrm{cg}}(P_n \,\square\, K_m) = 2n - 1 \,.$$

These results demonstrate that the bound of Theorem 2.2 is sharp.

Let next $r \geq 2$ and $n_1 \geq n_2 \geq \cdots \geq n_r \geq 2$. By the associativity of the Cartesian product we have $K_{n_1} \,\square\, K_{n_2} \,\square\, \cdots \,\square\, K_{n_r} = K_{n_1} \,\square\, (K_{n_2} \,\square\, \cdots \,\square\, K_{n_r})$, hence applying Theorem 2.2 we get

$$\gamma_{\mathrm{cg}}(K_{n_1} \,\square\, K_{n_2} \,\square\, \cdots \,\square\, K_{n_r}) \leq 2n_2 \cdots n_r - 1 \,.$$

Since it is proved in [20, Proposition 2.3] that the equality holds here provided that $n_1 \geq 2n_2 \cdots n_r$, this gives another example for the sharpness of Theorem 2.2.

3 Products of stars with paths or cycles

In this section we determine the connected game domination number for the Cartesian product of stars with paths or with cycles. In both cases the bound of Theorem 2.2 is sharp.

Theorem 3.1. *If $n \geq 3$ and $m \geq 1$, then*

$$\gamma_{\mathrm{cg}}(K_{1,n} \,\square\, P_m) = 2m - 1 \,.$$

Proof. The statement is clear for $m = 1$ and easily verified for $m = 2$. For $m = 3$ it holds $\gamma_{\mathrm{cg}}(K_{1,n} \,\square\, P_3) = 5$. Indeed, the upper bound follows from Theorem 2.2. The lower bound can be shown with a simple case analysis. In the remaining part of the proof, we consider $m \geq 4$.

Consider a path $P_m = u_1 \dots u_m$ ($m \geq 4$) and a star $K_{1,n}$ on the vertex set $v_1, \dots, v_{n+1}$ ($n \geq 3$) where v_{n+1} is the center. In $K_{1,n} \,\square\, P_m$, the layer $K_{1,n}^{u_j}$ will be denoted simply by $K_{1,n}^j$. In a connected domination game on $K_{1,n} \,\square\, P_m$, let A_i (resp. A_i') denote the set of integers $j \in [m]$ for which at least one vertex from the layer $K_{1,n}^j$ was played within the moves $d_1, s_1, \dots, d_i$ (resp. $d_1, s_1, \dots, d_i, s_i$). By connectivity, both A_i and A_i' are sets of consecutive integers. We will use the notation $A_i = [a_i, b_i]$ and $A_i' = [a_i', b_i']$. By Theorem 2.2, we have $\gamma_{\mathrm{cg}}(K_{1,n} \,\square\, P_m) \leq 2m - 1$. To prove the lower bound we describe a strategy of Staller that maintains the following property for every i.

($\star$) After the move s_i of Staller, either at least two vertices are played from every layer $K_{1,n}^j$ with $j \in [a_i', b_i'] \backslash \{1, m\}$, or there is only one layer $K_{1,n}^{j^*}$ ($j^* = a_i' > 1$ or $j^* = b_i' < m$) among them which is played only once. In the latter case, the vertex played in the layer $K_{1,n}^{j^*}$ is a leaf of the layer, and there are at least two leaves played from the neighboring layer.

Property ($\star$) and the strategy of Staller will ensure that from every $K_{1,n}^j$, with $2 \leq j \leq m - 1$, at least two vertices will be played during the game. The below rule (S1) in Staller's strategy clearly establishes ($\star$) after s_1. Then, we will suppose that it is true before Dominator's move d_i ($i \geq 2$). Note that, under this condition, we may have at most two layers $K_{1,n}^j$ with $2 \leq j \leq m - 1$ which are played exactly once after the move d_i.

(S1) If Dominator plays the center of a layer $K_{1,n}^j$ as his first move d_1, Staller replies by playing the leaf (v_1, u_j) of the same star; if Dominator plays a leaf from $K_{1,n}^j$, Staller picks the center (v_{n+1}, u_j) as s_1.

Later, if Dominator plays the first vertex from $K_{1,n}^{b_i}$ as d_i, that is, if $b_i = b_{i-1}' + 1$, and $b_i < m$, then Staller replies according to the following rules:

(S2) If at least two vertices from the neighboring layer $K_{1,n}^{b_i-1}$ have been played in the game so far, Staller chooses a vertex from $K_{1,n}^{b_i}$. This clearly can be done as at most one vertex from $K_{1,n}^{b_i+1}$ has been dominated so far.

(S3) If only one vertex from $K_{1,n}^{b_i-1}$ has been played so far, Staller chooses a leaf from $K_{1,n}^{b_i-1}$. Such a legal move exists as ($\star$) was true after the move s_{i-1}.

The rules are analogous (and we do not repeat them) for the case when Dominator plays the first vertex from $K_{1,n}^{a_i}$ as d_i (and $a_i > 1$). If Dominator plays a vertex from $K_{1,n}^m$ or $K_{1,n}^1$ as d_i, or he plays a vertex from a layer $K_{1,n}^j$ which has been played earlier (i.e., $j \in [a_{i-1}', b_{i-1}']$), Staller replies as follows:

(S4) If $b_i < m$ and $K_{1,n}^{b_i}$ is played only once, Staller chooses a vertex from $K_{1,n}^{b_i}$. This clearly can be done as at most one vertex from $K_{1,n}^{b_i+1}$ has been dominated so far. The rule is similar for the case when $a_i > 1$ and $K_{1,n}^{a_i}$ is played only once.

(S5) If both $K_{1,n}^{b_i}$ and $K_{1,n}^{a_i}$ are played at least twice, Staller plays a vertex from $K_{1,n}^{b_i}$ or $K_{1,n}^{a_i}$, if such a legal move exists.

(S6) If there are no playable vertices in $K_{1,n}^{b_i}$ (and $b_i < m$), then all vertices of the layer have been played. If the situation is similar for $K_{1,n}^{a_i}$, Staller plays the leaf (v_1, u_{b_i+1}) if $b_i < m - 1$, and she plays (v_1, u_{a_i-1}) if $b_i \geq m - 1$ and $a_i > 2$.

What remains is only the case when all the layers $K_{1,n}^j$ with $2 \leq j \leq m - 1$ have been played at least twice.

(S7) Staller plays a vertex from $K_{1,n}^{m-1}$ or from $K_{1,n}^2$. Note that it is always possible if the game is not yet over.

One can check that property (⋆) is preserved if Staller applies the above strategy. Remark that, by (S5) and (S6), Staller plays a first vertex from a $K_{1,n}^j$ only if all vertices of the neighboring layer have already been played. Moreover, if Staller plays the first vertex in a layer, then this vertex corresponds to a leaf of the star. Consequently, if we have two neighboring layers which are played only once, then these played vertices must be leaves. In particular, we may have the following two situations before the end of the game:

– If Staller finishes the game and in the last turn a vertex from $K_{1,n}^m$ becomes dominated, then the center of this star is not played and, therefore, at least $n + 1$ vertices were played from these two stars. Moreover, every layer $K_{1,n}^j$, $2 \leq j \leq m - 2$, is played at least twice and the domination of $K_{1,n}^1$ needs at least one further vertex. We may infer that at least $n + 1 + 2(m - 3) + 1 \geq 2m - 1$ vertices were played during the game. (The argumentation is similar if a vertex from $K_{1,n}^1$ was dominated in the last turn.)

– If Dominator finishes the game, at least two vertices are chosen from every $K_{1,n}^j$, if $2 \leq j \leq m - 1$. The domination of $K_{1,n}^1$ and $K_{1,n}^m$ needs at least two further vertices. Hence, at least $2m - 2$ vertices were played. On the other hand, the number of chosen vertices must be odd as, by our assumption, Dominator finishes the game. This proves $\gamma_{cg}(K_{1,n} \square P_m) \geq 2m - 1$.

This finishes the proof of the equality. ☐

Theorem 3.2. *If $n \geq 3$ and $m \geq 3$, then*

$$\gamma_{cg}(K_{1,n} \square C_m) = 2m - 1.$$

Proof. For $m = 3$, it can easily be checked that $\gamma_{cg}(K_{1,n} \square C_3) = 5$. Thus we only consider $m \geq 4$.

By Theorem 2.2, $\gamma_{cg}(K_{1,n} \square C_m) \leq 2m - 1$. To prove the lower bound, we use a similar strategy of Staller as in the proof of Theorem 3.1. We also use the same notation, but now the addition is done modulo m (with values in $[m]$), so the sets A_i and A_i' are sets of consecutive integers modulo m. The main difference is only that in the strategy on $K_{1,n} \square P_m$ layers $K_{1,n}^1$ and $K_{1,n}^m$ are studied differently, but on the $K_{1,n} \square C_m$ the special $K_{1,n}$-layers are the ones which are played last or not played at all. Similarly, all the conditions like $1 < a_i'$ and $b_i' < m$ must be replaced with $|b_i' - a_i'| \leq m - 3$. Apart from that, Staller's strategy and the final counting of the moves remains the same. ☐

4 Products of P_3 with paths or cycles

In Theorems 3.1 and 3.2 we have considered Cartesian products of stars $K_{1,n}$, $n \geq 3$, with paths or cycles. For the star $K_{1,2} = P_3$ the situation is slightly different, in particular, the bound of Theorem 2.2 is no longer sharp.

Theorem 4.1. *If $m \geq 4$, then*

$$\gamma_{cg}(P_3 \square P_m) = 2m - 2.$$

Proof. Let $P_3 = v_1 v_2 v_3$ and $P_m = u_1 \ldots u_m$. We will also use the notations introduced in the proof of Theorem 3.1.

To prove that $\gamma_{cg}(P_3 \square P_m) \leq 2m - 2$, we consider the following strategy of Dominator in a connected domination game with Chooser and apply the Chooser Lemma 1.1. Let $d_1 = (v_2, u_2)$. Then, Staller has four different legal replies.

- If Staller responds with the move $s_1 = (v_1, u_2)$, then Chooser picks the vertices $(v_2, u_3), \ldots, (v_2, u_{m-1})$. Dominator's strategy is to play $d_2 = (v_2, u_m)$. Then, only the vertex (v_3, u_1) remains undominated and the game finishes with a set of $m + 1$ played vertices. By Chooser Lemma, we have $t \leq (m+1) + (m-3) = 2m - 2$, where t is the length of the connected domination game under the described strategies of the players.
- The reply $s_1 = (v_3, u_2)$ is treated analogously as the move $s_1 = (v_1, u_2)$.
- If $s_1 = (v_2, u_3)$, Chooser picks $(v_2, u_4), \ldots, (v_2, u_m)$ and Dominator plays $d_2 = (v_2, u_1)$. By Chooser Lemma, we have $t \leq m + (m - 3) = 2m - 3$.
- If $s_1 = (v_2, u_1)$, Chooser picks $(v_2, u_3), \ldots, (v_2, u_{m-1})$ and Dominator plays $d_2 = (v_2, u_m)$. By Chooser Lemma, we have $t \leq m + (m - 3) = 2m - 3$, again.

This proves that Dominator can ensure $t \leq 2m - 2$ under any strategy of Staller, that is, $\gamma_{cg}(P_3 \square P_m) \leq 2m - 2$.

To prove the other direction, that is, $\gamma_{cg}(P_3 \square P_m) \geq 2m - 2$, we consider the strategy (S1)–(S7) of Staller and property (⋆) as these are given in the proof of Theorem 3.1. In fact, as $P_3 = K_{1,2}$, we have the same line of the proof except the final counting which is the following for the case of $P_3 \square P_m$. By Staller's strategy, at least two vertices were played from every P_3^j with $2 \leq j \leq m - 1$. Moreover, the domination of $V(P_3^1) \cup V(P_3^2)$ and $V(P_3^{m-1}) \cup V(P_3^m)$ needs at least two further vertices. Hence, we get $\gamma_{cg}(P_3 \square P_m) \geq 2(m-2) + 2 = 2m - 2$ and the theorem follows. □

We remark in passing that $\gamma_{cg}(P_3 \square P_3) = 3$, an optimal first move of Dominator being the vertex of $P_3 \square P_3$ of degree 4.

Theorem 4.2. *If $m \geq 4$, then*
$$\gamma_{cg}(P_3 \square C_m) = 2m - 2 \, .$$

Proof. Let $m \geq 4$ and let $V(C_m) = [m]$. All the exponents of the P_3-layers will be taken modulo m.

First, we show that $\gamma_{cg}(P_3 \square C_m) \leq 2m - 2$. The basic strategy of Dominator is to play a legal move in the center of some P_3-layer. Clearly this is possible in the D-game. Inductively, let $P_3^i, \ldots, P_3^j$, $i \leq j$, be the P_3-layers in which at least one vertex has been played so far. We may assume without loss of generality that Staller's last move was in P_3^j. If this move was in the center of P_3^j, then Dominator replies with the move on the center of P_3^{j+1}. Otherwise, Dominator selects the center of P_3^j. After $m - 1$ moves of Dominator we have two cases. If all the P_3-layers were played, then by the strategy of Dominator, the whole graph is dominated, hence at most $2m - 3$ moves were played. Otherwise, one P_3-layer has not been played, say P_3^j. In this case Dominator has played all the centers of the remaining $m - 1$ P_3-layers. But then Staller has played at least one non-center vertex from the layers P_3^{j-1} and P_3^{j+1}. Consequently, at most one vertex from P_3^j is not yet dominated after the $(m-1)$st move of Dominator, which means that in the next move Staller must finish the game. Hence the game finishes in no more than $2m - 2$ moves.

Proving that $\gamma_{cg}(P_3 \square C_m) \geq 2m - 2$ goes along the same lines as in the proof for $P_3 \square P_m$ in Theorem 4.1, where the strategy of Staller is modified as in the proof of Theorem 3.2. □

The upper bound in Theorem 4.2 can also be proven using the game with Chooser as it is done in the proof of Theorem 4.1. We have given an alternative proof instead, because it yields an exact strategy of Dominator.

We have thus seen that in many cases the upper bound of Theorem 2.2 is sharp or close to be sharp. On the other hand, the bound can also be arbitrary weak. For this sake let G be a graph with $\gamma_c(G) \gg n(G)/2$ and consider $G \square G$. Then a rough estimate for the the right hand side of the inequality of Theorem 2.2 is that it is $\gg n(G)^2 = n(G \square G)$.

5 Products with complete graphs

A large class of graphs which are extremal with respect to the general upper bound in Theorem 2.2 can be obtained when we consider the Cartesian products $K_k \square G$ with sufficiently large k. In fact, for every graph G, there exists a threshold $k_0(G)$ such that $k \geq k_0(G)$ ensures

$$\gamma_{\mathrm{cg}}(K_k \square G) = 2n(G) - 1 = \min\{2\gamma_{\mathrm{c}}(G)n(K_k), 2\gamma_{\mathrm{c}}(K_k)n(G)\} - 1 .$$

It is not difficult to prove that $k_0(G) = 2n(G) - 1$ is always appropriate, see the proof of Theorem 5.1. On the other hand, we also establish a threshold $k_0(G)$ in terms of the maximum degree $\Delta(G)$ and independence number $\alpha(G)$ of G.

Theorem 5.1. *If G is a connected graph and $k \geq \min\{4\Delta(G) + \alpha(G), 2n(G) - 1\}$, then*

$$\gamma_{\mathrm{cg}}(K_k \square G) = 2n(G) - 1 .$$

Proof. Let G be a connected graph on n vertices with maximum degree Δ and with independence number α. During the game, a K_k-layer will be called played if at least one vertex has been already played from it. A K_k-layer will be called dominated if all of its vertices have been dominated, otherwise, it is undominated. Clearly, if a K_k-layer is played, then it is dominated. Observe that, by Theorem 2.2, we have $\gamma_{\mathrm{cg}}(K_k \square G) \leq 2n-1$.

First suppose that $k \geq 2n - 1$ and consider the following simple strategy of Staller: she always chooses a (legal) vertex from a dominated K_k-layer. While the game is not over, Staller can follow this strategy. Indeed, by the connectivity of G, if there is an undominated K_k-layer in $K_k \square G$, then we have two neighboring layers K_k^g and $K_k^{g'}$ (i.e., $gg' \in E(G)$) such that K_k^g is dominated but a vertex $(h, g') \in V(K_k^{g'})$ is undominated. Then, Staller may play (h, g). Hence, after the $(n - 1)$st move of Staller, at most $n - 1$ K_k-layers are played. Each remaining unplayed layer is certainly not dominated since it would need at least $k \geq 2n - 1$ vertices played in the neighboring layers. This proves $\gamma_{\mathrm{cg}}(K_k \square G) \geq 2n - 1$ under the condition $k \geq 2n - 1$.

Now, suppose that $k \geq 4\Delta + \alpha$ and consider a connected domination game on $K_k \square G$. We assign blue, green or red color to those vertices which were played in the game. If Dominator plays a vertex $d_i = v$ from a K_k-layer such that the layer was undominated before his move, the vertex d_i is colored blue. If d_i is blue and Staller plays s_i from the same K_k-layer as her next move, then s_i is blue, otherwise s_i is green. If Dominator plays d_i from a dominated layer, then both d_i and s_i are red. We say that a K_k-layer is a blue layer if it contains a blue vertex.[1] A K_k-layer is white if it remains unplayed at the end of the game and also if it becomes dominated before it is played. When the game finishes, each K_k-layer is either blue or white and, denoting the number of blue or white layers by ℓ_b or ℓ_w respectively, we have $\ell_b + \ell_w = n$.

Consider the following strategy of Staller: After Dominator's move d_i, Staller plays a legal vertex from the same K_k-layer if possible; otherwise, Staller plays a legal vertex from any dominated layer. As we have seen earlier, the latter choice is always possible (if the game is not over) as there must be an undominated layer which has a neighboring dominated layer.

Now, we prove the following claims.

Claim 1. *At the end of the game, the number of green vertices is at most α.*

Proof. Suppose that s_i, which is the ith move of Staller, is a green vertex. Then d_i, the previous move of Dominator, was the first vertex played from the undominated layer K_k^g and no further vertex could be played from K_k^g after his move. The latter property means that every neighboring layer was already dominated (either played or not) right after the move d_i. Consequently, no blue vertex can be played from the neighboring layers

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[1] We will see that under the described strategy of Staller a layer is blue if and only if it was undominated right before the first vertex was played from it.

in the game after d_i. Therefore, if every green vertex s_j of the product is associated with the vertex $f(s_j)$ from $V(G)$ such that $d_j \in V(K_k^{f(s_j)})$ holds for the previous move of Dominator, the set $F = \{f(s_j) : s_j \text{ is green}\}$ is an independent vertex set in G and $|F| \le \alpha$ follows. $\square$

Claim 2. *If K_k^g is a white layer, then at least 2Δ red vertices were played from the neighboring layers.*

Proof. Since K_k^g is white, its k vertices are dominated by k different (blue, green, red) vertices of the neighboring layers. (This remains true, even if later some vertices from K_k^g are played.) By definition, every neighboring K_k-layer may contain at most two blue vertices and there are at most Δ neighboring layers. By Claim 1, there exist at most α green vertices. Hence, the number of red vertices in the neighboring layers is at least $k - 2\Delta - \alpha$. Since $k \ge 4\Delta + \alpha$, the number of such red vertices is at least 2Δ. $\square$

Now, denote by r and p the number of red vertices and played vertices, respectively, at the end of the game. The number of edges between the red vertices and the white layers is at least $2\Delta \cdot \ell_w$ by Claim 2. On the other hand, since Δ is the maximum vertex degree, the number of these edges is at most $\Delta \cdot r$. This yields $2\ell_w \le r$. Moreover, $p - r$, which is the total number of blue and green vertices, equals either $2\ell_b$ or $2\ell_b - 1$. (The latter occurs if Dominator finishes the game by playing a blue vertex.) Hence, $2\ell_b - 1 \le p - r$. We have the following inequalities:

$$2\ell_b - 1 \le p - r \le p - 2\ell_w$$

and, since $\ell_b + \ell_w = n$, we may conclude $2n - 1 \le p$. This finishes the proof of $\gamma_{cg}(K_k \square G) = 2n(G) - 1$ under the condition $k \ge 4\Delta + \alpha$. $\square$

In the following theorem we prove that if $\gamma_{cg}(K_{k_0} \square G) = 2n(G) - 1$ holds for some k_0, then $\gamma_{cg}(K_k \square G) = 2n(G) - 1$ holds for every $k \ge k_0$. This result might appear obvious, however, it does not generalize to general subgraphs. If G is a graph with a large $\gamma_{cg}(G)$ and H is a graph obtained from G by adding a universal vertex, then it holds $1 = \gamma_{cg}(H) \ll \gamma_{cg}(G)$, even though $G \subseteq H$.

Theorem 5.2. *If $n \ge 1$, then*

$$\gamma_{cg}(K_{n+1} \square G) \ge \gamma_{cg}(K_n \square G).$$

Proof. We prove the theorem using imagination strategy. Two games will be played in the proof, one on $K_{n+1} \square G$ and one on $K_n \square G$. In order to avoid confusion between vertices of those two graphs, we introduce the notation $x_{[n]} = \{x_1, \dots, x_n\}$.

The real game is played on $K_{n+1} \square G$ with vertex set $v_{[n+1]} \times V(G)$, while Staller imagines a game on $K_n \square G$ with vertex set $u_{[n]} \times V(G)$. Define a function $f : v_{[n]} \times V(G) \to u_{[n]} \times V(G)$ by $f : (v_i, g) \mapsto (u_i, g)$ for every $i \in [n]$ and $g \in V(G)$, with the inverse $f^{-1} : (u_i, g) \mapsto (v_i, g)$. Even though $f(v_{n+1}, g)$ is not formally defined, we will simply say that it is not a legal move in the imagined game on $K_n \square G$. Additionally, in the real game the label v_{n+1} is given to a vertex of K_{n+1} such that the underlying G-layer was played last among all G-layers (or not played at all).

Dominator plays optimally in the real game and Staller plays optimally in the imagined game. Let D_R and D_I denote the set of already played vertices in the real and in the imagined game, respectively. Staller's strategy is to ensure that until the imagined games ends it holds (i) $|D_R| = |D_I|$ and (ii) $f(N[D_R] \cap (v_{[n]} \times V(G))) \subseteq N[D_I]$. If this is true, then the number of moves needed to finish the real game is at least the number of moves needed in the imagined game. As Staller plays optimally in the imagined game and Dominator plays optimally in the real game, we have $\gamma_{cg}(K_n \square G) \le \gamma_{cg}(K_{n+1} \square G)$.

Consider the strategy of Staller. She always replies optimally in the imagined game and tries to copy her move to the real game. By the property (ii), this move surely dominates new vertices in the real game, but might not be connected to a previous move in the real game. Let $x = s_i$ be the optimal move of Staller in the imagined game. If $f^{-1}(x)$ is a legal move in the real game, then Staller copies it to the real game (clearly preserving properties (i) and (ii)). But if $f^{-1}(x)$ is not a legal move in the real game, then let y be the move in the imagined game that first dominated x. Now set $x = y$ and iteratively repeat the above procedure until Staller reaches a vertex x such that $f^{-1}(x)$ is a legal move in the real game. Clearly, the procedure stops (as the graph is finite) and properties (i) and (ii) are preserved.

Let $d_i = (v_k, g)$, $k \in [n+1]$, $g \in V(G)$, be an (optimal) move of Dominator in the real game. Staller copies his move to the imagined game by the following rules.

(D1) If $f(d_i)$ is a legal move in the imagined game, then Staller imagines Dominator played $f(d_i) = (u_k, g)$ in the imagined game.

(D2) If $f(d_i)$ is not a legal move, but there exists a playable vertex p in K_n^g in the imagined game, then Staller imagines Dominator played p.

(D3) If neither (D1) nor (D2) can be applied, then Staller imagines Dominator played any legal move in the imagined game.

All the rules clearly preserve (i) and the rule (D1) also preserves (ii).

If $k \in [n]$, but $f(d_i)$ is not a legal move, then either $f(d_i)$ dominates no new vertex in the imagined game or it is not connected to a previously played vertex. In the first case, the property (ii) still holds after Staller imagines the ith move of Dominator by the rule (D2) or (D3). We now explain that the second case is not possible. Let x be the already played neighbor of d_i in the real game. By the property (ii), $N[x]$ and $f(N[x] \cap (v_{[n]} \times V(G)))$ are both dominated, in particular, d_i and $f(d_i)$ are both dominated. Thus a neighbor of $f(d_i)$ was already played in the imagined game.

Next, consider the case $k = n+1$. By the definition of the vertex v_{n+1}, the move d_i cannot be the first move of the game. By induction on the number of moves on $^{v_{n+1}}G$, we will prove that the property (ii) is preserved and also that if Staller cannot apply (D2) at Dominator's move d_i, then the whole set $u_{[n]} \times N[g]$ is already dominated in the imagined game.

Firstly, consider the first move $d_{i_0} = (v_{n+1}, g_0)$ played on $^{v_{n+1}}G$ in the real game. Property (ii) is preserved no matter where Staller imagines his move. Suppose that Staller cannot apply (D2) after the move d_{i_0}. If a vertex was already played in $K_n^{g_0}$ in the imagined game, then Staller cannot apply (D2) only if $u_{[n]} \times N[g_0]$ is already dominated (otherwise she could play another move on $K_n^{g_0}$). So from now on, suppose also that no vertex was already played in $K_n^{g_0}$ in the imagined game. As d_{i_0} is a legal move of Dominator in the real game, it is adjacent to an already played vertex $x = (v_k, g_0)$, $k \in [n]$, in the real game. Then x was a move of Dominator in the real game, $N[(u_k, g_0)]$ is already dominated in the imagined game, in particular $K_n^{g_0}$ is already dominated. Thus all vertices in $K_n^{g_0}$ are adjacent to an already played vertex in the imagined game. As none of them is a legal move, it means that $u_{[n]} \times N[g_0]$ is already dominated.

Next, consider the case when d_i is some later move on $^{v_{n+1}}G$. If Staller can apply (D2), then the property (ii) is clearly satisfied. Suppose now that Staller cannot apply (D2). If d_i is adjacent to an already played vertex in $v_{[n]} \times \{g\}$ in the real game, then the proof goes along the same lines as above for d_{i_0}. So the only remaining case is if d_i is not adjacent to an already played vertex in $v_{[n]} \times \{g\}$ in the real game. This means it is adjacent to an already played vertex $d' = (v_{n+1}, g')$, $g \neq g' \in V(G)$, in the real game and no vertex was played so far in $v_{[n]} \times \{g\}$ in the real game. We distinguish two cases.

The first case is that d_i dominates no new vertex in $v_{[n]} \times \{g\}$. Then this set is already dominated in the real game and $u_{[n]} \times \{g\}$ is dominated in the imagined game, thus the connectedness condition is satisfied for every vertex in $u_{[n]} \times \{g\}$. As Staller cannot apply (D2), $u_{[n]} \times N[g]$ is already dominated and property (ii) holds.

The second case is that d_i dominates a new vertex in $v_{[n]} \times \{g\}$. If $u_{[n]} \times \{g\}$ is already dominated in the imagined game, then Staller could apply rule (D2), unless $u_{[n]} \times N[g]$ is already dominated. Otherwise, there is an undominated vertex in $u_{[n]} \times \{g\}$ in the imagined game. This means that a move (u_l, g') was played in the imagined game on $u_{[n]} \times \{g'\}$ (otherwise $u_{[n]} \times \{g\} \subseteq u_{[n]} \times N[g']$ would be already dominated). But then Staller can imagine Dominator plays (u_l, g) in the imagined game to dominate the undominated vertex in $u_{[n]} \times \{g\}$, hence applying (D2). □

6 A lower bound

In [13] an upper bound but no lower bound is given on $\gamma_{cg}(G \square K_n)$. To give (general) lower bounds on $\gamma_{cg}(G \square H)$ appears more difficult than giving upper bounds. An intuitive reason for this is the intrinsic structure of the Cartesian product that, for instance, makes a possible proof of the lower bound on the domination number of $G \square H$ as suggested by Vizing a notorious open problem [22]. Nonetheless, we can give the following general lower bound.

Theorem 6.1. *If G and H are connected graphs on at least two vertices, then*

$$\gamma_{cg}(G \square H) \geq \begin{cases} 2\gamma_c(G); & n(H) = 2, \\ 2\gamma_c(G) + 1; & n(H) \geq 3. \end{cases}$$

Proof. The vertices of $V(G \square H)$ will be denoted as $\{(g, h); \ g \in G, h \in H\}$. Let A_i (resp. A_i') denote the set of all $g \in G$ such that at least one vertex in the layer ${}^gH$ has been played within moves $d_1, s_1, \ldots, d_i$ (resp. $d_1, s_1, \ldots, d_i, s_i$). Let $B_i \subseteq A_i$ be the smallest connected subset of A_i that dominates $N[A_i]$. We similarly define $B_i' \subseteq A_i'$. Note that at each step (and at the end of the game) it holds $|A_i| \geq |B_i|$ and $|A_i'| \geq |B_i'|$.

Staller's strategy is to play s_i on a vertex above B_i, meaning that $s_i \in B_i \times V(H)$. If this is not possible, she plays any legal move. We prove that this strategy assures that after each move of Staller it holds $i \geq |B_i'|$.

As $n(H) \geq 2$, Staller can make her first move in the same H-layer where Dominator played, thus $1 \geq |B_1'|$. Suppose the property holds for the first $i - 1$ moves of Staller and consider the situation after Dominator's ith move, say $d_i = (g, h)$. We now distinguish two cases.

1. The vertex d_i newly dominates only vertices above A_i (i.e. $d_i \in A_i \times V(H)$).
   In this case, g is not added to B_i, so $B_i = B_{i-1}'$. No matter where Staller replies, we have $|B_i| \leq |B_i'| \leq |B_i| + 1 = |B_{i-1}'| + 1$. But then $i - 1 \geq |B_{i-1}'| \geq |B_i'| - 1$, which implies $i \geq |B_i'|$.
2. The vertex d_i dominates at least one vertex not above A_i, say a vertex (g', h).

   (a) B_{i-1}' dominates $N[A_i]$.
       In this case $B_i = B_{i-1}'$ and $i \geq |B_i'|$ as in the previous case.
   (b) B_{i-1}' does not dominate $N[A_i]$.
       This means that at least the vertex $g' \in N[A_i]$ is not dominated by B_{i-1}', so no vertex in ${}^{g'}H$, except for (g', h), is dominated after Dominator's ith move. We have $B_i = B_{i-1}' \cup \{g\}$ and d_i is the only already played vertex in ${}^gH$. As $n(H) \geq 2$, Staller can reply in the same H-layer to newly dominate a vertex in ${}^{g'}H$. Thus $|B_i'| = |B_i| = |B_{i-1}'| + 1 \leq (i - 1) + 1 = i$.

If Dominator is the one who finishes the game in his jth move, then his last move is not as in the case 2.(b) above (otherwise Staller has a legal reply and the game is not over). Thus $j - 1 \geq |B_{j-1}'| = |B_j| \geq \gamma_c(G)$, which means $j \geq \gamma_c(G) + 1$. Hence the number of moves is at least $j + (j - 1) \geq 2\gamma_c(G) + 1$.

On the other hand, if Staller ends the game with her jth move, then her strategy assures that $j \geq |B_j'| \geq \gamma_c(G)$, thus the number of moves is at least $2\gamma_c(G)$. This completes the proof in the case $n(H) = 2$.

If $n(H) \geq 3$, consider again the situation where Staller ended the game. We distinguish two cases. If the last vertex was added to the set B_j' not in the last two moves of the game but earlier, then clearly the number of moves is greater than $2\gamma_c(G) + 1$. Otherwise, in the last two steps, one of the players added the last vertex to the set B_j'. If the move d_j added the last vertex to $B_j' = B_j$, then this situation is as in case 2.(b). Thus after the moves d_j, s_j, at most $2 < n(H)$ vertices in the layer ${}^{g'}H$ are dominated. But then the game is not yet finished and at least one more move is needed. On the other hand, if the move s_j added a vertex to $B_j' \neq B_j$ (and it follows from the above cases that this can only occur if Dominator did not add a vertex to B_j), then in the newly dominated H-layer only $1 < n(H)$ vertex is dominated, hence at least one more move is needed. It follows that if $n(H) \geq 3$, the game ends after at least $2\gamma_c(G) + 1$ moves. $\qquad\square$

If G and H are graphs of order at least 3, then Theorem 6.1 can be rephrased as follows:

$$\gamma_{cg}(G \square H) \geq \max\{2\gamma_c(G), 2\gamma_c(H)\} + 1 \, .$$

Particular examples on which the bound of Theorem 6.1 is sharp are $P_3 \square P_3$, $C_3 \square K_2$, and $C_3 \square C_3$. The bound is also sharp for $P_m \square K_2$ and $P_m \square K_3$, $m \geq 4$ [13, Theorem 7], and for Circular ladders $C_m \square K_2$, $m \geq 4$ [20, Theorem 5.3]. Using similar reasoning as in the proof of [13, Theorem 7], we obtain another equality case

$$\gamma_{\mathrm{cg}}(C_m \square K_3) = \begin{cases} 3; & m = 3, \\ 2m - 3; & m \geq 4. \end{cases}$$

7 Concluding remarks

As far as we can recall, the (total) domination game was studied on the Cartesian product only in [1], where a connection with Vizing's conjecture was established and a lower bound in terms of 2-packings proved; in [23] where Cartesian products of complete graphs were investigated; and in [24], where (with a considerable effort) the total domination game critical graphs were characterized among the products $K_2 \square C_n$. Relating this small number of known results to the waste bibliography on the (total) domination game indicates that the game is intrinsically difficult on the Cartesian product. On the other hand, the results from this paper as well as from [13, 20] demonstrate that on Cartesian products it is somehow easier to investigate the connected domination game. An intuitive reason for this could be that the players have more restrictive possibilities for their moves than in the standard game. This intuition does not hold in general when comparing the connected domination game with the domination game. For instance and as already noted earlier, the domination game admits Continuation Principle while the connected domination game does not.

The results of this paper lead to several open problems, we list some of them.

Problem 7.1. *Find additional infinite families of graphs which attain the upper bound of Theorem 2.1.*

In view of Theorems 4.1 and 4.2 we pose:

Problem 7.2. *Determine $\gamma_{\mathrm{cg}}(P_n \square P_m)$, $\gamma_{\mathrm{cg}}(P_n \square C_m)$, and $\gamma_{\mathrm{cg}}(C_n \square C_m)$.*

For Theorem 5.1 we strongly feel that the condition $4\Delta + \alpha$ is replaceable with some smaller value. More precisely:

Problem 7.3. *Does there exist a constant $c \in \mathbb{R}^+$, such that Theorem 5.1 remains valid provided that $4\Delta + \alpha$ is replaced with $c\Delta$?*

If the answer is positive for Problem 7.3, one may be interested in finding the smallest coefficient $c = c(\mathcal{G})$ over the class $\mathcal{G}$ of all connected graphs or over any specified subclass of it. In view of the monotonicity stated in Theorem 5.2, this question can be formulated as follows.

Problem 7.4. *Given a graph class $\mathcal{G}$ that does not contain disconnected graphs, determine the constant*

$$c(\mathcal{G}) = \sup_{G \in \mathcal{G}} \, \min \left\{ \frac{k}{\Delta(G)} \; : \; \gamma_{\mathrm{cg}}(G \square K_k) = 2n(G) - 1, k \in \mathbb{Z}^+ \right\} .$$

For the lower bound of Theorem 6.1 we do not know many sharpness examples, hence we also pose:

Problem 7.5. *Do there exist families of graphs for which the lower bound of Theorem 6.1 is sharp and none of the factors is K_2 or K_3?*

Finally, in this paper we have focused on the D-game played on Cartesian products. It seems likewise interesting to investigate the S-game on Cartesian products.

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