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Connectedness and Stability of the Julia Sets of the Composition of Polynomials of Type E_d

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Abstract

For a sequence (c_n) of complex numbers, we consider composition of polynomials of type E_d. The aim of this article is to study the topological properties and stability of the Julia set $\mathcal{J}_{(c_n)}$ of the composition of polynomials of type E_d.

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1 Introduction

We first recall some terminology and definitions in holomorphic dynamics(see[**?**]). let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial self-map of the complex plane. For each $z \in \mathbb{C}$, the orbit of z is

$$\mathrm{Orb}_f(z) = \{z, f(z), f(f(z)), \cdots, f^n(z), \cdots\}.$$

The dynamical plane $\mathbb{C}$ is decomposed into two complementary sets: the *filled Julia set*

$$K(f) = \{z \in \mathbb{C} : \quad \mathrm{Orb}_f(z) \quad \text{is bounded}\},$$

and its complementary, the *basin of infinity*

$$A_f(\infty) = \mathbb{C} - K(f).$$

The boundary of $K(f)$, called the *Julia set*, is denoted by $J(f)$.
When f is a quadratic polynomial, say $f(z) = P_c = z^2 + c$, the Mandelbrot set

$\mathcal{M}_2$ is defined as the set of parameter values c, for which $K(P_c)$ is connected, that is

$$\mathcal{M}_2 = \{c \in \mathbb{C} : K(P_c) \quad \text{is connected}\}.$$

More generally, for the family $f_c(z) = z(z+c)^d$, the *Connectedness locus* $\mathcal{M}_d$, or what is the same, the *Mandelbrot set*, is defined by

$$\mathcal{M}_d = \{c \in \mathbb{C} : K(f_c) \quad \text{is connected}\}.$$

2 Polynomials of type E_d

Let us recall (see[**?**]) terminology and definitions in monic family of higher degree polynomials. Consider the monic family of complex polynomials $f_c(z) = z(z+c)^d$, where $c \in \mathbb{C}^* = \mathbb{C} - 0$ and $d \geq 2$. Each f_c has degree $d+1$ and has exactly two critical points: $-c$ with multiplicity $d-1$, and $\frac{-c}{d+1}$ with multiplicity one. Moreover, $f_c(-c) = 0$ and 0 is fixed. It is proved (see[**?**]) that polynomials with these features, can always be expressed in the form f_c:

Definition.[**?**] *A monic polynomial f of degree $d \geq 2$ is of type E_d if it satisfies the following properties:*

1. f has two critical points: $-c$ of multiplicity $(d-1)$ and $\frac{-c}{d+1}$ of multiplicity one.

2. f has a fixed point at $z = 0$.

3. $f(-c) = 0$.

Proposition 1. *Any monic polynomial $f(z)$ of degree $d+1$ which is of type E_d is of the form*

$$f_c(z) = z(z+c)^d.$$

Moreover, if f_c and $f_{c'}$ of type E_d are affine conjugate with $d \geq 2$, then $c = \omega c'$ for some $\omega^d = 1$.

The proof is straightforward and is omitted. □

2.1 The Composition of Polynomials of type E_d

For a sequence (c_n) of complex numbers, we consider the polynomials $f_{c_n}(z) = z(z+c_n)^d$, and the sequence (F_n) of iterates $F_n := f_{c_n} \circ \cdots \circ f_{c_1}$. we say that a point $z \in \mathbb{C}$ belongs to the Fatou set if (F_n) is normal in some neighborhood of z. We call the set

$$\mathcal{F}_{(c_n)} := \{z \in \mathbb{C}; (F_n)\text{is normal in some neighborhood of } z\}$$

the *Fatou set* and its complement $\mathcal{J}_{(c_n)} := \mathbb{C} - \mathcal{F}_{(c_n)}$ the *Julia set*, according to the classical iteration theory. Similar to the classical case, one can easily prove that the Julia set is not empty. Components of the Fatou set are called *stable domains*. If ∞ belongs to the Fatou set, we denote the corresponding stable domain by $\mathcal{A}_{(c_n)}(\infty)$. This domain need not be invariant(i.e. $f_{c_n}(\mathcal{A}_{(c_n)}(\infty)) \subset \mathcal{A}_{(c_n)}(\infty)$ for all n) or back ward invariant (i.e. $f_{c_n}^{-1}(\mathcal{A}_{(c_n)}(\infty)) \subset \mathcal{A}_{(c_n)}(\infty)$ for all n). but there exists an invariant domain $M = M_{(c_n)} \subset \mathcal{A}_{(c_n)}(\infty)$ which contains the point ∞ and which satisfies $\bigcup_{k=1}^{\infty} F_k^{-1}(M) = \mathcal{A}_{(c_n)}(\infty)$. Therefore, the *filled Julia set* $\mathcal{K}_{(c_n)} := \mathbb{C} - \mathcal{A}_{(c_n)}(\infty)$ and the Julia set $\mathcal{J}_{(c_n)}$ are compact in $\mathbb{C}$, and $\mathcal{K}_{(c_n)}$ is the set of all $z \in \mathbb{C}$ such that $(F_k(z))_{k=1}^{\infty}$ is bounded. Furthermore, we have $\mathcal{J}_{(c_n)} = \partial\mathcal{A}_{(c_n)}(\infty) = \partial\mathcal{K}_{(c_n)}$. Also, $\mathcal{J}_{(c_n)}$ and $\mathcal{K}_{(c_n)}$ are perfect sets.
The function $g_{(c_n)}$ defined by

$$g_{(c_n)}(z) := \lim_{k\to\infty} \tfrac{1}{(d+1)^k} \log^+ |F_k(z)|$$

is continuous in $\mathbb{C}$, $g_{(c_n)}(z) = 0$ for $z \in \mathcal{K}_{(c_n)}$, and it is the Green's function of $\mathcal{K}_{(c_n)}$ with pole at infinity. Furthermore, the functional equation

$$g_{(c_n)}(z) = \lim_{k\to\infty} \frac{1}{(d+1)^k} g_{(c_n)}(F_k(z)) \tag{1}$$

holds.

3 Main Results

Denote $\rho_k = \{\frac{-c_k}{d+1}, -c_k\}$, the critical points of $f_{c_k}(z) = z(z+c_k)^d$. Let $\mathcal{C}_{(c_n)} := \bigcup_{k=1}^{\infty} F_k^{-1}(\rho_{k+1})$, be the critical set of sequence (F_k).

Theorem 3.1 *The Julia set $\mathcal{J}_{(c_n)}$ is connected if and only if $\mathcal{C}_{(c_n)} \subset \mathcal{K}_{(c_n)}$.*

proof. From equation (1) we obtain

$$\mathtt{grad}\, g_{(c_n)}(z) = \lim_{k\to\infty} \tfrac{1}{(d+1)^k} \mathtt{grad}\, g_{(c_n)}(F_k(z)) F_k'(z).$$

For a given $z \in \mathcal{A}_{(c_n)}(\infty)$ we have $\mathtt{grad}\, g_{(c_n)}(F_k(z)) \neq 0$ for k large enough and, therefore, $\mathcal{J}_{(c_n)}$ is connected if and only if $F_k'(z) \neq 0$ on any compact subset of $\mathcal{A}_{(c_n)}(\infty)$ for k sufficiently large. For every $k \in \mathbb{N}$ we have

$$F_k'(z) = \textstyle\prod_{j=0}^{k-1} (F_j(z) + c_{j+1})^{d-1}((d+1)F_j(z) + c_{j+1}).$$

therefore, $F'_k(z_0) = 0$ for some $z_0 \in \mathbb{C}$ if and only if $((d+1)F_j(z_0) + c_{j+1}) = 0$ or $(F_j(z_0) + c_{j+1}) = 0$ for some $j \in \{0, 1, \cdots, k-1\}$, i.e. $z_0 \in \mathcal{C}_{(c_n)}$. $\square$

Consider the following sets:

$$K_\delta^{\mathbb{N}} := K_\delta \times \cdots \times K_\delta^{\mathbb{N}} \text{ (N times)},$$

such that

$K_\delta := \{z \in \mathbb{C}; |z| \leq \delta\}$ for some $\delta > 0, \mathcal{D} := \{(c_n) \in K_\delta^{\mathbb{N}} : \mathcal{J}_{(c_n)}$ is connected ,

and γ such that $(d+2)\gamma \leq 1$

Theorem 3.2 *The set $\mathcal{D}$ is a dense open subset of $K_\delta^{\mathbb{N}}$ provided that $\delta > \gamma$.*

Proof. Let $(c_n^0) \in \mathcal{D}$. There exists $z_0 \in \mathbb{C}$ such that $F_m(z_0) = (f_{c_m^0} \circ \cdots \circ f_{c_1^0})(z_0) = \rho_{m+1} \in \mathcal{C}_{m+1}$ for some $m \in \mathbb{N}$ and $F_n(z_0) \to \infty (n \to \infty)$ which implies that $(f_{c_n^0} \circ \cdots \circ f_{c_{m+1}^0}(\rho_{m+1})) \to \infty (n \to \infty)$. Therefore we may choose R so large and $N \in \mathbb{N}, N > m$ such that $|(f_{c_N^0} \circ \cdots \circ f_{c_{m+1}^0})(\rho_{m+1})| > R$. Since $(f_{c_N^0} \circ \cdots \circ f_{c_{m+1}^0})(\rho_{m+1})$ depends continuously on $c_{m+1}, \cdots, c_N$ there exists a neighborhood $U = U_{m+1} \times \cdots \times U_N \subset K_\delta^{N-m}$ of $(c_{m+1}^0, \cdots, c_N^0)$ such that $|(f_{c_N} \circ \cdots \circ f_{c_{m+1}})(\rho_{m+1})| > R$ for all $(c_{m+1}, \cdots, c_N) \in U$. We set $\mathcal{U} := K_\delta^m \times U \times K_\delta^{\mathbb{N}}$. Then $\mathcal{U}$ is a neighborhood of (c_n^0) with respect to the product topology of $K_\delta^{\mathbb{N}}$.
In order to show that $\mathcal{U} \subset \mathcal{D}$, let $(c_n) \in \mathcal{U}$. We choose $\zeta \in \mathbb{C}$ with $(f_{c_m} \circ \cdots \circ f_{c_1})(\zeta) = \rho_{m+1}$. We have $(c_{m+1}, \cdots, c_N) \in U$ and thus $|(f_{c_N} \circ \cdots \circ f_{c_{m+1}}(\rho_{m+1})| > R$ which means that $|(f_{c_N} \circ \cdots \circ f_{c_1})(\zeta)| > R$. This implies that $|(f_{c_N} \circ \cdots \circ f_{c_1})(\zeta)| \to \infty (n \to \infty)$ and thus $(c_n) \in \mathcal{D}$.
Finally, to show that $\mathcal{D}$ is dense in $K_\delta^{\mathbb{N}}$, Let $(c_n^0) \in K_\delta^{\mathbb{N}}$. We define a sequence $\{c_n^m)\}_{m=1}^\infty$ in $K_\delta^{\mathbb{N}}$ by

$$c_n^m := \begin{cases} c_n^0 \text{ for } n = 1, \cdots, m, \\ c \text{ for } n > m, \end{cases}$$

where $c \in K_\delta^{\mathbb{N}} - \mathcal{M}_d$. Then $(c_n^m) \to (c_n^0)(m \to \infty)$ and $\mathcal{J}(f_c)$ is disconnected. Since $\mathcal{J}_{(c_n^m)} = (f_{(c_m^0)} \circ \cdots \circ f_{(c_1^0)})^{-1}(\mathcal{J}(f_c))$, the Julia sets $\mathcal{J}_{(c_n^m)}$ are also disconnected which means that $(c_n^m) \in \mathcal{D}$ for all $m \in \mathbb{N}$. $\square$

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