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Connecting Singular Control with Optimal Switching

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Abstract— **We summarize our recent work [1], [2] on a new theoretical connection between singular control of finite variation and optimal switching problems. This correspondence not only provides a novel method for analyzing multi-dimensional singular control problems, but also builds links among singular controls, Dynkin games, and sequential optimal stopping problems.**

I. INTRODUCTION

In our recent work ([1]), we established a generic theoretical connection between singular control and optimal switching problems: we defined a consistency property for collections of switching controls, and proved that there is an exact correspondence between the set of finite variation càglàd processes and the set of consistent collections of switching controls.

This correspondence allows one to analyze multi-dimensional control problem under a general setting for the regularity properties and the smooth fit principle *directly*: one can obtain an integral representation for the value function of a general class of singular control problem in terms of the values of corresponding optimal switching problems.

As a byproduct, we showed that the value of a Dynkin game can be represented as the difference between the values of two related switching problems, thereby linking the general reversible investment problem, the Dynkin game, and the optimal switching problem.

Continuing our analysis on singular control problems with possible non-smooth payoff functions, we ([2]) analyzed a class of singular control problems for which value functions are not necessarily smooth. Necessary and sufficient conditions for the well-known smooth fit principle, along with the regularity of the value functions, are given. Explicit solutions for the optimal policy and for the value functions are provided. In particular, when payoff functions satisfy the usual Inada conditions, the boundaries between action and no-action regions are smooth and strictly monotonic as postulated and exploited in the existing literature ([3], [4], [5], [6], [7], [8], [9], [10]). Illustrative examples for both smooth and non-smooth cases are discussed to highlight the pitfall of solving singular control problems with *a priori* smoothness assumptions.

a) Previous work: Singular control problems have been studied extensively in both the mathematics and economics, starting from the well-known monotone fuel follower problem, for which explicit solutions can be found in [11], [12], [13] and [14]. In mathematical economics, a typical (ir)reversible investment problem can be formulated as a singular control problem in which a company, by adjusting its production capacity through expansion and contraction according to market fluctuations, wishes to maximize its overall expected net profit over an infinite horizon. This problem has been investigated by numerous authors (See for instance [4], [5], [6], [15], [7], [8], [16], [17], [18], [19], and [9]). For a standard reference on irreversible investment, see [3].

Our approach of connecting singular control problems and related optimal stopping problems dates back to the seminal paper of [11], and has since been developed and applied to monotone singular control problems by [20], [21], [22], [23], and [15]. Indeed, our integral representation theorem for the reversible investment problem is in part inspired by the elegant integration arguments of [15] for irreversible investment. Another closely related body of work is [24], [25], [26]. However, the connections between the singular control problem, the entry-exit problem, and Dynkin's game in their works are established within the framework of forward backward stochastic differential equations and require a finite time horizon with the restrictive assumption that the control has only an additive affect on the diffusion.

b) Our contribution: Compared to all previous works and approaches, the correspondence between singular controls and switching controls in our paper *does not depend on the specific form of the control problem.* Thus, our methodology may be applied to cases for which the *underlying randomness is not necessarily captured by a diffusion* and *the payoff function is not necessarily smooth.*

II. CORRESPONDENCE BETWEEN SINGULAR CONTROLS AND SWITCHING CONTROLS

The correspondence established in [1] is analogous to the well-known correspondence between a non-decreasing, $\mathbb{F}$-adapted, càglàd singular control $(\xi_t)_{t\geq 0}$ and a collection of stopping times $(\tau^{\xi}(z))_{z\in\mathbb{R}}$, given by

$$\tau^{\xi}(z) = \inf\{t \geq 0 : \xi_t > z\}, \xi_t = \sup\{z \in \mathbb{R} : \tau^{\xi}(z) < t\}.$$

A. Definitions

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $\mathbb{F} = \{\mathcal{F}_t; 0 \leq t < \infty\}$ a filtration satisfying the usual hypotheses. Let $\mathcal{I} \subset \mathbb{R}$ be an open (possibly unbounded) interval, and $\bar{\mathcal{I}}$ be its closure.

Definition 2.1: Given $y \in \bar{\mathcal{I}}$, an admissible singular control is a pair $(\xi_t^+, \xi_t^-)_{t\geq 0}$ of $\mathbb{F}$-adapted, non-decreasing càglàd processes such that $\xi^+(0) = \xi^-(0) = 0, Y_t :=$

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$y + \xi_t^+ - \xi_t^- \in \bar{\mathcal{I}}, \forall t \in [0, \infty)$, and $d\xi^+$, $d\xi^-$ are supported on disjoint subsets.

We denote here $\mathcal{A}_y$ to be the set of admissible strategies corresponding to an initial capacity level of y.

Since $d\xi^+$, $d\xi^-$ are supported on disjoint subsets, ξ^+ and ξ^- are the positive and negative variation of Y, respectively. By the uniqueness of the variation decomposition, there is a one-to-one correspondence between strategies $(\xi^+, \xi^-) \in \mathcal{A}_y$ and $\mathbb{F}$-adapted càglàd finite variation processes Y with $Y_0 = y$ and $Y_t \in \bar{\mathcal{I}}$ for all t.

Throughout the paper, $(Y_t)_{t\geq 0}$ *is a finite variation control process with* $Y_0 = y$.

Definition 2.2: A switching control $\alpha = (\tau_n, \kappa_n)_{n\geq 0}$ consists of an increasing sequence of stopping times $(\tau_n)_{n\geq 0}$ and a sequence of new regime values $(\kappa_n)_{n\geq 0}$ that are assumed immediately after each stopping time.

When there are only two distinct regimes, an optimal switching problem is often referred to as the starting and stopping problem ([27], [28], etc.) or the entry and exit problem ([25], [29], etc.). Following convention, we label the two regimes 0 and 1.

Definition 2.3: A switching control $\alpha = (\tau_n, \kappa_n)_{n\geq 0}$ is admissible if the following hold almost surely: $\tau_0 = 0$, $\tau_{n+1} > \tau_n$ for $n \geq 1$, $\tau_n \to \infty$, and for all $n \geq 0$, $\kappa_n \in \{0, 1\}$ is $\mathcal{F}_{\tau_n}$ measurable, with $\kappa_n = \kappa_0$ for even n and $\kappa_n = 1 - \kappa_0$ for odd n.

Alternatively, an admissible switching control has a more mathematically convenient representation given by its regime indicator function.

Proposition 2.4: There is a one-to-one correspondence between admissible switching controls and the regime indicator function $I_t(\omega)$, which is an $\mathbb{F}$-adapted càglàd process of finite variation, so that $I_t(\omega) : \Omega \times [0, \infty) \to \{0, 1\}$, with

$$I_t := \sum_{n=0}^{\infty} \kappa_n 1_{\{\tau_n < t \leq \tau_{n+1}\}}, \quad I_0 = \kappa_0. \tag{1}$$

Definition 2.5: Let $y \in \bar{\mathcal{I}}$ be given, and for each $z \in \mathcal{I}$, let $\alpha(z) = (\tau_n(z), \kappa_n(z))_{n\geq 0}$ be a switching control. The collection $(\alpha(z))_{z\in\mathcal{I}}$ is consistent if

$$\alpha(z) \text{ is admissible for Lebesgue-almost every } z \in \mathcal{I}, \tag{2}$$

$$I_0(z) := \kappa_0(z) = 1_{\{z\leq y\}}, \text{ for Lebesgue-almost every } z \in \mathcal{I}, \tag{3}$$

and for all $t < \infty$,

$$\int_{\mathcal{I}} (I_t^+(z) + I_t^-(z))dz < \infty, \text{ almost surely, and} \tag{4}$$

$$I_t(z) \text{ is decreasing in } z \text{ for } \mathbb{P} \otimes dz\text{-almost every } (\omega, z). \tag{5}$$

Here $I_t(z)$, $I_t^+(z)$ and $I_t^-(z)$ are $I_t = \kappa_0 + I_t^+ - I_t^-$, and I_t^+ (I_t^-) is the positive (negative) variation of the corresponding regime indicator function such that $I_t^+ := \sum_{n>0,\kappa_n=1}^{\infty} 1_{\{\tau_n<t\}}$, $I_0^+ = 0$, $I_t^- := \sum_{n>0,\kappa_n=0}^{\infty} 1_{\{\tau_n<t\}}$, and $I_0^- = 0$.

For $I_t(z)$ to be decreasing in z for $\mathbb{P} \otimes dz$-almost every (ω, z), it means there exists a set $E \subset \Omega \times \bar{\mathcal{I}}$ such that $\mathbb{P} \otimes dz(E) = 0$ and if $(\omega, z_0), (\omega, z_1) \in (\Omega \times \bar{\mathcal{I}}) \backslash E$ with $z_0 \leq z_1$, then $I_t(\omega, z_0) \geq I_t(\omega, z_1)$.

B. Bijection

The bijection between the singular control and the switching control was established based on a relatively old result in analysis [30, Theorem 5.5.1].

Proposition 2.6 (From Singular to Switching Controls): Given $(\xi^+, \xi^-) \in \mathcal{A}_y$, define a switching control $\alpha(z) = (\tau_n(z), \kappa_n(z))_{n\geq 0}$ for each $z \in \mathcal{I}$ through the regime indicator function $I_t(z) := \lim_{s\uparrow t} 1_{\{Y_s>z\}}$. Then, the resulting collection $(\alpha(z))_{z\in\mathcal{I}}$ of switching controls is consistent.

Proposition 2.7 (From Switching to Singular Controls): Given $y \in \bar{\mathcal{I}}$ and a consistent collection of switching controls $(\alpha(z))_{z\in\mathcal{I}}$, define two processes ξ^+ and ξ^- by setting $\xi_0^+ = 0$, $\xi_0^- = 0$, and for $t > 0$: $\xi_t^+ := \int_{\mathcal{I}} I_t^+(z)dz$, $\xi_t^- := \int_{\mathcal{I}} I_t^-(z)dz$. Then

(1) The pair $(\xi^+, \xi^-) \in \mathcal{A}_y$ is an admissible singular control,

(2) Up to indistinguishability, $Y_t = y + \int_y^{\infty} I_t(z)1_{\{z\in\mathcal{I}\}}dz + \int_{-\infty}^{y} (I_t(z) - 1)1_{\{z\in\mathcal{I}\}}dz$, and

(3) For all t, we almost surely have

$$Y_t = \operatorname{ess\,sup}\{z \in \mathcal{I} : I_t(z) = 1\} = \operatorname{ess\,inf}\{z \in \mathcal{I} : I_t(z) = 0\},$$

where $\operatorname{ess\,sup} \emptyset := \inf \mathcal{I}$ and $\operatorname{ess\,inf} \emptyset := \sup \mathcal{I}$.

Proposition 2.8 (One-to-One Mapping): The mapping from consistent collections of switching controls to singular controls defined by Proposition 2.7 is one-to-one.

Theorem 2.9 (Bijection): The mappings in Propositions 2.6 and 2.7 define a bijection between admissible singular controls $(\xi^+, \xi^-) \in \mathcal{A}_y$ and consistent collections of switching controls (up to equivalence).

C. Change of Variable Formula

With the bijection established in Theorem 2.9, we established a change of variable formula for integration with respect to the variation of a singular control.

Proposition 2.10: Let $(\xi^+, \xi^-) \in \mathcal{A}_y$ be an admissible singular control and $(\alpha(z))_{z\in\mathcal{I}}$ be the corresponding collection of switching controls. For every càdlàg process $g : \Omega \times [0, \infty] \to [0, \infty)$ with $g(\infty) \equiv 0$,

$$\int_{[0,\infty)} g(t)d\xi_t^+ = \int_{\mathcal{I}} \sum_{\substack{n>0 \\ \kappa_n=1}} g(\tau_n(z))dz, \quad a.s., \text{ and}$$

$$\int_{[0,\infty)} g(t)d\xi_t^- = \int_{\mathcal{I}} \sum_{\substack{n>0 \\ \kappa_n=0}} g(\tau_n(z))dz, \quad a.s.$$

In particular, when Y is non-decreasing (i.e. $\xi^- \equiv 0$), $\bar{\mathcal{I}} = [0, \infty)$ and $y \geq 0$, we have $\tau_n(z) \equiv 0$ for all $n > 1$, and for $n = 1$ when $z \leq y$. In this case, our change of variable formula reduces to the one for monotone controls in [15], after adjusting for notational differences,

$$\int_{[0,\infty)} g(t)d\xi_t^+ = \int_y^{\infty} g(\tau_1(z))dz.$$

III. Application: Analysis of a Class of Singular Control Problems

Having established the correspondence between singular controls and consistent collections of switching controls, we showed how this theory can be applied to analyzing singular control problems.

A. The Singular Control Problem

Consider the following class of singular control problems from economics named *reversible* investment problem: a company adjusts its reversible production capacity (or investment) level by proper controls of expansion and contraction in the presence of a stochastic economic environment. The net profit of such an investment depends on the running production function of the actual capacity, the economic uncertainty such as price or demand for the product, the benefits of contraction (e.g. via spinning off part of the business), and the cost of expanding and reducing the capital. The company's objective is to maximize the expected profit over an infinite time horizon by controlling expansion and contraction.

Let the unit cost of increasing the capacity at time t be $\gamma_+(\omega, t) : \Omega \times [0, \infty) \to \mathbb{R}$, the unit cost of decreasing capacity be $\gamma_-(\omega, t) : \Omega \times [0, \infty) \to \mathbb{R}$, with both γ_+ and γ_- adapted to $\mathbb{F}$. Let ξ_t^+ and ξ_t^- represent the cumulative expansion and reduction of capital until time t respectively, both $\mathbb{F}$-adapted, non-decreasing càglàd processes. Let $Y_t = y + \xi_t^+ - \xi_t^- \in \bar{\mathcal{I}}$ with $\xi^+(0) = \xi^-(0) = 0$. The objective is to solve the following optimization problem defined as:

$$V(y) := \sup_{(\xi^+,\xi^-)\in\mathcal{A}'_y} J(y, \xi^+, \xi^-), \tag{6}$$

Here

$$J(y, \xi^+, \xi^-) := \mathbb{E}\left[\int_0^\infty \Pi(t, Y_t)dt - \int_{[0,\infty)} \gamma_+(t)d\xi_t^+ - \int_{[0,\infty)} \gamma_-(t)d\xi_t^-\right], \tag{7}$$

with $\Pi(\omega, t, z) : \Omega\times[0, \infty)\times\bar{\mathcal{I}} \to \mathbb{R}$ being the instantaneous operating profit and the maximization is over the set of integrable strategies $\mathcal{A}'_y \subset \mathcal{A}_y$. Note that for any $y \in \bar{\mathcal{I}}$, $\mathcal{A}'_y$ is not empty, as the expected profit of not investing at all (i.e. $\xi^+ \equiv 0 \equiv \xi^-$) is finite and is given by

$$R(y) := J(y, 0, 0) = \mathbb{E}\left[\int_0^\infty \Pi(t, y)dt\right]. \tag{8}$$

c) Standing assumptions:

$\mathbb{A}1)$ Π is concave in y and continuous at the boundary of $\mathcal{I}$, so that for $y_1 < y_2 \in \bar{\mathcal{I}}$,

$$\Pi(t, y_2) - \Pi(t, y_1) := \int_{y_1}^{y_2} \pi(t, z)dz, \tag{9}$$

where π is decreasing in z a.s. and adapted to $\mathbb{F}$. Furthermore,

$$\mathbb{E}\left[\int_0^\infty |\Pi(t, z)|dt\right] < \infty, \quad \forall z \in \bar{\mathcal{I}}, \tag{10}$$

$$\mathbb{E}\left[\int_0^\infty |\pi(t, z)|dt\right] < \infty, \quad \forall z \in \mathcal{I}. \tag{11}$$

$\mathbb{A}2)$ γ_+ and γ_- are adapted to $\mathbb{F}$, $\gamma^\pm(\infty) := 0$ and

$$\gamma_+(t) + \gamma_-(t) > 0, \quad \text{for all } t, \text{ a.s.} \tag{12}$$

$\mathbb{A}3)$ If $\mathcal{I}$ is not bounded above, then $\gamma_+(t) \geq 0$ for all t almost surely. And, if $\mathcal{I}$ is not bounded below, $\gamma_-(t) \geq 0$ for all t almost surely.

B. Optimal Control from its Corresponding Optimal Switching Problem

The key to using the connection between singular controls and switching controls to solve problem (6) in Section III-A is to write the payoff of this problem in terms of the payoffs of its corresponding optimal switching problems.

1) Switching Controls from Singular Controls: First, given the running profit and cost functions from the singular control problem (6), define a collection of optimal switching problems, indexed by $z \in \mathcal{I}$.

Definition 3.1: The switching cost process $\gamma : \Omega \times [0, \infty) \times \{0, 1\} \to \mathbb{R}$ is given by $\gamma(t, \kappa) := \gamma_+(t)1_{\{\kappa=1\}} + \gamma_-(t)1_{\{\kappa=0\}}$. Here $\gamma(t, \kappa)$ represents the cost of switching to regime κ at time t.

Proposition 3.2: Assume $(\xi^+, \xi^-) \in \mathcal{A}'_y$. Let $(\alpha(z))_{z\in\mathcal{I}}$ be the corresponding consistent collection of switching controls with regime indicator functions $I(z)$, then $J(y, \xi^+, \xi^-) - R(y) = \int_y^\infty m_+(z, \alpha(z))1_{\{z\in\mathcal{I}\}}dz + \int_{-\infty}^y m_-(z, \alpha(z))1_{\{z\in\mathcal{I}\}}dz$, where

$$m_+(z, \alpha) := \mathbb{E}\left[\int_0^\infty \pi(t, z)I_tdt - \sum_{n=1}^\infty \gamma(\tau_n, \kappa_n)\right], \tag{13}$$

$$m_-(z, \alpha) := \mathbb{E}\left[\int_0^\infty -\pi(t, z)(1 - I_t)dt - \sum_{n=1}^\infty \gamma(\tau_n, \kappa_n)\right]. \tag{14}$$

Here $m_+(z, \alpha), m_-(z, \alpha)$ are two expected payoffs for the switching controls for each $z \in \mathcal{I}$ and $\alpha \in \mathcal{B}$, with $\kappa_0 = k \in \{0, 1\}$.

2) Representation Theorem: Now, for each $z \in \mathcal{I}$, the optimal switching control problem is to maximize the expected payoff over possible switching controls $\alpha \in \mathcal{B}$ such that $\kappa_0 = k \in \{0, 1\}$. This leads to value functions given by

$$m_+^*(z, k) := \sup_{\substack{\alpha\in\mathcal{B}\\ \kappa_0=k}} m_+(z, \alpha), \tag{15}$$

$$m_-^*(z, k) := \sup_{\substack{\alpha\in\mathcal{B}\\ \kappa_0=k}} m_-(z, \alpha), \tag{16}$$

where $m_+(z, \alpha)$ and $m_-(z, \alpha)$ are given by (13) and (14).

Theorem 3.3 (Representation): Fix $y \in \bar{\mathcal{I}}$, let $V(y)$ and $R(y)$ be given from (6), $m_+^*(z, k)$ and $m_-^*(z, k)$ be given by (15) and (16), and $(\hat{\xi}^{j+}, \hat{\xi}^{j-}) \in \mathcal{A}_y$ be the corresponding singular control as per Proposition 2.7. Assume there is a sequence of consistent collections of switching controls $(\alpha_j(z))_{z\in\mathbb{R}}$ so that as $j\to\infty$,

$$\int_y^\infty m_+(z,\alpha_j(z))1_{\{z\in\mathcal{I}\}}dz+\int_{-\infty}^y m_-(z,\alpha_j(z))1_{\{z\in\mathcal{I}\}}dz$$
$$\to\int_y^\infty m_+^*(z,0)1_{\{z\in\mathcal{I}\}}dz+\int_{-\infty}^y m_-^*(z,1)1_{\{z\in\mathcal{I}\}}dz.$$

Assume also $(\hat{\xi}^{j+},\hat{\xi}^{j-})\in\mathcal{A}'_y$ for all j. Then, $V(y)-R(y)=\int_y^\infty m_+^*(z,0)1_{\{z\in\mathcal{I}\}}dz+\int_{-\infty}^y m_-^*(z,1)1_{\{z\in\mathcal{I}\}}dz$.

With stronger assumptions, one can further establish the existence of an optimal control strategy.

Assumption 3.4:

1) [Existence of consistent controls] Fix $y\in\bar{\mathcal{I}}$ and let $m_+^*(z,k)$ and $m_-^*(z,k)$ be given by (15) and (16). For almost all $z\in\mathcal{I}$, there exists an optimal admissible switching control $\alpha(z)\in\mathcal{B}$ such that

$$m_+^*(z,0)=m_+(z,\alpha(z)),\qquad \text{for } z>y,$$
$$\text{and,}\quad m_+^*(z,1)=m_+(z,\alpha(z)),\qquad \text{for } z\le y.$$

Furthermore, the collection $(\alpha(z))_{z\in\mathbb{R}}$ is consistent.

2) [Integrability of singular control] Let $(\hat{\xi}^+,\hat{\xi}^-)\in\mathcal{A}_y$ be the corresponding singular control as per Proposition 2.7, then $(\hat{\xi}^+,\hat{\xi}^-)\in\mathcal{A}'_y$.

Theorem 3.5 (Representation and Existence): Under Assumption 3.4, the Representation Theorem 3.3 holds. Moreover, the strategy $(\hat{\xi}^+,\hat{\xi}^-)$ is optimal.

Theorem 3.6 (Sufficient Condition for Integrability): Let $\mathcal{I}$ be bounded, assume 3.4.1 and let $(\hat{\xi}^+,\hat{\xi}^-)$ be the corresponding singular control as per Proposition 2.7. Furthermore, suppose

(1) $\sup_{0\le t\le T}\sup_{z\in\mathcal{I}}|\Pi(\omega,t,z)|<\infty$, almost surely, for all $T>0$,

(2) $\limsup_{T\to\infty}\mathbb{E}\left[|\gamma_+(T)|+|\gamma_-(T)|\right]<\infty$, and

(3) For every strategy $(\xi^+,\xi^-)\in\mathcal{A}_y$, either $(\xi^+,\xi^-)\in\mathcal{A}'_y$; Or, there exists an $\mathbb{F}$-adapted process Z such that $U_.\le Z_.$ almost surely, $\mathbb{E}[|Z_T|]<\infty$ for all $T\ge 0$, and $\limsup_{T\to\infty}\mathbb{E}[Z_T]=-\infty$, where $U_T(y,\xi^+,\xi^-):=\int_0^T\Pi(t,Y_t)dt-\int_{[0,T)}\gamma_+(t)d\xi_t^+-\int_{[0,T)}\gamma_-(t)d\xi_t^-$.

Then $(\hat{\xi}^+,\hat{\xi}^-)\in\mathcal{A}'_y$. Hence Assumption 3.4 holds, yielding Theorem 3.5.

C. Regularity of the Value Function and Dynkin's Game

Based on the representation theorem, we provided conditions under which the value function of the switching controls is not only continuous, but also continuously differentiable.

Theorem 3.7 (Regularity): Suppose that for some open interval $\mathcal{J}\subset\mathcal{I}$ and any $y\in\mathcal{J}$,

$$\lim_{z\to y}\mathbb{E}\left[\int_0^\infty|\pi(t,z)-\pi(t,y)|dt\right]=0. \qquad (17)$$

Suppose also that on $\mathcal{J}$, the value function has the representation $V(y)-R(y)=\int_y^\infty m_+^*(z,0)1_{\{z\in\mathcal{I}\}}dz+\int_{-\infty}^y m_-^*(z,1)1_{\{z\in\mathcal{I}\}}dz$. Then V is C^1 on $\mathcal{J}$. And for any $y\in\mathcal{J}$, $V'(y)=\mathbb{E}\left[\int_0^\infty\pi(t,y)dt\right]+m_-^*(y,1)-m_+^*(y,0)=m_+^*(y,1)-m_+^*(y,0)$.

1) Dynkin Games: A Dynkin game is a game of timing between two players, whom we call MAX and MIN, following [26]. We fix some level $z\in\mathcal{I}$. While the game is in progress, MIN pays MAX at rate $\pi(t,z)$ and the game ends when one player chooses to stop. Thus, MAX and MIN each chooses strategies on when to exit the game (the stopping times σ_- and σ_+ respectively). The player to exit first receives an amount from her opponent equal to $\gamma_-(\sigma_-)$ if MAX exits first, and $\gamma_+(\sigma_+)$ if MIN exits first. If both players exit at the same time, we treat it as though MIN exited first. Furthermore, each player may choose never to exit, i.e. $\sigma=\infty$. MAX chooses her strategy σ_- to maximize her payoff, and MIN chooses σ_+ in order to minimize MAX's payoff.

This game is formally described below. To ensure that the payoff of the game is well defined, we assume in this section that for every stopping time σ, $\mathbb{E}[|\gamma_-(\sigma)|]<\infty$ and $\mathbb{E}[|\gamma_+(\sigma)|]<\infty$.

Definition 3.8: Given $z\in\mathcal{I}$ and $\mathbb{F}$-stopping times σ_- and σ_+, the payoff of the Dynkin game is $D(\sigma_-,\sigma_+;z)=\int_0^{\sigma_-\wedge\sigma_+}\pi(t,z)dt+\gamma_+(\sigma_+)1_{\{\sigma_+\le\sigma_-\}}-\gamma_-(\sigma_-)1_{\{\sigma_-<\sigma_+\}}$. The game has a value if $\sup_{\sigma_-}\inf_{\sigma_+}\mathbb{E}\left[D(\sigma_-,\sigma_+;z)\right]=\inf_{\sigma_+}\sup_{\sigma_-}\mathbb{E}\left[D(\sigma_-,\sigma_+;z)\right]$.

We have,

Theorem 3.9: Given any $z\in\mathcal{I}$ such that conditions (11) and (12) hold, the value of the Dynkin game exists, and is equal to $m_+^*(z,1)-m_+^*(z,0)=\sup_{\sigma_+}\inf_{\sigma_-}\mathbb{E}\left[D(\sigma_-,\sigma_+;z)\right]=\inf_{\sigma_-}\sup_{\sigma_+}\mathbb{E}\left[D(\sigma_-,\sigma_+;z)\right]$.

IV. Explicit Solutions and Smooth Fit Principle

To further analyze regularity properties and establish necessary and sufficient conditions for the smooth fit principle, we ([2]) studied the following specific problem:

$$V(x,y):=\sup_{(\xi^+,\xi^-)\in\mathcal{A}'_y}J(x,y;\xi^+,\xi^-), \qquad (18)$$

with

$$J(x,y;\xi^+,\xi^-):=\mathbb{E}\left[\int_0^\infty e^{-\rho t}H(Y_t)X_t^x dt-\int_0^\infty e^{-\rho t}K_1 d\xi_t^+-\int_0^\infty e^{-\rho t}K_0 d\xi_t^-\right], \qquad (19)$$

subject to

$$Y_t:=y+\xi_t^+-\xi_t^-,\quad y\in[a,b],$$
$$dX_t^x:=\mu X_t^x dt+\sqrt{2}\sigma X_t^x dW_t,\quad X_0:=x>0,$$
$$H:[a,b]\to\mathbb{R}\text{ is concave with }H(y)=H(a)+\int_a^y h(z)dz,$$
$$K_1+K_0>0,\mu<\rho,\quad K_1>0.$$

The supremum is taken over all strategies $(\xi^+,\xi^-)\in\mathcal{A}'_y$, where

$$
\begin{aligned}
\mathcal{A}'_y := \{(\xi^+, \xi^-) : \ & \xi^\pm \text{ are left continuous,} \\
& \text{non-decreasing processes, } \ \xi_0^\pm = 0; \\
& y + \xi_t^+ - \xi_t^- \in [a, b]; \\
& \mathbb{E}\left[\int_0^\infty e^{-\rho t} d\xi_t^+ + \int_0^\infty e^{-\rho t} d\xi_t^-\right] < \infty\}.
\end{aligned}
$$

Theorem 4.1: **[Value function]**

$$V(x, y) = \eta H(a)x + \int_a^y v_1(x, z)dz + \int_y^b v_0(x, z)dz, \tag{20}$$

where v_0 and v_1 are given explicitly based on K_0:

Case I ($K_0 \geq 0$):

1) For each $z \in (a, b)$ such that $h(z) = 0 : v_0(x, z) = v_1(x, z) = 0$.
2) For each $z \in (a, b)$ such that $h(z) > 0$:

$$\begin{cases} v_0(x,z) & = \begin{cases} A(z)x^n, & x < G(z), \\ \eta h(z)x - K_1, & x \geq G(z), \end{cases} \\ v_1(x,z) & = \eta h(z)x, \end{cases}$$

where $G(z) = \nu h(z)^{-1}$, and $A(z) = \frac{K_1}{(n-1)}(\frac{h(z)}{\nu})^n$, with $\nu = K_1\sigma^2 n(1-m)$.

3) For each $z \in (a, b)$ such that $h(z) < 0$:

$$\begin{cases} v_0(x,z) & = 0, \\ v_1(x,z) & = \begin{cases} B(z)x^n + \eta h(z)x, & x < F(z), \\ -K_0, & x \geq F(z), \end{cases} \end{cases}$$

where $F(z) = -\frac{\kappa}{h(z)}$, and $B(z) = \frac{K_0}{(n-1)}\kappa^{-n}(-\frac{h(z)}{\kappa})^n$, with $\kappa = K_0\sigma^2 n(1-m)$.

Case II ($K_0 < 0$):

1) For each $z \in (a, b)$ such that $h(z) \leq 0$: $v_0(x, z) = 0, v_1(x, z) = -K_0$.
2) For each $z \in (a, b)$ such that $h(z) > 0$:

$$v_0(x,z) = \begin{cases} A(z)x^n, & x < G(z), \\ B(z)x^m + \eta h(z)x - K_1, & x \geq G(z), \end{cases} \tag{21}$$

$$v_1(x,z) = \begin{cases} A(z)x^n - K_0, & x \leq F(z), \\ B(z)x^m + \eta h(z)x, & x > F(z). \end{cases} \tag{22}$$

Here

$$
\begin{aligned}
A(z) &= \frac{h(z)^n}{(n-m)\nu^n}\left(\frac{\nu}{\sigma^2(n-1)} + mK_1\right) \\
&= \frac{h(z)^n}{(n-m)\kappa^n}\left(\frac{\kappa}{\sigma^2(n-1)} - mK_0\right); \qquad (23) \\
B(z) &= \frac{-h(z)^m}{(n-m)\nu^m}\left(\frac{\nu}{\sigma^2(1-m)} - nK_1\right) \\
&= \frac{-h(z)^m}{(n-m)\kappa^m}\left(\frac{\kappa}{\sigma^2(1-m)} + nK_0\right). \qquad (24)
\end{aligned}
$$

The functions F and G are non-decreasing with $F(z) = \frac{\kappa}{h(z)}$ and $G(z) = \frac{\nu}{h(z)}$, with $\kappa < \nu$ being the unique solutions to $\frac{1}{1-m}\left[\nu^{1-m} - \kappa^{1-m}\right] = -\frac{\rho}{m}\left[K_1\nu^{-m} + K_0\kappa^{-m}\right], \frac{1}{n-1}\left[\nu^{1-n} - \kappa^{1-n}\right] = \frac{\rho}{n}\left[K_1\nu^{-n} + K_0\kappa^{-n}\right]$. Here, $m < 0 < 1 < n$ are the roots of $\sigma^2x^2 + (\mu - \sigma^2)x - \rho = 0$ and $\eta := \frac{1}{\rho-\mu} = \frac{-mn}{(n-1)(1-m)\rho} = \frac{1}{\sigma^2(n-1)(1-m)}$.

Theorem 4.2: **[Optimal control]** The optimal singular control $(\hat{\xi}^+, \hat{\xi}^-) \in \mathcal{A}'_y$ exits. For each $z \in (a, b)$, the optimal control is described in terms of $F(z)$ and $G(z)$ from Theorem 4.1 such that

- (Case I, $K_0 \geq 0$): For z such that $h(z) > 0$, it is optimal to invest in the project past level z when $X_t^x \in [G(z), \infty)$, and never disinvest. When $h(z) < 0$, it is optimal to disinvest below level z when $X_t^x \in [F(z), \infty)$, and it is never optimal to invest. When $h(z) = 0$, it is optimal to neither invest nor disinvest (i.e. $F(z) = \infty = G(z)$).
- (Case II, $K_0 < 0$): For z such that $h(z) > 0$, it is optimal to invest in the project past level z when $X_t^x \in [G(z), \infty)$, and to disinvest below level z when $X_t^x \in (0, F(z)]$. For z such that $h(z) \leq 0$, it is always optimal to disinvest.

Theorem 4.3: **[Optimally controlled process]** The resulting optimal control process $\hat{Y}_t$ is give by:

Case I: (up to indistinguishability) for $t > 0$,

- If $h(y^+) > 0$ then $\hat{Y}_t = \max\{G^{\rightarrow}(M_t), y\}$,
- If $h(y+) = 0$ or $h(y-) = 0$ then $\hat{Y}_t = y$,
- If $h(y-) < 0$ then $\hat{Y}_t = \min\{F^{\rightarrow}(M_t), y\}$.

Here $M_t = \max\{X_s^x : s \in [0, t]\}$, and $F^{\rightarrow}$ and $G^{\rightarrow}$ are respectively the left-continuous inverses of F (non-increasing) and G (non-decreasing).

Case II: (up to indistinguishability) for $t > 0$,

$$\hat{Y}_t = \begin{cases} G^{\rightarrow}(M_t^0) \vee y, & \text{on } \{t \leq S_1\}, \\ F^{\leftarrow}(m_t^n) \wedge \hat{Y}_{S_n}, & \text{on } \{S_n < t \leq T_n\}, \\ G^{\rightarrow}(M_t^n) \vee \hat{Y}_{T_n}, & \text{on } \{T_n < t \leq S_{n+1}\}, \end{cases} \tag{25}$$

and $\lim_{n\to\infty} S_n = \infty = \lim_{n\to\infty} T_n$ almost surely.

Here $F^{\leftarrow}(x)$ and $G^{\rightarrow}(x)$ are respectively the right continuous inverse of F and the left-continuous inverse of G. Moreover, the stopping times (S_n) and (T_n) are given by $S_1 = \inf\{t > 0 : (X_t^x, \hat{Y}_t) \in \mathcal{S}_0\}$, $T_1 = \inf\{t > S_1 : (X_t^x, \hat{Y}_t) \in \mathcal{S}_1\}$, $S_n = \inf\{t > T_{n-1} : (X_t^x, \hat{Y}_t) \in \mathcal{S}_0\}$, and $T_n = \inf\{t > S_n : (X_t^x, \hat{Y}_t) \in \mathcal{S}_1\}$. Lastly, the processes M_t^n, m_t^n are defined by $M_t^0 = \max\{X_t^x : 0 \leq s \leq t\}$, and $m_t^n = \min\{X_t^x : S_n \leq s \leq t\}1_{\{S_n \leq t\}}$, and $M_t^n = \max\{X_t^x : T_n \leq s \leq t\}1_{\{T_n \leq t\}}$.

Theorem 4.4: **[Sufficient Conditions]** $V(x, y)$ is C^1 in x for all $(x, y) \in (0, \infty) \times [a, b]$, and

$$\frac{\partial}{\partial x}V(x, y) = \eta H(a) + \int_a^y \frac{\partial}{\partial x}v_1(x, z)dz + \int_y^b \frac{\partial}{\partial x}v_0(x, z)dz.$$

Moreover, if H is C^1 on an open interval $\mathcal{J} \subset [a, b]$, then $V(x, y)$ is C^1 in y on $(0, \infty) \times \mathcal{J}$; that is, $V(x, y)$ is $C^{1,1}$ on $(0, \infty) \times \mathcal{J}$.

Theorem 4.5: **[Necessary and Sufficient Conditions for Smooth Fit]** $V(x, y)$ is continuously differentiable in x for all $(x, y) \in (0, \infty) \times [a, b]$. $V(x, y)$ is differentiable in y at the point (x, y) if and only if

$$(x, y) \in \{(x, y) \in (0, \infty) \times (a, b) : H \text{ is differentiable at } y\} \cup \mathcal{S}_0 \cup \mathcal{S}_1,$$

where $\mathcal{S}_0$ and $\mathcal{S}_1$ are given in Eq. (26). Alternatively, it is not differentiable in y at the point (x, y) if and only if

$$(x, y) \in \{(x, y) \in (0, \infty) \times (a, b) : H \text{ not differentiable at } y\} \cap \mathcal{C}.$$

Theorem 4.6: **[Region characterization]** Under the optimal singular control $(\hat{\xi}^+, \hat{\xi}^-) \in \mathcal{A}'_y$, define the corresponding investment ($\mathcal{S}_1$), disinvestment ($\mathcal{S}_0$), and continuation ($\mathcal{C}$) regions by

$$\begin{cases} \mathcal{S}_0 & := \begin{cases} \{(x, z) \in (0, \infty) \times [a, b] : x \geq \lim_{w \uparrow z} F(w)\}, \\ \quad \text{if } K_0 \geq 0 \text{ (Case I)}, \\ \{(x, z) \in (0, \infty) \times [a, b] : x \leq \lim_{w \uparrow z} F(w)\}, \\ \quad \text{if } K_0 < 0 \text{ (Case II)}, \end{cases} \\ \mathcal{S}_1 & := \{(x, z) \in (0, \infty) \times [a, b] : x \geq \lim_{w \downarrow z} G(w)\}, \\ \mathcal{C} & := (0, \infty) \times [a, b] \setminus (\mathcal{S}_0 \cup \mathcal{S}_1). \end{cases} \tag{26}$$

Then, the action and continuation regions can be characterized as

$$\begin{cases} \mathcal{S}_0 = \{(x, y) \in (0, \infty) \times [a, b] : V_y(x, y) = -K_0\}, \\ \mathcal{S}_1 = \{(x, y) \in (0, \infty) \times [a, b] : V_y(x, y) = K_1\}, \\ \mathcal{C} = \{(x, y) \in (0, \infty) \times [a, b] : V_{y^-}(x, y) > -K_0, \\ \qquad V_{y^+}(x, y) < K_1\}. \end{cases} \tag{27}$$

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