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ORIGINAL ARTICLE

Connecting theories in mathematics education: challenges and possibilities

Luis Radford

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Abstract This paper is a commentary on the problem of networking theories. My commentary draws on the papers contained in this ZDM issue and is divided into three parts. In the first part, following semiotician Yuri Lotman, I suggest that a network of theories can be conceived of as a *semiosphere*, i.e., a space of encounter of various languages and intellectual traditions. I argue that such a networking space revolves around two different and complementary "themes"—integration and differentiation. In the second part, I advocate conceptualizing theories in mathematics education as triplets formed by a system of theoretical principles, a methodology, and templates of research questions, and attempt to show that this tripartite view of theories provides us with a morphology of theories for investigating differences and potential connections. In the third part of the article, I discuss some examples of networking theories. The investigation of limits of connectivity leads me to talk about the *boundary* of a theory, which I suggest defining as the "limit" of what a theory can legitimately predicate about its objects of discourse; beyond such an edge, the theory conflicts with its own *principles*. I conclude with some implications of networking theories for the advancement of mathematics education.

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L. Radford (✉)
École des sciences de l'éducation, Université Laurentienne,
Sudbury, ON, Canada P3E 2C6
e-mail: lradford@laurentian.ca

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1 The semiosphere as a theory networking space

The goal of this ZDM issue is not to present a catalogue of different theories in mathematics education. It is rather about finding ways to connect theories. Although it is not wrong to trace the origins of this *problématique* back to the need to deal with the diversity of current theories in our field, it might be more accurate to trace it to the rapid contemporary growth of forms of communication, increasing international scientific cooperation and some local attenuations of political and economical barriers around the world, a clear example being, of course, the European Community. Hence, it comes as no surprise that this ZDM issue was preceded by several meetings at the Congress of the European Society for Research in Mathematics Education (CERME), where the topic of connecting or networking theories was one of recurrent interest (Bosch, 2006; Pitta-Pantazi & Philippou, 2007; see in particular Artigue, Bartolini-Bussi, Dreyfus, Gray, & Prediger, 2006; Arzarello, Bosch, Lenfant, & Prediger, 2007).

Certainly, a condition for the implementation of a network of theories is the creation of a new conceptual space where the theories and their connections become objects of discourse and research. This space is one of networking practice and its language, or better still, its meta-language. In particular, the meta-language has to make possible the objectification of and reference to new conceptual "connecting" entities, such as "combining" or "synthetizing" theories (Prediger, Bikner-Ahsbahs, & Arzarello, 2008). It might be helpful, I think, to look at this social networking practice and its meta-language as located in a conceptual semiotic space that cultural theorist Yuri Lotman, in the context of the encounter of various languages and intellectual traditions, called a *semiosphere* (Lotman, 1990), i.e., an uneven multi-cultural space of meaning-making processes and understandings generated by individuals as they come to know and interact with each other.

One of the striking characteristics of the semiosphere is its *heterogeneity*. For Lotman (1990, p. 125) "Heterogeneity is defined both by the diversity of elements and by their different functions." In this context, the role of a meta-language connecting two or more theories is not to erase them through uniform assimilation. Rather, it is to ensure possible forms of connecting different heterogeneous elements. The crucial problem for the new *problématique* set out in this ZDM issue can hence be formulated as follows: to uncover the goals, possibilities, modalities and limits of networking theories. To say it in terms of their semiosphere, the problem is to characterize the *types* of connections that are expressible in the new meta-language as these connections become objectified in their social practice.

Now, even the simplest connection requires *dialogue*. Dialogue is indeed the door for entering the semiosphere. However, a dialogue between theories is much more complex than it may appear at first sight. To talk to another theory means indeed to make an effort to be understood and to understand what the other theory says. In order to understand what is said in the language L of a theory τ, a theory τ' has to translate it (at least at the beginning) into its own language L'. Because of the differences between theories, most of the time, the concepts and structures of L and L' do not coincide, which means that a sentence s of L is transformed into a sentence s' of L' which, if translated back into L, does not necessarily coincide with s. But even in cases where L and L' are fundamentally different, communication becomes possible because dialogical interactions are entailed by imagination and the semiotic flexibility of understanding and language reference. We can visualize Christopher Columbus' first dialogue with the natives of the Caribbean. We may very well imagine that the lack of minimal linguistic reference was compensated for by a profusion of acts of pointing or iconic gestures, accompanied by the slow pronunciation of words, like "gold," which had to be understood and translated into the natives' language through great efforts of the imagination (does the man with the funny thing on his head mean colour, weight, form or something else?).[1]

[1] It is this flexible characteristic of communication that the Russian literary critic Mikhail Bakhtin seems to have been referring to when he asserted that any language can always in principle be deciphered, that is, translated into other languages (Bakhtin, 1986).

Naturally, the authors engaged in the networking of theories share a general cultural background from the outset, which simplifies—at least to a certain extent—the task of translation. But polysemy (i.e., the plurality of meanings of a single word) cannot be ignored, for, what in one theory may be called "epistemic," for instance, does not seem to correspond to what is called "epistemic" in another. We can even ask whether or not the use of the term "learning" has a same referent across the theories included in this ZDM issue.

This remark leads us to ask the following question: is compatibility a condition for connectivity? Kidron, Lenfant, Bikner-Ahsbahs, Artigue, and Dreyfus (2008) argue that it is vain to search for the smallest common denominator between the conceptual categories of two or more theories, for it might turn out to be inexistent. Gellert (2008) in fact provides an example of two such incompatible theories in his discussion about Interactionism and socio-structural approaches in mathematics education. Gellert cogently shows that the theoretical grounds of these two approaches are essentially different—if not contradictory. Yet, a certain form of connectivity is still possible (I will come back to Gellert's example later).

The possible forms of connectivity hence do not seem to be constrained and afforded solely by the nature of the theories. In general terms, a network N of theories τ_1, τ_2, τ_3,...can be seen as a set of connections c_1, c_2, c_3,..., where c_k involves at least two theories τ_i, τ_j. A connection c_k will depend at least on two parameters: (1) the *structure* of the theories involved in the connection, and (2) the *goal* of the connection.

Depending on the goal, connections may take several forms. Prediger, Bikner-Ahsbahs, and Arzarello (2008) identify some of them, like "comparing" and "contrasting," "coordinating" and "combining," "integrating locally" and "synthesizing." As stated by Prediger, Bikner-Ahsbahs, and Arzarello (2008) in "comparing," the goal is finding out similarities and differences, while in "contrasting," the goal is "stressing differences." In coordinating theories, elements from different theories are chosen and put together in a more or less harmonious way to investigate a certain research problem. Halverscheid's paper (2008) is a clear example of an attempt at coordinating theories, in that, the goal is to study a particular educational problem (the problem of modeling a physical situation) through the use of elements from two different theories (a modeling theory and a cognitive one). In combining theories, the chosen elements do not necessarily show the coherence that can be observed in coordinating theories. It is rather a "juxtaposition" of theories (Prediger, Bikner-Ahsbahs, & Arzarello, 2008). Maracci (2008) and Bergsten (2008) furnish examples of combining theories.

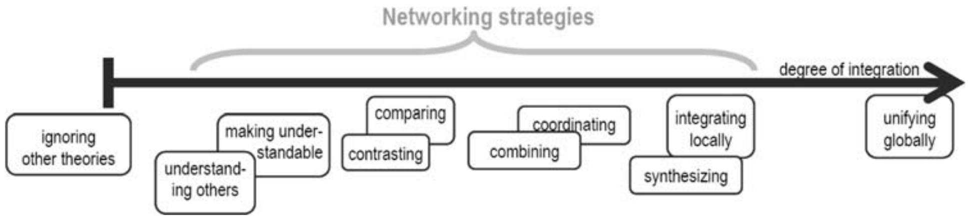

Fig. 1 Prediger, Bikner-Ahsbahs, and Arzarello (2008). Landscape of strategies for connecting theoretical approaches

At least in principle, "comparing" and "contrasting" theories are always possible: given two mathematics education theories τ and τ', it is possible to seek out their similarities and/or differences. In contrast, to "coordinate," to "locally integrate" or to "synthesize" theories seems to be a more delicate task.

Prediger, Bikner-Ahsbahs, and Arzarello (2008) order their typology of connections of theoretical approaches in a linear way. The types of connections are indeed ordered in terms of the *degree* of "integration" (thus, the couple "comparing/contrasting" appear to the left of the couple "coordinating/combining," for they present a lesser degree of integration (see Fig. 1). In terms of the ideas discussed previously, I prefer to see the network of theories as a dynamic set of connections subsumed in the semiosphere where *integration* is only one of the possible themes or "plots" (to use another of Lotman's terms) of the meta-language of the semiosphere. Another interesting "plot," I want to submit, is *identity*.

Indeed, one of the interesting aspects in the networking of theories is that it not only leads to a deeper acquaintance and understanding of the theories τ', τ'',... with which our theory τ is dialoguing; it also leads to a better understanding of our own theory τ as well. Surely, as mentioned earlier, we have to make our own theory understandable for others. But at a more subtle level, by reacting to our own theory and our claims about it, the other theories make visible some elements that may have remained in the background of our theory τ. This is why, in dialoguing, we enter into a process of *extraction*: we pull out things from the brackets of common sense (the brackets of things that we take for granted in our theory to the extent that we no longer even notice them) and, with the help of other theories and in the course of dialoguing, we subject these things to renewed scrutiny. As a result of this connection (that may fit into Prediger et al.'s category of "comparing /contrasting"), the connection may result in a better *self-understanding* of one's own theory. Although the plot of identity may later on give rise to better activities of integration, it may also constitute a theme of the meta-language of the semiosphere on its own, a theme centred on a better understanding of identity, and the recognition of new dimensions of compatibility and incompatibility vis-à-vis other theories.

The self-inquisitionist character of the plot of identity should not be understood as a contemplative gesture, like someone scrutinizing himself in front of a mirror; for, the semiosphere is about interacting and dialoguing, and both interaction and dialogue are transformative relationships. Thus, in the 1980s, in a dialogue that lasted several years and that can be considered as an instance of what we are calling a connection of theories, American Constructivism and German Interactionism influenced each other. In the case of Constructivism, an interest in the social aspects of learning became a focus of attention (see Cobb & Bauersfeld, 1995), leading Constructivism to set new research questions, methodologies, and to revisit and expand their theoretical constructs (Radford, 2008a).

It is in the nature of the semiosphere that theories change to lesser or greater extent. These changes mean that the semiosphere is in constant motion, but its *telos* (finality) should not necessarily be sought in a sort of global unification. In all likelihood, a global unification would mean the constitution of a very general abstract meta-language incapable of doing justice to the identity of the theories it tried to speak of. The challenge in the constitution of the semiosphere's meta-language is precisely this: it should be general enough so that all theories are genuinely objects of discourse, without, at the same time, being too abstract and losing sight of the particularities of its theories. Success in the constitution of the semiosphere may reside, I want to suggest, in the dialectical tension between its plots of identity and integration.

In what follows, I want to reflect on the possibilities and limits of networking theories. Since I am suggesting that connectivity depends on the structure of theories and the connecting goal, in the following section, I will revisit some ideas about what is usually understood by "theory" in our field. Then, I will deal in some detail with the plots of identity and integration, dwelling, in particular, upon the idea of *theoretical boundaries*.

2 Theories

Literary critic Terry Eagleton suggests that "theory is just a practice forced into a new form of self-reflectiveness on account of certain grievous problems it has encountered" (Eagleton, 1990, p. 26); that is, theory is something we start doing when common sense is no longer of help. To carry out the actions entailed by this social praxis of theorizing and self-reflection, some theoretical and methodological normative principles need to be adopted. Niss (1999) suggests that a theory in math education entails a descriptive purpose, aimed at increasing understanding of the phenomena studied, and a normative purpose, aimed at developing instructional design. More specifically, I want to suggest that a theory can be seen as a way of producing understandings and ways of action based on:

- A system, P, of *basic principles*, which includes implicit views and explicit statements that delineate the frontier of what will be the universe of discourse and the adopted research perspective.
- A *methodology*, M, which includes techniques of data collection and data-interpretation as supported by P.[2]
- A set, Q, of paradigmatic *research questions* (templates or schemas that generate specific questions as new interpretations arise or as the principles are deepened, expanded or modified).

2.1 The system P of principles

To refer to P, I use the term *system* instead of *set*, for the elements of P do not have the same weight; there is a *hierarchy* that organizes and prioritizes them. This important hierarchical configuration of P (pointed out by Niss, 1999) has certainly to be kept in mind; otherwise, we can get the impression that two theories τ_1 and τ_2 are not really different, given that the "ingredients" of one can be found in the other. Kidron et al.'s paper (2008) makes this point clear: social interaction is an important component of the three theories that they examine (the Theory of Didactic Situations, the Nested Epistemic Actions Model for Abstraction in Context, and the Theory of Interest-dense Situations). However, social interaction does not play the same role in each of them. As we shall see below, the hierarchical position of an item in the system of principles P endows such an item with a specific theoretical *meaning*.

Another example that can illustrate the importance of considering P as a system instead of as a set is provided by the fundamental concept of cognition. Because P is a system and not a set, there is a strong relationship *between many of its items*. For instance, the relationship between cognition and the social realm may change substantially from theory to theory. For constructivist approaches, cognition is related to mental adaptive structures organized in a logical-mathematical manner; for this theory, the social realm is considered as a mere facilitator of the individual's development of these structures. In the Theory of Didactic Situations, cognition is also conceived of in an adaptive manner, but the social realm is thematized as a "milieu" and a game that the individual plays with it. In Vygotskian approaches, the relationship between cognition and the social realm is worked out in a different way. In Vygotskian approaches, cognition cannot be described in terms of biological adaptations: cognition is a cultural and historically constituted form of reflection and action embedded in social praxes and mediated by language, interaction, signs and artifacts. Instead of constituting the material space that intellectual mechanisms of adaptation need in order to become manifest and active, the social realm is the very material of cognition (Radford, 2003; Radford, in press).

Thus, although the ingredients "social interaction" and "cognition" are part of the items of the corresponding principles P of Constructivism, the Theory of Didactic Situations and Vygotskian approaches, these ingredients have a different meaning.

It is precisely because of the different structure of the systems P and P' of the Theory of Didactic Situations and Vygotskian approaches (as found in Engeström's (1987) version of Activity Theory) that the interesting differences put into evidence by Cerulli, Georget, Maracci, Psycharis, and Trgalova (2008) are accounted for. As these authors persuasively show, the ideas conveyed by these theories about cognition and the role of the social lead one to conceive of the role of the teacher and cultural artifacts in different ways.[3]

2.2 The methodology

The system of principles P of a theory, we just said, is characterized by its *hierarchical structure* and the ensuing *meaning* of its key concepts. The theoretical characterization and functioning of the methodology M of a theory τ is different. The minimal requirements of M are *operability* and *coherence* vis-à-vis P.

For instance, Piaget's methodological data production and interpretation were informed by:

[2] Data collection is not necessarily intended here in the positivist empirical sense of the natural sciences; data collection can also refer to hermeneutical, phenomenological, epistemological and other processes of producing and endowing data with relevance and meaning.

[3] For the sake of clarity, let me add that theoretical principles can be of various sorts. Among others, they include (interrelated) principles of psychological, epistemological and ontological natures. Among the psychological principles, we find ideas about the "cognizing subject," the role of others in knowledge acquisition, etc. Epistemological principles include ideas about what the theory understands by learning, the role of cultural institutions and society, ways of understanding and interpreting the teaching and learning of mathematics, etc. Ontological principles have to do with the status that the theory attributes to mathematical knowledge and the realities the theory deals with.

1. The principles of his Genetic Epistemology, in particular by the principle about the assumed legitimacy of expressing a subject's behaviour in the language of logical calculus, and
2. The idea that the vast array of actions that children display can be seen as "coordination" between actions in a limited number of ways.

In this context, Piaget's methodology sought to produce "meaningful" data (e.g. through clinical interviews) and to find in the child's actions and utterances traces of thought translatable in logical formulas such as $\overline{p} \vee q$ and $p \wedge \overline{q}$. Particular factors in the objects handled by the child, the role of the adult, the import of interaction, etc. were simply dismissed.

The unavoidably selective manner in which methodologies operate can also be seen in Maracci's paper (2008). In order to investigate the difficulties that students encounter when they solve vector space problems, Maracci uses two different theoretical frameworks, namely Fischbein's Theory of Tacit Models and Sfard's process/object duality theory. As in Piaget's case, in his protocol analysis, Maracci searched data that could be "consistent with the possible activation of some *intuitive tacit models*" taking into account, according to Fischbein's Theory, a tacit dimension that "is beyond one's consciousness and control, and influences one's thinking processes." The axiomatic approach adopted in Italy to the teaching of Vector Space Theory, and its emphasis on the underlying algebraic structure, made Sfard's work suitable for understanding Maracci's data. In other words, the theoretical framework and its methodology provided Maracci with a grid for making a distinction between relevant and irrelevant data.

In the most general terms, data "relevance" is dictated by the exigency of coherence between the principles P and the methodology M of a theory τ. As Gardner (1972) suggested in his critique of Piaget's protocol analysis, informed by the principles of his Genetic Epistemology, Piaget focused, at the methodological level, on verbs while dismissing the role of deictics, adverbs and other linguistic markers. Gardner shows how, in protocol analyses, while Piaget looked for action and linguistic clues that could be interpreted in terms of logical sum, multiplication, reversion, etc. the anthropologist Claude Lévi-Strauss looked in his own protocols for traces of opposites like "raw and cooked," "hollow and filled," and "strong and feeble" (Gardner, 1972). Thus, it is through a methodological design that data is first produced; then the methodology helps the researcher to "select" some data among the data that was produced but also helps the researcher to "forget" or to leave some other data unattended.

2.3 The set of paradigmatic research questions

Because theories emerge as forms of understanding and action, and because they emerge as responses to particular problems, they bear the imprint of the initial questions that they sought to answer. The role of the famous problem, "The race to 20," in the shaping of the Theory of Didactic Situations (Brousseau, 1997) is an emblematic example of this phenomenon. As theories evolve, the original questions become generalized in the form of templates or schemas. This means that in order to tackle a particular question, the question still has to be framed in a form that the theory can deal with.

Prediger (2008) presented a general question to various theories: "How is it that some students can learn to tackle a particular type of mathematical problem successfully (as shown by their performance in the class), but be unable to do so two weeks or months later? What strategies can the teacher use to reduce the likelihood of this occurring?"

Researchers working within the Theory of Didactic Situations and the Anthropological Theory of Didactics reframed the problem in terms of didactical choices about the mathematical content and organization of the learning environment: for them, the answer was related to how tasks were dealt with in the classroom, the type of knowledge that resulted from the students' engagement in these tasks, the negotiation of responsibilities between teacher and students, the mechanisms of knowledge institutionalization, etc. Researchers working within emergent embodied theories of cognition (see Arzarello, Bosch, Gascón, & Sabena, 2008) resorted to a distinction between: (1) traditional methodologies based essentially on a symbolic-reconstructive approach, which may produce analytical thinking, and (2) methodologies based on a perceptuo-motor favouring spatio-motoric thinking, arguing that the latter, in contrast to the former, produces long-term effects. Vygotskian theorists would perhaps have re-framed the question in terms of designs of zones of proximal development, interiorization of knowledge and meaningful participation in social praxes.

The previous discussion suggests that theories can be conceived of as organized (implicitly or explicitly) in accordance with three main components (P, M, Q) and that these components are *interrelated* in specific ways. For instance, the methodology has to fulfill at least two conditions: *operability* and *coherence*. Operability means that the methodology must be able to produce and deal with the data in such a way that "satisfactory" answers to the research questions are provided. "Satisfactory" answers may rest on e.g. statistical methods, interviews, discourse analyses, classroom episodes, etc. Coherence means that the rhetoric of the argumentation of the methodology (be it statistical, discursive or other) is consistent with, and rests on, the chosen principles. The research questions must be clearly stated within the conceptual apparatus of the theory. A research question, indeed, already presupposes a "theoretical perspective" from which to state it. Only very general questions (like that asked by Prediger about the students' mathematical memory) can be stated in what seems to be a free-theoretical stance.[4] It is precisely because research questions are dependent on their theories that abstracts for articles are very often difficult to write: the author is required to state her research questions without having had the chance to lay down the theory from which the research questions borrow their meaning.

3 Networking theories

The interest of considering theories as flexible triples $\tau = (P, M, Q)$ of principles, methodologies and paradigmatic research questions for the research problem of networking theories resides in the fact that a connection between theories can happen at *several levels*. For example, a connection can happen at the level of principles, at the level of methodologies, at the level of questions or as combinations of these. But the conceptualization of theories in terms of triplets can also shed some light on the question of the limits of networking theories. Let us discuss here two short examples.

3.1 Connecting the principles P_1 of a theory τ_1 and the methodology M_2 of a theory τ_2

As mentioned previously, in his paper, Halverscheid (2008) used elements from the principles P_1 of the theory of the Modelling Cycle developed by Mason (1988) and continued by Blum, Galbraith, Henn, and Niss (2006) and used the methodology M_2 of the Nested Epistemic Actions Model for abstraction in context (Hershkowitz, Schwarz, & Dreyfus, 2001). Although the methodology was initially developed to be used under different theoretical principles, Halverscheid's work nonetheless shows that a combination of principles of a theory and a different methodology still makes sense. What seems to make the connection possible is, on the one hand, a certain "compatibility" or "proximity" between the methodology M_2 of the Nested Epistemic Actions Model for abstraction in context and Halverscheid's research question (as he says, he was "interested in mental constructs"). On the other hand, the connection seems to be ensured by the generality of the principles P_1 of the theory of the Modelling Cycle. It may be conjectured that had Halverscheid wanted to use the methodology of the Theory of Didactic Situations (or any other non-mental oriented or cognitive theory), the task would have been much more difficult, if not impossible. It may also be conjectured that as research problems become specific (i.e., conceptually formulated within the specific theoretical principles of the theory), the space left for manoeuvring combinations becomes smaller and, after a certain point, a combination is no longer possible. Beyond a certain point, it would be like formulating an economic problem within a Marxist Theory of value while borrowing methods of Liberal economy to solve it.

3.2 Connecting methodologies

Gellert (2008) employed the methodologies M_1 of Interactionism and M_2 of socio-structural approaches in mathematics education to analyze a short classroom episode. As is well known, Interactionism focuses on the negotiations, argumentations and interactions of small group of students (eventually with their teacher), and is interested in reconstructing "the emerging rationality in the development of a collective argument" (Gellert), without bringing into the analysis any social or cultural elements from the more encompassing context in which the classroom is situated. Socio-structural approaches, in contrast, bring the sociocultural dimension in which the classroom is embedded to the fore. As Gellert (2008) argues, "External to the school, there exists a hierarchy of social groups and differential power. The fundamental assumption of structuralist studies in mathematics education is that this *structure* translates into the hierarchies of knowledge, possibility and value within the classroom." Although there is no simple way to connect these micro- and macro-sociological perspectives, even though both are interested in social issues, Gellert suggests that a form of connectivity can still be envisioned: it is possible to start with the identification of "relevant" data (where "relevance" is understood in the sense of the macro-structuralist approach); the relevant data can then be investigated through the interpretation techniques of the interactionists' analyses. The aim of these meticulously fine-grained analyses, Gellert claims, is not to study the products of interaction as emerging products without constraints but rather "to perceive the contingencies of interactions *within* structurally framed classroom situations." In the next step, the question is to re-interpret the interactionist account through the structural lens.

The success of this interesting way of connecting two a priori different (and seemingly contradictory) approaches rests on an asymmetrical use of their methodologies. The data that is submitted to the interactionist methodology has

[4] We could easily imagine the difficulties that would have arisen had Prediger asked this question in terms of, say, Arzarello, Bosch, Gascón, and Sabena's embodied perspective.

already been processed by the structuralist conceptual principles in such a way that the emerging rationality and other classroom discursive products are already tainted with the social structures that the school, willingly or not, reproduces. The asymmetrical role played by the methodologies may be the result of a re-ordering, at the levels of the principles P_1 and P_2 of their corresponding theories, induced by the sequence in which the methodologies operate. The point is, then, that the hierarchical use of methodologies induces a hierarchical organization in the principles of the theories without leading to an apparent theoretical fundamental inconsistency. A corollary of Gellert's example may be that, on closer inspection, Interactionism and structuralist approaches are not, in the semiosphere of networking theories, as far from each other as one might think initially.

4 Boundaries

Although connections are always possible, as asserted earlier, there is nonetheless a *limit* to what can be connected. This limit is determined by the *goal* of the connection, but also by the *specificities* of the components (P, M, Q) of the theories that are being connected. This limit has to do with the *boundary* of each theory under consideration.

For Lotman (1990), a boundary is one of the primary mechanisms of semiotic individuation, something that marks the limits of a first-person form ("I," "us") in opposition to non-first person forms ("you," "them"). Drawing on this idea, I suggest calling the boundary of a theory the "edge" that a theory cannot cross without a substantial loss of its own identity. The boundary sets the "limit" of what a theory can legitimately predicate about its objects of discourse; beyond such an edge, the theory conflicts with its own principles.

Now, the principles of a theory always arise under historical and cultural conditions, many times in competition or in dialogue with other theories. This means that the principles that we find at the foundation of a theory bear the traces of a distinction "us"/"them" that shapes the boundary in question. It is this distinction of cultural entities that led Lotman (1990, p. 142) to claim that a first-person form "creates not only its own type of internal organization but also its own type of external 'disorganization.'"[5]

[5] "It is entirely to be expected" says Lotman, giving an example of the dynamics of internal organization and external disorganization, "that the rational positivistic society of nineteenth-century Europe should create images of the 'pre-logical savage,' or the irrational subconscious as anti-spheres lying beyond the rational space of culture" (Lotman, 1990, p. 142).

The identification of boundaries is still a work in progress. Some of the authors featured in this ZDM issue have started to pave the way ahead through their contributions. One clear example is provided by Kidron et al. (2008). Their primary interest was not to search for a general metalanguage to connect their theories (i.e., the Theory of Didactic Situations, the Nested Epistemic Actions Model for Abstraction in Context, and the Theory of Interest-dense Situations). Their primary interest was rather to explore the corresponding theoretical boundaries. Even if the three theories involved examined a common classroom episode (a technique used by other contributors as well), the goal was not to investigate the kind of complementarity that could result from studying the same data from three theoretical perspectives. The goal was to determine what can and cannot be said about social interaction in each of these theories. The exercise was something like this: in theory τ social interaction plays the roles $r_1, r_2,\ldots,r_n$. Can something "similar" be found in theories τ' and τ'' ? In other words, is there a meaningful translation of all (or some) of the $r_1, r_2,\ldots,r_n$ into τ' and τ''?

Within an interest-dense situation, students are supposed to get intensely involved in the mathematical activity and progressively construct mathematical meanings that reach farther and farther. Moreover, the activity leads them to highly appreciate the mathematics to which they resorted. The analysis presented in Kidron et al.'s paper (2008) shows that for the Theory of Interest-dense Situations, social interaction, while considered as an epistemic process, is deeply related to the evolution of a *psychological* construct—*interest*, i.e., a personal or social feature reflecting a genuine engagement in the mathematical activity. This is why the Theory of Interest-dense Situations focuses on the students' involvement and tracks verbal utterances and other elements of interaction that reveal the motivational and affective dimensions embedded in knowing (dimensions that carry little weight, if any, in the analyses conducted in the Theory of Didactic Situations). Thus, in the Theory of Interest-dense Situations, it is only after analyzing data protocols that the researcher can say whether or not a situation captured the students' interest.

Although social interaction is omnipresent in the Theory of Didactic Situations, its role is not exactly to produce conditions that will raise the students' interest in a mathematical task. Of course, a student's interest can be enhanced by social interaction (like in the puzzle problem; Brousseau, 1997, pp. 177–179) but "true" interest is fostered by the student's *intellectual need* for providing an optimal solution to a problem or situation. This is why, in the Theory of Didactic Situations, social interaction is not really concerned by its affective or volitive dimensions; its main concern is in the design of "adidactic situations" and its "devolution." In short, while social interaction plays an *epistemic* role in the

Theory of Didactic Situations, in the Theory of Interest-dense Situations, because of its focus on interest, social interaction plays, above all, a *psychological* role.

Here, we reach an irreducible difference between these two theories as far as social interaction is concerned. Irreducible differences also appeared when the role of social interaction was investigated in the Nested epistemic actions model for abstraction in context. As noted by Kidron et al. (2008), the latter, in contrast to the two other theories, is a student-centred theory that seeks to provide fine-grained analyses of the processes of knowledge construction. Although not automatically excluded, peer interaction is not necessarily a part of its experimental setting. Social interaction in this theory therefore plays a different role, perhaps something closer to a kind of heuristic tool for the investigation of individual knowledge construction, a role that clashes with the role it plays in the Theory of Interest-dense Situations, where "social interactions generate the emergence of mathematical knowledge" (Kidron et al., 2008).

Trying to understand the roots of these differences the authors suggested that

> the different views the three theories have in relation to social interactions force us to reconsider the theories in all their details. The reason for this is that the social interactions, as seen by the different frameworks, intertwine with the other characteristics of the frameworks (Kidron et al., 2008).

Furthermore, these characteristics may not only be responsible for these differences but may point to "possible contradictions between the underlying assumptions of the theoretical approaches" (Kidron et al., 2008), i.e., between what I call here the principles of the theories.

In terms of the notion of theory $\tau = (P, M, Q)$ conveyed in this paper, social interaction is positioned in a different place in the system P of principles of the corresponding theories and therefore has a different meaning. Its meaning can only be revealed through a detailed investigation of the relationship that it has with the other elements of its corresponding system P.

5 Synthesis and concluding reflections

Since it is impossible to have an all-encompassing theory, a dialogue between theories in mathematics education, with an emphasis on the possible connections between them, is more than an appropriate and welcomed task. In a recent paper, Frank Lester argued that "a grand 'theory of everything' cannot ever be developed and efforts to develop one are very likely to keep us from making progress toward the goals of our work" (Lester, 2005, p. 460). The current European effort to create what I have called here, following the Russian semiotician and scholar Yuri Lotman, a *semiosphere* is certainly an important and commendable endeavour.

In the first part, drawing explicitly and implicitly on some previous ideas and discussions (e.g., Bikner-Ahsbahs & Prediger, 2006; Cobb, 2007; Lerman, 2006; Lester, 2005; Niss, 1999; Silver & Herbst, 2007; Sriraman & English, 2005), I argued for the need to create "topics" or "plots" in the constitution of the semiosphere and its metalanguage. While Prediger, Bikner-Ahsbahs, and Arzarello (2008) pointed out an ordering of networking activities in terms of the degree of "integration" (Fig. 1), I suggested that this ordering could also be complemented by another "plot"—the "identity" of theories. These two "plots" (imagined perhaps as spatial axes) can inform us about the dynamic structure of the semiosphere.

In the second part, I suggested that a theory in mathematics education can be seen as a triplet $\tau = (P, M, Q)$ of principles, methodologies and research questions. The distinctive features of the components P, M, and Q of a theory were illustrated through reference to papers in this ZDM issue. In an effort to put into evidence some interrelationships between these components, I commented on how the choice of principles P influences M (in terms of operability and coherence) and Q (in terms of the manner in which research questions are formulated). The prominent role played by P in my account does not mean that Q and M are mere logical consequences of P. Nor does it mean that theories are essentially characterized by the system of their principles P. Such a view would lead us think that all theories in mathematics education are talking about the same research questions and using the same methodologies, which obviously is not the case.

In the third part of the article, I discussed some examples of networking theories. I argued that, although comparison of theories is always possible, we will nevertheless find some limits to the connectivity of theories. The investigation of limits of connectivity led me to talk about the *boundary* of a theory, which I suggested defining as the "limit" of what a theory can legitimately predicate about its objects of discourse; beyond such an edge, the theory conflicts with its own *principles*.

Although in asking questions about the connectivity or non-connectivity of theories, the three components P, M and Q should be taken into account, I am inclined to think that the possibilities of connectivity rest, in the end, on the goal of the connectivity and the possibilities afforded by the principles of the theories under consideration. Gellert's example shows how two methodologically incompatible theories (one focused on macro-socio economic and political structures, the other on contextual classroom discourse analysis) can nevertheless be interconnected. Thus, methodological incompatibility does not seem to be an impediment for connectivity.

At this moment, my conjecture is that, if we dig deep enough, we will find that difficult to connect theories are more likely to have fundamental differences in their system of principles. Kidron et al.'s paper lends some support to this conjecture. Another example is provided by a comparison between the Theory of Didactic Situations and Constructivism. Although, from the outset, these theories were committed to improving the teaching and learning of mathematics and both started out as alternatives to previous models of learning without meaning, since the beginning both also addressed research questions of *different* natures. Constructivism dealt with problems of mental structures and formulated questions related to the students' behaviour in classroom situations (Cobb, 1988). The Theory of Didactic Situations dealt with the problem of students' acquisition of institutional mathematical knowledge (Brousseau, 1986). The differences are thus clear at the level of their respective research questions. Now, are these research questions incompatible or just different?

My argument here is that we cannot answer this question by looking at the theories' research questions alone and that we need to look into the principles as well. For, research questions are not stated in a conceptual vacuum: research questions are stated within a world-view and this world-view is defined by the explicit and implicit principles of any given theory.

If we turn to the principles of these theories, at first glance we note a seemingly compatibility. Indeed, both theories conceive of knowledge in a Piagetian adaptive manner and both value the autonomy of the cognizing subject vis-à-vis the teacher. However, if we dig a bit further, we realize that each of these theories resorts to a *principle of autonomy* that is *not* the same: while in Constructivism, the principle of the autonomy of the cognizing subject is framed by an ethics of personal constructions and self-determination, in the Theory of Didactic Situations, the autonomy of the cognizing subject is conceived of as an epistemic condition of knowledge attainment (Radford, 2008a, b). If we dig further still, we find that, behind the way the research questions were asked, lies the difference between what political economists call the paradigms of the *right* and the *good* (Mouffe, 1993; Kymlicka, 1989). In the paradigm of the *right*, which goes back to Kant's work (Rawls, 1999), people are conceived of as endowed with the freedom to secure the conditions under which they can best make judgments and decisions, imagine, tackle and solve problems. Individuals are thought of as endowed with the capacity to form and revise their rational plans of life and produce what von Glasersfed (1995) calls their "viable" knowledge. Everything that does not come from within the individual is often seen as a kind of coercion and an attempt at trespassing the individual's right to self-determination. In the other paradigm, in contrast, a common *good* (in this case, a culturally and historically constituted mathematical knowledge) is put forth, from the outset, as something valued as being attainable by individuals: it becomes the goal of the teaching and learning of mathematics. Here, the question is not really the construction of "viable" knowledge, but the attainment of a knowledge that pre-exists the learning activity of the students. While Constructivism draws on the paradigm of the *right*, the Theory of Didactic Situations draws rather on the paradigm of the *good*. This is why the phase of institutionalization (where the normative nature of cultural knowledge is brought to the fore by the teacher) is of such great importance in the Theory of Didactic Situations. These differences, however, are hard (if not impossible) to notice by looking at the research questions only. This is why I want to maintain that, if we dig deep enough, divergences between theories are accounted for not by their methodologies or research questions but by their principles.

To cast this discussion in terms of the semiosphere, let us think of the latter as a topological space. Connectivity between theories depends on the compatibility of principles and the goals of the connections. In terms of integration, it might be conjectured that theories are more likely to be connected if their theoretical principles (or at least some of them) are "close" to each other. Thus, two principles p_i and p_j' of two theories τ and τ' that are "close enough" to each other can give rise to "integrative and synthetic connections" (in Prediger et al.'s sense). In contrast, principles p_i and p_j' that are not close enough point to irreducible aspects of the theories. It can be the case, of course, that for the same two theories τ and τ' one (or more) couples of principles (p_i, p_j') are "close enough," while one (or more) couples of principles (p_k, p_m') are not. In this case, only "local connections" will be found.

We can visualize a sector or region R_t of the semiosphere at a certain time t, as a Cartesian product of the principles of the linking theories (let us say theories τ and τ'). The product $\tau \times \tau'$ is provided with a "topology" defined by a system V of "neighbourhoods" where the principles p_i and p_j' are said to be "close enough" if a neighbourhood of V contains the couple (p_i, p_j'). Irreducible differences shown between two principles (p_i, p_j') mean that there is no "neighbourhood" connecting them: the theories are disconnected in the neighbourhood of these couples (p_i, p_j') of principles. But again, this does not mean that these theories are disconnected everywhere in the semiosphere.

By way of conclusion, I want to point out that the problématique of networking theories is interesting for the advancement of research in mathematics education on at least two counts. The "integrative plot" of its semiosphere can lead to new problems of linking theories and give rise to new multi-theoretical routes, capable of revealing the complexity of teaching and learning. The "identity plot" can enhance our understanding of theories in our field, their similarities and differences. Further research on these "plots," would require the elaboration of new tools and concepts and the corresponding meta-language to describe them. They may emerge from the study of a same set of classroom data as seen from the point of view of various theories. But they can also emerge from the investigation of theoretical problems (e.g. the role of "social interaction" as in Kidron et al.'s paper). A list of these problems might include items such as theoretical assumptions about development versus learning, the conceptual categories employed to account for the processes of learning, as well as the epistemic link between cultural and students' knowledge. We should not forget, however, that mathematical classroom practices are subsumed in cultural forms of signifying that convey, in the discourses that mediate them, attitudes and values about gender, race, inequalities, etc. Our list should hence also include items about power, intersubjectivity and a clear sensibility to other epistemologies (e.g. aboriginal and marginalized ones); for, as Gellert (2008) reminds us, in one way or another, mathematics education must attend to the ethical and political domains of the practices it investigates.

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