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Connection of Dual Models to Electrodynamics and Gravidynamics

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It is shown that the Virasoro-Shapiro model contains Einstein's theory of gravity as a zero-slope limit. It is also shown that the conventional dual model contains the scalar electrodynamics as a zero-slope limit. The connection between the generating functionals for the scattering matrices of these dual models and the corresponding field theories is demonstrated.

§ 1. Introduction and summary

It is widely believed that the slope parameter α in dual models has a fundamental meaning as a universal constant characterizing the unit of length. It is then of great interest to see what happens if one takes the limit $\alpha\to 0$. This limit was first investigated by Scherk[1] who showed that the conventional model reduces to the ϕ^3 Lagrangian theory in the limit with the ground-state mass and the quantity $g/\sqrt{\alpha}$ being fixed, where g is a dimensionless coupling constant of the model. This connection clarified in particular the correspondence[2] between the usual Feynman diagrams and the duality diagrams. We believe that Scherk's work is an important step in understanding field theoretical foundations of dual models.

On the other hand, an essential feature of dual models is that the ghost states which appeared in the manifestly covariant factorization do decouple[3] because of the built-in gauge invariance[4] provided the intercepts are taken appropriately. The intercepts are such that the physical spectra of the models contain a massless spin 1^- or 2^+ state ("photon" or "graviton"). The Virasoro conditions[4] are the reminiscences of the usual subsidiary conditions in quantum electrodynamics and quantum gravidynamics. In fact, in appropriate zero-slope limits fixing the "photon" or the "graviton" states, some matrix elements of the former conditions are equivalent to the latter conditions.

It is also remarkable that the photon vertices which play a crucial role in generating the algebras[5] of the spectra are derived through a generalized local-gauge principle.[6]

For these reasons, it will be more interesting to investigate the zero-slope limit fixing the intercepts at appropriate values. In fact, Neveu and Scherk[7] pointed out that in such a limit the conventional model with the Chan-Paton isospin factor yields a massless Yang-Mills field theory of tree approximation.

In view of the recent trends of gauge field theories such a systematic connection between a dual model and a gauge field would guide us to construct a more realistic dual theory of hadrons.

Now, the main purpose of the present paper is to extend such a connection to the gravitational field which is also an important example of the gauge fields. We show that the Einstein theory of gravity is obtained from the "graviton"-scalar amplitudes of the Virasoro-Shapiro model in the zero-slope limit with the quantity $g\sqrt{\alpha}$ being fixed. We also discuss a similar connection between the scalar electrodynamics and the conventional model without internal symmetry by considering the zero-slope limit of the "photon"-scalar amplitudes with g being fixed.

It is well known that in the S-matrix approach[8] the photon and the graviton can be treated in a closely parallel way inspite of the nonlinearity of the graviton interactions. It turns out that this is also the case with the dual models. This becomes clear when we make a comparison of the generating functionals of the scattering matrices in the dual models and the corresponding field theories. We point out that if the lowest-order interaction terms of the latter theories are given, the dual model generating functionals are obtained from the field theoretical ones by reinterpreting the latters in terms of the string variables.

The next two sections will be devoted to the discussion of the zero-slope limits. In the last section, we make the comparison for the generating functionals. Appendices, where the derivation of the generating functionals and a formulation of the dynamics of a closed string are given, are the preliminaries for the last section.

§ 2. The theory of gravity as a zero-slope limit of the Virasoro-Shapiro model

It was shown by Shapiro[9] that Virasoro's four-point function,[10] if its intercept equals two, can be generalized to a factorizable and ghost-free n-point fnuction. The Virasoro-Shapiro model thus contains a "graviton", just as the conventional model contains a "photon". In this section, we shall show that this model may be conceived as an extension of the Einstein gravidynamic in the sense that the latter is obtained from the former in an appropriate zero-slope limit. We take the limit keeping the quantity $g\sqrt{\alpha}\equiv\kappa$ finite. Clearly, the only surviving pole terms are those of a massless symmetrical tensor state of rank two. It turns out that there are infinitely many interaction terms describing the limit. However, it is possible to determine the corresponding Lagrangian using a uniqueness theorem for the Einstein theory of gravity. This situation is very similar to in Ref. 11) where the Ellis-Osborn theorem[12] was used to obtain the nonlinear σ model as a zero-slope limit of the dual pion model.

2·1 *The uniqueness theorem*

This theorem asserts the uniqueness of the Einstein theory of gravity as a

Lagrangian theory of the massless symmetrical tensor field of second rank under several conditions. A very clear derivation of this theorem has been given by Wyss.[13] The conditions can be stated as follows: (1) In the lowest-order approximation the source of the tensor field $h_{\mu\nu}$ is the energy momentum tensor of the free matter field. (2) The equation of motion does not contain the space-time derivatives whose degrees are higher than two. (3) The Lagrangian is invariant under a certain gauge transformation which is in the lowest-order approximation given by $h_{\mu\nu}\to h_{\mu\nu}+\partial_\mu\xi_\nu+\partial_\nu\xi_\mu$, where ξ_μ is an arbitrary vector function.

The consistency requirement for the Lagrangian formulation under these conditions determines uniquely the Lagrangian in the second-order approximation. The second-order Lagrangian is sufficient to obtain a unique extension of the linear gauge group $h_{\mu\nu}\to h_{\mu\nu}+\partial_\mu\xi_\nu+\partial_\nu\xi_\mu$. This extended gauge group is isomorphic to Einstein's general transformation group and can uniquely determine the higher-order terms of the Lagrangian under the above conditions. If one defines the metric tensor by $g_{\mu\nu}=\eta_{\mu\nu}+\kappa h_{\mu\nu}$,*) the Lagrangian is precisely the Einstein Lagrangian

$$\mathcal{L}=\frac{1}{\kappa^2}R\sqrt{-g}+\sqrt{-g}\,(g^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi-m^2\phi^*\phi), \tag{2·1}$$

where we only consider the complex scalar field as the matter field.

Now we must translate these conditions in the language of scattering amplitudes. Condition (2) is satisfied if every irreducible vertex in the tree approximation is bilinear in momenta. We note that the Einstein Lagrangian leads to amplitudes with this property. Condition (3) is equivalent to vanishing of the amplitudes with the wave function $\epsilon_\mu\epsilon_\nu$ of an external graviton being replaced by $\epsilon_\mu k_\nu+\epsilon_\nu k_\mu$ where k_μ is the four-momentum of the graviton. The first condition can be checked directly by examining the structure of an appropriate amplitude.

2·2 "*Gravition*" *amplitudes in the Virasoro-Shapiro model*

We consider the Virasoro-Shapiro amplitudes with several gravitons and matter particles as the external lines. We take the matter particle to be scalar. The mass (m) of the scalar particle can be fixed, independently of the value of the α, by introduction of the familiar fifth dimension. The fifth momenta of the gravitons are, of course, taken to be zero. Then, in addition to the massless-tensor state, there remain poles corresponding to that scalar state in the zero-slope limit.

As can be easily checked, the limiting amplitudes can be described by means of a Lagrangian composed of a symmetrical tensor field and a scalar field

*) Here $\eta_{\mu\nu}$ is the Minkowskian metric tensor given by $(1, -1, -1, -1)$.

because of the bootstrap property of the amplitudes with respect to those poles for an arbitrary value of α. Indeed, this property ensures that the irreducible vertices of the limiting amplitudes do not depend on the number of the external lines. Thus we can apply the above theorem to our problem.

Let us consider the process of n gravitons and two scalar particles. An operatorial expression for the corresponding amplitude is given below where the graviton vertex is derived by a method similar to one used in the conventional model in obtaining excited vertices.[14)]

$$T_n^G = g^n (\alpha)^{(n-2)/2} \left(\frac{1}{4\pi}\right)^{n-1} F_n^G (p, q; \epsilon_1 k_1, \epsilon_2 k_2, \cdots, \epsilon_n k_n), \tag{2·2}$$

$$F_n^G (p, q; \epsilon_1 k_1, \cdots, \epsilon_n k_n) = \frac{1}{C}\left(\prod_{i=1}^{n} \int \frac{d^2 z_i}{|z_i|^2}\right) \langle\!\langle p | T \prod_{i=1}^{n} V(\epsilon_i, k_i; z_i, z_i^*) | q \rangle\!\rangle, \tag{2·3}^{*)}$$

where

$$V(\epsilon, k; z, z^*) = :\frac{1}{i} z \frac{d\epsilon \cdot Q(z)}{dz} \frac{1}{i} z^* \frac{d\epsilon \cdot \bar{Q}(z^*)}{dz^*}$$
$$\times \exp[i\sqrt{\alpha} k \{Q(z) + \bar{Q}(z^*)\}]:, \tag{2·4}$$

$$Q(z) = q_0 + i p_0 \log z + \sum_{n=1}^{\infty} \sqrt{\frac{1}{n}} (a_n z^n + a_n^+ z^{-n}), \tag{2·5}$$

$$\bar{Q}(z^*) = \bar{q}_0 + i \bar{p}_0 \log z^* + \sum_{n=1}^{\infty} \sqrt{\frac{1}{n}} (b_n z^{*n} + b_n^+ z^{*-n}) \tag{2·6}$$

with

$$[q_0, p_0] = [\bar{q}_0, \bar{p}_0] = -i, \tag{2·7}$$

$$[a_n, a_m^+] = [b_n, b_m^+] = -\delta_{nm}, \tag{2·8}$$

$$|p\rangle\!\rangle = \exp[i\sqrt{\alpha} p (q_0 + \bar{q}_0)] |0) \otimes |0\rangle, \quad (q^2 = p^2 = m^2) \tag{2·9}$$

$$p_0 |0) = \bar{p}_0 |0) = a_n |0\rangle = b_n |0\rangle = 0. \tag{2·10}$$

The differences from the conventional model are that the operators are extended to mutually commuting double sets and the range of the integrations covers the whole Gauss plane.

2·3 *The condition* (2)

The bilinearity is proved by showing that the power-series expansion**) of the dimensionless amplitude F_n^G in the α begins with the first power. Since T_n^G is proportional to $(\kappa^n/\alpha) F_n^G$, only this first-power term contributes to the

*) T denotes the ordered product with respect to $-\log|z|$ and C is an infinite constant for normalization. In this expression, the fifth dimension does not appear explicitly because the z-variables of two scalar lines are taken to be 0 and ∞.

**) In spite of the square root in the expressions (2·4) and (2·9), the power-series expansion in α is possible because α appears only in the form $\sqrt{\alpha} \times$ (momentum) and the momentum variables always appear in pair in the non-operatorial expressions. This is not true for (3·2).

zero-slope limit. Therefore the total power in momenta of the zero-slope is two. This clearly means that every irrededucible vertex is bilinear in momenta since the power of the propagator is -2.

On setting $\alpha=0$, the F_n^G becomes

$$F_n^G|_{\alpha=0}=\frac{(-1)^n}{C}\int\prod_{i=1}^{n}\frac{d^2z_i}{|z_i^2|}\langle\!\langle 0|T\prod_{i=1}^{n}:z_i\frac{d\epsilon_i\cdot Q(z_i)}{dz_i}z_i{}^*\frac{d\epsilon_i\cdot\bar{Q}(z_i{}^*)}{dz_i{}^*}:|0\rangle\!\rangle\,. \quad (2\cdot11)$$

If n is odd, this expression is zero, since the integrand vanishes identically. If n is even, we have

$$F_n^G|_{\alpha=0}\propto\frac{(-1)^n}{C}\left(\int d^2u\int d^2v\frac{1}{|u-v|^4}\right)^{n/2}.$$

Although this is a divergent integral, it is really defined by analytic continuation. It is an easy exercise to show that the following identification should be made:*)

$$\int d^2u\int d^2v\frac{1}{|u-v|^4}\propto\lim_{\delta\to0}\frac{\Gamma(1+a\delta)\Gamma(1+b\delta)\Gamma(-1+c\delta)}{\Gamma(d\delta)\Gamma(e\delta)\Gamma(2+f\delta)}=0\,,$$

where the coefficients $a, b, \cdots, f$ are linear functions of the Mandelstam variables. This completes the proof for the validity of condition (2).

2·4 *The proof of gauge invariance*

We prove that (2·3) vanishes, if one of the vertices, say $V(\epsilon_1, k_1, z_1, z_1{}^*)$, is replaced by $:1/i\ dk\cdot Q(z_i)/dz_1\ 1/i\ d\epsilon\cdot\bar{Q}(z_1{}^*)/dz_1{}^*\ \exp[i\sqrt{\alpha}k\{Q(z_1)+\bar{Q}(z_1{}^*)\}]:$. Using the relation

$$:\frac{1}{i}\frac{dk\cdot Q(z_1)}{dz_1}\frac{1}{i}\frac{d\epsilon\cdot\bar{Q}(z_1{}^*)}{dz_1{}^*}\exp[i\sqrt{\alpha}k\{Q(z_1)+\bar{Q}(z_1{}^*)\}]:=z_1\frac{d}{dz_1}W\,, \quad (2\cdot12)$$

where

$$W=W(\epsilon_1,k_1,z_1,z_1{}^*)=\frac{-1}{\sqrt{\alpha}}:z_1{}^*\frac{d\epsilon\cdot\bar{Q}(z_1{}^*)}{dz_1{}^*}\exp[i\sqrt{\alpha}k\{Q(z_1)+\bar{Q}(z_1{}^*)\}]:, \quad (2\cdot13)$$

it is easy to see that under the above replacement Eq. (2·3) reduces to

$$-\frac{2^{n-1}}{C}\left(\prod_{i=1}^{n}\int_{-\infty}^{\infty}d\tau_i\int_0^{2\pi}d\theta_i\right)\langle\!\langle p|T\prod_{j=2}^{n}\delta(\tau_i-\tau_j)\,[W(\epsilon_1,k_1,e^{\tau_1+i\theta_1},e^{\tau_1-i\theta_1}),$$

$$V(\epsilon_j,k_j,e^{\tau_j+i\theta_j},e^{\tau_j-i\theta_j})]\prod_{i=2(\neq j)}^{n}V(\epsilon_i,k_i,e^{\tau_i+i\theta_i},e^{\tau_i-i\theta_i})|q\rangle\!\rangle\,. \quad (2\cdot14)$$

By the commutation relation

$$[Q(e^{\tau+i\theta}),Q(e^{\tau+i\theta'})]=-[\bar{Q}(e^{\tau-i\theta}),\bar{Q}(e^{\tau-i\theta'})]=i\pi\eta(\theta-\theta')\,, \quad (2\cdot15)$$

where $\eta(\theta)=1(\theta>0)$, $0(\theta=0)$, $-1(\theta<0)$, we can prove**)

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*) Note that our procedure is equivalent to taking the finite part of the integral.

**) Use the techniques given in Ref. 14).

$$\delta(\tau_1-\tau_j)\,[W(\epsilon_1, k_1, e^{\tau_1+i\theta_1}, e^{\tau_1-i\theta_1}),\ V(\epsilon_j, k_j, e^{\tau_j+i\theta_j}, e^{\tau_j-i\theta_j})]=0\,. \qquad (2\cdot16)$$

Thus (2·14) vanishes.

2·5 *Second-order calculation of the graviton amplitude*

We show that condition (1) is satisfied by the zero-slope limit of (2·2), by exhibiting that to second order in κ it coincides with the amplitude obtained from the Einstein Lagrangian (2·1). Since in this order of approximation a vertex in which one off-the-mass shell graviton is attached to the scalar line appears, this is sufficient*) for our purpose.

By lengthy calculations, we obtain the two-graviton amplitude in the Virasoro-Shapiro model:

$$\begin{aligned}
T_2^G=&(\epsilon_1\cdot\epsilon_2)^2\Gamma\begin{pmatrix}1&1&-1\\0&0&2\end{pmatrix}+2\alpha(\epsilon_1\cdot p)(\epsilon_2\cdot p)(\epsilon_1\cdot\epsilon_2)\Gamma\begin{pmatrix}1&1&-1\\0&0&2\end{pmatrix}\\
&+2\alpha(\epsilon_1\cdot q)(\epsilon_2\cdot p)(\epsilon_1\cdot\epsilon_2)\left\{\Gamma\begin{pmatrix}1&1&-1\\0&0&2\end{pmatrix}-\Gamma\begin{pmatrix}1&1&0\\1&0&2\end{pmatrix}\right\}\\
&+2\alpha(\epsilon_1\cdot p)(\epsilon_2\cdot q)(\epsilon_1\cdot\epsilon_2)\left\{\Gamma\begin{pmatrix}1&1&-1\\0&0&2\end{pmatrix}-\Gamma\begin{pmatrix}1&1&0\\0&1&2\end{pmatrix}\right\}\\
&+2\alpha(\epsilon_1\cdot q)(\epsilon_2\cdot q)(\epsilon_1\cdot\epsilon_2)\,\Gamma\begin{pmatrix}1&1&-1\\0&0&2\end{pmatrix}\\
&+\{\alpha^2(\epsilon_1\cdot p)^2(\epsilon_2\cdot p)^2+\alpha^2(\epsilon_1\cdot q)^2(\epsilon_2\cdot q)^2\}\,\Gamma\begin{pmatrix}1&1&-1\\0&0&2\end{pmatrix}\\
&+2\alpha^2(\epsilon_1\cdot p)^2(\epsilon_2\cdot p)(\epsilon_2\cdot q)\left\{\Gamma\begin{pmatrix}1&1&-1\\0&0&2\end{pmatrix}-\Gamma\begin{pmatrix}1&1&0\\1&0&2\end{pmatrix}\right\}\\
&+2\alpha^2(\epsilon_1\cdot p)(\epsilon_2\cdot p)^2(\epsilon_1\cdot q)\left\{\Gamma\begin{pmatrix}1&1&-1\\0&0&2\end{pmatrix}-\Gamma\begin{pmatrix}1&1&0\\0&1&2\end{pmatrix}\right\}\\
&+2\alpha^2(\epsilon_1\cdot q)^2(\epsilon_2\cdot p)(\epsilon_2\cdot q)\left\{\Gamma\begin{pmatrix}1&1&-1\\0&0&2\end{pmatrix}-\Gamma\begin{pmatrix}1&1&0\\0&1&2\end{pmatrix}\right\}\\
&+2\alpha^2(\epsilon_1\cdot p)(\epsilon_2\cdot q)^2(\epsilon_1\cdot q)\left\{\Gamma\begin{pmatrix}1&1&-1\\0&0&2\end{pmatrix}-\Gamma\begin{pmatrix}1&1&0\\1&0&2\end{pmatrix}\right\}\\
&+\alpha^2(\epsilon_1\cdot p)^2(\epsilon_2\cdot q)^2\Gamma\begin{pmatrix}2&0&-1\\-1&1&2\end{pmatrix}+\alpha^2(\epsilon_1\cdot q)^2(\epsilon_2\cdot p)^2\Gamma\begin{pmatrix}0&2&-1\\1&-1&2\end{pmatrix}\\
&+\alpha^2(\epsilon_1\cdot p)(\epsilon_2\cdot q)(\epsilon_1\cdot q)(\epsilon_2\cdot p)\left\{2\Gamma\begin{pmatrix}1&1&-1\\0&0&2\end{pmatrix}+\Gamma\begin{pmatrix}0&2&-1\\1&-1&2\end{pmatrix}\right.\\
&\left.+\Gamma\begin{pmatrix}2&0&-1\\-1&1&2\end{pmatrix}-2\Gamma\begin{pmatrix}1&0&0\\0&1&1\end{pmatrix}-2\Gamma\begin{pmatrix}0&1&0\\1&0&1\end{pmatrix}+\Gamma\begin{pmatrix}0&0&1\\1&1&0\end{pmatrix}\right\},
\end{aligned}\qquad (2\cdot17)^{**)}$$

*) It is known[15] that if the anomalous interaction is included, one cannot introduce gravitational interaction consistently even to this order of approximation under the conditions stated in subsection 2·1.

**) This result is already contained in a previous note of the present author.[16] The preliminary results of this paper are given there.

where, with $s=(p+q)^2$, $t=(p+k_1)^2$ and $u=(p+k_2)^2$,

$$\Gamma\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}=\frac{g^2}{2}\frac{\Gamma(a-(\alpha/2)(u-m^2))\Gamma(b-(\alpha/2)(t-m^2))\Gamma(c-(\alpha/2)s)}{\Gamma(d+(\alpha/2)(u-m^2))\Gamma(e+(\alpha/2)(t-m^2))\Gamma(f+(\alpha/2)s)}. \quad (2\cdot18)$$

Taking the limit $\alpha\to0$ and fixing κ finite, we see that $T_2{}^G$ reduces to

$$\begin{aligned}
&\kappa^2\frac{1}{s}\Big[(\epsilon_1\cdot\epsilon_2)^2\Big\{-\frac{s^2}{4}+\frac{(t-m^2)(u-m^2)}{4}\Big\}+(\epsilon_1\cdot\epsilon_2)\frac{t-m^2}{2}\{(\epsilon_1\cdot p)(\epsilon_2\cdot p)\\
&+(\epsilon_1\cdot q)(\epsilon_2\cdot q)+2(\epsilon_1\cdot q)(\epsilon_2\cdot p)\}+(\epsilon_1\epsilon_2)\frac{u-m^2}{2}\{(\epsilon_1\cdot p)(\epsilon_2\cdot p)\\
&+(\epsilon_1\cdot q)(\epsilon_2\cdot q)+2(\epsilon_1\cdot p)(\epsilon_2\cdot q)\}+(\epsilon_1\cdot\epsilon_2)\frac{s}{2}\{(\epsilon_1\cdot p)(\epsilon_2\cdot p)\\
&+(\epsilon_1\cdot q)(\epsilon_2\cdot q)+2(\epsilon_1\cdot p)(\epsilon_2\cdot q)+2(\epsilon_1\cdot q)(\epsilon_2\cdot p)\} \qquad (2\cdot19)\\
&-(\epsilon_1\cdot p)^2(\epsilon_2\cdot q)^2-(\epsilon_1\cdot q)^2(\epsilon_2\cdot p)^2+2(\epsilon_1\cdot p)(\epsilon_1\cdot q)(\epsilon_2\cdot q)(\epsilon_2\cdot p)\}\Big]\\
&+\kappa^2\Big\{(\epsilon_1\cdot\epsilon_2)^2\frac{s}{4}-(\epsilon_1\cdot\epsilon_2)(\epsilon_1\cdot p)(\epsilon_2\cdot q)-(\epsilon_1\cdot\epsilon_2)(\epsilon_1\cdot q)(\epsilon_2\cdot p)\Big\}\\
&-\frac{\kappa^2}{t-m^2}(\epsilon_1\cdot p)^2(\epsilon_2\cdot q)^2-\frac{\kappa^2}{u-m^2}(\epsilon_2\cdot p)^2(\epsilon_1\cdot q)^2\,.
\end{aligned}$$

This result exactly coincides with the one obtained from the Einstein Lagrangian expanded in $h_{\mu\nu}$ to second order in κ on setting $g_{\mu\nu}=\eta_{\mu\nu}+\kappa h_{\mu\nu}$.

This also implies that to second order in κ the spin-zero component of the symmetrical transversal tensor state does not couple on-the-mass shell in the zero-slope limit. We can also check this fact by directly calculating the corresponding residue of the amplitude (2·17). Because of the Wyss theorem, the decoupling of the spin-zero component is valid to all orders in the zero-slope limit of (2·2). It should be noted, however, that for *non-zero* α the spin-zero component does couple even on-the-mass shell and is not a ghost in contrast with the pure-gravity theory.

Now we have established that the three properties which are required in the application of the Wyss theorem are indeed satisfied by the zero-slope limit of the Virasoro-Shapiro amplitude (2·2). We thus conclude that the Virasoro-Shapiro model contains the Einstein gravidynamics as a zero-slope limit.

§ 3. Scalar electrodynamics as a zero-slope limit of the conventional dual model without internal symmetry

In this section we consider the amplitude corresponding to the process of two scalar partices and n dual photons in the conventional dual model and show that it reduces to the one of the ordinary scalar electrodynamics.

The amplitude is given by

$$T_n{}^E = g^n(\alpha)^{(n-2)/2}2^{-n/2}\sum_{(1,2,\cdots,n)} F_n{}^E(p,q;\epsilon_1 k_1,\epsilon_2 k_2,\cdots,\epsilon_n k_n), \tag{3·1}$$ *)

$$F_n{}^E(p,q;\epsilon_1 k_1,\cdots,\epsilon_n k_n)=\frac{1}{C}\left(\prod_{i=1}^{n}\int_0^1\frac{dz_i}{z_i}\right)\left(\prod_{i=1}^{n-1}\Theta(z_{i+1}-z_i)\right)$$

$$\times\langle\!\langle p|\prod_{i=1}^{n}V(\epsilon_i,k_i,z_i)|q\rangle\!\rangle, \tag{3·2}$$

where

$$V(\epsilon,k,z)=:\left\{\frac{1}{i}z\frac{d\epsilon\cdot Q(z)}{dz},\exp[i\sqrt{2\alpha}kQ(z)]\right\}:, \tag{3·3}$$

$$Q(z)=q_0+ip_0\log z+\sum_{n=1}^{\infty}\sqrt{\frac{1}{n}}(c_n z^n+c_n{}^+z^{-n}) \tag{3·4}$$

with

$$[q_0,p_0]=-i, \tag{3·5}$$

$$[c_n,c_m{}^+]=-\delta_{nm}, \tag{3·6}$$

$$|p\rangle\!\rangle=\exp[i\sqrt{2\alpha}pq_0]|0)\otimes|0\rangle,\quad (p^2=q^2=m^2) \tag{3·7}$$

$$p_0|0)=0,\quad c_n|0\rangle=0 \tag{3·8}$$

and the constant C is an infinite normalization factor. In the quark picture, the amplitude (3·2) reflects the situation that only one of the two constituent quarks of the scalar partice is charged.

If $n=2$, (3·2) is given by

$$\frac{1}{4}F_2{}^E(p,q;\epsilon_1,k_1,\epsilon_2,k_2)=(\epsilon_1\cdot\epsilon_2)\frac{\Gamma(1-\alpha(t-m^2))\Gamma(-1-\alpha s)}{\Gamma(-\alpha(t-m^2)-\alpha s)}$$

$$+2\alpha\{(\epsilon_1\cdot p)(\epsilon_2\cdot p)+(\epsilon_2\cdot q)(\epsilon_1\cdot q)\frac{\Gamma(1-\alpha(t-m^2))\Gamma(-1-\alpha s)}{\Gamma(-\alpha(t-m^2)-\alpha s)}$$

$$+2\alpha(\epsilon_1\cdot p)(\epsilon_2\cdot q)\frac{\Gamma(1-\alpha(t-m^2))\Gamma(-1-\alpha s)}{\Gamma(-1-\alpha(t-m^2)-\alpha s)}$$

$$+2\alpha(\epsilon_1\cdot q)(\epsilon_2\cdot p)\frac{\Gamma(2-\alpha(t-m^2))\Gamma(-1-\alpha s)}{\Gamma(1-\alpha(t-m^2)-\alpha s)}, \tag{3·9}$$

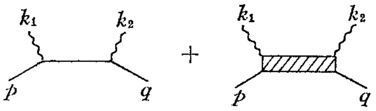
where $s=(p+q)^2$, $t=(p+k_1)^2$ and $u=(p+k_2)^2$. In the limit $\alpha\to 0$ with g being fixed, (3·9) reduces to

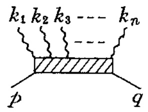
$$\frac{1}{2}F_2{}^E=-4(\epsilon_1\cdot p)(\epsilon_2\cdot q)\frac{1}{t-m^2}$$

$$+\left[-2(\epsilon_1\cdot\epsilon_2)\left(1+\frac{t-m^2}{4s}\right)-\frac{4}{s}\{(\epsilon_1\cdot p)(\epsilon_2\cdot q)-(\epsilon_1\cdot q)(\epsilon_2\cdot p)\}\right] \tag{3·10}$$

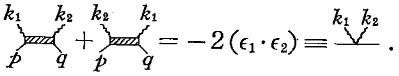
which is graphically represented as

*) Here $(1, 2,\cdots, n)$ indicates any permutation of the external photons.



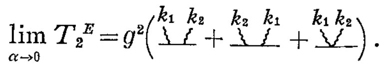
with obvious notations. In general, by the graph 

we denote the zero-slope limit of $(\alpha)^{(n-2)/2}2^{-n/2}F_n{}^E(p, q; \epsilon_1 k_1, \cdots, \epsilon_n k_n)$ after subtracting the scalar-pole (of mass m) contributions. We note that



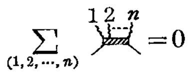
(3·11)

Thus, we obtain



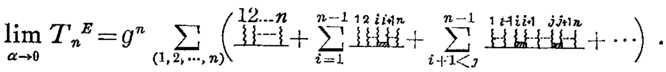
(3·12)

To proceed to the general case, we first note that



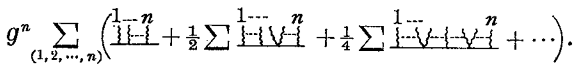
(3·13)

provided $n \geq 3$. The reasons for this are that (1) on dimensional grounds*) there do not exist any contact terms in which $n(\geq 3)$ photons are directly attached to a scalar line; (2) the photon cannot couple internally because of its oddness under the twisting. Therefore the ingredients of the graphs that can survive in the zero-slope limit are the propagator (——), the one-photon vertex () and the two-photon part ().
We have then



(3·14)

Because of the property (3·11), this reduces to



(3·15)

This result coincides with the Born term derived from the Lagrangian of the scalar electrodynamics

$$\mathcal{L} = (\partial_\mu \phi^* + ieA_\mu \phi^*)(\partial^\mu \phi - ieA^\mu \phi) - m^2 \phi^* \phi - \tfrac{1}{4} F^{\mu\nu} F_{\mu\nu}\,, \tag{3·16}$$

if one sets $g = e$.

§ 4. Comparison of generating functionals

Next we shall make a comparison between the two dual models and the

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*) Note that there does not exist any constant with inverse-mass dimension in the zero-slope limit.

corresponding Lagrangian field theories from a more formal standpoint. In particular we wish to know whether there exist in the dual models any traces of the Lagrangian. For this purpose, it is useful to study the generating functionals of the scattering matrices which are very simply related to the Lagrangian.

Let us define the generating functional $S_{fi}[\psi]$ by

$$\langle f, b_{m+1}, \cdots, b_n|S|i, b_1, \cdots, b_m\rangle = \frac{1}{\sqrt{m!(n-m)!}}\left(\prod_{i=0}^{n}\int d^4x_i\right)$$

$$\times\prod_{i=1}^{m} f_i(x_i) \prod_{i=m+1}^{n} f_i{}^*(x_i) \frac{\delta^n}{\delta\psi(x_1)\cdots\delta\psi(x_n)} S_{fi}[\psi]|_{\psi=0}, \tag{4·1}$$

where the initial state consists of a scalar particle in the state i (momentum q) and m photons or gravitons, and the final state consists of a scalar particle in the state f (momentum p) and $n-m$ photons or gravitons. The $f(x)$ is the wave function for the photon or the graviton.

We write the Lagrangian in the form

$$\mathcal{L} = \mathcal{L}(\psi) + \mathcal{L}(\psi, \phi) + \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi , \tag{4·2}$$

where ψ denotes the photon or graviton field. Then the generating functional in the tree approximation is given by

$$S_{fi}[\psi] = N^{-1}(p|T \exp\left\{\int_{-\infty}^{\infty} dt\Gamma[\Psi(x(t))]\right\}|q), \tag{4·3}$$

where

$$\Gamma[\psi(x)] = \frac{\partial^2}{\partial\phi^*(x)\partial\phi(x)}\mathcal{L}(\psi(x), \phi(x)), \tag{4·4}$$

$$|q) = \exp[iqx_0]|0), \quad (p| = (0|\exp[ipx_0] \tag{4·5}$$

$$x(t) = x_0 - 2ip_0 t , \tag{4·6}$$

$$[x_0, p_0] = -i , \quad p_0|0) = 0 , \tag{4·7}$$

and $\Psi(x)$ is defined by the integral equation

$$\Psi(x) = \psi(x) + \int S(x-y)\frac{\delta(\mathcal{L}(\Psi) - \mathcal{L}_{\text{free}}(\Psi))}{\delta\Psi(y)}d^4y . \tag{4·8}$$

The N is a normalization factor. In (4·8), $S(x-y)$ is the Green function corresponding to the free Lagrangian $\mathcal{L}_{\text{free}}(\Psi)$. For the derivation of (4·3) see Appendix A.

The $\Gamma[\psi(x)]$ is given by

$$\Gamma[A(x(t))] = e\{p_0{}^\mu, A_\mu(x(t))\} + e^2 A(x(t))^2 \tag{4·9}$$

in the electrodynamic case, and

$$\Gamma[h(x(t))] = \kappa p_\mu{}^0 h^{\mu\nu}(x(t))p_\nu{}^0 + \frac{\kappa}{2}p_\mu{}^0 h_\nu{}^\nu(x(t))p^{0\mu} - \frac{1}{2}m^2\kappa h_\nu{}^\nu(x(t)) + O(\kappa^2) \tag{4·10}$$

in the gravidynamic case, respectively.

On the other hand, the generating functionals for the dual amplitudes (2·2) and (3·1) are respectively given by

$$S_{fi}[A]=N^{-1}\langle\langle p|T\exp[g\int_{-\infty}^{\infty}dt\int_0^{2\pi\alpha}\frac{d\theta}{2\pi\alpha}\rho_E(\theta)\{p^\mu(t,\theta),A_\mu(x(t,\theta))\}]|q\rangle\rangle \quad (4\cdot11)^{*)}$$

and

$$S_{fi}[h]=N^{-1}\langle\langle p|T\exp[\kappa\int_{-\infty}^{\infty}dt\int_0^{2\pi\alpha}\frac{d\theta}{2\pi\alpha}\rho_G(\theta)p_+{}^\mu(t,\theta)h_{\mu\nu}(x(t,\theta))p_-{}^\nu(t,\theta)]|q\rangle\rangle, \quad (4\cdot12)^{*)}$$

where we have extensively used the notations of the string picture;

$$x(t,\theta)=x_0-2ip_0t+\sum_{n=1}^{\infty}\sqrt{\frac{\alpha}{n}}(a_ne^{-n(t+i\theta)/\alpha}+a_n{}^+e^{n(t+i\theta)/\alpha}+b_ne^{-n(t-i\theta)/\alpha}+b_n{}^+e^{n(t-i\theta)/\alpha}), \quad (4\cdot13)$$

$$p(t,\theta)=\frac{1}{2}p_+(t,\theta)+\frac{1}{2}p_-(t,\theta)=\frac{1}{2}i\frac{\partial}{\partial t}x(t,\theta), \quad (4\cdot14)$$

$$p_+(t,\theta)=p_0-i\sum_{n=1}^{\infty}\sqrt{\frac{\alpha}{n}}(a_ne^{-n(t+i\theta)/\alpha}-a_n{}^+e^{n(t+i\theta)/\alpha}), \quad (4\cdot15)$$

$$p_-(t,\theta)=p_0-i\sum_{n=1}^{\infty}\sqrt{\frac{n}{\alpha}}(b_ne^{-n(t-i\theta)/\alpha}-b_n{}^+e^{n(t-i\theta)/\alpha}. \quad (4\cdot16)$$

The integration variables used in the previous sections are related to t and θ by $z=\exp[-(t+i\theta)/\alpha]$. The relation of the operators c_n used in (3·1) to a_n and b_n is $c_n=(a_n+b_n)/\sqrt{2}$. The zero-mode operators are reinterpreted as $\sqrt{2\alpha}q_0\to x_0$, $(1/\sqrt{2\alpha})p_0\to p_0$ in (4·11) and $\sqrt{\alpha}(q_0+\bar{q}_0)\to x_0$, $(1/\sqrt{2\alpha})(p_0+\bar{p}_0)\to p_0$ in (4·12). The form factors $\rho_E(\theta)$ and $\rho_G(\theta)$ are defined by $\rho_E(\theta)=2\pi\alpha\delta(\theta)$ and $\rho_G(\theta)=1$, respectively. It is a familiar exercise to show that in (4·11) and (4·12) the on-the-mass shell physical states $|\varphi\rangle$ and $|\psi\rangle$ satisfy the coordinate condition

$$\langle\varphi|:p_+(t,\theta)^2:|\psi\rangle=\langle\varphi|:p_-(t,\theta)^2:|\psi\rangle=m^2\langle\varphi|\psi\rangle. \quad (4\cdot17)$$

For more details, see Appendix B.

Now, let us compare (4·3) with (4·11) and (4·12). We can formulate their relation in the following way. If the Lagrangian is given, we can know $\Gamma[\Psi(x)]$ by (4·4). Let us expand $\Gamma[\Psi(x)]$ in $\psi(x)$ and only retain the first term denoted by $\Gamma^{(1)}[\psi(x)]$. Then, the dual-model generating functional is obtained from the field-theory one (4·3) by replacing p_0 and $x(t)$ by $p_+(t,\theta)$ or $p_-(t,\theta)$ and $x(t,\theta)$ respectively in the $\Gamma^{(1)}[\psi]$, and by averaging it with respect to θ with a weight factor $\rho(\theta)$. The $\rho(\theta)$ should be determined so that the coordinate conditions (4·17) be satisfied.

*) Since the external photon and graviton are on-the-mass shell and transversal, the normal ordering is not necessary.

For the electrodynamic case, the $\Gamma^{(1)}[\psi]$ modified by the above prescriptions is given by

$$\frac{1}{2\pi\alpha}\int_0^{2\pi\alpha} d\theta\rho_E(\theta)\,\{a_+p_+{}^\mu(t,\theta)+a_-p_-{}^\mu(t,\theta),A^\mu(x(t,\theta))\}.$$

The conventional dual model corresponds to the solution $a_+=a_-=\frac{1}{2}$ and $\rho_E(\theta)=2\pi\alpha\delta(\theta)$. Similarly, the modified $\Gamma^{(1)}[\psi]$ in the gravidynamic case is given by

$$\frac{\kappa}{2\pi\alpha}\int_0^{2\pi\alpha} d\theta\rho_G(\theta)[a_{++}p_+{}^\mu(t,\theta)h_{\mu\nu}(x(t,\theta))p_+{}^\nu(t,\theta)+a_{+-}p_+{}^\mu(t,\theta)h_{\mu\nu}(x(t,\theta))p_-{}^\nu(t,\theta)$$
$$+a_{-+}p_-{}^\mu(t,\theta)h_{\mu\nu}(x(t,\theta))p_+{}^\nu(t,\theta)+a_{--}p_-{}^\mu(t,\theta)h_{\mu\nu}(x(t,\theta))p_-{}^\nu(t,\theta)].$$

Since the external graviton is a pure spin-two particle, the second and the third terms in (4·10) do not contribute. The Virasoro-Shapiro model corresponds to the solution $a_{++}=a_{--}=0$, $a_{+-}=a_{-+}=\frac{1}{2}$ and $\rho_G(\theta)=1$.

Thus, once the lowest-order interaction Lagrangian is given, the corresponding dual model can be obtained by the above re-interpretation without knowing more detailed properties of the original Lagrangian, which is obtained from the former by taking the zero-slope limit. This is not very surprising, for, in the gauge field theories, the knowledge of the lowest-order interaction enables us to derive more or less uniquely the higher-order contact terms and the self-interaction terms because of the requirement of gauge invariance. In the dual models, the role of the gauge-invariance requirement is played by the requirement of the coordinate conditions. The automatic appearence of the higher-order interaction terms in the zero-slope limit implies that for non-zero α the exchanges of the higher resonances replace the effects of such higher interactions.

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Appendix A

We derive the generating functional of field theory defined in §4. If we first neglect the self-interaction of Ψ, the $S_{fi}[\psi]$ is given by

$$S_{fi}[\psi]=\langle f|T^*\exp\{i\int d^4x\phi^*(x)\Gamma[\psi(x)]\phi(x)\}|i\rangle$$
$$=\sum_{n=0}^{\infty}i^n\int d^4x_1\cdots\int d^4x_n\langle f|\phi^*(x_1)|0\rangle\Gamma[\psi(x_1)]\Delta(x_1,x_2)\Gamma[\psi(x_2)]\cdots\Delta(x_{n-1},x_n)$$
$$\times\Gamma[\psi(x_n)]\langle 0|\phi(x_n)|i\rangle, \qquad (\mathrm{A}\cdot 1)$$

where $\Delta(x_1,x_2)=\langle 0|T\phi(x_1)\phi(x_2)|0\rangle$. On using the integral representation

$$\Delta(x_1, x_2) = -i\int_0^\infty ds\langle x_1|e^{-sH^0}|x_2\rangle\,,$$

where $H^0 = \Box + m^2$, (A·1) becomes

$$S_{fi}[\psi] = \sum_{n=0}^{\infty}\left(\prod_{j=1}^{n-1}\int_0^\infty ds_j\right)(p|\Gamma[\psi(x(s_1+\cdots+s_{n-1}))]\Gamma[\psi(x(s_2+\cdots+s_{n-1}))]$$
$$\times\cdots\times\Gamma[\psi(x(0))]|q), \qquad (\mathrm{A}\cdot 2)$$

where $\Gamma[\psi(x(s))] = e^{sH^0}\Gamma[\psi(x)]e^{-sH^0}$, $(p| = (0|\exp[ipx_0]$.
By the translational invariance of the integrand with respect to s,

$$S_{fi}[\psi] = \lim_{L\to\infty}\frac{1}{2L}\sum_{n=0}^{\infty}\left(\prod_{j=1}^{n-1}\int_0^\infty ds_j\right)\int_{-L}^{L}ds_n$$
$$\times(p|\Gamma[\Psi(x(s_1+\cdots+s_n))]\Gamma[\psi(x(s_2\cdots+s_n))]\cdots\Gamma[\psi(x(s_n))]|q)$$
$$= N^{-1}(p|T\exp\left\{\int_{-\infty}^{\infty}dt\Gamma[\psi(x(t))]\right\}|q),$$

where N is an infinite constant for normalization. The effect of the self-interaction is taken into account, in the tree approximation, by replacing $\psi(x)$ by the sum of all the connected ψ-tree graphs with one external ψ-line, which we denote by $\Psi(x)$. The $\Psi(x)$ satisfies[17)]

$$\Psi(x) = \psi(x) + \int S(x-y)\frac{\partial(\mathcal{L}(\Psi) - \mathcal{L}_{\mathrm{free}}(\Psi))}{\partial\Psi(y)}d^4y\,,$$

where $S(x-y)$ is the Green function for the free field. Thus, the complete generating functional is given by (4·3) in the tree approximation.

Appendix B

We consider a relativistic *closed* string. The action[18)] is given by

$$S = -\frac{1}{4\pi\alpha}\int d\tau\int_0^{2\pi\alpha}d\theta\sqrt{\left(\frac{\partial x(\tau,\theta)}{\partial\tau}, \frac{\partial x(\tau,\theta)}{\partial\theta}\right)^2 - \left(\frac{\partial x(\tau,\theta)}{\partial\tau}\right)^2\left(\frac{\partial x(\tau,\theta)}{\partial\theta}\right)^2}, \qquad (\mathrm{B}\cdot 1)$$

where $x_\mu(\tau,\theta) = x_\mu(\tau,\theta+2\pi\alpha)$ represents the space-time coordinate of the world tube at a parametrized point (τ,θ). The Euler equation is

$$\left(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\theta^2}\right)x(\tau,\theta) = 0 \qquad (\mathrm{B}\cdot 2)$$

in the presence of the coordinate conditions $\partial x/\partial\tau\cdot\partial x/\partial\theta = 0$ and $(\partial x/\partial\tau)^2 + (\partial x/\partial\theta)^2 = 0$. The quantization is done by postulating the canonical commutation relation $[x_\mu(\tau,\theta), p_\nu(\tau,\theta')] = -i\eta_{\mu\nu}\delta(\theta-\theta')$ where $p(\tau,\theta) = (4\pi\alpha)^{-1}(\partial/\partial\tau)x(\tau,\theta)$. Then we have $[x_0, p_0] = -i$, $[a_n, a_m{}^+] = [b_n, b_m{}^+] = -\delta_{nm}$ with the expansion*)

$$x(\tau,\theta) = x_0 + 2p_0\tau + \sum_{n=1}^{\infty}\sqrt{\frac{\alpha}{n}}(a_n e^{-in(\tau+\theta)/\alpha} + a_n{}^+ e^{in(\mu+\theta)/\alpha} + b_n e^{-in(\tau-\theta)/\alpha} + b_n{}^+ e^{in(\tau-\theta)/\alpha})\,.$$

---

*) In the text, τ is taken to be pure imaginary.

The coordinate conditions are interpreted as the subsidiary conditions on physical states

$$\langle\varphi|:p_+{}^2(\tau,\theta):|\psi\rangle=\langle\varphi|:p_-{}^2(\tau,\theta):|\psi\rangle=m^2\langle\varphi|\psi\rangle\,, \tag{B·3}$$

where $p_+=\frac{1}{2}(\partial/\partial\tau+\partial/\partial\theta)x$ and $p_-=\frac{1}{2}(\partial/\partial\tau-\partial/\partial\theta)x$. Since we can identify the operators a_n and b_n with those of the Virasoro-Shapiro model, (B·3) is satisfied. Note that the usual generators of the conformal transformation are given by $L_n{}^\pm=(2\pi\alpha)^{-1}\oint d\theta e^{i(n\theta/\alpha)}:p_\pm(\tau,\theta)^2:$. The same conditions are satisfied also in the conventional model. In fact, the operators of the conventional model c_n are related to a_n and b_n by $c_n=(a_n+b_n)/\sqrt{2}$. Any state in the conventional model is vacuum with respect to the operators $d_n\equiv(a_n-b_n)/\sqrt{2}$. Thus, for arbitrary physical states $|\varphi\rangle$ and $|\psi\rangle$,

$$\langle\varphi|:p(\tau+\theta)^2:|\psi\rangle=m^2\langle\varphi|\psi\rangle\,, \tag{B·4}$$

$$\langle\varphi|:Q(\tau-\theta)^2:|\psi\rangle=0\,, \tag{B·5}$$

where $p(\tau+\theta)=\frac{1}{2}(p_+(\tau,\theta)+p_-(\tau,\theta))$ and $Q(\tau-\theta)=\frac{1}{2}(p_+(\tau,-\theta)-p_-(\tau,\theta))$. (B·4) and (B·5) are equivalent to the conditions (B·3).

References

1) J. Scherk, Nucl. Phys. **B31** (1971), 222.
   See also N. Nakanishi, Prog. Theor. Phys. **48** (1972), 355.
2) N. Nakanishi, Prog. Theor. Phys. **45** (1971), 436, 451, 919.
   T. Yoneya, Prog. Theor. Phys. **48** (1972), 2044.
3) P. Goddard and C. B. Thorn, Phys. Letters **40B** (1972), 235.
   R. C. Brower, Phys. Rev. **D6** (1972), 1655.
4) M. A. Virasoro, Phys. Rev. **D1** (1970), 2933.
   S. Fubini and G. Veneziano, Ann. of Phys. **63** (1971), 12.
5) E. DelGiudice, P. DiVecchia and S. Fubini, Ann. of Phys. **70** (1972), 378.
6) P. Ramond, Phys. Rev. **D5** (1972), 2078.
   T. Yoneya, Prog. Theor. Phys. **48** (1972), 616.
7) A. Neveu and J. Scherk, Nucl. Phys. **B36** (1972), 155.
8) S. Weinberg, Phys. Rev. **135** (1964), B1049.
9) J. Shapiro, Phys. Letters **33B** (1970), 361.
   E. DelGiudice and P. DiVecchia, Nuovo Cim. **5A** (1971), 90.
   M. Yoshimura, Phys. Letters **34B** (1971), 79.
10) M. A. Virasoro, Phys. Rev. **177** (1969), 2309.
11) K. Kawarabayashi and S. Kitakado, Nuovo Cim. **14A** (1973), 190.
12) J. Ellis, Nucl. Phys. **B21** (1970), 217.
    E. Osborn, Lett. Nuovo Cim. **2** (1969), 717.
13) W. Wyss, Helv. Phys. Acta **38** (1965), 469.
    See also S. N. Gupta, Rev. Mod. Phys. **29** (1957), 334; W. Thirring, Fort. der Phys. **7** (1959), 79.
14) P. Compagna, E. Napolitano, S. Sciuto and S. Fubini, Nuovo Cim. **2A** (1971), 911.
15) T. Kimura, Prog. Theor. Phys. **36** (1966), 394.
16) T. Yoneya, Lett. Nuovo Cim. **8** (1973), 951.
17) Y. Nambu, Phys. Letters **26B** (1968), 626.
18) P. Goddard, J. Godstone, C. Rebbi and C. Thorn, Nucl. Phys. **B56** (1973), 109; and the references therein.