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CONNECTIONS AMONG DIFFERENT CRITERIA FOR ASYMMETRICAL FRACTIONAL FACTORIAL DESIGNS

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Abstract: In recent years, there has been increasing interest in the study of asymmetrical fractional factorial designs. Various new optimality criteria have been proposed from different principles for design construction and comparison, such as generalized minimum aberration, minimum moment aberration, minimum projection uniformity and the $\chi^2(D)$ (for design D) criteria. In this paper, these criteria are reviewed and the $\chi^2(D)$ criterion is generalized to the so-called minimum χ^2 criterion. Connections among different criteria are investigated. These connections provide strong statistical justification for each of them. Some general optimality results are developed, which not only unify several results (including results for the symmetrical case), but also are useful for constructing asymmetrical supersaturated designs.

Key words and phrases: Generalized minimum aberration, minimum moment aberration, orthogonal array, supersaturated design, uniformity.

1. Introduction

Fractional factorial designs (FFDs) are arguably the most widely used designs in scientific investigations. Practical success is due to efficient use of experimental runs to study many factors simultaneously. A fundamental and practical question for FFDs is how to choose a "good" design from a set of candidates. From different viewpoints, various optimality criteria have been proposed for design construction and comparison. There have been extensive studies on the criteria for symmetrical FFDs. Recently, there has been increasing interest in the study of asymmetrical FFDs, and several new kinds of criteria have been proposed for assessing them. Among those criteria, *generalized minimum aberration* (GMA, Tang and Deng (1999), Ma and Fang (2001) and Xu and Wu (2001)) considers the confounding situation between treatment effects under the ANOVA decomposition; *minimum moment aberration* (MMA, Xu (2003)) investigates the relationship between runs, and offers tremendous savings in computation over GMA; $\chi^2(D)$ (for design D, Yamada and Matsui (2002)) measures two-factor non-orthogonality combinatorially; *minimum projection uniformity*

(MPU, Hickernell and Liu (2002)) considers the uniformity of low-dimensional projections of a design. It should be noted that the GMA, MMA and $\chi^2(D)$ criteria are suitable mainly for qualitative factors. The MPU criterion is suitable for both qualitative and quantitative factors, but here we only consider MPU for the qualitative case. Refer to Cheng and Ye (2004) for criteria developed especially for quantitative factors.

Each criterion mentioned above has its own merits. A natural question now arises: what connections or equivalencies exist among those criteria? This article aims to study the question for criteria for asymmetrical FFDs, and to provide some optimality results. In Section 2, the existing criteria are described, the $\chi^2(D)$ criterion is generalized to the so-called minimum χ^2 criterion, and connections among the different criteria are investigated. The design criteria turn out to be closely related to each other. These connections provide statistical justification for each of them from other viewpoints. Section 3 contains some general optimality results. We develop several lower bounds, along with sufficient and necessary conditions for optimality. Most of these results apply to balanced designs, and they not only unify several results (including results for the symmetrical case), but also are useful for constructing asymmetrical supersaturated designs. For ease of presentation, proofs are deferred to the appendix.

2. Design Criteria and Connections

Some facts and notation are as follows. An *asymmetrical* (or mixed-level) design of n runs, m factors and levels $q_1, \ldots, q_m$ is denoted by $D(n; q_1, \ldots, q_m)$; when some q_j's are equal, it is denoted by $D(n; q_1^{r_1}, \ldots, q_l^{r_l})$ with $\sum_{j=1}^{l} r_j = m$. A design $D(n; q_1, \ldots, q_m)$ can be expressed as an $n \times m$ matrix $D = (d_{ij})$ with d_{ij} from a set of q_j symbols, say, $\{1, \ldots, q_j\}$. A $D(n; q_1, \ldots, q_m)$ is called an *orthogonal array of strength* s, denoted by $D(n; q_1, \ldots, q_m; s)$, if in any s columns all possible level-combinations appear equally often. A balanced design is an orthogonal array of strength 1, which is also called a *U-type design* and denoted by $U(n; q_1, \ldots, q_m)$ (Fang, Lin, Winker and Zhang (2000)). When $\sum_{j=1}^{m}(q_j - 1) = n-1$, the design $D(n; q_1, \ldots, q_m)$ is called *saturated*. When $\sum_{j=1}^{m}(q_j - 1) > n-1$, orthogonality is not obtainable and the design is called *supersaturated*. Next, we describe the criteria mentioned in the introduction, and we generalize the $\chi^2(D)$ criterion.

2.1. GMA, MMA and MPU criteria

Regular FFDs are often constructed to be of minimum aberration (Fries and Hunter (1980)), since this criterion limits the adverse effects of aliasing. Aberration has been generalized to nonregular FFDs (Tang and Deng (1999), Ma and Fang (2001) and Xu and Wu (2001)). Especially, GMA due to Xu and

Wu (2001) for the asymmetrical case covers all the other generalizations. Based on the ANOVA decomposition model, for a $D(n; q_1, \ldots, q_m)$ design D, let $X_j = (x_{ik}^j)$ be the matrix consisting of all j-factor contrast coefficients, for $j = 0, \ldots, m$. If

$$A_j(D) = \frac{1}{n^2} \sum_k \Big| \sum_{i=1}^n x_{ik}^j \Big|^2, \tag{1}$$

the GMA criterion is to sequentially minimize $A_j(D)$ for $j = 1, \ldots, m$.

For a design $D = (d_{ij})$, an integer $t > 0$ and some weights $w_k > 0$, let

$$\delta_{ij}(D) = \sum_{k=1}^m w_k \delta_{ij}^{(k)}, \tag{2}$$

where $\delta_{ij}^{(k)} = 1$ if $d_{ik} = d_{jk}$, and 0 otherwise. Thus $\delta_{ij}(D)$ is the weighted coincidence number between the ith and jth rows of D. Define the tth power moment to be

$$M_t(D) = \Big[\frac{n(n-1)}{2}\Big]^{-1} \sum_{1 \le i < j \le n} [\delta_{ij}(D)]^t. \tag{3}$$

The MMA criterion is to sequentially minimize $M_t(D)$ for $t = 1, \ldots, m$. The choice of $w_k = \lambda q_k$ is called a natural weight.

Now let us introduce the MPU criterion defined for asymmetrical FFDs and qualitative factors, which is developed from the uniformity viewpoint. Interested readers are referred to Hickernell and Liu (2002, Sec. 5) for a general definition and some discussion about MPU. For a $D(n; q_1, \ldots, q_m)$ design D, define the t-dimensional projection discrepancy $\mathcal{D}_{(t)}(D; K)$ as the non-negative square root of

$$\mathcal{D}_{(t)}^2(D; K) = \frac{1}{n^2} \sum_{i,j=1}^n \sum_{1 \le l_1 < \cdots < l_t \le m} \prod_{g=1}^t \left(-1 + q_{l_g} \delta_{ij}^{(l_g)}\right). \tag{4}$$

The MPU criterion is to sequentially minimize $\mathcal{D}_{(t)}(D; K)$ for $t = 1, \ldots, m$.

For symmetrical FFDs, the linear combination relationship between the $A_i(D)$'s and the $M_i(D)$'s had been thoroughly presented in several papers, such as Xu (2003, 2005). For asymmetrical designs, Hickernell and Liu (2002) proved that MPU and GMA are equivalent, and Xu (2003) showed that GMA and MMA are weakly equivalent.

Lemma 1. (i) *For a $D(n; q_1, \ldots, q_m)$ design D, $\mathcal{D}_{(j)}^2(D; K) = A_j(D)$, i.e., the MPU is equivalent to the GMA defined by* Xu and Wu (2001) (Hickernell and Liu (2002, Theorem 2)).

(ii) *For a $D(n; q_1, \ldots, q_m; s)$ design D, if $w_k = \lambda q_k$ for all k, then $M_t(D) = \lambda^t [n(n-1)^{-1} t! A_t(D) + \gamma_t]$ for $t = 1, \ldots, s+1$, where γ_t's are constants depending on n, m, t and the levels $q_1, \ldots, q_m$* (Xu (2003, Theorem 7)).

Thus, by replacing $A_t(D)$ in (ii) of this lemma with $\mathcal{D}^2_{(t)}(D;K)$, the weak equivalency between MPU and MMA follows directly, and provides a justification for using MPU as an optimality criterion for choosing asymmetrical designs.

2.2. Minimum χ^2 criterion

Supersaturated design (SSD) is an important nonregular FFD. Most studies have focused on symmetrical SSDs. As for asymmetrical SSDs, Yamada and Matsui (2002) used the $\chi^2(D)$ as a measure of two-factor non-orthogonality. Here we generalize it to assess the non-orthogonality among any t factors combinatorially.

For any t columns of a $D(n;q_1,\ldots,q_m)$ design D, say $(c_{l_1},\ldots,c_{l_t})$, let $n^{(l_1\cdots l_t)}_{\alpha_1\cdots\alpha_t}$ be the number of runs in which $(c_{l_1},\ldots,c_{l_t})$ takes the level-combination $(\alpha_1\cdots\alpha_t)$, let

$$\chi^2(c_{l_1},\ldots,c_{l_t}) = \frac{\sum_{\alpha_1,\ldots,\alpha_t}\left(n^{(l_1\cdots l_t)}_{\alpha_1\cdots\alpha_t} - \frac{n}{\prod_{i=1}^t q_{l_i}}\right)^2}{\left(\frac{n}{\prod_{i=1}^t q_{l_i}}\right)}, \tag{5}$$

where the summation is taken over all possible level-combinations, and then define

$$\chi^2_t(D) = \sum_{1\le l_1<\cdots<l_t\le m} \chi^2(c_{l_1},\ldots,c_{l_t}). \tag{6}$$

Note that $\chi^2(c_{l_1},\ldots,c_{l_t})$ is analogous to the χ^2 statistic, and it is clear that $\chi^2_t(D)$ is a measure of t-dimensional non-orthogonality of the design. Under this measure, an optimal design should minimize $\chi^2_t(D)$ for $t=1,\ldots,m$ sequentially. We call this criterion the *minimum χ^2 criterion*.

Take $t=2$ in (5), then the $\chi^2_2(D)$ in (6) is just the $\chi^2(D)$ defined by Yamada and Matsui (2002). Recently, Fang, Lin and Liu (2003) proposed the $E(f_{NOD})$ criterion for choosing asymmetrical SSDs, defined as minimizing

$$E(f_{NOD}) = \Big[\frac{m(m-1)}{2}\Big]^{-1} \sum_{1\le i<j\le m} \frac{\chi^2(c_i,c_j)n}{q_iq_j}.$$

Note that the $\chi^2(D)$ considers different weights for factors with different levels, while $E(f_{NOD})$ does not do so.

It has been shown that the $\chi^2(D)$, $E(f_{NOD})$ and $M_2(D)$ criteria are extensions of existing criteria for symmetrical SSDs, see Fang, Lin and Liu (2003), Xu (2003) and Li, Liu and Zhang (2004) for details.

2.3. Connections

Throughout this subsection, $w_k = \lambda q_k$ for all k in (2). First let us see some basic properties of the χ^2 statistic in (5) and $\chi^2_t(D)$ in (6):

a. $\chi^2(c_{l_1},\ldots,c_{l_t})=0$ if and only if $c_{l_1},\ldots,c_{l_t}$ have all possible level-combinations appear equally often.

b. $\chi^2(c_{l_1},\ldots,c_{l_t})=(1/n)\prod_{i=1}^t q_{l_i}\sum_{\alpha_1,\ldots,\alpha_t}\left(n^{(l_1\cdots l_t)}_{\alpha_1\cdots\alpha_t}\right)^2-n.$

c. If $\chi^2_t(D)=0$, then $\chi^2_j(D)=0$ for $j<t$.

d. $\chi^2_t(D)=0$ for some $t\geq 1$ if and only if the strength of D is at least t.

From the above statements, we conjecture that the minimum χ^2 criterion should be closely related to MPU and GMA. In fact we get the following result.

Theorem 1. (i) *For any $D(n;q_1,\ldots,q_m)$ design D,*

$$\chi^2_t(D)=\frac{1}{n}\sum_{i,j=1}^{n}\Big[\sum_{1\leq l_1<\cdots<l_t\leq m}\prod_{g=1}^{t}\left(q_{l_g}\delta_{ij}^{(l_g)}\right)\Big]-n\binom{m}{t} \textit{for } 1\leq t\leq m.$$

(ii) *Furthermore, if D is a $D(n;q_1,\ldots,q_m;s)$, $\mathcal{D}^2_{(s+1)}(D;K)=A_{s+1}(D)=\chi^2_{s+1}(D)/n$.*

This theorem provides another statistical justification for MPU/GMA from the χ^2 statistic point of view. We note that Tang (2001) proposed a criterion, called the V-criterion, which is similar to the minimum χ^2 criterion, and obtained the equivalency between the V-criterion and GMA for 2-level FFDs. Our Theorem 1 generalizes his result from 2-level FFDs to asymmetrical FFDs. Combining Lemma 1 and Theorem 1, we have

Corollary 1. *For any $D(n;q_1,\ldots,q_m;s)$ design D, all the values of $\mathcal{D}^2_{(j)}(D;K)$, $A_j(D)$, $\chi^2_j(D)$ and $M_j(D)$ for $j\leq s$ are minimized, and*

$$\mathcal{D}^2_{(s+1)}(D;K)=A_{s+1}(D)=\frac{\chi^2_{s+1}(D)}{n}=\frac{n-1}{n\lambda^{s+1}(s+1)!}\left[M_{s+1}(D)-\lambda^{s+1}\gamma_{s+1}\right].$$

This result tells us that though the GMA, MMA, MPU, minimum χ^2 criteria are raised from distinct considerations, they are strongly connected to each other: a $D(n;q_1,\ldots,q_m;s)$ design D minimizing one of $\mathcal{D}^2_{(s+1)}(D;K)$, $A_{s+1}(D)$, $\chi^2_{s+1}(D)$, $M_{s+1}(D)$, minimizes all of them. This conclusion is important for constructing asymmetrical SSDs, where the goal is to minimize one of these values for $s=1$ in balanced designs.

Now, we have set up the connection between MMA and the minimum χ^2 criterion, i.e., for any asymmetrical design D with strength s, $M_{s+1}(D)$ and $\chi^2_{s+1}(D)$ can be minimized at the same time. But we are uncertain whether $M_j(D)$ and $\chi^2_j(D)$, $j>s+1$, can be minimized simultaneously. From their definitions, we know that the power moments investigate the relationship between runs (i.e., rows) of a design, while the χ^2 statistics in (5) study the relationship between factors (i.e., columns). The following theorem shows the equivalency of $M_t(D)$ and a summation of $\chi^2(c_{l_1},\ldots,c_{l_t})$, regardless of the strength s.

Theorem 2. *For any $D(n;q_1,\ldots,q_m)$ design D,*

$$M_t(D)=\frac{\lambda^t}{n-1}\Big\{\sum_{1\le l_1,\ldots,l_t\le m}\chi^2(c_{l_1},\ldots,c_{l_t})+nm^t-\Big(\sum_{k=1}^m q_k\Big)^t\Big\}.$$

Note that in this theorem, equality is valid for any asymmetrical design. This theorem provides another statistical justification for MMA from the viewpoint of minimizing non-orthogonality among design columns: an MMA design is a design sequentially minimizing $\sum_{1\le l_1,\ldots,l_t\le m}\chi^2(c_{l_1},\ldots,c_{l_t})$ for $t=1,\ldots,m$.

3. Optimality Results

This section provides some optimality results for the various design criteria mentioned in the above section. Our discussions focus on balanced asymmetrical designs, i.e., $U(n;q_1,\ldots,q_m)$ designs. For such designs, our objective is to minimize the components $\mathcal{D}^2_{(2)}(D;K)$, $A_2(D)$, $M_2(D)$ or $\chi^2(D)$. As they are equivalent to each other, we study only the power moment measure, because of its conceptual simplicity and usefulness in the theoretical development.

3.1. Some lower bounds

Majorization theory (Marshall and Olkin (1979)) is a suitable tool for studying the properties of $M_t(D)$ for $t\ge 2$. Recent application of majorization to FFDs includes Cheng and Mukerjee (1998) and Cheng, Steinberg and Sun (1999) on estimation capacity, as well as Zhang, Fang, Li and Sudjianto (2005) on pairwise coincidences for assessing symmetrical balanced FFDs.

Recall that for two distinct vectors $\boldsymbol{x}=(x_1,\ldots,x_k)$ and $\boldsymbol{y}=(y_1,\ldots,y_k)$ with nonnegative components and the same sum of components ($\sum_{i=1}^k x_i=\sum_{i=1}^k y_i$), $\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$ if $\sum_{i=1}^r x_{[i]}\ge\sum_{i=1}^r y_{[i]}$, for all $1\le r\le k-1$, where $x_{[1]}\le x_{[2]}\le\cdots\le x_{[k]}$ and $y_{[1]}\le y_{[2]}\le\cdots\le y_{[k]}$ are the ordered components of $\boldsymbol{x}$ and $\boldsymbol{y}$ respectively. A real-valued function f of $\boldsymbol{x}$ is said to be *Schur-convex* if $f(\boldsymbol{x})\le f(\boldsymbol{y})$ whenever $\boldsymbol{x}$ is majorized by $\boldsymbol{y}$.

From the definition of the tth power moment, we know that it is a function of the weighted coincidence numbers between distinct rows of the design. For a $U(n;q_1,\ldots,q_m)$ design D and any given weights w_k,

$$\sum_{j=1,j\ne i}^n\delta_{ij}(D)=\sum_{k=1}^m w_k\Big(\frac{n}{q_k}-1\Big),\ \text{for } i=1,\ldots,n. \tag{7}$$

It is easy to see that $M_t(D)$ is a Schur-convex function of the vector

$$\boldsymbol{\delta}(D) = (\delta_{12}(D), \ldots, \delta_{1n}(D), \delta_{23}(D), \ldots, \delta_{2n}(D), \ldots, \delta_{(n-1)n}(D)), \tag{8}$$

for any $t \geq 2$.

Lemma 2. *Let D and D^* be two $U(n; q_1, \ldots, q_m)$ designs, $\boldsymbol{\delta}(D)$ and $\boldsymbol{\delta}(D^*)$ be the respective vectors defined in* (8). *If $\boldsymbol{\delta}(D^*)$ is majorized by $\boldsymbol{\delta}(D)$, then $M_t(D^*) \leq M_t(D)$, for any $t \geq 2$. In particular, if $\boldsymbol{\delta}(D^*)$ is majorized by any $\boldsymbol{\delta}(D)$, then $M_t(D^*)$ is minimized, i.e., D^* is an MMA design.*

From this lemma and (7), we have the following result, also given by Xu (2003, Theorem 6).

Lemma 3. *For a $U(n; q_1, \ldots, q_m)$ design D and $t \geq 2$, $M_t(D) \geq \delta^t$, and equality holds if and only if $\delta_{ij}(D)$ defined in* (2) *is a constant δ for all $i < j$, where $\delta = \sum_{k=1}^m w_k(n/q_k - 1)/(n-1)$.*

Remark 1. From Corollary 1, by letting $w_k = \lambda q_k$, lower bounds for $\mathcal{D}^2_{(2)}(D)$, $A_2(D)$ and $\chi^2(D)$ can be obtained in a straightforward manner, and these bounds are tight for the same condition as in Lemma 3. In particular, for $\chi^2(D)$, its lower bound obtained in this way is the same as that obtained by Yamada and Matsui (2002), but they did not give sufficient and necessary conditions for achieving the lower bound.

Remark 2. Lemma 3 provides a condition for which the lower bound can be achieved. For some values of $(n, m, q_1, \ldots, q_m, w_1, \ldots, w_k)$, this lower bound is attainable. For example, when D is a saturated $D(n; q_1, \ldots, q_m; 2)$, and natural weights $w_k = \lambda q_k$ for $1 \leq k \leq m$ are assumed, the lower bound is attained as $\boldsymbol{\delta}(D) = (\lambda(m-1), \ldots, \lambda(m-1))$ (Mukerjee and Wu (1995)). In cases where some $\delta_{ij}(D)$'s for $i < j$ can not equal $\delta = \sum_{k=1}^m w_k(n/q_k - 1)/(n-1)$, the lower bound can be improved.

For given m, q_k and w_k, let

$$\Delta = \Big\{ \sum_{k=1}^m w_k \delta_{ij}^{(k)} : \delta_{ij}^{(k)} = 0, 1, \text{ for } k = 1, \ldots, m \Big\}.$$

In Δ, let δ_L and δ_U be the two nearest values to $\delta = \sum_{k=1}^m w_k(n/q_k - 1)/(n - 1)$, satisfying $\delta_L \leq \delta < \delta_U$. It can be easily observed that if there exists a $U(n; q_1, \ldots, q_m)$ design D^* whose $\delta_{ij}(D^*)$ for $i < j$ take values from δ_L and δ_U, then for any other $U(n; q_1, \ldots, q_m)$ design D, $\boldsymbol{\delta}(D^*)$ is majorized by $\boldsymbol{\delta}(D)$, i.e., D^* is an MMA design. Condition (7) determines the numbers of times δ_L and δ_U appear in $\boldsymbol{\delta}(D^*)$. Explicitly, this result can be expressed as follows.

Theorem 3. *Given w_k for all k, then for a $U(n; q_1, \ldots, q_m)$ design D and $t \geq 2$,*

$$M_t(D) \geq \frac{\delta_U - \delta}{\delta_U - \delta_L}\delta_L^t + \frac{\delta - \delta_L}{\delta_U - \delta_L}\delta_U^t. \tag{9}$$

Equality holds if and only if for any i, among the $(n-1)$ values of $\delta_{1i}(D), \ldots,$ $\delta_{(i-1)i}(D)$, $\delta_{i(i+1)}(D), \ldots, \delta_{in}(D)$, there are $(n-1)[(\delta_U - \delta)/(\delta_U - \delta_L)]$ with the value δ_L and $(n-1)[(\delta - \delta_L)/(\delta_U - \delta_L)]$ with the value δ_U.

Remark 3. If equality holds in (9) for a certain t, then $M_t(D)$ is minimized, and all other $M_i(D)$'s for $i \geq 2$ and $i \neq t$ are uniquely determined by the values of δ, δ_L and δ_U, thus the design D is an MMA design. Moreover, the lower bound in this theorem includes the one in Lemma 3 as a special case when $\delta_L = \delta$.

Based on the connections developed in last section, we have

Corollary 2. *Suppose $w_k = \lambda q_k$ for all k, then for a $U(n; q_1 \cdots q_m)$ design D,*

$$M_2(D) \geq (\delta_U + \delta_L)\delta - \delta_U\delta_L, \text{ and}$$
$$\mathcal{D}^2_{(2)}(D; K) = A_2(D) = \frac{\chi^2(D)}{n} \geq \frac{n-1}{2n\lambda^2}\Big\{(\delta_U + \delta_L)\delta - \delta_U\delta_L - \lambda^2\gamma_2\Big\}.$$

The sufficient and necessary condition for the equalities to hold is the same as that of Theorem 3, except for replacing w_k by λq_k when calculating δ, δ_L and δ_U.

This corollary, along with Remark 3, tells us that when the lower bound of $M_2(D)$ is achieved by a design D, then D is an MMA design. It is also optimal according to $\mathcal{D}^2_{(2)}(D; K)$, $A_2(D)$ and $\chi^2(D)$. Recently, Li, Liu and Zhang (2004) obtained a lower bound for $\chi^2(D)$ that is a special case of the lower bound given in the corollary. Note that the optimality results developed here also unify the results obtained, e.g. by Liu and Hickernell (2002), Fang, Ge, Liu and Qin (2003, 2004b), Xu (2003), etc., for symmetrical FFDs.

3.2. Optimal designs

We should notice that most of the designs achieving those lower bounds provided in Corollary 2 are SSDs. As for the construction of asymmetrical SSDs, Yamada and Matsui (2002) and Yamada and Lin (2002) proposed two methods for constructing 2- and 3-level SSDs through computer searches. However their resulting designs cannot always achieve the lower bound of $\chi^2(D)$. Fang, Lin and Liu (2003) proposed a method for constructing $E(f_{NOD})$-optimal asymmetrical SSDs, called the *fractions of saturated orthogonal arrays* (FSOA) method, which is an extension of Lin's (1993) half fraction of a Hadamard matrix method. Recently, Li, Liu and Zhang (2004) extended the FSOA method to the construction of $\chi^2(D)$-optimal asymmetrical SSDs and studied the properties of the resulting designs. The designs constructed from their method are also optimal according to MMA, $\mathcal{D}^2_{(2)}(D)$ and $A_2(D)$.

Another paper concerning the construction of asymmetrical SSDs is Fang, Ge, Liu and Qin (2004a). In that paper, they set up an important bridge between SSDs and *uniformly resolvable designs*, a kind of combinatorial design, and obtained several new infinite classes of $E(f_{NOD})$-optimal SSDs. From Fang, Ge, Liu and Qin's (2004a) concluding remarks, we know that all their designs are of *one coincidence position* between any two distinct rows. Also we can see that most of their designs are of the form $D(n; p^1, q^{m-1})$. For $D(n; p^1, q^{m-1})$ designs, we have the following result.

Theorem 4. *Let D be a $D(n; p^1, q^{m-1})$ design, where $p \le q$ and $n/p + (m-1)n/q - m = n - 1$. If there exists exactly* one coincidence position *between any two distinct rows of D, then D is an MMA design with natural weights, and it is also optimal according to $\mathcal{D}^2_{(2)}(D)$, $A_2(D)$ and $\chi^2(D)$.*

From this theorem, we can easily draw the conclusion that the $E(f_{NOD})$-optimal $D(n; p^1, q^{m-1})$ designs with $p \le q$ due to Fang, Ge, Liu and Qin (2004a) are still optimal according to $\chi^2(D)$, $\mathcal{D}^2_{(2)}(D)$, $A_2(D)$ and MMA.

The *column juxtaposition* method can also be used to construct asymmetrical SSDs. Li, Liu and Zhang (2004) applied it to construct $\chi^2(D)$ optimal and MMA designs, where they assumed $w_k = \lambda q_k$ for all k. Obviously, the resulting designs are also $\mathcal{D}^2_{(2)}(D)$ and $A_2(D)$ optimal.

Corollary 3. *Let D_t, for $1 \le t \le l$, be balanced designs with the same number of runs. Given the weights w_k for all k, if the weighted coincidence numbers $\delta_{ij}(D_t)$ for $i < j$ are constant for each design D_t, then $D = (D_1, \ldots, D_l)$ is an MMA design. In particular, if the natural weights $w_k = \lambda q_k$ for all k are assumed, D is also optimal according to $\mathcal{D}^2_{(2)}(D)$, $A_2(D)$ and $\chi^2(D)$.*

Based on this corollary, many optimal SSDs can be constructed, not only from saturated orthogonal arrays of strength 2, but also from SSDs with the given property as shown in the corollary, such as the designs due to Liu and Zhang (2000), Fang, Lin and Ma (2000), Fang, Ge and Liu (2002, 2004) and Fang, Ge, Liu and Qin (2003, 2004b).

Besides those above methods, the construction of asymmetrical SSDs still needs to be investigated further.

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Appendix

Proof of Theorem 1. It is easy to verify that for any $D(n;q_1,\ldots,q_m)$ design D,

$$\sum_{\alpha_1,\ldots,\alpha_t}\left(n_{\alpha_1\cdots\alpha_t}^{(l_1\cdots l_t)}\right)^2=\sum_{i,j=1}^{n}\prod_{g=1}^{t}\delta_{ij}^{(l_g)}. \tag{10}$$

So from the result (**b**) in Subsection 2.3,

$$\begin{aligned}\chi_t^2(D)&=\frac{1}{n}\sum_{1\le l_1<\cdots<l_t\le m}\Big[\prod_{g=1}^{t}q_{l_g}\sum_{\alpha_1,\ldots,\alpha_t}\left(n_{\alpha_1\cdots\alpha_t}^{(l_1\cdots l_t)}\right)^2\Big]-n\binom{m}{t}\\&=\frac{1}{n}\sum_{1\le l_1<\cdots<l_t\le m}\Big[\sum_{i,j=1}^{n}\prod_{g=1}^{t}\left(q_{l_g}\delta_{ij}^{(l_g)}\right)\Big]-n\binom{m}{t},\end{aligned} \tag{11}$$

hence the expression for $\chi_t^2(D)$ follows. As for $\mathcal{D}_{(t)}^2(D;K)$, note that

$$\begin{aligned}&\prod_{g=1}^{t}\left(-1+q_{l_g}\delta_{ij}^{(l_g)}\right)\\&\quad=(-1)^t+\sum_{h=1}^{t-1}\Big[(-1)^{t-h}\sum_{1\le t_1<\cdots<t_h\le t}\prod_{g=1}^{h}\left(q_{l_{t_g}}\delta_{ij}^{(l_{t_g})}\right)\Big]+\prod_{g=1}^{t}q_{l_g}\delta_{ij}^{(l_g)}\end{aligned} \tag{12}$$

and, for any $D(n;q_1,\ldots,q_m;s)$ design D,

$$\sum_{i,j=1}^{n}\prod_{g=1}^{h}\left(q_{l_{t_g}}\delta_{ij}^{(l_{t_g})}\right)=n^2 \text{ for } 1\le h\le s, \text{ and any different } l_{t_g}\text{'s}. \tag{13}$$

Thus from (4) we can express $\mathcal{D}_{(t)}^2(D;K)$ in terms of (12), then by exchanging the order of summations in the expression and using (13), we get

$$\begin{aligned}\mathcal{D}_{(s+1)}^2(D;K)&=\binom{m}{s+1}\Big[(-1)^{s+1}+\sum_{h=1}^{s}(-1)^{s+1-h}\binom{s+1}{h}\Big]\\&\qquad+\frac{1}{n^2}\sum_{1\le l_1<\cdots<l_{s+1}\le m}\sum_{i,j=1}^{n}\prod_{g=1}^{s+1}\left(q_{l_g}\delta_{ij}^{(l_g)}\right)\\&=-\binom{m}{s+1}+\frac{1}{n^2}\sum_{1\le l_1<\cdots<l_{s+1}\le m}\sum_{i,j=1}^{n}\prod_{g=1}^{s+1}\left(q_{l_g}\delta_{ij}^{(l_g)}\right).\end{aligned}$$

Combing with (11) and Lemma 1, the proof is completed.

Proof of Theorem 2. For a $D(n; q_1, \ldots, q_m)$ design D, $\delta_{ii}(D)/\lambda = \sum_{k=1}^{m} q_k$, and thus

$$\begin{aligned}
\frac{n(n-1)M_t(D)}{\lambda^t} &= \sum_{i,j=1}^{n} \Big[\frac{\delta_{ij}(D)}{\lambda}\Big]^t - n\Big(\sum_{k=1}^{m} q_k\Big)^t \\
&= \sum_{i,j=1}^{n} \Big[\sum_{l=1}^{m} q_l \delta_{ij}^{(l)}\Big]^t - n\Big(\sum_{k=1}^{m} q_k\Big)^t \\
&= \sum_{i,j=1}^{n} \Big[\sum_{1\le l_1,\ldots,l_t\le m} \prod_{k=1}^{t} \Big(q_{l_k}\delta_{ij}^{(l_k)}\Big)\Big] - n\Big(\sum_{k=1}^{m} q_k\Big)^t \\
&= \sum_{1\le l_1,\ldots,l_t\le m} \Big[\sum_{i,j=1}^{n} \prod_{k=1}^{t} \Big(q_{l_k}\delta_{ij}^{(l_k)}\Big)\Big] - n\Big(\sum_{k=1}^{m} q_k\Big)^t.
\end{aligned}$$

Hence from (10),

$$\begin{aligned}
\frac{(n-1)M_t(D)}{\lambda^t} &= \sum_{1\le l_1,\ldots,l_t\le m} \Big[\frac{1}{n}\prod_{g=1}^{t} q_{l_g} \sum_{\alpha_1,\ldots,\alpha_t} \Big(n_{\alpha_1\cdots\alpha_t}^{(l_1\cdots l_t)}\Big)^2\Big] - \Big(\sum_{k=1}^{m} q_k\Big)^t \\
&= \sum_{1\le l_1,\ldots,l_t\le m} \Big[\frac{1}{n}\prod_{g=1}^{t} q_{l_g} \sum_{\alpha_1,\ldots,\alpha_t} \Big(n_{\alpha_1\cdots\alpha_t}^{(l_1\cdots l_t)}\Big)^2 - n\Big] + nm^t - \Big(\sum_{k=1}^{m} q_k\Big)^t.
\end{aligned}$$

Then the equality in the theorem follows from the result (**b**) in Subsection 2.3.

Proof of Theorem 4. For parameters n, m, p, q satisfying $p \le q$ and $n/p + (m-1)n/q - m = n - 1$, and the natural weights, it can be easily observed that

$$\lambda p \le \delta = \frac{(\lambda p)(n/p-1) + (m-1)(\lambda q)(n/q-1)}{n-1} \le \lambda q,$$

and the two nearest values δ_L and δ_U in Corollary 2 can only be λp and λq.

For the $D(n; p^1, q^{m-1})$ design with exactly one coincidence position between any two distinct rows, its natural weighted coincidence numbers $\delta_{ij}(D)$ for $i < j$ just take the two values λp and λq, hence the optimality of this design follows from Corollary 2.

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