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Connections between Hyers-Ulam stability and uniform exponential stability of discrete evolution families of bounded linear operators over Banach spaces

Tongxing Li[1,2*] and Akbar Zada[3]

*Correspondence: litongx2007@163.com
[1]LinDa Institute of Shandong Provincial Key Laboratory of Network Based Intelligent Computing, Linyi University, Linyi, Shandong 276005, P.R. China
[2]School of Informatics, Linyi University, Linyi, Shandong 276005, P.R. China
Full list of author information is available at the end of the article

Abstract
In this article, we prove that the ω-periodic discrete evolution family $\Gamma := \{\rho(n,k) : n,k \in \mathbb{Z}_+, n \geq k\}$ of bounded linear operators is Hyers-Ulam stable if and only if it is uniformly exponentially stable under certain conditions. More precisely, we prove that if for each real number γ and each sequence $(\xi(n))$ taken from some Banach space, the approximate solution of the nonautonomous ω-periodic discrete system $\theta_{n+1} = \Lambda_n\theta_n$, $n \in \mathbb{Z}_+$ is represented by $\phi_{n+1} = \Lambda_n\phi_n + e^{i\gamma(n+1)}\xi(n+1)$, $n \in \mathbb{Z}_+$; $\phi_0 = \theta_0$, then the Hyers-Ulam stability of the nonautonomous ω-periodic discrete system $\theta_{n+1} = \Lambda_n\theta_n$, $n \in \mathbb{Z}_+$ is equivalent to its uniform exponential stability.

MSC: 39A30

Keywords: Hyers-Ulam stability; uniform exponential stability; discrete evolution family of bounded linear operators; periodic sequence

1 Introduction

The stability theory is an important research area of the qualitative analysis of differential equations and difference equations. Ulam [1] proposed a question regarding the stability of functional equations for homomorphism as follows: *when can an approximate homomorphism from a group G_1 to a metric group G_2 be approximated by an exact homomorphism*? Assuming that G_1 and G_2 are Banach spaces, Hyers [2] brilliantly gave the first result to this question. Aoki [3] and Rassias [4] generalized this result. In particular, Rassias [4] improved the condition for the bound of the norm of the Cauchy difference $f(x+y) - f(x) - f(y)$. Obłoza [5] established the connections between Hyers and Lyapunov stability of ordinary differential equations. Later on, Alsina and Ger [6] investigated the stability of the differential equation $y'(x) = y(x)$, which was then extended to the Banach space-valued differential equation $y'(x) = \lambda y(x)$ by Takahasi *et al.* [7]. We also refer the reader to [8–11] regarding the study of Hyers-Ulam stability of differential equations and differential operators.

The stability and nonstability of different classes of recurrences were studied by Brzdęk *et al.* [12–14] and Popa [15, 16]. Jung [17] proved the Hyers-Ulam stability of a first-order linear homogeneous matrix difference equation. Note that the investigation of the difference equations $\theta_{n+1} = \Lambda_n\theta_n$ or $\theta_{n+1} = \Lambda_n\theta_n + \xi_n$ leads to the idea of discrete evolution family. The main interest in this area is the asymptotic properties and stability of solutions to such systems. In recent years, the exponential stability of such systems has received a great deal of attention since it has been widely applied in research of control theory and engineering; see, *e.g.*, [18–25] and the references cited therein.

In this paper, we are concerned with the first-order linear system

$$\theta_{n+1} = \Lambda_n\theta_n, \quad n \in \mathbb{Z}_+, \tag{1.1}$$

where $\mathbb{Z}_+$ is the set of all nonnegative integers and (Λ_n) is an ω-periodic sequence of bounded linear operators on Banach space Ω. We proved that system (1.1) is Hyers-Ulam stable if and only if it is uniformly exponentially stable under certain conditions.

2 Notation and preliminaries

Throughout, $\mathbb{R}$ stands for the set of all real numbers, Ω denotes a real or complex Banach space, $\mathcal{L}(\Omega)$ is the Banach algebra of all linear and bounded operators over Ω, $\mathcal{L}(\mathbb{Z}_+, \Omega)$ is the space of all Ω-valued bounded sequences endowed with the sup norm denoted by $\|\cdot\|_\infty$, $P_0^\omega(\mathbb{Z}_+, \Omega)$ stands for the space of all ω-periodic bounded sequences $(\xi(n))$ satisfying $\xi(0) = 0$, and we denote by $\|\cdot\|$ the norms in Ω and $\mathcal{L}(\Omega)$.

Let $\mathcal{H}$ belong to $\mathcal{L}(\Omega)$ and $\sigma(\mathcal{H})$ be its spectrum. The spectral radius of $\mathcal{H}$ is denoted by $r(\mathcal{H}) := \sup\{|\lambda| : \lambda \in \sigma(\mathcal{H})\} = \lim_{n\to\infty} \|\mathcal{H}^n\|^{\frac{1}{n}}$.

Definition 2.1 The operator $\mathcal{H}$ is said to be power bounded if there exists a positive constant M such that $\|\mathcal{H}^n\| \le M$ for all $n \in \mathbb{Z}_+$.

We need the following auxiliary lemma.

Lemma 2.2 (See [21]) *If $\mathcal{H} \in \mathcal{L}(\Omega)$ and*

$$\sup_{\gamma\in\mathbb{R}} \sup_{n\in\mathbb{Z}_+} \left\| \sum_{k=0}^{n} e^{i\gamma k}\mathcal{H}^k \right\| < \infty,$$

then $r(\mathcal{H}) < 1$.

The family $\Gamma := \{\rho(n,m) : n,m \in \mathbb{Z}_+, n \ge m\}$ of bounded linear operators is called an ω-periodic discrete evolution family for a fixed integer $\omega \in \{2,3,\dots\}$ if it satisfies the following properties:

- $\rho(n,n) = I$ for all $n \in \mathbb{Z}_+$.
- $\rho(n,m)\rho(m,r) = \rho(n,r)$ for all $n \ge m \ge r$, $n,m,r \in \mathbb{Z}_+$.
- $\rho(n+\omega, m+\omega) = \rho(n,m)$ for all $n \ge m$, $n,m \in \mathbb{Z}_+$.

It is well known that any ω-periodic discrete evolution family Γ is exponentially bounded, that is, there exist a $\tau \in \mathbb{R}$ and an $M_\tau \ge 0$ such that

$$\|\rho(n,m)\| \le M_\tau e^{\tau(n-m)} \quad \text{for all } n \ge m, n,m \in \mathbb{Z}_+. \tag{2.1}$$

When the family Γ is exponentially bounded, its growth bound $\tau_0(\Gamma)$ is the infimum of all $\tau \in \mathbb{R}$ for which there exists an $M_\tau \ge 1$ such that inequality (2.1) is fulfilled. It is well known that

$$\tau_0(\Gamma) = \lim_{n\to\infty} \frac{\ln\|\rho(n,0)\|}{n} = \frac{1}{\omega}\ln\big(r\big(\rho(\omega,0)\big)\big).$$

The family Γ is uniformly exponentially stable if $\tau_0(\Gamma)$ is negative or, equivalently, there exist an $M > 0$ and a $\tau > 0$ such that $\|\rho(n,m)\| \le Me^{-\tau(n-m)}$ for all $n \ge m \in \mathbb{Z}_+$. Thus, we have the following lemma.

Lemma 2.3 *The discrete evolution family* Γ *is uniformly exponentially stable if and only if* $r(\rho(\omega,0)) < 1$.

The map $\rho(\omega,0)$ is also called the Poincaré map of the evolution family Γ.

Consider the following discrete Cauchy problem:

$$\begin{cases} \phi_{n+1} = \Lambda_n\phi_n + e^{i\gamma(n+1)}\xi(n+1), & n \in \mathbb{Z}_+, \\ \phi_0 = \theta_0, \end{cases} \qquad (\Lambda_n,\gamma,\theta_0)$$

where $\gamma \in \mathbb{R}$, the sequence (Λ_n) is ω-periodic, *i.e.*, $\Lambda(n+\omega) = \Lambda(n) = \Lambda_n$ for all $n \in \mathbb{Z}_+$ and a fixed $\omega \in \{2,3,\ldots\}$. Let

$$\rho(n,k) := \begin{cases} \Lambda_{n-1}\Lambda_{n-2}\cdots\Lambda_k & \text{if } k \le n-1, \\ I & \text{if } k = n. \end{cases}$$

Then the family $\{\rho(n,k)\}_{n\ge k\ge 0}$ is a discrete ω-periodic evolution family. Over finite dimensional spaces, the uniform exponential stability of the Cauchy problem $(\Lambda_n,\gamma,\theta_0)$ in discrete and continuous autonomous cases has been investigated in [20, 23].

Definition 2.4 System (1.1) is said to be Hyers-Ulam stable if

$$\|\phi_{n+1} - \Lambda_n\phi_n\| \le \epsilon \quad \text{for any } n \in \mathbb{Z}_+ \text{ and } \epsilon > 0,$$

and there exist an exact solution θ_n of (1.1) and a constant $L \ge 0$ such that

$$\|\phi_n - \theta_n\| \le L\epsilon \quad \text{for any } n \in \mathbb{Z}_+ \text{ and } \epsilon > 0.$$

Remark 2.5 If ϕ_n is an approximate solution of (1.1), then $\phi_{n+1} \approx \Lambda_n\phi_n$. Therefore, letting $(\xi(n))$ be an error sequence, then ϕ_n is the exact solution of $\phi_{n+1} = \Lambda_n\phi_n + \xi(n)$.

With the help of Remark 2.5, Definition 2.4 can be modified as follows.

Definition 2.6 System (1.1) is termed Hyers-Ulam stable if $\|\xi(n)\| \le \epsilon$ holds for any $n \in \mathbb{Z}_+$ and $\epsilon > 0$, and there exist an exact solution θ_n of (1.1) and a constant $L \ge 0$ such that

$$\|\phi_n - \theta_n\| \le L\epsilon \quad \text{for any } n \in \mathbb{Z}_+ \text{ and } \epsilon > 0.$$

3 Main results

Consider the Cauchy problem $(\Lambda_n, \gamma, \theta_0)$. The solution of the Cauchy problem $(\Lambda_n, \gamma, \theta_0)$ is given by

$$\phi_n = \rho(n,0)\theta_0 + \sum_{k=0}^{n} e^{i\gamma k}\rho(n,k)\xi(k).$$

Let us divide n by ω, *i.e.*, $n = l\omega + r$ for some $l \in \mathbb{Z}_+$, where $r \in \{0,1,\dots,\omega-1\}$. We consider the following sets, which will be useful in this paper:

$$\mathcal{A}_j := \{1 + j\omega, 2 + j\omega, \dots, (j+1)\omega - 1\} \quad \text{for all } j \in \mathbb{Z}_+.$$

If $r \in \{1,2,\dots,\omega-1\}$, then define

$$B_l := \{l\omega + 1, l\omega + 2, \dots, l\omega + r\}$$

and

$$C := \{0, \omega, 2\omega, \dots, l\omega\}.$$

It is clear that

$$\bigcup_{j=0}^{l-1} A_j \bigcup B_l \bigcup C = \{0,1,2,\dots,n\}. \tag{3.1}$$

On the basis of partition (3.1), we construct the space $\mathcal{W}$, which consists of all the sequences of the form

$$\xi(k) := \begin{cases} (k - j\omega)[(1+j)\omega - k]\rho(k - j\omega, 0) & \text{if } k \in \mathcal{A}_j, \\ k(\omega - k)\rho(k,0) & \text{if } k \in B_l, \\ 0 & \text{if } k \in C. \end{cases} \tag{3.2}$$

That is,

$$\mathcal{W} := \{(\xi(n)) : \xi(n) \text{ satisfies (3.2)}\}.$$

Obviously, $\mathcal{W}$ is the subspace of $P_0^\omega(\mathbb{Z}_+, \Omega)$.

Now, we state and prove the main results.

Theorem 3.1 *Let* $\Gamma := \{\rho(n,k) : n \geq k \in \mathbb{Z}_+\}$ *be the* ω*-periodic discrete evolution family on* Ω *and let* $\phi_{n+1} = \Lambda_n\phi_n + e^{i\gamma(n+1)}\xi(n+1)$, $\phi_0 = \theta_0$ *be the approximate solution of* (1.1) *with the error term* $e^{i\gamma(n+1)}\xi(n+1)$, *where* $\gamma \in \mathbb{R}$ *and* $(\xi(n)) \in P_0^\omega(\mathbb{Z}_+, \Omega)$. *Then the following statements are true.*

(1) *If system* (1.1) *is uniformly exponentially stable, then it is Hyers-Ulam stable.*

(2) *If system* (1.1) *is Hyers-Ulam stable for each* $\gamma \in \mathbb{R}$ *and each* ω*-periodic sequence* $(\xi(n)) \in \mathcal{W} \subset P_0^\omega(\mathbb{Z}_+, \Omega)$, *then system* (1.1) *is uniformly exponentially stable, i.e.,* Γ *is uniformly exponentially stable.*

Proof (1) Let $\epsilon > 0$ and ϕ_n be the approximate solution of (1.1) such that $\sup_{n\in\mathbb{Z}_+} \|\phi_{n+1} - \Lambda_n\phi_n\| = \sup_{n\in\mathbb{Z}_+} \|e^{i\gamma(n+1)}\xi(n+1)\|$, $\phi_0 = \theta_0$, and $\sup_{n\in\mathbb{Z}_+} \|\xi(n)\| \leq \epsilon$, and let θ_n be the exact solution of (1.1). Taking into account that Γ is uniformly exponentially stable, we conclude that there exist two positive constants M and ν such that

$$\begin{aligned}
\sup_{n\in\mathbb{Z}_+} \|\phi_n - \theta_n\| &= \sup_{n\in\mathbb{Z}_+} \left\| \rho(n,0)\theta_0 + \sum_{k=0}^{n} e^{i\gamma k}\rho(n,k)\xi(k) - \rho(n,0)\theta_0 \right\| \\
&= \sup_{n\in\mathbb{Z}_+} \left\| \sum_{k=0}^{n} e^{i\gamma k}\rho(n,k)\xi(k) \right\| \\
&\leq \sum_{k=0}^{n} \sup_{n\in\mathbb{Z}_+} \left\| e^{i\gamma k}\rho(n,k)\xi(k) \right\| \\
&= \sum_{k=0}^{n} \left\| e^{i\gamma k} \right\| \sup_{n\in\mathbb{Z}_+} \left\| \rho(n,k) \right\| \left\| \xi(k) \right\| \\
&= \sum_{k=0}^{n} \sup_{n\in\mathbb{Z}_+} \left\| \rho(n,k) \right\| \left\| \xi(k) \right\| \\
&\leq \sum_{k=0}^{n} Me^{-\nu(n-k)}\epsilon \\
&= Me^{-\nu n} \sum_{k=0}^{n} e^{\nu k}\epsilon \\
&= Me^{-\nu n} \left(\frac{1 - e^{(n+1)\nu}}{1 - e^{\nu}} \right)\epsilon \\
&= L\epsilon,
\end{aligned}$$

where $L := Me^{-\nu n}(1 - e^{(n+1)\nu})/(1 - e^{\nu})$. Thus, system (1.1) is Hyers-Ulam stable.

(2) Let $(\xi(n)) \in \mathcal{W}$. Then

$$\begin{aligned}
\sum_{k=1}^{n} e^{i\gamma k}\rho(n,k)\xi(k) &= \sum_{k=1}^{l\omega+r} e^{i\gamma k}\rho(l\omega+r,k)\xi(k) \\
&= \sum_{k\in\bigcup_{j=0}^{l-1} A_j \cup B_l \cup C} e^{i\gamma k}\rho(l\omega+r,k)\xi(k) \\
&= \sum_{k\in\bigcup_{j=0}^{l-1} A_j} e^{i\gamma k}\rho(l\omega+r,k)\xi(k) \\
&\quad + \sum_{k\in B_l} e^{i\gamma k}\rho(l\omega+r,k)\xi(k) \\
&\quad + \sum_{k\in C} e^{i\gamma k}\rho(l\omega+r,k)\xi(k) \\
&= \sum_{j=0}^{l-1} \sum_{k=1+j\omega}^{\omega-1+j\omega} e^{i\gamma k}\rho(l\omega+r,k)\xi(k)
\end{aligned}$$

$$+\sum_{k=l\omega+1}^{l\omega+r} e^{i\gamma k}\rho(l\omega+r,k)\xi(k)$$
$$+\sum_{k\in C} e^{i\gamma k}\rho(l\omega+r,k)\xi(k).$$

By virtue of (3.2), we have

$$\begin{aligned}\sum_{k=1}^{n} e^{i\gamma k}\rho(n,k)\xi(k) &= \sum_{j=0}^{l-1}\sum_{k=1+j\omega}^{\omega-1+j\omega} e^{i\gamma k}\rho(l\omega+r,k)(k-j\omega)\big[(1+j)\omega-k\big]\rho(k-j\omega,0)\\ &\quad+\sum_{k=l\omega+1}^{l\omega+r} e^{i\gamma k}\rho(l\omega+r,k)k(\omega-k)\rho(k,0)\\ &=\sum_{j=0}^{l-1}\sum_{k=1+j\omega}^{\omega-1+j\omega} e^{i\gamma k}\rho(l\omega+r,k)(k-j\omega)\big[(1+j)\omega-k\big]\rho(k-j\omega,0)\\ &\quad+\sum_{k=l\omega+1}^{l\omega+r} e^{i\gamma k}\rho(l\omega+r,k)k(\omega-k)\rho(k,0)\\ &=L_1+L_2,\end{aligned}$$

where

$$L_1 := \sum_{j=0}^{l-1}\sum_{k=1+j\omega}^{\omega-1+j\omega} e^{i\gamma k}\rho(l\omega+r,k)(k-j\omega)\big[(1+j)\omega-k\big]\rho(k-j\omega,0)$$

and

$$L_2 := \sum_{k=l\omega+1}^{l\omega+r} e^{i\gamma k}\rho(l\omega+r,k)k(\omega-k)\rho(k,0).$$

We write L_1 in the form

$$\begin{aligned}L_1 &= \sum_{j=0}^{l-1}\sum_{k=1+j\omega}^{\omega-1+j\omega} e^{i\gamma k}\rho(l\omega+r,k)(k-j\omega)\big[(1+j)\omega-k\big]\rho(k-j\omega,0)\\ &=\sum_{j=0}^{l-1}\sum_{k=1+j\omega}^{\omega-1+j\omega} e^{i\gamma k}\rho(l\omega+r,k)(k-j\omega)\big[(1+j)\omega-k\big]\rho(k,j\omega)\\ &=\sum_{j=0}^{l-1}\sum_{k=1+j\omega}^{\omega-1+j\omega} e^{i\gamma k}\rho(l\omega+r,j\omega)(k-j\omega)\big[(1+j)\omega-k\big]\\ &=\sum_{j=0}^{l-1}\rho(l\omega+r,j\omega)\sum_{k=1+j\omega}^{\omega-1+j\omega} e^{i\gamma k}(k-j\omega)\big[(1+j)\omega-k\big]\\ &=\sum_{j=0}^{l-1}\rho(l\omega+r,j\omega)\sum_{k=1+j\omega}^{\omega-1+j\omega} e^{i\gamma k}(k-j\omega)\big[\omega-(k-j\omega)\big]\end{aligned}$$

$$= \sum_{j=0}^{l-1} \rho(r,0)\rho^{l-j}(\omega,0)e^{i\gamma j\omega} \sum_{\nu=1}^{\omega-1} e^{i\gamma\nu}\nu(\omega-\nu)$$

$$= \rho(r,0) \sum_{\nu=1}^{\omega-1} e^{i\gamma\nu}\nu(\omega-\nu) \sum_{j=0}^{l-1} e^{i\gamma j\omega}\rho^{l-j}(\omega,0)$$

$$= \rho(r,0) \sum_{\nu=1}^{\omega-1} e^{i\gamma\nu}\nu(\omega-\nu) \sum_{\alpha=1}^{l} e^{i\gamma\omega(l-\alpha)}\rho^{\alpha}(\omega,0)$$

$$= \rho(r,0) \sum_{\nu=1}^{\omega-1} e^{i\gamma\nu}\nu(\omega-\nu)e^{i\gamma l\omega} \sum_{\alpha=1}^{l} e^{-i\gamma\omega\alpha}\rho^{\alpha}(\omega,0)$$

$$= G(\gamma,\omega) \sum_{\alpha=1}^{l} e^{-i\gamma\omega\alpha}\rho^{\alpha}(\omega,0),$$

where $G(\gamma,\omega) := \rho(r,0)\sum_{\nu=1}^{\omega-1} e^{i\gamma\nu}\nu(\omega-\nu)e^{i\gamma l\omega} \neq 0$. Furthermore,

$$L_2 = \sum_{k=l\omega+1}^{l\omega+r} e^{i\gamma k}\rho(l\omega+r,k)k(\omega-k)\rho(k,0)$$

$$= \sum_{k=l\omega+1}^{l\omega+r} e^{i\gamma k}\rho(l\omega+r,0)k(\omega-k)$$

$$= \rho(l\omega+r,0) \sum_{k=l\omega+1}^{l\omega+r} e^{i\gamma k}k(\omega-k).$$

Therefore,

$$\sum_{k=0}^{n} e^{i\gamma k}\rho(n,k)\xi(k) = G(\gamma,\omega)\sum_{\alpha=1}^{l} e^{-i\gamma\omega\alpha}\rho^{\alpha}(\omega,0) + \rho(l\omega+r,0)\sum_{k=l\omega+1}^{l\omega+r} e^{i\gamma k}k(\omega-k).$$

Since (1.1) is Hyers-Ulam stable,

$$\sup_{n\in\mathbb{Z}_+} \|\phi_n-\theta_n\| = \sup_{n\in\mathbb{Z}_+} \left\| \sum_{k=1}^{n} e^{i\gamma k}\rho(n,k)\xi(k) \right\|$$

is bounded, and so L_1 is bounded, *i.e.*,

$$\sup_{l\geq 0} \left\| \sum_{\alpha=0}^{l} e^{-i\gamma\omega\alpha}\rho^{\alpha}(\omega,0) \right\| < \infty.$$

Using Lemma 2.2, we deduce that $r(\rho(\omega,0)) < 1$. Hence, by Lemma 2.3, Γ is uniformly exponentially stable. This completes the proof. □

On the basis of Theorem 3.1, we obtain the following corollary as the main result of this paper.

Corollary 3.2 *Assume that for each $\gamma \in \mathbb{R}$ and each sequence $(\xi(n)) \in \mathcal{W}$, the approximate solution of the nonautonomous ω-periodic discrete system* (1.1) *is presented by* $\phi_{n+1} =$

$\Lambda_n\phi_n + e^{i\gamma(n+1)}\xi(n+1)$, $n \in \mathbb{Z}_+$; $\phi_0 = \theta_0$. *Then the Hyers-Ulam stability of* (1.1) *is equivalent to its uniform exponential stability.*

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally to this work and are listed in alphabetical order. They both read and approved the final version of the manuscript.

Author details
[1]LinDa Institute of Shandong Provincial Key Laboratory of Network Based Intelligent Computing, Linyi University, Linyi, Shandong 276005, P.R. China. [2]School of Informatics, Linyi University, Linyi, Shandong 276005, P.R. China. [3]Department of Mathematics, University of Peshawar, Peshawar, 25000, Pakistan.

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