Journal of Applied Mathematics and Stochastic Analysis, **15**:1 (2002), 53-69.

CONNECTIONS BETWEEN THE CONVECTIVE DIFFUSION EQUATION AND THE FORCED BURGERS EQUATION

NEJIB SMAOUI and FETHI BELGACEM
Kuwait University
Department of Mathematics and Computer Science
P.O. Box 5969, Safat 13060, Kuwait
E-mail: smaoui@mcs.sci.kuniv.edu.kw; belgacem@mcs.sci.kuniv.edu.kw

(Received September, 2000; Revised October, 2001)

The convective diffusion equation with drift $b(x)$ and indefinite weight $r(x)$,

$$\frac{\partial \phi}{\partial t} = \frac{\partial}{\partial x}\left[a\frac{\partial \phi}{\partial x} - b(x)\phi\right] + \lambda r(x)\phi, \tag{1}$$

is introduced as a model for population dispersal. Strong connections between Equation (1) and the forced Burgers equation with positive frequency $(m \geq 0)$,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u\frac{\partial u}{\partial x} + mu + k(x), \tag{2}$$

are established through the Hopf-Cole transformation. Equation (2) is a prime prototype of the large class of quasilinear parabolic equations given by

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial (f(v))}{\partial x} + g(v) + h(x). \tag{3}$$

A compact attractor and an inertial manifold for the forced Burgers equation are shown to exist via the Kwak transformation. Consequently, existence of an inertial manifold for the convective diffusion equation is guaranteed. Equation (2) can be interpreted as the velocity field precursed by Equation (1). Therefore, the dynamics exhibited by the population density in Equation (1) show their effects on the velocity expressed in Equation (2). Numerical results illustrating some aspects of the previous connections are obtained by using a pseudospectral method.

Key words: Convective Diffusion Equation, Indefinite Weights, Burgers Equation, Hopf-Cole Transformation, Kwak Transformation.

AMS subject classifications: 35F25, 35K05, 34K45, 35K57.

1. Introduction

The indefinite logistic type with drift vector $b(x)$,

$$\frac{\partial \phi}{\partial t} = \nabla \cdot [a\nabla\phi - b(x)\phi] + r(x)\phi - e(x)\phi^2 \text{ in } \Omega \times (0,\infty), \tag{4}$$

with initial density $\phi(x,0) = \phi_0(x) \geq 0$ and boundary values $\phi(x,t) = 0$ on $\partial\Omega \times (0,\infty)$, describes a population with density ϕ inhabiting a bounded environment Ω in $\mathbb{R}^n$, where $n = 1, 2$, or 3, with deadly boundary $\partial\Omega$. The matrix $a = [a_{ij}]$ measures the diffusion rate of the population while the function $e(x)$ measures the self-limiting effects such as crowding. The coefficient $r(x)$ represents a local growth rate and may be considered a food source. The region Ω may be endowed with favorable subregions $(r(x) > 0)$, unfavorable ones $(r(x) < 0)$, and neutral subregions $(r(x) = 0)$. If Ω contains both favorable and unfavorable subhabitats, we say that r is indefinite.

Many quantitative and qualitative aspects of the analysis of (4) depend heavily on the size of the first positive eigenvalue for the Dirichlet indefinite linear elliptic problem

$$\nabla \cdot [a\nabla\phi - b(x)\phi] + \lambda r(x)\phi = 0 \text{ in } \Omega \times (0,\infty). \tag{5}$$

When a is a constant, $b(x) \equiv 0$ and $r \in L^\infty(\Omega)$, Equation (4) has a unique positive steady state solution ϕ provided that the principal eigenvalue λ^* of Equation (5) satisfies $\lambda^* < \frac{1}{a}$ (Cantrell and Cosner [12-14]). Furthermore, if $\lambda^* < \frac{1}{a}$, the steady state solution of Equation (5) is globally asymptotically stable. A population whose dynamics are modeled by Equation (4), with $\lambda^* < \frac{1}{a}$, will persist and will go extinct if $\lambda^* > \frac{1}{a}$. The estimation of λ^* for linear indefinite weight problems is then necessary to comprehend more complicated dynamics (see [5]).

The paper is organized as follows: In Section 2, we show the existence of the principal eigenvalue for the convective diffusion equation. A connection via the Hopf-Cole transformation between the Burgers equation and the convective diffusion equation is presented in Section 3. Section 4 describes the original Burgers equation, the forced Burgers equation, and shows that the latter admits an absorbing all and hence has a compact attractor. In that section, a nonlinear transformation mapping the forced original Burgers equation into a reaction diffusion system that admits an inertial manifold is also given. In Section 5, some numerical results based on a pseudospectral method are obtained for the forced Burgers equation, the reaction diffusion system, and the convective diffusion equation. The computational results illustrate and reinforce many of the connections obtained analytically among the three separate systems.

2. Existence of the Principal Eigenvalue

The Langevin equation

$$dx = \sigma dw(t) + bdt, \tag{6}$$

is the stochastic equation governing the motion of each individual in a population undergoing a diffusion $a = \frac{\sigma^2}{2}$ and a drift b. Belgacem [7] showed that using the Ito chain rule on Equation (6) yields the convection diffusion equation

$$\frac{\partial\phi}{\partial t} = \nabla \cdot [a\nabla\phi - b(x)\phi]. \tag{7}$$

The dynamics of a population inhabiting a region Ω in $\mathbb{R}^n$, subjected to indefinite reactions, moving about under the influence of diffusion and drift, were studied in [5 8, 18-20]. The effects of the drift on the survival of the population described can be established by considerations of the weighted, steady state eigenvalue problem

$$L(\phi) = -\nabla \cdot [a\nabla\phi - b(x)\phi] = \lambda r(x)\phi. \tag{8}$$

Belgacem and Cosner [8] considered situations when the drift is along the negative gradient of the reaction growth potential r, $(\alpha \geq 0)$:

$$b = -\alpha \nabla r. \tag{9}$$

For Equation (8), existence of a positive principal eigenvalue $\lambda^* = \lambda^*(a, b, r, \Omega)$, having a unique positive density ϕ^*, can be deduced from the works of Belgacem [5], Castro-Lazer [15] or those of Hess [26-28].

Theorem 2.1: *Let a be positive definite in Ω, with $a_{ij}(x)$, $b_i(x) \in C^2(\overline{\Omega}\,)$, $r(x) \in C^1(\overline{\Omega}\,)$, and $r(x_0) >\,)$ for some $x_0 \in \Omega$. Then for Dirichlet boundary conditions, Equation (8) admits a positive principal eigenvalue λ^*, having a unique positive density eigenfunction ϕ^*. Furthermore, λ^* is connected to ϕ^* via a generalized Raleigh-Ritz quotient.*

Proof: For $b(x) \equiv 0$ in Ω, Equation (8) is self-adjoint, and hence the conclusion is reached by using the Manes-Micheletti [32] results. For $b \neq 0$, multiplying Equation (8) by ϕ and integrating yields:

$$\lambda \int_\Omega r\phi^2 dx = \int_\Omega \nabla \cdot [b\phi - a\phi_x]\phi dx. \tag{10}$$

Integrating the right-hand side of equation (10) by parts yields:

$$\begin{aligned} \lambda \int_\Omega r\phi^2 dx &= \int_\Omega \nabla \cdot [(b(x)\phi - a\phi_x)\phi] dx + \int_\Omega (\nabla\phi) a (\nabla\phi)^T dx \\ &\quad - \int_\Omega b(x) \nabla\left(\tfrac{\phi^2}{2}\right) dx. \end{aligned} \tag{11}$$

Now, using the boundary condition, and integrating by parts again, we finally get:

$$\lambda \int_\Omega r\phi^2 dx = \int_\Omega (\nabla\phi) a (\nabla\phi)^T dx + \int_\Omega \tfrac{\nabla \cdot b(x)}{2} \phi^2 dx. \tag{12}$$

It is obvious that the positivity of $\int_\Omega r\phi^2 dx$, and therefore of λ^*, is guaranteed if

$$\nabla \cdot b(x) \geq 0 \text{ in } \Omega, \tag{13}$$

or if for some $\epsilon \geq 0$,

$$\nabla \cdot b(x) \geq \epsilon r(x), \ \forall x \in \Omega. \tag{14}$$

Other coercivity conditions guaranteeing the positivity of λ^* are provided in [5-7]. Provided $r \in C^2(\Omega)$, joining Equation (9) and (14), we obtain the following condition on the weight r:

$$\alpha \Delta r + \epsilon r \leq 0. \tag{15}$$

Equation (15) with equality is readily satisfied, if for instance r is proportional to $\cos\left(\sqrt{\frac{\epsilon}{\alpha}}x\right)$ or $\sin\left(\sqrt{\frac{\epsilon}{\alpha}}x\right)$ in the one-dimensional case. Furthermore, when $n = 1$, taking

$$B = \int -\tfrac{b}{a} dx, V = \phi e^B, R = re^{-B}, \text{ and } A = ae^{-B}, \tag{16}$$

makes Equation (8) self-adjoint. Indeed, since

$$ae^{-B}V_x = a(\phi_x + B_x\phi) = a\phi_x - b(x)\phi, \tag{17}$$

Equation (8) becomes:

$$-\frac{\partial}{\partial x}[AV_x] = -\frac{\partial}{\partial x}[a\phi_x - b(x)\phi] = \lambda r\phi = \lambda re^{-B}V = \lambda RV. \tag{18}$$

Note that $R(x_0) > 0$, and $V(x) \equiv 0$ on $\partial\Omega$. The positivity of $R(x)$ at some point in Ω, insures the positivity and simplicity of a principal eigenvalue λ^* having a positive eigenfunction V^*, as in [32]. The eigenfunction ϕ^* for problem (8) is then obtained via the transformation:

$$\phi^* = V^* e^{-B}, \tag{19}$$

with λ^* being given by:

$$\lambda^* = \frac{\int_\Omega A(V_x^*)^2 dx}{\int_\Omega R(V^*)^2 dx}. \tag{20}$$

Now, set $\lambda_0 = \lambda^*(1, 0, 1, \Omega)$. Then from Equation (20), we see that the principal eigenfunction ϕ^* of Equation (8) satisfies

$$\lambda_0 \int_\Omega (\phi^*)^2 dx \le \int_\Omega (\phi_x^*)^2 dx. \tag{21}$$

There is no loss of generality in assuming that $|\, r(x) \,| \le 1$ and that $1 = \int_\Omega r(\phi^*)^2 dx \le \int_\Omega (\phi^*)^2 dx.$

Corollary 2.1: *Let a be a positive constant, and $b_x = \epsilon r$ with $\epsilon \ge 0$, then*

$$\lambda^* \le a\lambda_0 + \tfrac{\epsilon}{2}. \tag{22}$$

Proof: Using Equation (12) together with Equation (21), we get

$$\lambda^* \ge \int_\Omega a\lambda_0(\phi^*)^2 dx + \tfrac{\epsilon}{2} \ge a\lambda_0 + \tfrac{\epsilon}{2}. \tag{23}$$

From a control viewpoint, such as in animal refuge design, taking $b = -\alpha r_x$ is expected to be detrimental to the population survival, since it raises the energy (λ^*) required for that survival. Therefore, one may think that in general,

$$\lambda^*(a, b, r, \Omega) \ge \lambda^*(a, 0, r, \Omega). \tag{24}$$

At first look, it appears that, should we choose a drift in the positive direction of the gradient of r, we expect the chances of the population to be improved. However, Belgacem [5] showed that the direction of the inequality in Equation (24) depends on the nature of the weight r and its interplay with the absorbative Dirichlet boundary condition. In fact, the sign and value of $\frac{\partial\lambda^*}{\partial\alpha}(0) = (\lambda^*)_0'$ highly depend on the proximity of the food source r to the deadly boundary. *However, in the Neumann boundary condition it remains a conjecture of Cosner (see [5-8]) that inequality (24) is always true, provided $b = \alpha r_x$ with $\alpha \ge 0$.*

Conjecture 2.1: $\mathrm{sign}(\lambda^*)_0' = -1.$

At this point, we show:

Corollary 2.2: *If $b = \alpha r_x$ and $b_x = \epsilon r$ with ϵ, $\alpha \ge 0$ then*

$$\lambda^*(\alpha) = \int_\Omega a(\phi_x^*)^2 dx + \alpha(\lambda^*)_0'. \tag{25}$$

Proof: Belgacem [5] has already shown that in this case $(\lambda^*)_0' = \int_\Omega r_{xx}\phi^2 dx$. Using this result along with Equation (12), we obtain Equation (25).

Now suppose that Ω is initially infinite and that the coefficients a, b, and r in Equation (8) are periodic with period L. We may then limit our problem to $\Omega = [0, L]$, with $\int_0^L r(\phi^*)^2 dx = 1$ and periodic boundary conditions $\phi(0) = \phi(L)$.

Corollary 2.3: *The eigenvalue of Equation* (8) *with periodic coefficients and periodic boundary conditions with period L is given by*

$$\lambda^* = \int_0^L \left[a(\phi_x^*)^2 + \tfrac{b_x}{2}(\phi^*)^2\right]dx + a\phi^*(0)[\phi_x^*(0) - \phi_x^*(L)]. \tag{26}$$

Proof: Equation (26) is a simple recomputation of Equation (11). The surface term on the RHS of Equation (26) disappears, and we recover Equation (12) should we use the Dirichlet condition $\phi^*(0) = \phi^*(L) = 0$.

3. The Hopf-Cole Transformation

The Hopf-Cole transformation [29] has been used extensively in the area of nonlinear parabolic and hyperbolic partial differential equations [1, 24, 25, 33, 38] and in the study of viscous conservation laws [16, 30, 40]. In this work, the Hopf-Cole transformation is used on the one-dimensional time dependent family of equations:

$$\phi_t = \tfrac{\partial}{\partial x}[a\phi_x - b\phi] + \lambda r(x)\phi \text{ in } \Omega \times [0, \infty), \tag{27}$$

where $\lambda \in [0, \infty)$. Letting $u = -2a\frac{\phi_x}{\phi}$, ($\phi(x,t) = \exp[\int -\frac{u(x,t)}{2a}dx]$), we get

$$u_t = -2\left(\tfrac{\phi_x}{\phi}\right)_t = -2a\left(\tfrac{\phi_t}{\phi}\right)_x \tag{28}$$

and

$$u_t = -2a\left[\tfrac{\phi_{xx}}{\phi} - \left(\tfrac{\phi_x}{\phi}\right)^2\right] = -2a\left[\tfrac{\phi_{xx}}{\phi} - \left(\tfrac{u}{-2a}\right)^2\right]. \tag{29}$$

Therefore,

$$au_{xx} = -2a\left[a\tfrac{\phi_{xx}}{\phi} - a\left(\tfrac{u}{2a}\right)^2\right]_x, \tag{30}$$

and

$$\begin{aligned} au_{xx} - 2a^2\left[\left(\tfrac{u}{2a}\right)^2\right]_x &= -2a\left(a\tfrac{\phi_{xx}}{\phi}\right)_x \\ &= -2a\left[\left(\tfrac{\phi_t}{\phi}\right)_x + \left(\tfrac{(b\phi)_x}{\phi}\right)_x - \lambda r_x\right] \\ &= u_t - 2a\left[b_x + \tfrac{b}{-2a}u\right]_x + 2a\lambda r_x. \end{aligned} \tag{31}$$

Rearranging terms, we get:

$$u_t = au_{xx} - uu_x - (bu)_x + 2a[b_x - \lambda r]_x. \tag{32}$$

Equation (32) is Burgers equation with a cross velocity-drift gradient and a forcing term. The forcing term is generated from the drift and the weight r. Note that if we choose $b_x = \lambda r$ (compare with Corollary 2.1), then the forcing term disappears from Equation (32) and we would have

$$u_t = au_{xx} - uu_x - (bu)_x. \tag{33}$$

In this case, Equation (33) is simply the Hopf-Cole transform to the linear diffusion equation

$$\phi_t = a\phi_{xx} - b(x)\phi_x. \tag{34}$$

Setting $v(x,t) = u(x,t) + b(x)$, then $u_t = v_t$, and Equation (33) can be transformed to

$$v_t = (av_{xx} - vv_x) - (ab_{xx} - bb_x) = BF(v) - BF(b) = BF(v) + k(x), \tag{35}$$

where (BF) is denoted by the Burger Functional:

$$BF(v) = av_{xx} - vv_x. \tag{36}$$

The Hopf-Cole transformation allows for the clear separation of events related to v, and those related to b, so the cross term $(Bu)_x$ in Equation (33) is avoided in Equation (35). Equation (35) also indicates that the velocity field is augmented linearly in the presence of an external drift b, when Equation (34) is satisfied. So, the acceleration in the absence of a drift is parallel to that with a drift $b(x)$ having $BF(b) = 0$. This is in particular true if $b(x)$ is constant.

4. The Burgers Equation

The viscous Burgers equation

$$\tfrac{\partial v}{\partial t} = \nu \tfrac{\partial^2 v}{\partial x^2} - 2v\tfrac{\partial v}{\partial x}, \tag{37}$$

introduced by Burgers [9, 10] as a simple model for a turbulent flow through a channel, has received a lot of attention in recent years [1, 11, 21, 29, 34]. Originally, Burgers introduced two different types of models for studying the behavior of hydrodynamic equations. The first model consisted of a system of nonlinear ODEs,

$$\tfrac{dU}{dt} = P - \nu U - v^2 \tag{38}$$

$$\tfrac{dv}{dt} = Uv - \nu v, \tag{39}$$

where U and v are velocities connected with the primary and secondary motions, respectively. The quantities P and ν are positive constants representing the external force and the kinematic viscosity, respectively. A simple linear stability analysis shows that if $\nu^2 \geq P$, then the system above has only one stable steady solution $(\frac{P}{\nu}, 0)$. However, if $\nu^2 < P$, then the system has three steady state solutions: $(\frac{P}{\nu}, 0)$, $(\nu, \sqrt{P-\nu^2})$, and $(\nu, -\sqrt{P-\nu^2})$. The first steady state solution is unstable, the second and third solutions are asymptotically stable if $\nu^2 < \frac{8}{9}P$ and stable if $\nu^2 \in [\frac{8}{9}P, P)$.

Burgers realized that Equations (38) and (39) fail to incorporate the spatial nature of turbulence. Therefore, he proposed an improved mathematical model for turbulence given by

$$L\frac{dU}{dt} = P - \nu\frac{U}{L} - \frac{1}{L}\int_0^L v^2 dx \tag{40}$$

$$\frac{\partial v}{\partial t} = \nu\frac{\partial^2 v}{\partial x^2} - 2v\frac{\partial v}{\partial x} + \frac{U}{L}v. \tag{41}$$

Here $U,\ v$, P and ν are the same variables as in (38) and (39), except that $v = v(x,t)$ is now a function of space and time. When v differs from zero, it is said that there is turbulence in the system. The space variable x is the coordinate in the direction of the cross dimension of the channel and extends from 0 to L.

If turbulence is not activated by energy transmission from a primary motion (i.e., $U = 0$), Equation (41) simplifies to the viscous Burgers equation (37). However, when there is a constant transmission of energy from the primary motion to the secondary motion (i.e., $U \neq 0$), Burgers equations simplify to the original Burgers equation (41). The steady state solutions of Equation (41) also satisfy Equation (40) provided an appropriate value of P is given.

The forced original Burgers equation,

$$\frac{\partial v}{\partial t} = \nu\frac{\partial^2 v}{\partial x^2} - 2v\frac{\partial v}{\partial x} + \frac{1}{L}Uv + h(x), \tag{42}$$

with periodic boundary conditions $v(0,t) = v(L,t)$ and zero mean $\int_0^L v dx = 0$, can be easily transformed to

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u\frac{\partial u}{\partial x} + mu + k(x). \tag{43}$$

Equation (43) is of the type of Equation (35) if $m = 0$ and $a = 1$ (see Smaoui [34]). Furthermore, the relation $k(x) = -BF(b)$ clarifies the source and type of the forcing term in the Burgers equation. *That is, an external drift* b *in the diffusion equation induces an external forcing acceleration term* k *in the Burgers equation.* For instance, in $[0, 2\pi]$ if $b(x) = r_x = \sin x$ and $a = 1$, then $k(x) = \sin x + \frac{1}{2}\sin 2x$ (compare with Belgacem [5], page 106).

In the absence of the drift b, we observe that the transform of the diffusion equation

$$w_t = aw_{xx}, \tag{44}$$

is identical to that of the diffusion equation with constant growth frequency r,

$$\phi_t = a\phi_{xx} + r\phi. \tag{45}$$

Indeed, taking $w = \phi e^{-rt}$, yields $u = -2a\frac{w_x}{w} = -2a\frac{\phi_x}{\phi}$. Substituting $b \equiv 0$ in Equation (32), we obtain the classical Burgers equation

$$u_t = au_{xx} - uu_x = BF(u). \tag{46}$$

The growth term $r\phi$, with constant frequency r in the diffusion equation (45) has no effect on the velocity profile $u(x,t)$ in the Burgers equation. However, if r is dependent on x, then its influence becomes apparent through its derivative (or gradient) in the forcing term of the velocity field.

Recall that for $x(0) = 0$ and $w(x, 0) = N\delta(x)$ (see for instance, Belgacem [6]), the solution $w(x, t)$ in Equation (44) is given by:

$$w(x,t) = \frac{N}{2\sqrt{\pi a t}} \ \exp\Big(- \frac{x^2}{4at}\Big). \tag{47}$$

Hence, for $t > 0$,

$$\phi(x,t) = \frac{N}{2\sqrt{\pi a t}} \ \exp\Big(\frac{4art^2 - x^2}{4at}\Big) \tag{48}$$

and

$$u(x,t) = \ - 2a\big(\tfrac{-2x}{4at}\big) = \tfrac{x}{t}. \tag{49}$$

This is the solution to the Burgers equation (46) with $u(x, 0) = \ - 2a\delta'(x)/\delta(x)$. Recall that $u(x, t)$ has the dimensions of a velocity. *Intuitively, this means that if $w(x, t)$ is the population density, then $u(x, t)$ describes the population instantaneous velocity field*. So, if in the long run the population goes extinct, its motion must cease, and its velocity field must vanish as indicated by Equation (49).

The steady state Burgers equation, $BF(u) = 0$, yields yet another facet in this rich connection. A possible solution with $u(0) = u_0$ would be

$$u(x) = u_0 \tan\big(\tfrac{x}{2a} + \tfrac{\pi}{4}\big). \tag{50}$$

Since the Hopf-Cole setup in this case dictates

$$\phi(x) = \exp\Big[\int - \tfrac{u(x)}{2a} dx\Big], \tag{51}$$

we have

$$\phi(x) = D\big[\cos\big(\tfrac{x}{2a} + \tfrac{\pi}{4}\big)\big]^{u_0}, \tag{52}$$

where D is an integration constant. Choosing $\phi(0) = 1$, and substituting at $x = 0$ yields:

$$u_0 = \ - \frac{2\ell n(D)}{\ell n(2)}. \tag{53}$$

4.1 Existence of the Attractor

In this section, we show that in the Hilbert space $H = H^2[0, L]$, the forced original Burgers equation admits an absorbing ball.

Theorem 4.1: *Let c denote the Poincare constant, then*

$a)$ *Every solution to the original Burgers equation*

$$u_t = u_{xx} - uu_x + mu \tag{54}$$

satisfies the inequality

$$\| \ u \ \| \ \leq \ \| \ u_0 \ \| \ e^{(m-\frac{1}{c})(t-t_0)}, \textit{ for } t \geq t_0. \tag{55}$$

$b)$ *Every solution to the forced original Burgers equation*

$$u_t = u_{xx} - uu_x + mu + k(x) \tag{56}$$

satisfies the inequality

$$\| u \| \leq \sqrt{\frac{2c^2}{1-2mc}} \| k \| \text{, for } t \geq t_0, \tag{57}$$

with $t_0 = \frac{c}{1-2mc} ln\left(\frac{(1-2mc)\|u_0\|^2}{c^2\|k\|^2}\right)$, *provided* $m < \frac{1}{2c}$.

Proof: If we multiply Equation (56) by u and integrate, we get

$$\frac{d}{dt}\left(\frac{1}{2}\int_0^{2\pi} u^2 dx\right) = \int^{2\pi} uu_{xx} dx - \frac{1}{3}\int_0^{2\pi} (u^3)_x dx + m\int_0^{2\pi} u^2 dx + \int_0^{2\pi} uk\, dx. \tag{58}$$

Using the periodicity of u, Equation (58) becomes

$$\frac{d}{dt}\left(\frac{1}{2}\int_0^{2\pi} u^2 dx\right) = -\int_0^{2\pi} u_x^2 dx + m\int_0^{2\pi} u^2 dx + \int_0^{2\pi} uk\, dx. \tag{59}$$

Then, using the Poincare inequality on Equation (59) and the zero mean condition of u, we get

$$\frac{d}{dt}\left(\frac{1}{2}\int_0^{2\pi} u^2 dx\right) \leq (m - \tfrac{1}{c})\int_0^{2\pi} u^2 dx + \int_0^{2\pi} \frac{1}{\sqrt{c}} u\sqrt{c}k\, dx. \tag{60}$$

If $k(x) = 0$, we use the Gronwall inequality on Equation (60) to obtain the first part of Theorem 4.1,

$$\| u \| \leq \| u_0 \| e^{(m-\frac{1}{c})(t-t_0)}, \text{ for } t \geq t_0. \tag{61}$$

However, in case $k(x) \neq 0$, we apply the Cauchy-Schwartz inequality on Equation (60) to obtain

$$\frac{1}{2} \frac{d}{dt}(\| u \|^2) + (\tfrac{1}{c} - m) \| u \|^2 \leq \| \frac{u}{\sqrt{c}} \| \cdot \| \sqrt{c}k \| . \tag{62}$$

Using the inequality $ab < \frac{a^2}{2} + \frac{b^2}{2}$, Equation (62) becomes

$$\frac{d}{dt}(\| u \|^2) + (\tfrac{1}{c} - 2m) \| u \|^2 \leq c \| k \|^2. \tag{63}$$

Again, using the Gronwall inequality on Equation (63), we obtain

$$\| u \|^2 \leq e^{-(\frac{1}{c}-2m)t} \| u_0 \|^2 + \frac{c^2}{1-2mc}(1 - e^{-(\frac{1}{c}-2m)t}) \| k \|^2. \tag{64}$$

Finally, given $\| u_0 \|$, if we choose $t \geq t_0$ with $t_0 = \frac{c}{1-2mc} ln\left(\frac{(1-2mc)\|u_0\|^2}{c^2\|k\|^2}\right)$, and $m < \frac{1}{2c}$ then

$$\| u \| \leq \sqrt{\frac{2c^2}{1-2mc}} \| k \| . \tag{65}$$

It follows from Equation (65) that the original Burgers equation admits an absorbing ball in the Hilbert space $H^2[0, L]$. Using the machinery developed in [39], one can prove the existence of a compact attractor, Hence, we will not repeat the analysis, but quote the result.

Proposition 4.1: *There exists a compact set in H^2, called an attractor, to which any solution to the forced original Burgers Equation, starting from any initial value of H^2 converges.*

4.2 Existence of an Inertial Manifold

The notion of inertial manifold was introduced by Foias, Sell and Temam [22] as a way to obtain a system of ODEs that has the same dynamics as the PDEs. Various attempts have been made to exhibit inertial manifolds for a large class of PDEs [2-4, 17, 35, 37]. Smaoui and Armbruster [36] have found a system of ODEs that mimics the dynamics of Kolmogorov flow for a given Reynolds number. Kwak [31] introduced a nonlinear transformation that embeds any quasilinear parabolic equation given by

$$u_t = u_{xx} + f(u))_x + g(u) + h(x), \tag{66}$$

on the interval $[0, L]$, into a reaction-diffusion system that admits an inertial manifold. The transformation is defined by

$$J(u) = (u, u_x, f(u)). \tag{67}$$

The Kwak triple $(u, v, w) = J(u)$, where u, v and w are representing the velocity, frequency, and kinetic energy, respectively, must satisfy the system of equations

$$\begin{aligned} u_t &= u_{xx} + w_x + g(u) + h(x), \\ v_t &= v_{xx} + w_{xx} + g'(u)v + h'(x), \\ w_t &= w_{xx} - f''(u)v^2 + f'(u)^2 v + f'(u)\{g(u) + h(x)\}, \end{aligned} \tag{68}$$

with the periodic boundary condition given by $J(u(0,t)) = J(u(L,t))$ and initial values given by $J(u_0(x))$. In (68), the prime denotes the derivative with respect to the corresponding argument.

We apply the transformation defined in (67) to the forced original Burgers equation

$$u_t = u_{xx} - uu_x + mu + k(x). \tag{69}$$

By setting $u = u$, $v = u_x$ and $w = -\frac{1}{2}u^2$, we obtain

$$\begin{aligned} u_t &= u_{xx} + w_x + mu + k(x), \\ v_t &= v_{xx} + w_{xx} + mv + k'(x), \\ w_t &= w_{xx} + v^2 + u^2 v - mu^2 - uk(x), \end{aligned} \tag{70}$$

with periodic boundary conditions $u(L,t) = u(0,t)$, $v(L,t) = v(0,t)$, and $w(L,t) = w(0,t)$. The initial conditions are $u(x,0) = u_0(x)$, $v(x,0) = v_0(x)$, and $w(x,0) = w_0(x)$.

Lemma 4.1: *a) If $v(x,0) = u_x(x,0)$ and $u(x,t)$ is a solution of (66), then $v(x,t) = u_x(x,t)\ \forall t \geq 0$.*

b) For any steady state solutions of (70), $u_x = v$.

Proof: $a)$ Let $\eta = v - u_x$. Then $\eta_t = \eta_{xx}$ with $\eta(x,0) = 0$. The uniqueness property of solutions to the diffusion equation with periodic boundary conditions and zero mean implies that $\eta \equiv 0$; hence $v(x,t) = u_x(x,t)$.

$b)$ Let $\eta(x) = u_x - v$. Then η satisfies $\eta_{xx} = 0$. Since η is periodic in space with zero mean, $\eta \equiv 0$.

Proposition 4.2: *The steady state solution of the forced original Burgers equation* (69) *is also the steady state solution to the transformed reaction diffusion system* (70). *Conversely, any steady solution* (u,v,w) *of system* (70) *is necessarily of the form* $v = u_x$, $w = -\frac{1}{2}u^2$, *where* u *is a steady state solution of Equation* (69).

Proof: At the steady state, $v_t = w_t = 0$. Subtracting the last two equations in (70), we get

$$v_{xx} - u^2 v - v^2 + m(v + u^2) + k'(x) + uk(x) = 0. \tag{71}$$

Since $v = u_x$, Equation (71) becomes

$$u_{xxx} - u^2 u_x - u_x^2 + m(u_x + u^2) + k'(x) + uk(x) = 0, \tag{72}$$

which implies

$$(u_{xx} - uu_x + mu + k(x))_x + u(u_{xx} - uu_x + mu + k(x))0. \tag{73}$$

However,

$$u_{xx} - uu_x + mu + k(x) = 0, \tag{74}$$

since u is a steady state solution of the forced Burgers equation.

To prove the converse, first observe that the steady state solution of (70) satisfies

$$\begin{aligned} u_{xx} + w_x + mu + k(x) &= 0, \\ v_{xx} + w_{xx} + mv + k'(x) &= 0, \\ w_{xx} + u^2 v + v^2 - mu^2 - uk(x) &= 0. \end{aligned} \tag{75}$$

From the first part of the proof, this implies that

$$(u_{xx} - uu_x + mu + k(x))_x + u(u_{xx} - uu_x + mu + k(x)) = 0. \tag{76}$$

Let

$$\phi = u_{xx} - uu_x + mu + k(x). \tag{77}$$

Then, we have

$$\phi_x + u\phi = 0. \tag{78}$$

If

$$\theta = \exp\left(\int_0^x u(s)ds\right) \tag{79}$$

then

$$(\phi\theta)_x = 0, \tag{80}$$

which implies that $\phi = c_1/\theta$,

$$\begin{aligned}\int_0^{2\pi} \phi(x)dx &= \int_0^{2\pi} (u_{xx} - uu_x + mu + k(x))dx \\ &= \int_0^{2\pi} (u_x - \tfrac{u^2}{2})_x dx + \int_0^{2\pi} (mu + k(x))dx.\end{aligned} \tag{81}$$

Using the periodicity of u and the fact that $\int_0^{2\pi} k(x)dx = 0$, we get

$$\int_0^{2\pi} \phi(s)ds = 0, \tag{82}$$

which implies $\int_0^{2\pi} \frac{c_1}{\theta} dx = 0$. Since $\theta > 0$, we have $c_1 = \phi = 0$ and

$$u_{xx} - uu_x + mu + k(x) = 0. \tag{83}$$

5. Numerical Results

If we set

$$u(x,t) = \sum_{-\infty}^{\infty} e^{ilx} \widehat{u}(l,t), \tag{84}$$

and restrict the calculations to 2π-periodic solutions, then the forced original Burgers equation (43) in Fourier space can be written as

$$\widehat{u}_t(l,t) = (m - l^2)\widehat{u}(l,t) - il \sum_{p+q=l} \widehat{u}(p,t)\widehat{u}(q,t) + \widehat{k}(l), \tag{85}$$

where $\widehat{u}(l,t)$ is the Fourier transform of $u(x,t)$ and $m \geq 0$. A computer program that uses a spectral Galerkin method with $N = 256$ was written to solve Equation (85). The “slaved-frog" scheme was used [23].

Figure 1 presents the steady state solution of the forced Burgers equation for different values of m with $k(x) = 0$ and $k(x) = \sin x + \frac{1}{2}\sin(2x)$ (recall that in this case, $k = -BF(b)$, where $b = 0$, and $b = \sin x$, respectively). Steady state solutions converge with at least four digits of accuracy in almost 10^4 time steps. From Figure 1a), one can observe that if $m = 0$ and $k(x) = 0$, then the trivial solution is the only stable solution. The steady state solution given in Equation (50) is numerically unstable for that particular case. In case when $m = 0$ and $k(x) \neq 0$ is shown in Figure 1b). Although the steady state solution in this case is nontrivial, it is numerically stable.

Figure 1: Steady state solutions for different values of m and different forcing terms with $u(x{,}0) = \sin x$: a) $k(x) = 0$; b) $\sin x + \frac{1}{2}\sin 2x$.

Figure 2 shows the triplet steady state solutions (u, v, w) of the Kwak reaction diffusion system. In this case, the steady state solution of the original Burgers equation with $m = 3$ and $k(x) = \sin x + \frac{1}{2}\sin 2x$ was used as an initial condition for the transformed reaction diffusion system. After only few time steps, four digits of accuracy was observed in the system. This result reinforces the fact that steady state solutions for the forced original Burgers equation and the transformed reaction diffusion system are the same.

Figure 2: Steady state triplet (u,v,w) of the Kwak system with initial conditions equal to the steady state original Burgers equation obtained from the case $m = 2$ and $k(x) = \sin x + \frac{1}{2}\sin 2x$.

Figure 3a depicts the time evolution of u when $m = 0$ and $k(x) = \sin x + \frac{1}{2}\sin 2x$. Using the transformation given in Equation (51), one can easily obtain the time evolution of the population density ϕ without numerically solving Equation (34) (see Figure 3b)).

Figure 3: a) The time evolution of u for the case when $m = 0$ and $k(x) = \sin x + \frac{1}{2}\sin 2x$ and initial condition $u(x,0) = \exp(-10(0.4x-1)^2)$. Time step is 5×10^{-4}. The output is every 100 time steps. Successive outputs are shifted by $\delta u = 0.025$. b) The time evolution of the population density ϕ obtained from Hopf-Cole transformation of equation (51).

6. Concluding Remarks

In this paper, we have shown that the convection diffusion equation with an indefinite weight can be transformed into the forced Burgers equation via the Hopf-Cole transformation. The latter can in turn be transformed into a system of reaction diffusion equations through the Kwak transformation. This ingenuous transformation allowed us to show the existence of inertial manifold for the forced Burgers equation. Transitively, this implies the existence of inertial manifold for the convection diffusion equation. Biologically, this is sound since a population whose density is described by the indefinite convective diffusion equation has a velocity field described by the corresponding forced Burgers equation. For instance, a population who is destined to go extinct must have a velocity that converges to zero, while a persistent population with density ϕ will have a velocity distribution given by the Hopf-Cole transform $u = -2a\frac{\phi_x}{\phi}$. Numerical results illustrate and support many of the aspects of the connections mentioned above. In particular, they confirm that the drift in the convection diffusion equation triggers the forcing in the Burgers equation.

References

[1] Ablowitz, M.J. and DeLillo, S., The Burgers equation under deterministic and stochastic forcing, *Physica D* **92** (1996), 245-259.

[2] Armbruster, D., Nicolaenko, B., Smaoui, N. and Chossat, P., Analyzing bifurcations in the Kolmogorov flow equations, In: *Dynamics, Bifurcation and Symmetry* (ed. by P. Chossat), NATO, ASI, Cargess, Kluwer, Dordrecht 1994.

[3] Armbruster, D., Nicolaenko, B., Smaoui, N. and Chossat, P., Symmetries and dynamics for 2-D Navier-Stokes flow, *Physica D* **95** (1996), 81-93.

[4] Babin, A.V. and Vishik, M.I., Attractors of partial differential equations and estimate of their dimension, *Uspeki Mat. Nauk.* **38** (1983), 133-187. (In Russia) *Russian Math. Surveys* **28** (1983), 151-213. (In English)

[5] Belgacem, F., *Elliptic Boundary Value Problems with Indefinite Weights: Variational Formulations of the Principal Eigenvalue and Applications*, Pitman Research Notes No. **368**, Addison-Wesley-Longman, London 1997.

[6] Belgacem, F., Boundary value problem with indefinite weight and applications, *Intern. J. of Prob. of Nonl. Anal. in Eng. Sys.* **10**:2 (1999), 51-58.

[7] Belgacem, F., Diffusion and drift models for population dispersal from stochastic and continuum views, *Intern. J. of Appl. Math.* **5**:1 (2001), 85-106.

[8] Belgacem, F. and Cosner, C., The effects of dispersal along environmental gradients on the dynamics of populations in heterogeneous environments, *Can. Appl. Math. Quart. J.* **3**:4 (1995), 379-397.

[9] Burgers, J.M., Mathematical examples illustrating relations occurring in the theory of turbulent fluid motion, *Verhandel. Kon. Nederl. Akad. Wetenshappen Amsterdam*, Atdeel. Natuurkunde (1st Sect.) **17**:2 (1939), 1-53.

[10] Burgers, J.M., A mathematical model illustrating the theory of turbulence, *Adv. Appl. Mech.* **1** (1948), 171-199.

[11] Burgers, J.M., *The Nonlinear Diffusion Equation*, Reidel, Boston 1974.

[12] Cantrell, R.S. and Cosner, C., Diffusive logistic equations with indefinite weights: Population models in disrupted environments, *Proc. Roy. Soc. Edinburgh Sect. A* **112** (1989), 293-318.

[13] Cantrell, R.S. and Cosner, C., The effects of spatial heterogeneity in population dynamics, *J. of Math. Biology* **29** (1991), 315-338.

[14] Cantrell, R.S. and Cosner, C., Diffusive logistic equations with indefinite weights: Population models in disrupted environments II, *SIAM J. Math. Anal.* **22** (1991), 1043-1064.

[15] Castro, A. and Lazer, A.C., Results on periodic solutions of parabolic equations suggested by elliptic theory, *Boll. della Unione math. Italiana*, Serie I-B **6** (1982), 1089-1104.

[16] Chern, I.-L. and Liu, T.-P., Convergence to diffusion waves of solutions for viscous conservation laws, *Commun. Math. Phys.* **110** (1987), 503-517.

[17] Constantin, P., Foias, C., Nicolaenko, B. and Temam, R., *Integral Manifold and Inertial Manifolds for Dissipative Partial Differential Equations*, App. Math. Sciences, Springer-Verlag, New York 1989.

[18] Cosner, C., *Eigenvalue Problems with Indefinite Weights and Reaction-Diffusion Models in Population Dynamics*, In: Reaction-Diffusion Equations (ed. by Brown and Lacey), Clamendon Press, Oxford 1990.

[19] Eckstein, E.C. and Belgacem, F., Model of platelet transport in flowing blood with drift and diffusion terms, *Biophy. J.* **60** (1991), 53-69.

[20] Eckstein, E.C., Yeh, C.J. and Belgacem, F., Inflow experiments showing lateral migration of platelet-sized latex beads, In: *Proc. of 3rd Mid-Atlantic Conf. on Bio-Mechanics* (1990), 99-106.

[21] Eden, A., On Burgers' original mathematical model for turbulence, *Nonlinearity* **3** (1990), 557-566.

[22] Foias, C., Sell, G. and Temam, R., Inertial manifolds for nonlinear evolutionary equations, IMA Preprint Series, *J. Differential Equations* **234** (1986).

[23] Frisch, U., Su, S.Z. and Thual, O., Viscoelastic behavior of cellular solutions to the Kuramoto-Sivashinsky model, *J. Fluid Mech.* **168** (1986), 221-240.

[24] Guil, F., Manas, M. and Alvarez, G., The Hopf-Cole transformation and the KP equation, *Phys. Lett. A.* **190**:1 (1994), 49-52.

[25] Gutkin, E., Propagation of chaos and the Hopf-Cole transformation, *Adv. Appl Math.* **6** (1985), 413-421.

[26] Hess, P. and Kato, T., On some linear and nonlinear eigenvalue problems with an indefinite weight function, *Comm. in PDE* **5** (1980), 999-1030.

[27] Hess, P., On the principal eigenvalue of a second order linear elliptic problem with an indefinite weight function, *Math. Z.* **179** (1982), 237-239.

[28] Hess, P., *Periodic-Parabolic Boundary Value Problems and Positivity*, Pitman Research Notes **247**, John Wiley and Sons, Inc., New York 1991.

[29] Hopf, E., The partial differential equation $u_t + uu_x = \nu u_{xx}$, *Comm. Pure Appl. Math.* **3** (1950), 201-230.

[30] Kreiss, G., Convergence to steady state of solutions of viscous conservation laws, *Lect. Notes Pure Appl. Math.* **130** (1990), 225-237.

[31] Kwak, M., Finite dimensional inertial form of the two-dimensional Navier Stokes equations, *Ind. Math. J.* **41** (1992), 927-981.

[32] Manes, A. and Micheletti, A.M., Unestensione della teoria variazionale classica degli autovalori per operatori ellittiei del secondo ordine, *Boll. UMI* **7** (1973), 285-301.

[33] Sachdev, P.L., A generalized Cole-Hopf transformation for nonlinear parabolic and hyperbolic equations, *Z. Angew. Math. Phys.* **29** (1978), 963-970.

[34] Smaoui, N., Analyzing the dynamics of the forced Burgers equation, *J. of Appl. Math. and Stoch. Anal.* **13**:3 (2000), 269-285.

[35] Smaoui, N., A model for the unstable manifold of the bursting behavior in the 2d Navier-Stokes flow, *SIAM J. Sci. Comput.* **23**:3 (2001), 824-840.

[36] Smaoui, N. and Armbruster, D., Symmetry and the Karhunen-Loeve analysis, *SIAM J. Sci. Comput.* **18**:5 (1997), 1526-1532.

[37] Smaoui, N. and Nicolaenko, B., Burgers equation: Does it have an inertial manifold?, *Bull. of the Amer. Phys. Soc.* **38**:12 (1993).

[38] Tasso, H. and Teichmann, J., On the generalization of the Cole-Hopf transformation, *Z. Angew. Math. Phys.* **30** (1979), 1023-1025.

[39] Temam, R., *Infinite Dimensional Dynamical System in Mechanics and Physics*, Springer-Verlag, New York 1988.

[40] Wang, W.C., On L^1-convergence rate of viscous and numerical approximate solutions of genuinely nonlinear scalar conservation laws, *SIAM J. Math. Anal.* **30**:1 (1999), 38-52.