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Connectivity Properties of Archimedean and Laves Lattices

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ABSTRACT

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An Archimedean lattice is a graph of a regular tiling of the plane, such that all corners are equivalent. A tiling is regular if all tiles are regular polygons: equilateral triangles, squares, et cetera. There exist exactly 11 Archimedean lattices. Being planar graphs, the Archimedean lattices have duals, 3 of which are Archimedean, the other 8 are called Laves lattices.

In the thesis, three measures of connectivity of these 19 graphs are studied: the connective constant for self-avoiding walks, and bond and site percolation critical probabilities. The connective constant measures connectivity by the number of walks in which all visited vertices are unique. The critical probabilities quantify the proportion of edges or vertices that can be removed, so that the produced subgraph has a large connected component.

A common issue for these measures is that they, although intensely studied by both mathematicians and scientists from other fields, have been calculated only for very few graphs. With the goal of comparing the induced orders of the Archimedean and Laves lattices under the three measures, the thesis gives improved bounds and estimates for many graphs.

A large part of the thesis focuses on the problem of deciding whether a given graph is a subgraph of another graph. This, surprisingly difficult problem, is considered for the set of Archimedean and Laves lattices, and for the set of matching Archimedean and Laves lattices.

Keywords: percolation, self-avoiding walks, Archimedean and Laves lattices.

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Acknowledgements

More or less exactly five years ago I took my first steps into the fascinating field of percolation, when I, looking for a master thesis subject, contacted Sven Erick Alm. When it later came to choosing an adviser for my PhD-studies, Sven Erick was the natural choice.

I have learned a lot from Sven Erick, most importantly the many different aspects of being a researcher, and I am truly grateful for these five years.

The work in this thesis is to a great extent influenced also by John C. Wierman, whom I first met in the summer of 2002. The outline for the thesis was formulated, somewhat implicitly, during a memorable visit of John in Uppsala that autumn, and it has been a pleasure collaborating with John ever since.

I would also like to thank my colleagues at the department, especially the always friendly "matstat"-group. The people who, perhaps unknowingly, have provided valuable CPU-time to my projects deserves a special little acknowledgment.

Life is not only mathematics, and I thank family, friends, neighbors, and past and present team mates of NNIF, IBK U:a Teknologer, MoPS, Gladpack and Oss Själva, for, among other things, filling my spare time with activities not even remotely connected to percolation.

Robert Parviainen,
Uppsala, 1 April 2004.

Included Papers

I. Robert Parviainen and John C. Wierman. Inclusions and Non-Inclusions of Archimedean and Laves Lattices. *U.U.D.M. Report* 2002:**13**, Department of Mathematics, Uppsala University. Submitted.

II. John C. Wierman and Robert Parviainen. Ordering Bond Percolation Critical Probabilities. *U.U.D.M. Report* 2002:**44**, Department of Mathematics, Uppsala University. Submitted.

III. Sven Erick Alm and Robert Parviainen. Bounds for the Connective Constant of the Hexagonal Lattice. *J. Phys. A* **37**(3), 549–560, 2004.

IV. Robert Parviainen. Pairs of Graphs with Connective Constants and Critical Probabilities in the Same Order. *U.U.D.M. Report* 2003:**26**, Department of Mathematics, Uppsala University. Submitted.

V. Robert Parviainen. Inclusions and Non-Inclusions of Matching Archimedean and Laves Lattices. *U.U.D.M. Report* 2004:**9**, Department of Mathematics, Uppsala University. Submitted.

VI. Robert Parviainen. Estimation of Bond Percolation Thresholds on the Archimedean Lattices. *U.U.D.M. Report* 2004:**11**, Department of Mathematics, Uppsala University. Submitted.

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Comments on my contributions

Paper I Joint paper with John C. Wierman. The research behind the paper was mostly done during an intense 3 week visit of Wierman in Uppsala. The final writing was done during a visit of mine at Johns Hopkins University. The individual contributions are indistinguishable.

Paper II Joint paper with John C. Wierman. The paper summarizes what is known about the bond percolation critical probabilities on the Archimedean and Laves lattices, as well as giving some new bounds. It also extends the results given in Paper I somewhat by incorporating two more lattices to the partial order — much of this work was done in connection with writing Paper I. Most writing was done by Wierman, who also is responsible for the majority of results referred to in the paper.

Paper III Joint paper with Sven Erick Alm. The idea of combining Kesten's method with calculations on one-dimensional lattices was Alm's. I did all programming and computations for the lower bound, as well as most of the writing.

Other Papers

Apart from the work included in this thesis, I have also (co-)authored the following papers:

[1] Sven Erick Alm and Robert Parviainen. Lower and upper bounds for the time constant of first-passage percolation. *Combin. Probab. Comput.* **11**, 433–445, 2002.

[2] Robert Parviainen. Random assignment with integer costs. *Combin. Probab. Comput.* **13**, 103–113, 2004.

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Konnektivitetsegenskaper för Arkimediska och Lavesgrafer

Inledning

Avhandlingen studerar 3 mått på hur sammanhängande en graf är, framför allt studeras hur måtten är relaterade till varann.

För transitiva grafer (sådana att alla noder är ekvivalenta) är gradtalet ett enkelt mått på konnektivitet. För kvasitransitiva grafer (grafer med ändligt antal nodklasser) kan man använda något slags genomsnittligt gradtal. Liksom de flesta andra enkla mått, där enkel ska tolkas som att det är möjligt att snabbt bestämma ett exakt värde, har dessa en svaghet: de tilldelar samma värde till många olika grafer.

I avhandligen studeras tre komplicerade mått, kritiska sannolikheter för nod- och kantperkolation, samt konnektiva konstanter för självundvikande vägar. Dessa är så komplicerade att beräkna att det endast har klarats av i ett fåtal fall.

De grafer som studeras är de Arkimediska graferna, samt deras duala grafer, de så kallade Lavesgraferna, 19 st. plana, oändliga, transitiva eller kvasitransitiva grafer. Låt oss kalla dessa 19 grafer för AL-graferna.

Ett mål har varit att besvara följande fråga. Låt $p_c^s(G), p_c^b(G)$ och $\mu(G)$ beteckna kritiska sannolikheten för nodperkolation, för kantperkolation, och konnektiva konstanten för grafen G.

Fråga 1. *Är det sant att det för alla AL-grafer G och H, gäller att*

$$\mu(G) \leq \mu(H) \Leftrightarrow p_c^b(G) \geq p_c^b(H) \Leftrightarrow p_c^s(G) \geq p_c^s(H)?$$

Ett rigoröst svar har inte hittats, men resultaten tyder, något förvånande, på att svaret är nej. Att det inte gäller allmänt för plana kvasitransitiva grafer har visats av Wierman, [29], och i Uppsats IV i denna avhandling.

Arkimediska och Lavesgrafer

Den första systematiska studien av de Arkimediska graferna gjordes av Kepler, [13], som visade att det finns exakt 11 stycken. En graf är Arkimedisk om den är transitiv, och kan ritas upp i planet så att alla områden är reguljära polygoner, dvs. alla kanter ska vara lika långa, och alla vinklar lika stora.

De duala graferna fås genom att se varje område som en nod, och säga att två noder är grannar om motsvarande områden delar en kant. Det kvadratiska punktgittret är en Arkimedisk graf, och dessutom sin egen dual, och dualen till det hexagonala gittret är det triangulära gittret (och vice versa), så det finns totalt 19 AL-grafer.

Självundvikande vägar

En självundvikande väg är en följd av parvis angränsande noder, sådana att alla ingående noder är olika. Om vi fixerar en startpunkt o, där alla självundvikande vägar ska starta, och låter f_n beteckna antalet vägar av längd n, defineras den konnektiva konstanten som

$$\mu = \lim_{n\to\infty} f_n^{1/n}.$$

Den konnektiva konstanten är okänd för alla tvådimensionella grafer, men Nienhuis, [20], har, via icke-rigorösa metoder, beräknat värdet $\sqrt{2+\sqrt{2}}$ för det hexagonala gittret. Den allmänna uppfattningen är dock att detta värde är det korrekta. Detta skulle också ge ett exakt värde för den Arkimediska grafen kallad $(3, 12^2)$.

Perkolation

Perkolationsprocessen introducerades 1954 av Broadbent, i en diskussion om Hammersley och Mortons uppsats *Poor man's Monte Carlo*, [10], som ett möjligt exempel för de tekniker som presenterades.

Perkolation har sedan dess använts som modell inom många olika områden, framför allt inom fysiken och geovetenskapen.

Modellen är följande. För en given oändlig graf, singla en slant med sannolikhet p för krona, för varje kant i grafen. De kanter som får klave tas bort. Vi har nu producerat en slumpmässig delgraf av den ursprungliga, och är intresserade av storleken på de sammanhängande komponenterna, klustrena, i denna delgraf.

Det visar sig att det finns ett kritiskt värde p_c, sådant att om $p < p_c$, så finns det endast ändligt stora kluster, medan det för $p > p_c$ existerar ett unikt oändligt stort kluster.

Modellen ovan kallas för kantperkolation, och det kritiska värdet p_c kallas för den kritiska sannolikheten. Analogt defineras nodperkolation; istället för kanter tas noder bort slumpmässigt.

De kritiska sannolikheterna är endast kända i ett fåtal fall. För AL-graferna är de kända för tre (kantperkolation), respektive sex grafer (nodperkolation).

Resultat

Ett användbart faktum för vår studie är följande. Om G är delgraf till H, kommer de kritiska sannolikheterna för G att vara högre än för H, och den konnektiva konstanten lägre. Det är därför intressant att reda ut vilka av AL-graferna som är delgrafer av varandra. Detta är gjort i Uppsats I.

Ytterligare grafer kan fås från AL-graferna genom att ta deras matchande grafer (fås genom att fylla i alla möjliga diagonaler). Dessa är användbara i studiet av kritiska sannolikheter för nodperkolation. Också för dessa har vi utrett vilka som är delgrafer av varandra, i Uppsats V.

När resultatet i Uppsats I kombineras med gränser för, och exakta värden på, kritiska sannolikheter fås en bild av vad som kan sägas exakt om ordningen av graferna, med avseende på de kritiska sannolikheterna. I Uppsats II presenterar vi ett sådant resultat, och härleder också nya gränser för den kritiska sannolikheten för kantperkolation för vissa grafer.

Då exakta värden på de kritiska sannolikheterna saknas för de flesta AL-graferna är skattningar ett komplement till de, oftast svaga, gränser som existerar. I Uppsats VI har vi skattat de kritiska sannolikheterna för kantperkolation, för de Arkimediska grafer där den kritiska sannolikheten är okänd.

Att det existerar par av grafer (dessa är inte AL-grafer) sådana att kritiska sannolikheterna och de konnektiva konstanterna ligger i samma ordning, visas i Uppsats IV.

Slutligen, har vi i Uppsats III beräknat förbättrade gränser för den konnektiva konstanten på det hexagonala gittret.

Slutsats

Som tidigare nämnts har inget rigoröst svar på Fråga 1 hittats. Dock tyder skattningar på att svaret är nej. Det finns troligen par av AL-grafer, sådana att de kritiska sannolikheterna och de konnektiva konstanterna ej uppfyller påståendet i Fråga 1.

Chapter 1

Introduction

The motivating question behind the thesis could, in retrospect, be stated as *What is the relation between different measures of connectivity of a graph?*

There are countless ways of measuring connectivity, ranging from simple to extremely complicated, in terms of evaluation of the measure on a given graph.

The simple measures tend to be coarse, they assign the same value to several different graphs. The complicated measures often avoid this problem, but as suggested above, the drawback is that they are seldom possible to compute exactly.

For a transitive graph, the common degree of the vertices is trivial to compute. For quasi-transitive graphs we may use some kind of mean degree, also easy to compute. A high (mean) degree indicates a highly connected graph. We could also use mean size of faces, combinations of degree and face size, so on and so forth.

Examples of complicated measures are critical probabilities for site and bond percolation, and connective constants for self-avoiding walks. These can be computed exactly only for very few special cases, and will be described in detail below.

All these measures certainly are more or less related, and an important question is to what extent the different measures give the same information. Put another way, if measure 1 says that the graph G is more connected than the graph H, what does measure 2 say?

A more tractable form of the question is the following.

Question 1.1. *For two measures μ_1 and μ_2 of connectivity, and a set $\mathcal{C}$ of graphs, is it true that for $G_1, G_2 \in \mathcal{C}$,*

$$\mu_1(G_1) \leq \mu_1(G_2) \Longleftrightarrow \mu_2(G_1) \leq \mu_2(G_2)?$$

Yet another way of looking at the problem is the prediction of complicated measures from simple ones, for example in the sense of "universal formulae"

that tries to predict percolation thresholds given simple graph statistics, such as mean degree.

The thesis addresses Question 1.1 for the Archimedean and Laves lattices, a set of 19 infinite planar quasi-transitive two-dimensional graphs. The focus is on bond and site percolation critical probabilities and connective constants.

A rigorous answer is not found, but simulation results indicate that the answer should be no.

Chapter 2

Background

2.1 Graph theory

A graph consists of a countable (finite or infinite) set V of *vertices*, and a set E of *edges* connecting pairs of vertices. We will only consider undirected graphs, so each edge $e \in E$ is an unordered pair of vertices (u, v), $u, v \in V$. We will also restrict to simple graphs, disallowing loops – edges joining one vertex with itself, and multiple edges – more than one edge between any pair of vertices.

If $e = (u, v)$ is an edge, u and v are said to be *adjacent*. Also, u and v are *incident* to e. Two edges sharing one vertex are also said to be adjacent. The degree of a vertex v is the number of vertices adjacent to v, or equivalently, the number of edges incident to v.

Two graphs $G = (V, E)$ and $G' = (V', E')$ are *isomorphic* if there exists a graph isomorphism $\phi : V \to V'$ such that $(u, v) \in E$ if and only if $(\phi(u), \phi(v)) \in E'$.

A graph G is *transitive* if all vertices play the same role, i.e. for every pair of vertices u and v, there exists a graph automorphism ϕ (a graph isomorphism from V to V, such that G is isomorphic to $\phi(G)$), such that $\phi(u) = v$.

A *quasi-transitive* graph is slightly more general; there are finitely many equivalence classes of vertices under graph automorphisms.

A graph is *planar* if it is possible to draw it in $\mathbb{R}^2$ so that no two edges intersect. A planar graph divides the plane into regions, separated by the edges, called *faces*. The graph in Figure 2.1 has three faces, one square, one triangular, and one unbounded face. A precise definition is the following: the *faces* of a planar graph are the connected regions in $\mathbb{R}^2$ that remain after the line segments that constitute the edges in the planar embedding of G in $\mathbb{R}^2$ are removed.

The *dual graph* $D(G)$ of a planar graph G is the graph in which every face of G is a vertex, and in which two vertices are adjacent if and only if the corresponding faces in G are adjacent, that is, share an edge.

See Figure 2.2 for an example of a graph and its dual. Note that in this case,

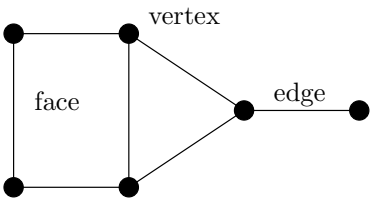


Figure 2.1: A small planar graph, with 6 vertices, 7 edges, and 3 faces (one unbounded).

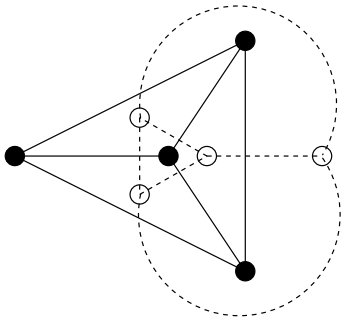


Figure 2.2: A graph and its dual.

the two graphs are isomorphic (the graph is said to be *self-dual*).

Given a planar infinite graph G, the *matching graph* of G, denoted $M(G)$, is the graph in which all diagonal edges in every face are added, that is, each face is substituted for a complete graph with order equal to the number of vertices in the face. In Figure 2.3 the matching graph of the Kagomé lattice is shown. Note that all triangulated graphs are *self-matching*.

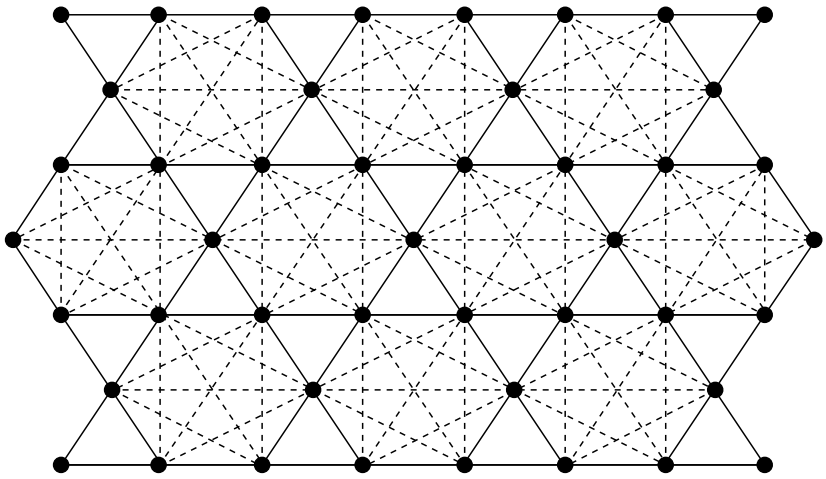


Figure 2.3: A finite subgraph of the matching graph of the Kagomé lattice. The added diagnonals are shown as dashed edges.

2.2 Archimedean and Laves lattices

The Archimedean lattices are the infinite transitive planar graphs, that can be drawn in the plane such that all faces are regular polygons. A regular polygon with n edges is a polygon where all edges have the same length, and the inner angle between any two adjacent edges is $(n-2)\pi/n$ radians. Kepler, [13], showed early in the 17th century that there are exactly 11 Archimedean lattices. The Archimedean lattices include 3 lattices with only one type of face, the square, triangular and hexagonal lattices, and 8 lattices with more than one type of face.

The lattices have systematic names, that also serve as formulas to produce them. For any given vertex, the incident faces are listed in, say, clockwise order, by the number of edges in the face, the list being the name of the lattice. Several lattices get multiple names, depending on the starting face, the convention is then to choose the lexicographically smallest name. If two or more consecutive entries agree, exponents may be used to abbreviate the name. Consequently, the square lattice is also known as $(4, 4, 4, 4)$, or (4^4).

The dual graphs of the Archimedean lattices are called the Laves lattices. Since the square lattice is self-dual, and the triangular and hexagonal lattices are each others duals, the union of the Archimedean and Laves lattices consists of 19 graphs. The graphs and their names are shown in Figure 2.4.

Some of the graphs also have more colorful names. The regular lattices (4^4), (6^3) and (3^6) are usually called the square, hexagonal and triangular lattices. The hexagonal lattice is also commonly called the honeycomb lattice. The $(3, 6, 3, 6)$ lattice have a name from Japanese, it is known as the Kagomé lattice (Kagomé means "woven bamboo pattern"). The $(3, 12^2)$ lattice is known as the extended Kagomé lattice. The $(3, 4, 6, 4)$ lattice is known as the ruby lattice, and in analogy with the Kagomé lattices, $(4, 6, 12)$ can be called the extended ruby lattice. The $(4, 8^2)$ is known as the bathroom tiling (the origin of the name should be obvious), or occasionally, the Briarwood lattice (named by one researcher after a local shopping mall).

The dual Kagomé lattice, $D(3, 6, 3, 6)$, is known as the dice lattice, and the dual of the bathroom tiling, $D(4, 8^2)$, is known as the Union Jack lattice. The lattice $D(3, 12^2)$ also has a Japanese name, Asanoha, meaning "hemp leaf". The lattice $D(3^2, 4, 3, 4)$ is sometimes called the Cairo lattice, as the pattern occurs frequently as tilings on the streets of Cairo.

2.3 The connective constant

The connective constant gives information about connectivity by counting the number of possible walks on the lattice that are self-avoiding; they are not allowed to visit the same vertex more than once.

Self-avoiding walks were proposed in the late 40's by Flory, [8], as a model for long linear polymer chains ("the excluded volume problem"). The mathematical

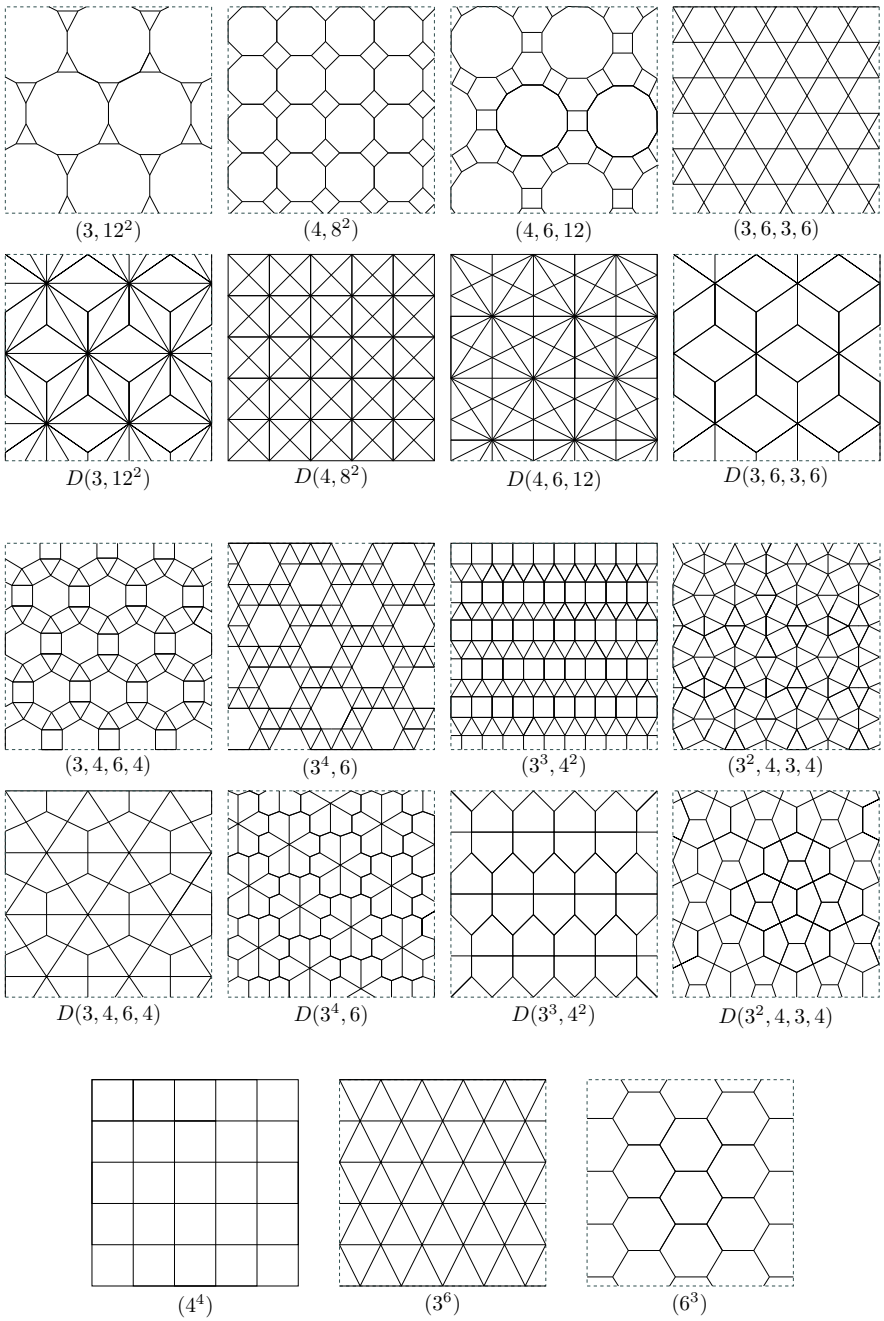


Figure 2.4: The 11 Archimedean lattices and their dual Laves lattices.

analysis took off in the 50's, by early work often involving Hammersley.

A *walk* from the vertex v_0 to the vertex v_n is an alternating sequence of vertices and edges, $\{v_0, e_1, v_1, e_2 \ldots, v_n\}$, such that $e_k = (v_{k-1}, v_k)$, $k = 0, \ldots, n-1$. A *self-avoiding walk* is a walk in which all vertices are distinct.

Let f_n denote the number of self-avoiding walks with n edges, starting at a predetermined vertex (which we will call the origin). The connective constant $\mu(G)$ of a graph G, which is independent of the chosen origin, is defined as

$$\mu(G) = \lim_{n\to\infty} f_n^{1/n}.$$

The connective constant is not known exactly for any non-trivial graph, with the possible exception of the hexagonal and $(3, 12^2)$ lattices. For the hexagonal lattice, Nienhuis, [20], derived the value $\sqrt{2+\sqrt{2}}$, through mathematically non-rigorous methods (but there is a very strong belief that this value is correct). If true, this also implies that the connective constant for the $(3, 12^2)$ lattice is 1.711041, [12], see also Paper III of this thesis.

For a more thorough introduction to self-avoiding walks, the book by Hughes, [11], contains a nice chapter on the subject, and the book by Madras and Slade, [18], gives an even more substantial treatment.

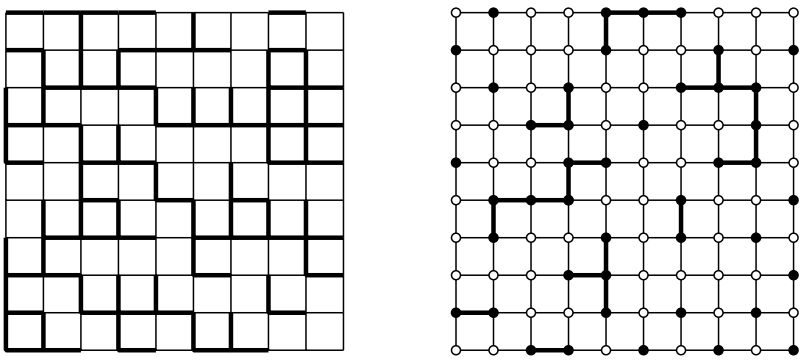


Figure 2.5: In the left figure, a realization of the bond percolation process on a finite subgraph of (4^4) is shown. The bold edges constitute the random subgraph. In the right figure, a realization of the of site percolation process on the same graph is shown. The vertices with unfilled circles are removed, and the random subgraph consists of the filled circles and the bold edges.

2.4 Percolation

Percolation critical probabilities measure connectivity by the proportion of edges or vertices necessary in a random subgraph of the lattice, in order to have a large connected component.

The percolation process was born in the 50's, in the discussion of Hammersley and Morton's paper *Poor man's Monte Carlo*[1], [10]. Broadbent suggested the bond percolation process as a possible problem to apply the presented methods on[2]. Broadbent and Hammersley went on to write the first paper on percolation, [6], published 1957.

In bond percolation, a random subgraph of a given graph is produced as follows. Let each edge remain in the subgraph with probability p, independently of all other edges. The remaining edges are also said to be open, the removed to be closed.

For the kind of graphs studied here, it is well known that there exists a critical value p_c^b, depending on the graph G, such that, for p below p_c^b, there exists only finite connected components in the random subgraph with probability 1, and for p above p_c^b, there exists a unique infinite connected component with probability 1.

The critical value p_c^b is called the *bond percolation critical probability* or *bond percolation threshold*. The *site percolation critical probability* or *site percolation threshold* p_c^s is the analogous constant, when vertices instead of edges are removed from the original graph (when a vertex is removed, the incident edges are also removed). In Figure 2.5, realizations of both processes are shown.

The value of the critical probability is known only for a few graphs. Although the correct values for the square, hexagonal and triangular lattices were conjectured in the 60's, it was not until 1980 that Kesten, [15], proved that the critical probability for the square lattice equals $1/2$. Soon thereafter, Wierman, [24], extended Kestens result to show that the critical probabilities for the hexagonal and triangular lattices equal $2\sin(\pi/18)$ and $1-2\sin(\pi/18)$. Wierman, [25], has also found the critical probabilities for another pair of dual lattices, the Bowtie lattice and its dual.

Except for derivatives of these lattices, these are the only 2-dimensional lattices for which the bond percolation critical probabilities are known.

The site percolation critical probability is known for a few planar graphs. It is known for triangulated graphs, for which it equals $1/2$ (proved by Kesten, [16]), and for two other Archimedean lattices - the $(3,6,3,6)$ (Kagomé) and $(3,12^2)$ (extended Kagomé) lattices - the latter two follow from the bond percolation threshold on the hexagonal lattice.

2.5 Degree based measures

In this section we look at some generalized mean degree measures, defined by Alm, [2].

For a quasi-transitive graph, with K vertex classes with a proportion of w_i

[1] Which also contains some early results on self-avoiding walks.

[2] The reply showed foresight: *"Mr. Broadbent's problem is very fascinating and difficult..."*

vertices of class i, the mean degree, $\bar{d}$, is usually defined as

$$\bar{d} = \sum_{i=1}^{K} w_i d_i,$$

where d_i is the degree of the vertices of class i.

To generalize $\bar{d}$, we study a finite Markov chain $\{X_n\}_{n=0}^{\infty}$, describing a random walk on the lattice, with vertex classes as state space. The unique stationary distribution $\pi = \{\pi_1, \pi_2, \ldots, \pi_K\}$ satisfies

$$w_i = C\frac{\pi_i}{d_i}, \text{ where } C = \left(\sum_{i=1}^{K} \frac{\pi_i}{d_i}\right)^{-1}.$$

Thus, $\bar{d} = \left(\sum_{i=1}^{K} \frac{\pi_i}{d_i}\right)^{-1} = d_h$, the weighted harmonic mean of the degrees d_i. Alternative mean degrees are the weighted arithmetic mean,

$$d_a = \sum_{i=1}^{K} d_i \pi_i,$$

and the weighted geometric mean,

$$d_g = \prod_{i=1}^{K} {d_i}^{\pi_i}.$$

Alm also defines a further mean degree, $\tilde{q}$, which improves separation between different lattices. Let $g_i(n)$ be the number of walks of length n, starting at a vertex of class i, and define

$$\tilde{d_i} = \lim_{n\to\infty} g_i(n)^{1/n}.$$

The value of $\tilde{d_i}$ is the same for all i, and this common value is the generalized mean degree $\tilde{d}$.

An equivalent definition uses the Markov chain defined earlier. Let P denote the chain's transition matrix, and D the diagonal matrix with entries d_i. Then $\tilde{d}$ is the largest eigenvalue of the matrix DP. Using this definition, numerical evaluation of $\tilde{d}$ is straightforward.

Note the similarity between $\tilde{d}$ and the connective constant (it is obvious that $\tilde{d}$ is an upper bound for the connective constant).

The mean degrees defined above only use information on the number of neighbors of different classes, and the neighbors degrees. Improved measures of connectivity along these lines should use more information (such as neighbor's neighbors and so on). For example, the definition of $\tilde{d}$ could be generalized in the direction of the connective constant;

$$\tilde{d}^{(m)} = \lim_{n\to\infty} \left(g_i^{(m)}(n)\right)^{1/n},$$

where $g_i^{(m)}$ is the number of walks starting at a vertex of class i, such that all subwalks of length m are self-avoiding.

2.6 Other measures

There are of course many other models that give measures of connectivity. Examples are time constants for first-passage percolation, the critical temperature for the Ising model, or more generally, critical probabilities for random cluster models, and probabilities related to random walks.

2.7 The main problem

Let the connective constant, bond percolation critical probability, and site percolation critical probability of a graph G be denote by, respectively, $\mu(G)$, $p_c^b(G)$ and $p_c^s(G)$.

Problem 2.1. *For which classes $\mathcal{G}$ of graphs, is it true that, if $G, H \in \mathcal{G}$,*

$$\mu(G) \leq \mu(H) \Leftrightarrow p_c^b(G) \geq p_c^b(H) \Leftrightarrow p_c^s(G) \geq p_c^s(H).$$

That the statement does not hold for all planar quasi-transitive graphs is shown by Wierman, [29], who gives a pair of planar graphs G and H, such that

$$p_c^b(G) > p_c^b(H) \text{ but } p_c^s(G) < p_c^s(H),$$

and in Paper IV of this thesis, where we give pairs of planar graphs (G_1, H_1) and (G_2, H_2) such that

$$p_c^b(G_1) > p_c^b(H_1) \text{ but } \mu(G_1) > \mu(H_1), \text{ and}$$

$$p_c^s(G_2) > p_c^s(H_2) \text{ but } \mu(G_2) > \mu(H_2).$$

The included papers in this thesis address various aspects of this problem for the set of Archimedean and Laves lattices. The Archimedean lattices are a natural choice as a class of graphs for this problem. The graphs have many pleasant features; they are transitive, planar, are relatively few, and include most graphs commonly studied in the literature on these subjects. The inclusion of their duals gives a larger class of graphs, which includes quasi-transitive lattices. There are two further reasons for including the duals. Since the bond percolation critical probabilities for a graph and its dual sum to 1, we gain much information about the bond percolation critical probabilities. Also, the effect of taking the dual graph on the other measures is an interesting problem in itself.

Similarly, since the site percolation critical probabilities for a graph and its matching graph sum to 1, inclusion of the matching graphs would also be interesting. Only a first step in this direction is taken in this thesis; the subgraph order of the matching graphs are studied in paper V.

Normally, any study of a probability model on graphs is much easier on trees (connected graphs without cycles). In the next section we summarize what is known about our problem on trees.

2.7.1 The tree case

Denote the root of a tree o, and for v in the vertex set $V(T)$, let $|v|$ denote the graph distance from o. The *branching number* of a tree is defined as

$$\mathrm{br}(T) = \sup\{\lambda : \inf_{\substack{o\in W\subset V(T)\\ |W|<\infty}} \sum_{v\in\partial W} \frac{1}{\lambda^{|v|}} > 0\}.$$

If the limit exists, the *growth* of T is defined as

$$\mathrm{gr}(T) = \lim_{n\to\infty} |T_n|^{1/n},$$

where T_n is the set of a vertices of distance n from the root. The lower (upper) growth of a tree is defined as

$$\underline{\mathrm{gr}}(T) = \liminf_{n\to\infty} |T_n|^{1/n} \quad (\overline{\mathrm{gr}}(T) = \limsup_{n\to\infty} |T_n|^{1/n}).$$

It is well known that both the site and bond critical probability for percolation on a tree is $1/\mathrm{br}(T)$. The number of self-avoiding walks, starting at the root, is obviously $|T_n|$, so the (lower) growth can be interpreted as the connective constant for trees.

Question 2.2. *If T and S are two infinite, locally finite trees, is it true that*

$$\mathrm{br}(T) \le \mathrm{br}(S) \Longleftrightarrow \underline{\mathrm{gr}}(T) \le \underline{\mathrm{gr}}(S)?$$

For a class of trees, called *sub-periodic* trees, the answer is positive. If T is a regular tree, where each vertex has k children, $\mathrm{br}(T) = k$ and $\mathrm{gr}(T) = k$. More generally, if T is spherically symmetric, so that the degree of a vertex only depends on the distance from the root, it is easy to check that $\mathrm{br}(T) = \mathrm{gr}(T)$. These are special cases of sub-periodic trees, for which Furstenberg, [9], showed that $\mathrm{br}(T) = \mathrm{gr}(T)$. T is N-sub-periodic if, for any vertex v, there exists a 1-1 adjacency preserving map $f : T^v \to T^{f(v)}$, with $|f(v)| \le N$, where T^v is the subtree rooted at v, which contains all descendants of v. T is sub-periodic if it is N-sub-periodic for some $N \ge 0$.

For trees T and S, for which the growths exist, we can prove the following.

Proposition 2.3. *If the growths* $\mathrm{gr}(T)$ *and* $gr(S)$ *exist for two trees T and S, with* $\mathrm{gr}(T) \le \mathrm{gr}(S)$*, then* $\mathrm{br}(T) \le \mathrm{br}(S)$*.*

Proof. Since $\lim_{n\to\infty} |T_n|^{1/n} \le \lim_{n\to\infty} |S_n|^{1/n}$, for all n greater than some finite number N, $T_n \le S_n$. The forest $T \setminus \bigcup_{0\le n\le N} T_n$ is then a subgraph of the forest $S \setminus \bigcup_{0\le n\le N} S_n$. Since the critical probability does not depend on finite subgraphs, $p_c(T) = \mathrm{br}(T)^{-1} \ge p_c(S) = \mathrm{br}(S)^{-1}$. □

Chapter 3

The included papers

3.1 Paper I

In Paper I, we determine which Archimedean and Laves lattices are subgraphs of each other. For each of the $19 \cdot 18 = 342$ ordered pairs of graphs (G, H), we either show how to get G from H by deleting edges, or prove that it is not possible.

The positive cases are proved simply by showing which edges to remove from the supergraph, or, which is the preferred viewpoint from the practical side, which to keep (however, finding the proof is not always trivial).

For the negative cases we have developed several general criteria for non-inclusions. That the maximum degree of the subgraph can not be higher than the maximum degree of the supergraph is a simple criterion, that handles many cases. Face sizes also give some useful criteria.

However, 21 cases are neither positive cases, nor handled by any of the criteria, and have to be handled case by case. The arguments for these special cases tend to follow the general lines of this example (although this is one of the easy special cases).

Example 3.1. $D(4, 8^2) \not\subset D(3, 12^2)$
In $D(4, 8^2)$, each vertex of degree 4 is adjacent to every vertex of a surrounding cycle of length 4. In $D(3, 12^2)$, the only vertices in the interior of a cycle of length 4 are vertices of degree 3. Therefore, $D(4, 8^2)$ cannot be a subgraph of $D(3, 12^2)$.

The resulting partial order is conveniently summarized by a Hasse diagram (Figure 3.1). In the figure, a line between two graphs indicate that the graph higher in the diagram is a subgraph of the graph lower in the diagram.

Remark 3.2. That the subgraph relation defines a partial order on a given set of graphs is not true in general. An example of two non-isomorphic infinite trees which are subgraphs of each other is given in [5, p. 231]. In our case it follows from the results.

For our purpose, knowing which graphs are subgraphs of each other is useful due to the following well known facts. If H is a subgraph of G, the site and bond critical probabilities are higher for H than for G. Also, the connective constant is lower for H than for G.

3.2 Paper II

The aim of Paper II is to give an, as complete as possible, ordering of the Archimedean and Laves lattices, with respect to the bond percolation critical probabilities. The main tool is the partial order from Paper I combined with exact values of and rigorous bounds for the critical probabilities. For some lattices we derive new bounds.

Since the critical probability also is known exactly for the Bowtie lattice and its dual (see Figure 3.2), the partial order from Paper I was extended to included these two graphs. The results are summarized in Section 4.2.

3.3 Paper III

The paper gives improved bounds for the connective constant on the hexagonal lattice.

For the upper bound we use the method of Alm, [1], a transfer matrix technique based on enumeration of "short" self-avoiding walks. The result is a large square matrix (in our case of dimension 17700×17700), each element is the number of self-avoiding walks of a certain length that start and end in specified ways. The largest eigenvalue of the matrix gives the bound. The whole computation took about 40 days on a standard desktop computer.

The lower bound is achieved by combining a method of Kesten, [14], based on a subclass of self-avoiding walks, called bridges, and by exact calculations on 1-dimensional lattices. Alm and Janson, [4], showed, in principle, how to compute the generating function for self-avoiding walks (and thus the connective constant) for *any* periodic 1-dimensional lattice. It is straightforward to extend this to get the generating function for bridges of a predetermined height, which is required for Kestens method.

The generating function is obtained as one element of the inverse of a large matrix. In the matrix, each element is a polynomial, representing certain possible configurations of edges, which can be found by inspection. However, for the largest case studied in the paper, the matrix has dimension 1024×1024, eliminating any hope of finding the matrix by hand.

The task is not trivial even with the help of a computer. The rules for generating the elements are quite easy to describe in words, but harder to implement as a computer program. (The source code has over 500 lines.)

Generating the matrix took several weeks of CPU-time. Next, (a bound of) the generating function had to be found, requiring days of further computation.

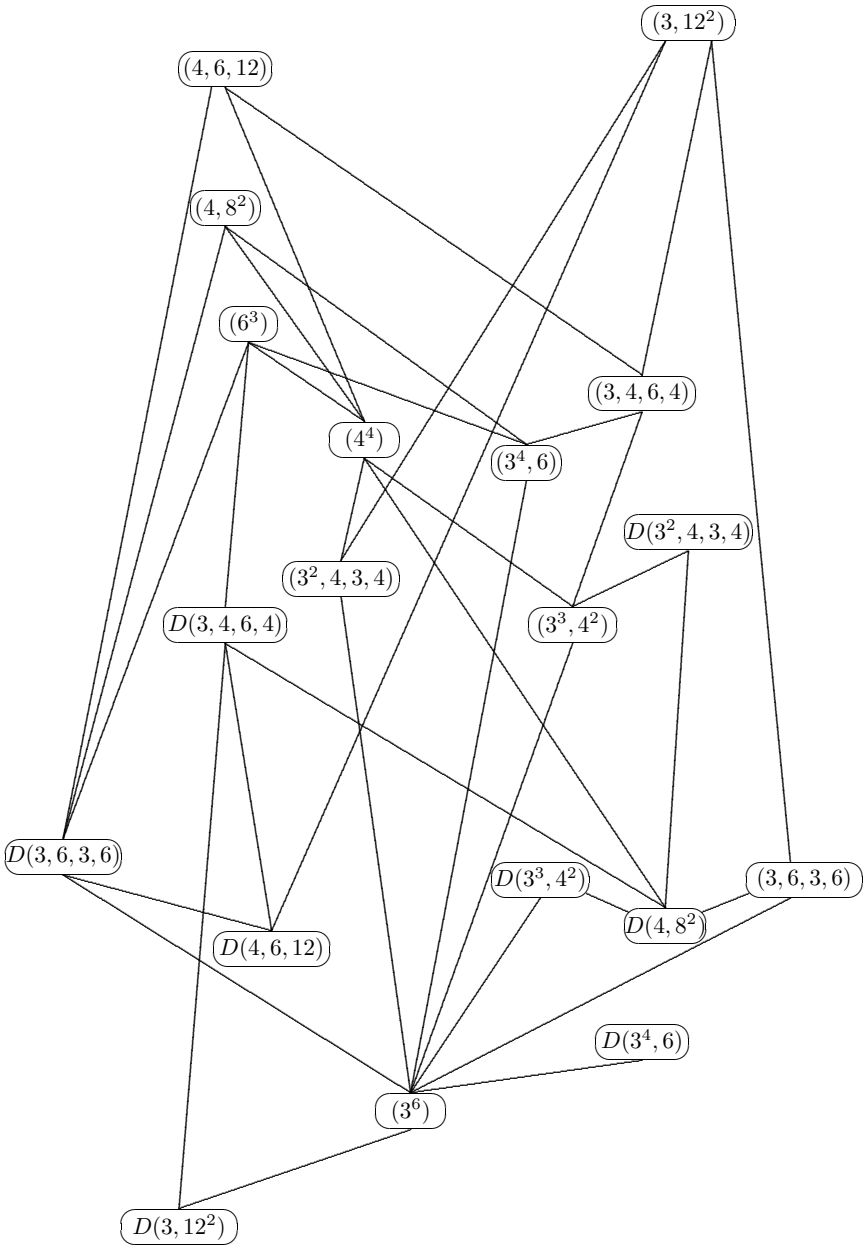
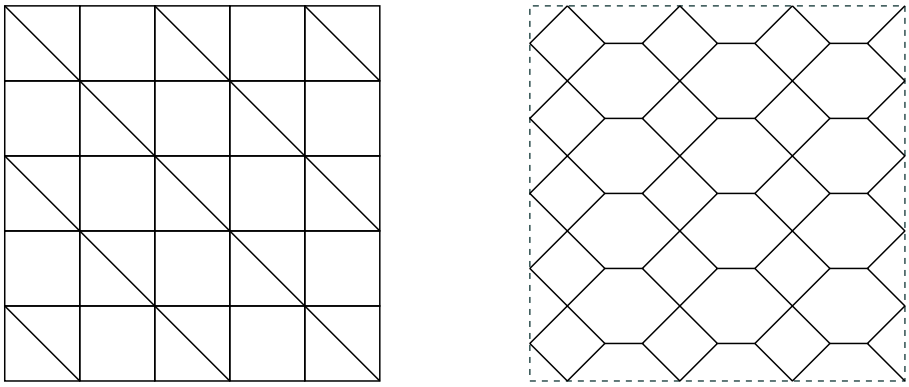


Figure 3.1: The Hasse diagram of the subgraph partial order of the Archimedean and Laves lattices. Edges of the diagram indicate covering relationships, in which the lattice higher in the diagram is a subgraph of the lattice lower in the diagram. Additional subgraph relationships, valid by transitivity, are implied, but not shown.



Bowtie D(Bowtie)

Figure 3.2: The Bowtie lattice and its dual.

The same method may be used to improve the lower bounds for other lattices as well, but we have not pursued that.

3.4 Paper IV

In Paper IV we find pairs of planar graphs with critical probabilities and connective constants in the same order, both for bond and site percolation.

The site percolation case is in fact almost trivial if multiple edges are allowed. For a suitable graph, replace each edge with k multiple edges. This does not change the site percolation critical probability, but increases the connective constant. Hence, we start with a pair of graphs G and H with $p_c^s(G) > p_c^s(H)$ and $\mu(G) < \mu(H)$. Replacing each edge in G with k multiple edges, to get G', we get $p_c^s(G') = p_c^s(G) > p_c(H)$ and $\mu(G') > \mu(H)$, if k is large enough.

Only minor modifications are necessary if multiple edges are not allowed. Replace each edge with k disjoint paths of length 2. This modification decreases the site percolation critical probability, but the graph will always be a subgraph of a triangulated graph, and thus have site percolation critical probability above $1/2$. It is clear that these methods give an infinite number of counterexamples.

To find a bond percolation example with planar graphs is a little harder (non-planar examples can be found quite easily by replacing vertices by complete graphs of some order). The example in the paper uses the Bowtie lattice and a subdivided triangular lattice (the triangular lattice with each edge replaced by two in series). For both of these, the bond percolation critical probabilities are known, and combined with bounds given in [3] for the connective constants, we draw the desired conclusion.

3.5 Paper V

For site percolation, the counterpart of dual graphs are matching graphs. The matching graph $M(G)$ of a planar graph G is the graph obtained by adding every diagonal edge in every face of the graph, that is, every face with k vertices is substituted for a complete graph with k vertices. Kesten, [16], showed that for a broad class of graphs,

$$p_c^s(G) + p_c^s(M(G)) = 1.$$

Consequently, since triangulated graphs are self-matching, the site percolation critical probability for triangulated graphs equals $1/2$.

In Paper V we give the (almost) complete partial order induced by the subgraph relation, for the matching graphs of the Archimedean and Laves lattices. That this partial order should extend the information given by the subgraph partial order for the graphs themselves, is not a priori clear. It might conceivably be the case that for the Archimedean and Laves lattices, $M(G) \subset M(H)$ implies $H \subset G$. However, examples show that neither the subgraph partial order, nor the matching subgraph partial order is a suborder of the other (of course one of the orders have to be reversed).

The presence of complete subgraphs of high order makes the analysis much more complicated than for the original graphs. For example, several of the non-inclusion criteria and many of the arguments for the special cases in Paper I are based on planarity and face structures.

Nevertheless, the results in Paper I are still useful in some aspects. In some cases it is true that $M(G) \subset M(H)$ implies $H \subset G$, so if $H \not\subset G$, it follows that $M(G) \not\subset M(H)$. Also, some criteria based on face structures have analogues based on complete subgraph structures.

Still, there are 36 cases that are neither inclusions nor handled by a general criterion, as compared to 21 cases for the original graphs (3 of the cases are common). The complexity of these special cases ranges from very easy to very hard.

Unfortunately, one case remains unsolved. We do not know at the time if $M(3^3, 4^2)$ is a subgraph of $M(4, 6, 12)$ or not. The feeling is that it could go either way, with a small bias towards a negative answer.

Hasse diagrams of the two possible partial orders are shown in Figure 3.3.

3.6 Paper VI

With the absence of exact values for the critical probability for most lattices, accurate estimates are a complement to the (sometimes crude) bounds.

The site percolation critical probabilities for the Archimedean lattices have previously been estimated with high precision, but for many of these lattices, no accurate estimates of the bond percolation critical probabilities were available.

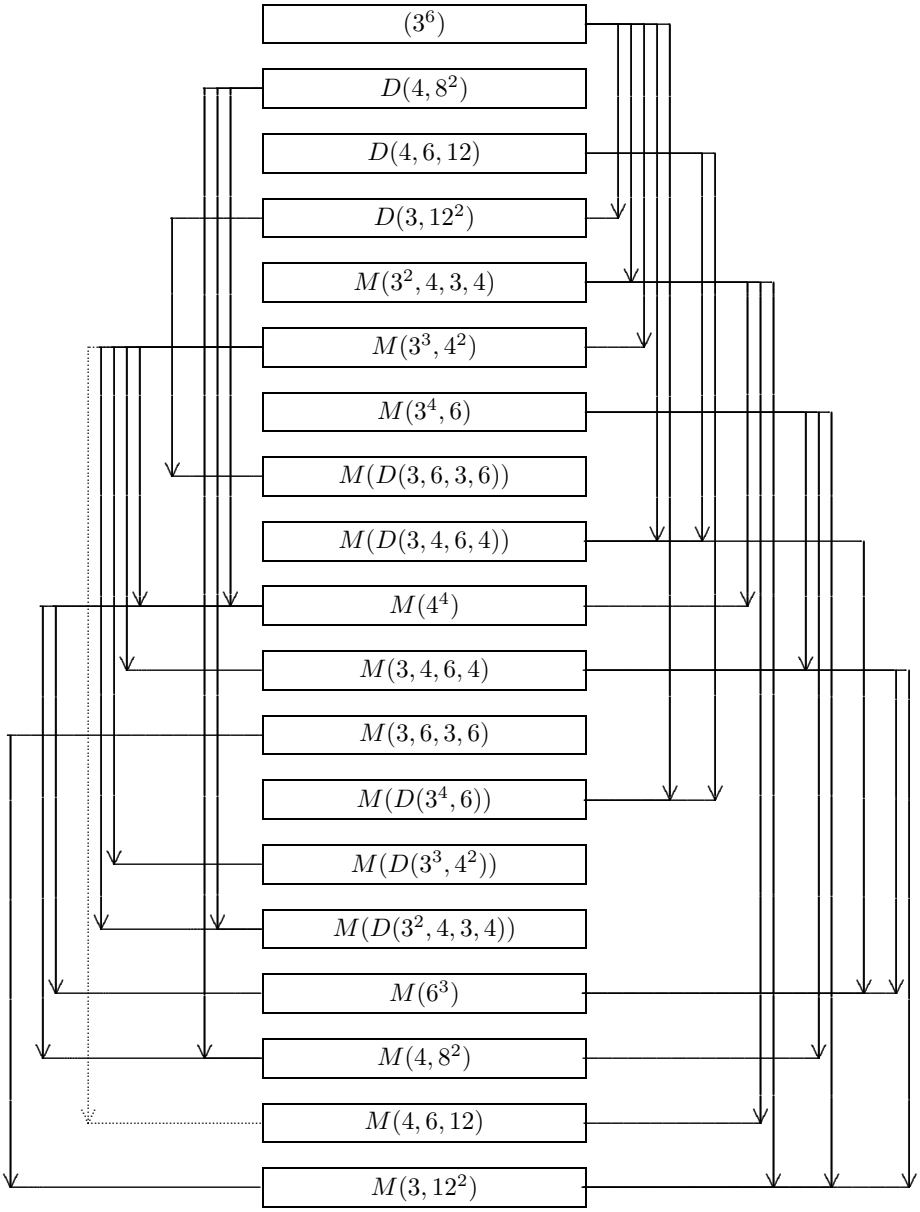


Figure 3.3: The Hasse diagram of the two possible subgraph partial orders of the matching Archimedean and Laves lattices. An arrow indicates that the lattice higher in the diagram is a subgraph of the lattice lower in the diagram. The dotted arrow represents the undecided inclusion.

3.6.1 The hull gradient method

The hull gradient method was derived for site percolation, [22, 30, 31], but it is well know that every bond percolation problem corresponds to an equivalent site percolation problem, via the line graph transformation. Each bond in the original graph is a vertex in the line graph, and the adjacency of edges in the original graph gives the adjacency of vertices in the line graph.

In the hull gradient method, percolation in a 1-dimensional, say horizontal, strip is considered. The probability of an open vertex depends linearly on the height, so at the top of the strip all vertices are open, and at the bottom, all vertices are closed. Thus there will be an infinite open cluster at the top, and an infinite closed cluster at the bottom. There exists a unique curve, called the interface, separating the two clusters. The hull gradient method is a method for generating the interface, via a random walk. Statistics of the interface then give information about the critical probability – heuristically, the mean height of the interface should correspond to the critical probability.

3.6.2 The method of Newman and Ziff

Newman and Ziff, [19], have developed a fast algorithm for estimating percolation thresholds. To show its true efficiency, we start by looking at an early estimation method.

A traditional method for estimating a percolation threshold p_c is the following. Consider a finite subgraph L of the lattice for which we wish to estimate p_c. Generate a large number of realizations of the percolation process on L, for a fixed value of p, and estimate, for example, the probability $Q(p)$ that a cluster spans L in the horizontal direction. If $p > p_c$ this probability should be large, if $p < p_c$ it should be small. Repeat for a large number of different values of p.

The inefficiency should be apparent. To estimate $Q(p)$ accurately for a single value of p we need a large number of realizations. In the example above, the standard error of the estimate of $Q(p)$ is of order $1/\sqrt{n}$, for n realizations. We then have to repeat for large enough number of values of p, to be able to get an accurate estimate of, in this case, the point where the derivate of $Q(p)$ has a maximum. Further, we should repeat the whole thing for different sizes of the sublattice, to be able to correct for finite size bias.

As the hull gradient method, Newman and Ziff's method removes the need to do simulations for fixed values of p. Combined with a clever way of detecting clusters wrapping around the lattice, it results in an impressively efficient method for estimating p_c (the method is however not restricted to estimating p_c; many other interesting statistics may also be estimated): the running time is in principle *linear* in the system size (the number of vertices or edges of the subgraph), while still producing accurate estimates.

The method will now be described in more detail for the case of bond percolation.

It turns out that it is profitable to modify the subgraphs, and use regular

boundary conditions to get torus shaped regions. We will consider $Q(p)$, the probability that a cluster wraps around the torus in one direction, but not both.

The key observation is the following. If we generate a percolation process on the subgraph L, with S edges, by adding edges in random order, we have $S+1$ different, but dependent, realizations of a percolation process, with unknown p, which gives (very crude) estimates of $Q_n(p)$, the probability that a cluster spans the torus in one direction but not the other, given that there are exactly n open edges.

Assume for a moment that the probabilities $Q_n(p)$ are exactly known. Then, by the law of total probability, we have

$$Q(p) = \sum_n Q_n(p) P(N = n),$$

where N is a random variable with the Binomial distribution with parameters S and p. Thus, from a single realization, we get an estimate of $Q(p)$ for *all* values of p, from 0 to 1.

In our case, p_c can be estimated by the value of p at which $Q(p)$ is maximized, since the probability $Q(p)$ tends to zero both for p above and below p_c, as the system size S grows.

For the method to achieve its impressive running time, it is necessary to keep track of clusters efficiently. This can be done by using a tree based union/find algorithm. When an edge is added, we *find* the clusters to which the endpoints belong. If the clusters are different, the *union* of the clusters is calculated. Both steps are rapidly done by representing the set of clusters as a directed forest. By a small modification of the union/find algorithm, detection of cluster wrapping is also easy.

The paper uses Newman and Ziff's method, with the number of realizations chosen such that the standard error of each estimate is approximately 5×10^{-7}, and with system sizes such that the finite size error is small compared to the standard error. This required almost a year of CPU time on a standard desktop computer, for each graph. Of course, many computers[1] were used simultaneously.

The simulations are complemented with short runs with the hull gradient method, to get independent checks on the estimates. The numerical results are summarized in Section 4.2.

We also study the algorithm's performance on the hexagonal lattice (for which the bond percolation critical probability is known). In particular, we find that a scaling law for the estimates standard deviation, proposed by Newman and Ziff, [19], does not hold.

[1] Clock frequencies between 1 and 1.6 GHz.

Chapter 4

Summary

In this section we summarize what is known about the percolation thresholds and connective constants for the Archimedean and Laves lattices. For percolation, most results are from the thesis, while most results for connective constants are from Alm, [3]. See the relevant subsections for more precise references.

4.1 Connective constants

Table 4.1 and Figure 4.1 summarize the current state of knowledge about the values of the connective constants. In the table, numerical values for bounds, exact values, and estimates are given. The table also gives values for the mean degrees discussed in Section 2.5.

All bounds are from Alm, [3], except for (4^4), [7] (lower bound), [21] (upper bound), and for (6^3), which are from Paper III. In the column for estimates, the conjectured values for the $(3, 12^2)$ and (6^3) lattices, [20, 12, Paper III] are given. The estimates for $(3^6), (4^4), (3, 6, 3, 6)$ and $(4, 8^2)$ are from [12], all other estimates are given in [3].

The lattices are ordered in the table by the mean degrees, by the following rule. If, for some mean degree $d \in \{\tilde{d}, \bar{d}_a, \check{d}_g, \bar{d}_h\}$, $d(G) < d(H)$, or if $d(D(G)) > d(D(H))$, then G is above H. It turns out that this rule gives a total order of the 19 lattices.

Note that this also gives the correct order with respect to the estimated connective constants, with the exception of $D(3, 4, 6, 4)$ and $D(3, 6, 3, 6)$. However, these estimates are quite unreliable.

The figure shows a Hasse diagram of the partial order induced by our information on the connective constants. An edge in the diagram indicates that the lattice higher in the diagram has a smaller connective constant than the lattice lower in the diagram.

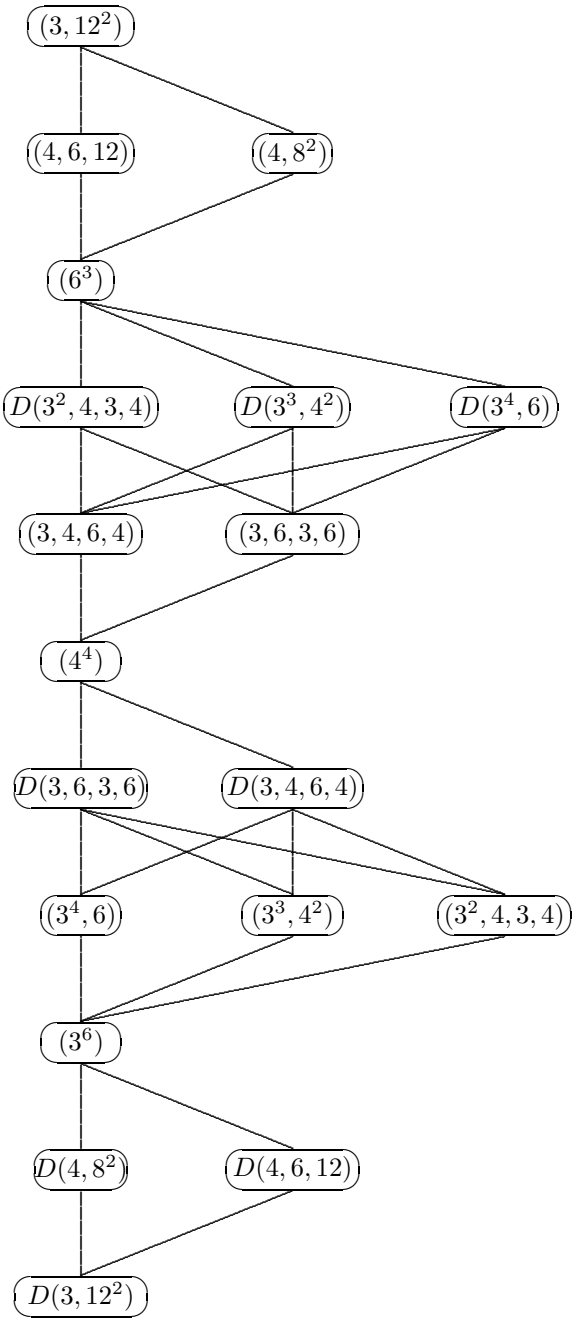


Figure 4.1: A diagram describing the current knowledge of the connective constant order of the Archimedean and Laves lattices.

Lattice	Lower bound	Upper bound	Estimate	$\tilde{d}$	$\bar{d}_a$	$\bar{d}_g$	$\bar{d}_h$
$(3, 12^2)$	1.70526	1.71926	1.71104*	3	3	3	3
$(4, 6, 12)$	1.76376	1.80907	1.787	3	3	3	3
$(4, 8^2)$	1.78564	1.82926	1.809	3	3	3	3
(6^3)	1.83300	1.86884	1.84776*	3	3	3	3
$D(3^2, 4, 3, 4)$	2.09257	2.16832	2.132	3.372	3.4	3.366	10/3
$D(3^3, 4^2)$	2.11289	2.18672	2.152	3.414	3.4	3.366	10/3
$D(3^4, 6)$	2.15481	2.23507	2.193	3.541	3.6	3.446	10/3
$(3, 6, 3, 6)$	2.50967	2.60493	2.561	4	4	4	4
$(3, 4, 6, 4)$	2.51125	2.61084	2.564	4	4	4	4
(4^4)	2.62006	2.67920	2.638	4	4	4	4
$D(3, 4, 6, 4)$	2.69342	2.82818	2.763	4.243	4.25	4.120	4
$D(3, 6, 3, 6)$	2.70423	2.81774	2.761	4.243	4.5	4.243	4
$(3^4, 6)$	3.20640	3.36912	3.293	5	5	5	5
$(3^3, 4^2)$	3.26640	3.42537	3.350	5	5	5	5
$(3^2, 4, 3, 4)$	3.28528	3.45144	3.374	5	5	5	5
(3^6)	4.03333	4.25142	4.151	6	6	6	6
$D(4, 8^2)$	4.30471	4.56537	4.442	6.472	6.667	6.350	6
$D(4, 6, 12)$	4.46305	4.78723	4.624	6.823	7.333	6.604	6
$D(3, 12^2)$	5.37715	5.79621	5.595	8.196	9	7.560	6

Table 4.1: Connective constant results. The entries with * are conjectured values.

4.2 Bond percolation thresholds

Table 4.2 and Figure 4.2 give the current state of knowledge about the values of the bond percolation critical probabilities. The lattices are given in the same order as in the table for connective constants.

There are four cases where the estimates disobeys the order, the estimated critical probability for $(3, 4, 6, 4)$ is higher than the estimate for $(3, 6, 3, 6)$, and the estimate for $(3^3, 4^2)$ is higher than the estimate for $(3^2, 4, 3, 4)$. The inverse is true for the dual lattices. Here, the estimates are very accurate (the standard errors are approximately 5×10^{-7}), so we believe the estimated order to hold for the true values as well.

In the Hasse diagram (Figure 4.2), the Bowtie lattice (B), and its dual ($D(B)$) are also included, as this gives fewer edges.

The bounds are given in Paper II. Many of these are previous results by Wierman, see the paper for exact references.

The exact values were derived by Kesten, [15], for the square lattice, and by Wierman, [24, 25] for the hexagonal and Bowtie dual pairs.

All estimates are from Paper VI, except the estimate for $(3, 6, 3, 6)$ (and its

dual), which is from Ziff and Suding, [31].

Lattice	Lower bound	Upper bound	Estimate	$\tilde{d}$	$\bar{d}_a$	$\bar{d}_g$	$\bar{d}_h$
$(3, 12^2)$	0.7393	0.7418	0.7404220	3	3	3	3
$(4, 6, 12)$	0.6430	0.7376	0.6937338	3	3	3	3
$(4, 8^2)$	0.6281	0.7201	0.6768023	3	3	3	3
(6^3)	0.6527036			3	3	3	3
$D(3^2, 4, 3, 4)$	0.5	0.6528	0.5803581	3.372	3.4	3.366	10/3
$D(3^3, 4^2)$	0.5	0.6528	0.5858626	3.414	3.4	3.366	10/3
$D(3^4, 6)$	0.4474	0.6528	0.5656938	3.541	3.6	3.446	10/3
$(3, 6, 3, 6)$	0.5209	0.5291	0.5244053	4	4	4	4
$(3, 4, 6, 4)$	0.4106	0.5955	0.5428326	4	4	4	4
(4^4)	0.5			4	4	4	4
$D(3, 4, 6, 4)$	0.4045	0.5894	0.4571674	4.243	4.25	4.120	4
$D(3, 6, 3, 6)$	0.4709	0.4791	0.4755947	4.243	4.5	4.243	4
$(3^4, 6)$	0.3472	0.5526	0.4343062	5	5	5	5
$(3^3, 4^2)$	0.3472	0.5	0.4141374	5	5	5	5
$(3^2, 4, 3, 4)$	0.3472	0.5	0.4196419	5	5	5	5
(3^6)	0.3472964			6	6	6	6
$D(4, 8^2)$	0.2799	0.3719	0.3231977	6.472	6.667	6.350	6
$D(4, 6, 12)$	0.2624	0.3570	0.3062662	6.823	7.333	6.604	6
$D(3, 12^2)$	0.2582	0.2607	0.2595780	8.196	9	7.560	6

Table 4.2: Bond percolation results.

4.3 Site percolation thresholds

Table 4.3 and Figure 4.5 give the current state of knowledge about the values of the site percolation critical probabilities. The lattices are again ordered in the table by the mean degrees.

The estimates for the Archimedean lattices are from Suding and Ziff, [22], and the estimates for the Laves lattices are given in Section 4.3.2. The exact values for triangulated graphs were proved by Kesten, [16]. The exact values for $(3, 6, 3, 6)$ and $(3, 12^2)$ easily follow from the bond percolation threshold on (6^3), by the line graph transformation. References for the bounds are given in the table. All other bounds are given in Section 4.3.1.

4.3.1 Bounds for the site percolation thresholds

As all Archimedean and Laves lattices are subgraphs of some of the 4 triangulated lattices, we have the lower bound 1/2 for all lattices. Unless otherwise

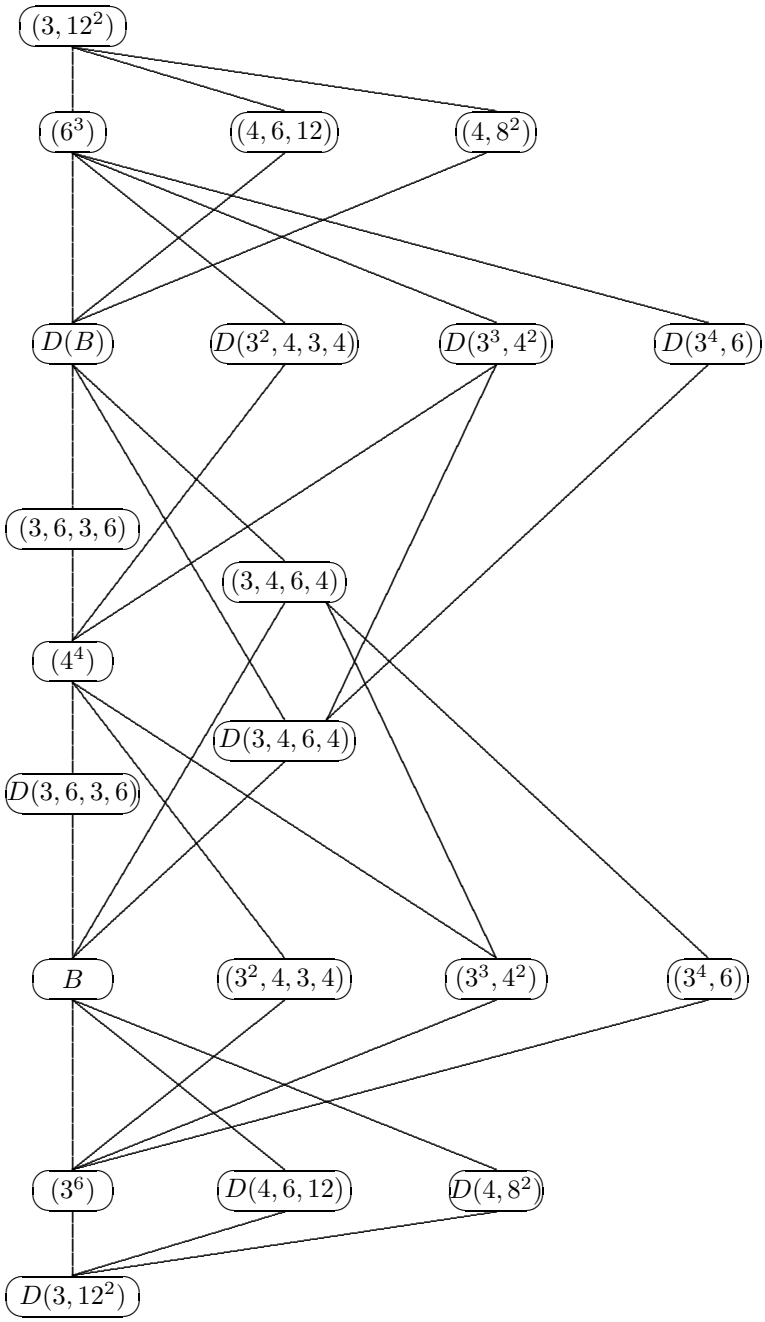


Figure 4.2: A diagram describing the current knowledge of the bond percolation threshold order of the Archimedean and Laves lattices.

stated below, this is the best known lower bound.

(6^3)

The lower bound follows from the general relation $p_c^s(G) \geq p_c^b(G)$.

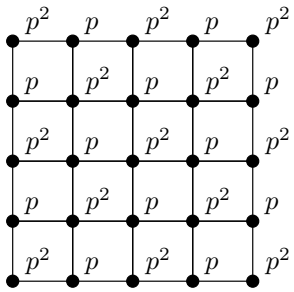


Figure 4.3: The lattice L_1.

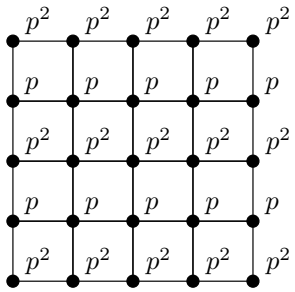


Figure 4.4: The lattice L_2.

$D(3^2, 4, 3, 4)$

The upper bound follows from a grouping argument, as in [17]. Group pairs of adjacent vertices with degree three. Consider the groups as new vertices, which are open if and only if both original vertices are open.

This gives a lattice L_1 with higher threshold. The lattice L_1 is the square lattice, in which every other vertex is open with probability p^2, the others open with probability p, see Figure 4.3. The threshold for L_1 is less than the threshold for (4^4), when all vertices are open with probability p^2. Thus, $p_c(D(3^2, 4, 3, 4)) \leq \sqrt{p_c(4^4)}$.

$D(3^3, 4^2)$

The upper bound is analogous to the previous case. In Figure 4.4 the grouped lattice L_2 is shown.

$D(3^4, 6)$

The upper bound follows from the relation $p_c^s(G) \leq 1 - (1 - p_c^b(G))^{\Delta - 1}$, where Δ is the maximum degree of G.

$(3, 4, 6, 4)$

The linegraph of the dice lattice $D(3, 6, 3, 6)$ is a subgraph of the matching graph of $(3, 4, 6, 4)$. Therefore

$$p_c^s(3, 4, 6, 4) = 1 - p_c^s(M(3, 4, 6, 4)) \geq 1 - p_c^b(D(3, 6, 3, 6)) = p_c^b(3, 6, 3, 6).$$

The upper bound follows from the subgraph relation $(6^3) \subset (3, 4, 6, 4)$.

$D(3,4,6,4)$, $(3^4,6)$, $(3^3,4^2)$ and $(3^2,4,3,4)$

The bounds follow from subgraph relationships. Both $(3^3,4^2)$ and $(3^2,4,3,4)$ are supergraphs to (4^4), while $(3^4,6)$ is a supergraph to $(3,4,6,4)$, and $D(3,4,6,4)$ is a supergraph to (6^3).

4.3.2 Estimates for the site percolation thresholds

For 5 of the Laves lattices, no estimates of the site percolation threshold were available. The same algorithm used in Paper VI, and described in Section 3.6.2, was used to estimate the thresholds. The simulations were run until the standard error of the estimate was approximately 5×10^{-6}, which took a couple of days of CPU-time per lattice (if run on a single computer). The finite size biases are roughly one order of magnitude smaller than the standard errors.

Lattice	Lower bound	Upper bound	Estimate	$\tilde{d}$	$\bar{d}_a$	$\bar{d}_g$	$\bar{d}_h$
$(3,12^2)$	0.807904			3	3	3	3
$(4,6,12)$	0.72173[1]	0.81898[1]	0.747806	3	3	3	3
$(4,8^2)$	0.70710[1]	0.79997[1]	0.729724	3	3	3	3
(6^3)	0.6527	0.79472[2]	0.697043	3	3	3	3
$D(3^2,4,3,4)$	0.5	0.824314	0.650184	3.372	3.4	3.366	10/3
$D(3^3,4^2)$	0.5	0.824314	0.647084	3.414	3.4	3.366	10/3
$D(3^4,6)$	0.5	0.994948	0.639447	3.541	3.6	3.446	10/3
$(3,6,3,6)$	0.652704			4	4	4	4
$(3,4,6,4)$	0.5209	0.79472	0.621819	4	4	4	4
(4^4)	0.556[3]	0.679472[4]	0.592746	4	4	4	4
$D(3,4,6,4)$	0.5	0.79472	0.582410	4.243	4.25	4.120	4
$D(3,6,3,6)$	0.5020	0.7937[5]	0.585040	4.243	4.5	4.243	4
$(3^4,6)$	0.5	0.79472	0.579498	5	5	5	5
$(3^3,4^2)$	0.5	0.679472	0.550213	5	5	5	5
$(3^2,4,3,4)$	0.5	0.679472	0.550806	5	5	5	5
(3^6)	0.5			5	5	5	5
$D(4,8^2)$	0.5			6.472	6.667	6.350	6
$D(4,6,12)$	0.5			6.823	7.333	6.604	6
$D(3,12^2)$	0.5			8.196	9	7.560	6

Table 4.3: Site percolation results. [1] Wierman, [27], [2] Wierman, [28], [3] van den Berg and Ermakov, [23], [4] Wierman, [26], [5] Łuczak and Wierman, [17].

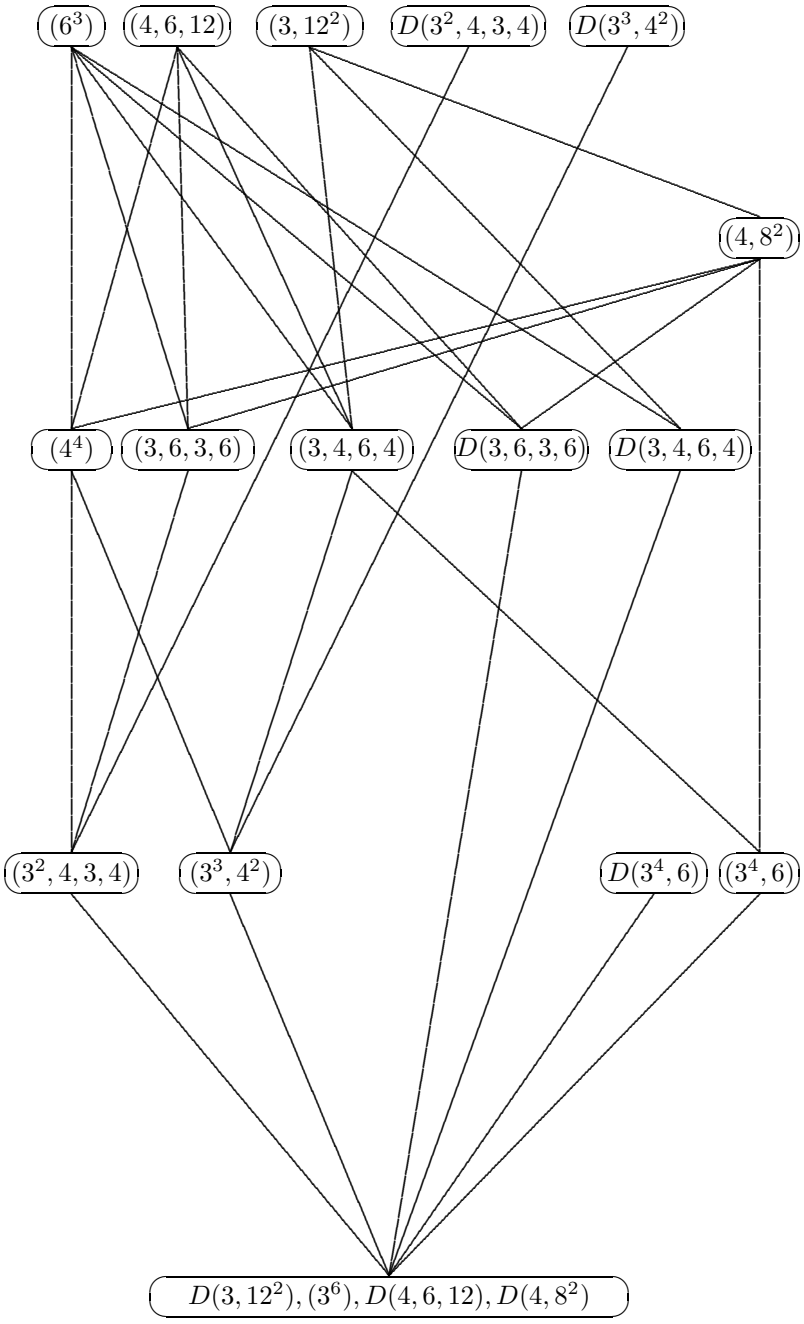


Figure 4.5: A diagram describing the current knowledge of the site percolation threshold order of the Archimedean and Laves lattices.

4.4 Comparison of the measures

Write $G \overset{d}{<} H$, if, for some mean degree $d \in \{\tilde{d}, \bar{d}_a, \bar{d}_g, \bar{d}_h\}$, $d(G) < d(H)$, or if $d(D(G)) > d(D(H))$, and call the resulting order the mean degree order. Recall that this is a total order on the set of Archimedean and Laves lattices.

Also consider the three orders defined by connective constants and bond and site percolation thresholds, for which $G \leq H$ if, respectively, $\mu(G) \leq \mu(H)$, $p_c^b(G) \geq p_c^b(H)$ and $p_c^s(G) \geq p_c^s(H)$.

The estimates and exact values indicates that no two of the four orders agree, although the numerical evidence for the case of connective constants versus mean degree order is weak. The cases for which the four orders do not agree are summarized in the first four of the following conjectures.

Conjecture 4.1. *For* $(3^3, 4^2)$ *and* $(3^2, 4, 3, 4)$*, for which* $(3^3, 4^2) \overset{d}{<} (3^2, 4, 3, 4)$*, it holds that*

$$\begin{aligned} \mu(3^3, 4^2) &< \mu(3^2, 4, 3, 4), \\ p_c^b(3^3, 4^2) &< p_c^b(3^2, 4, 3, 4), \\ p_c^s(3^3, 4^2) &< p_c^s(3^2, 4, 3, 4). \end{aligned}$$

Conjecture 4.2. *For* $D(3^2, 4, 3, 4)$ *and* $D(3^3, 4^2)$*, for which* $D(3^2, 4, 3, 4) \overset{d}{<} D(3^3, 4^2)$*, it holds that*

$$\begin{aligned} \mu(D(3^2, 4, 3, 4)) &< \mu(D(3^3, 4^2)), \\ p_c^b(D(3^2, 4, 3, 4)) &< p_c^b(D(3^3, 4^2)), \\ p_c^s(D(3^2, 4, 3, 4)) &> p_c^s(D(3^3, 4^2)). \end{aligned}$$

Conjecture 4.3. *For* $(3, 4, 6, 4)$ *and* $(3, 6, 3, 6)$*, for which* $(3, 6, 3, 6) \overset{d}{<} (3, 4, 6, 4)$*, it holds that*

$$\begin{aligned} \mu(3, 6, 3, 6) &< \mu(3, 4, 6, 4), \\ p_c^b(3, 6, 3, 6) &< p_c^b(3, 4, 6, 4), \\ p_c^s(3, 6, 3, 6) &> p_c^s(3, 4, 6, 4). \end{aligned}$$

Conjecture 4.4. *For* $D(3, 6, 3, 6)$ *and* $D(3, 4, 6, 4)$*, for which* $D(3, 4, 6, 4) \overset{d}{<} D(3, 6, 3, 6)$*, it holds that*

$$\begin{aligned} \mu(D(3, 4, 6, 4)) &> \mu(D(3, 6, 3, 6)), \\ p_c^b(D(3, 4, 6, 4)) &< p_c^b(D(3, 6, 3, 6)), \\ p_c^s(D(3, 4, 6, 4)) &< p_c^s(D(3, 6, 3, 6)). \end{aligned}$$

Conjecture 4.5. *For two Archimedean or Laves lattices G and H, except for the four pairs in Conjectures 4.1 – 4.4, it holds that*

$$G \overset{d}{<} H \iff \mu(G)) < \mu(H) \iff p_c^b(G) > p_c^b(H) \iff p_c^s(G) \geq p_c^s(H),$$

and equality holds in the last inequality if and only if both G and H are triangulated.

4.5 Conclusions

A rigorous answer to question of whether percolation thresholds and connective constants give the same order on the Archimedean and Laves lattices is not known today, although numerical estimates give compelling arguments for a negative answer.

The hope for a rigorous answer is scant — the methods for finding bounds for the critical constants, especially for percolation thresholds, are simply not good enough. Estimates show that there are lattices for which the critical constants are so close, that a radically new method of finding bounds would have to be found. Of course, we could also try to compute the constants exactly, a task for which little if any progress has been made since the mid 80's.

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