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CONSENSUS OF MULTI-AGENT LINEAR DYNAMIC SYSTEMS

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ABSTRACT

In this paper the consensus problem is considered for multi-agent systems, in which all agents have an identical linear dynamic mode that can be of any order. The main result is that if the adjacent topology of the graph is frequently connected then the consensus is achievable via local-information-based decentralized controls, provided that the linear dynamic mode is completely controllable. Consequently, many existing results become particular cases of this general result. In this paper, the case of fixed connected topology is discussed first. Then the case of switching connected topology is considered. Finally, the general case is studied where the graph topology is switching and only connected often enough.

Key Words: Multi-agent systems, consensus, higher order dynamics, decentralized control.

I. INTRODUCTION

In the past few years the study of multi-agent systems has attracted considerable attention from various research communities. It has been revealed that multi-agent systems appear in various areas, such as cooperative control of unmanned air vehicles [1], consensus problem of communication networks [2–4], formation control of mobile robots [5, 6] and flocking of birds [7, 8], etc.

In collective behaviors of multiple agents, consensus is one of the most interesting behaviors. There is already a large amount of literature concerning this, *e.g.*, [2–4, 9–11] and the references therein. An early work is [8], in which Vicsek *et al.* proposed a consensus scheme based on a simple discrete-time model for the headings of n autonomous agents moving in a plane. Then some theoretical explanations for the consensus behavior of the Vicsek model were given in [2] and other works, such as [4, 9]. With the first order model considered in [2] it was shown that under the assumption of "joint connection" of graphs, the headings of all agents converged to a steady-state value. More general cases of linear models and connection topologies were studied in [3, 5, 10, 11] for example. In particular, in [5], a necessary and sufficient condition was given for the solvability of consensus problems based on local information feedback control with fixed connection topology. A survey on consensus problems was given in [12]. A closely related problem that has been studied by many researchers is the synchronization problem of coupled oscillators [13, 14].

Nevertheless, the consensus problem that has been solved so far is mostly only for agents with first or second order dynamics. In this paper we consider a more general case, where the dynamics of each agent can be of any order. A decentralized high gain control law is provided, under which the consensus problem can be solved for both fixed and varying topology cases.

Let us consider a system with N agents. The dynamics of each agent is

$$\dot{x}^i = Ax^i + Bu^i, \quad x^i \in \mathbb{R}^n,\ u^i \in \mathbb{R}^m, \quad i = 1, \ldots, N \tag{1}$$

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where we assume $rank(B)=m$. Let $x^i=(x_1^i, x_2^i, \ldots, x_n^i)^T \in \mathbb{R}^n$ denote the state of agent i.

Define

$$z^i = \sum_{j\in\mathcal{N}_i}(x^i - x^j), \quad i=1,\ldots,N \tag{2}$$

where $\mathcal{N}_i$ denotes the set of neighbors of agent i. (More precise definition can be found in Section II.) z^i is considered as the local information available for agent i.

Definition 1. Consider system (1). The consensus is said to be achieved using local information if there is a local state error feedback control

$$u^i = Kz^i, \quad i=1,\ldots,N \tag{3}$$

such that

$$\lim_{t\to\infty}\|x^i - x^j\| = 0, \quad i,j=1,\ldots,N. \tag{4}$$

Naturally one can refine the problem by considering only *output error feedback control*. In this paper, however, we focus on the case where some full state error is available.

The rest of the paper is organized as follows: Section II provides some necessary preliminaries. Section III considers the consensus for connected varying graph topology. Section IV deals with the frequently connected case and the main result is presented there. In Section V we investigate the Laplacians first, and then give some simulation results. Section VI is the conclusion.

II. PRELIMINARIES

In this section, we recall some basic concepts and results on graph theory, which are often used in coordination problems of multiple agents and related to our later discussion. More details can be found in [15] for example.

An undirected graph $\mathcal{G}$ of order N consists of a vertex set $\mathcal{V}=\{1,2,\ldots,N\}$ and an edge set $\mathcal{E}=\{(i,j) : i,j \in \mathcal{V}\} \subset \mathcal{V}\times\mathcal{V}$. A weighted adjacency matrix $A=[a_{ij}] \in \mathbb{R}^{N\times N}$, where $a_{ii}=0$ and $a_{ij}=a_{ji}\geq 0$. $a_{ij}>0$ if and only if there is an edge between agent i and agent j (*i.e.*, $a_{ij}=a_{ji}>0 \Leftrightarrow (i,j)\in\mathcal{E}$) and the two agents are called adjacent (or they are mutual neighbors). In this paper for an unweighted graph $\mathcal{G}$, A is a $0-1$ matrix. The set of neighbors of vertex i is denoted by $\mathcal{N}_i=\{j\in\mathcal{V} : (i,j)\in\mathcal{E},\ j\neq i\}$. Throughout this paper, we assume the graph is undirected. If there is a path between any two vertices of a graph $\mathcal{G}$, then $\mathcal{G}$ is connected, otherwise disconnected.

Define the Laplacian $L_{\mathcal{G}}$ with respect to the graph $\mathcal{G}$ as

$$L_{\mathcal{G}}=[l_{ij}]_{N\times N}, \quad \text{where } l_{ij}=\begin{cases}|\mathcal{N}_i|, & i=j,\\ -1, & j\in\mathcal{N}_i,\\ 0, & \text{otherwise.}\end{cases} \tag{5}$$

By the definition, every row sum of L is zero.

Notations. Throughout this paper, let $\mathbf{1}_N=(1,1,\ldots,1)^T \in \mathbb{R}^N$ and $e_i=(0,\ldots,1,\ldots,0)^T \in \mathbb{R}^n$. $\|\cdot\|$ denotes the Euclidean norm and $\otimes$ denotes the Kronecker product. x^T represents the transpose of x.

The following lemma [3, 15] shows some basic properties of the Laplacian L.

Lemma 1. ([*3, 15*]) Let L be the Laplacian of an undirected graph $\mathcal{G}$ with N vertices, $\lambda_1\leq\cdots\leq\lambda_N$ be the eigenvalues of L. Then

1. 0 is an eigenvalue of L and $\mathbf{1}_N$ is the associated eigenvector, that is, $L\mathbf{1}_N=0$;
2. If $\mathcal{G}$ is connected, then 0 is the algebraically simple eigenvalue of L and $\lambda_2=\min_{\xi\neq 0,\xi\perp\mathbf{1}_N}\xi^T L\xi/\xi^T\xi>0$, which is called the algebraic connectivity of $\mathcal{G}$;
3. If 0 is the simple eigenvalue of L, then it is an n multiplicity eigenvalue of $L\otimes I_n$ and the corresponding eigenvectors are $\mathbf{1}_N\otimes e_i,\ i=1,\ldots,n$.

Now let us go back to the consensus problem. The following observations are basically from [5] with some trivial modification.

Tentatively, we assume the topology is fixed, then we can drop the subscript $\mathcal{G}$ of $L_{\mathcal{G}}$. Denote by x and z the concatenations of vectors $\{x^1,\ldots,x^N\}$ and $\{z^1,\ldots,z^N\}$, respectively. From (2), we have

$$z=(L\otimes I_n)x. \tag{6}$$

Then the closed-loop system of (1) with control (3) becomes

$$\dot{x}=[I_N\otimes A+(I_N\otimes BK)(L\otimes I_n)]x. \tag{7}$$

Since L is symmetric, there is an orthogonal matrix T such that

$$TLT^T=D=\mathrm{diag}(\lambda_1,\lambda_2,\ldots,\lambda_N)$$

is diagonal, where $\{\lambda_i\}=\sigma(L)$ is the spectrum of L. Now let

$$\tilde{x}=(T\otimes I_n)x, \tag{8}$$

then (7) becomes

$$\dot{\tilde{x}}^i=[A+\lambda_i BK]\tilde{x}^i, \quad i=1,\ldots,N. \tag{9}$$

Note that (9) is a special case of equation (13) of [5]. From (9) it is clear [5] that when the graph is connected the consensus problem would be solvable if there is a K such that (9) is stabilized for $i = 2, \ldots, n$.

III. GRAPH WITH CONNECTED TOPOLOGY

First, we consider the case when the adjacent topology is fixed. We want to find a common K that stabilizes the subsystems in (9) with $i = 2, \ldots, N$. We need

Lemma 2. ([*16*]) Let n be a positive integer and let $P(s)$ be a stable polynomial of degree $n-1$:

$$P(s) = p_0 + p_1 s + \cdots + p_{n-1}s^{n-1}$$

with all $p_i > 0$.

Then there exists an $\alpha > 0$ such that

$$Q(s) = P(s) + p_n s^n$$

is stable if and only if $p_n \in [0, \alpha)$.

Lemma 3. Consider a finite set of linear systems

$$\dot{x}^i = Ax^i + \lambda_i Bu^i, \quad i = 1, \ldots, k \tag{10}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, (A, B) is completely controllable, $\text{rank}(B) = m$, and $\lambda_i > 0$, $i = 1, \ldots, k$. Then there exists a K which simultaneously assign the poles of k systems as negative as possible. Precisely, for any $M > 0$ there exist

$$u^i = Kx^i, \quad i = 1, \ldots, k$$

such that

$$Re\sigma(A + \lambda_i BK) < -M, \quad i = 1, \ldots, k. \tag{11}$$

Proof. Without loss of generality, we can assume the pair (A, B) is in Brunovsky canonical form. We prove it in the following two cases:

Case 1. Assume $m = 1$. Then the characteristic polynomials for $A + \lambda_i BK$ are

$$P^i(s) = s^n - \lambda_i k_{n-1}s^{n-1} - \cdots - \lambda_i k_1 s - \lambda_i k_0 - p_a(s), \quad i = 1, \ldots, k \tag{12}$$

where $p_a(s) = a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$. Let

$$P_{n-1}(s) = d_{n-1}s^{n-1} + \cdots + d_1 s + d_0$$

be any Hurwitz polynomial. Using Lemma 2, there exist $a > 0$ such that when $d_n < a$ $s^n + 1/d_n P_{n-1}(s)$ is also Hurwitz.

Let $\lambda^* = \min_{1 \le i \le k} \lambda_i$, then

$$P_n^i(s) := s^n + \lambda_i \frac{2}{a\lambda^*} P_{n-1}(s)$$

is Hurwitz. Denote the roots of $P_n^i(s)$ by $\{-s_1^i, \ldots, -s_n^i\}$, then $Re(s_j^i) > 0$. Define

$$P_n^i(\mu, s) = s^n + \lambda_i \left(\mu \frac{2}{a\lambda^*} d_{n-1}\right) s^{n-1} + \cdots + \lambda_i \left(\mu^{n-1} \frac{2}{a\lambda^*} d_1\right) s + \lambda_i \left(\mu^n \frac{2}{a\lambda^*} d_0\right), \quad i = 1, \ldots, k. \tag{13}$$

Namely choosing

$$k_j^* = -\mu^{n-j} \frac{2}{a\lambda^*} d_j, \quad j = 0, \ldots, n-1. \tag{14}$$

It is easy to see by, for example, singular perturbation analysis that when μ is sufficiently large the effect of $p_a(s)$ on the roots of $P^i(s)$ is negligible. Thus when μ is sufficiently large and

$$\mu > \frac{M}{\min\{Re(s_j^i) | i = 1, \ldots, k;\ j = 1, \ldots, n\}}$$

inequality (11) is satisfied.

Case 2. We consider the multi-input case. When $m > 1$, the characteristic polynomials for $A + \lambda_i BK$ are

$$\begin{aligned} P^i(s) &= \prod_{j=1}^{m} P_j^i(s) + Q^i(s) \\ &= \prod_{j=1}^{m} (s^{r_j} - \lambda_i k_{r_j-1}^j s^{r_j-1} - \cdots - \lambda_i k_1^j s \\ &\quad - \lambda_i k_0^j) + Q^i(s), \quad i = 1, \ldots, k \end{aligned}$$

where r_j, $j = 1, \ldots, m$ are *controllability indices*, $\sum_{j=1}^m r_j = n$ and $Q^i(s) = p_{n-1}(\lambda_i, K)s^{n-1} + \cdots + p_1(\lambda_i, K)s + p_0(\lambda_i, K)$. Let

$$k_l^{j*} = \mu^{r_j - l} k_l^j, \quad j = 1, \ldots, m,\ l = 0, \ldots, r_j - 1$$

then by Leibniz formula one can easily see that

$$\lim_{\mu \to \infty} \frac{p_l(\lambda_i, K)}{\mu^{n-l}} = 0, \quad l = 0, \ldots, n-1.$$

For each $P_j^i(s)$, repeat the process of Case 1, we can find μ_j large enough, such that the real parts of the roots of $P_j^i(s)$ are as negative as possible. Choose

$$\mu > \max\{\mu_j,\ j = 1, \ldots, m\}.$$

It is easy to see by singular perturbation analysis that when μ is sufficiently large the effect of $Q^i(s)$ on the roots of $\prod_{j=1}^m P_j^i(s)$ is negligible. Thus when μ is sufficiently large, inequality (11) is satisfied. □

Corollary 1. Assume (A, B) is a controllable pair, $\lambda_i > 0$, $i = 1, \ldots, k$. Then for any $\tau > 0$ and $\varepsilon > 0$ there exists a K such that

$$\|e^{(A+\lambda_i BK)\tau}\| < \varepsilon, \quad i = 1, \ldots, k. \tag{15}$$

Proof. Using $\{k^*\}$ as defined in (14), we know that $A + \lambda_i BK$ has eigenvalues as

$$\sigma(A + \lambda_i BK) = \{-\mu s_j^i | j = 1, \ldots, n\},\ i = 1, \ldots k$$

where $Re(-s_j^i) \le -\sigma < 0$. It is well known that

$$\|e^{(A+\lambda_i BK)\tau}\| \le Q(\mu)e^{-\sigma\mu\tau} \tag{16}$$

where $Q(\mu)$ is a polynomial of μ (see for example [17]). Since

$$\lim_{\mu\to\infty} Q(\mu)e^{-\sigma\mu\tau} = 0$$

the result follows. □

Now we are ready to consider the consensus problem. First, if the adjacent topology is connected and fixed, the result is obvious.

Proposition 1. Consider system (1). Assume the adjacent topology is connected and fixed. If (A, B) is controllable, then the consensus is achieved via local state error feedback (3).

Proof. Since the graph is connected, we have

$$0 = \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \le \lambda_N.$$

Using (6), when $z(t) \to 0$, $t \to \infty$, $x \to \mathbf{1}_N \otimes s$, for some $s \in \mathbb{R}^n$. So the consensus is obtained. (A more precise argument can be found in the second part of this section.) Now, by the definition of z, it is clear that the consensus is achieved, if and only if $z(t) \to 0$, $t \to \infty$.

Using (6) and (8), we have

$$(T \otimes I_n)z = (D \otimes I_n)\tilde{x} = \begin{bmatrix} 0 \\ \lambda_2\tilde{x}^2 \\ \vdots \\ \lambda_N\tilde{x}^N \end{bmatrix}. \tag{17}$$

From Lemma 3 it is clear that we can find the feedback law K which simultaneously stabilizes $\tilde{x}^i$, $i = 2, \ldots, N$, and hence $z(t) \to 0$, $t \to \infty$. □

Next, we consider the case when the adjacent topology is time-varying and connected.

When the adjacent graph is switching, we define a switching signal $\sigma(t) : [0, +\infty) \to \{1, 2, \ldots, m\}$ which is a piecewise constant right continuous function. A switching system is said to have a non-vanishing dwell time, if there is a positive time period $\tau^* > 0$, such that the switching moments $0 < t_1 < \cdots < t_k < \cdots$ satisfy $\inf_k(t_{k+1} - t_k) = \tau^*$.

Throughout this paper, we assume that

Assumption 1. Admissible switching signals have a dwell time $\tau^* > 0$.

Let Λ be the set of all possible graphs and $\Lambda_c \subset \Lambda$ the set of connected graphs.

We give the first result for varying topology.

Theorem 1. Consider system (1) with varying topology and Assumption 1 holds. Assume (A, B) is controllable and its adjacent graph is connected, then the consensus can be achieved by local state error feedback (3).

Proof. Note that in this theorem the neighbor set of an agent i, denoted by $\mathcal{N}_i(t)$, is time-varying. So we consider a non-switched duration $[\alpha, \beta)$, and assume its graph is $\mathcal{G}_p$ with the Laplacian L_p, where $p \in \Lambda_c$. Denote by $T_p L_p T_p^T = D_p$, where $D_p = \mathrm{diag}(0, \lambda_2^p, \ldots, \lambda_N^p)$. Using (17), we have

$$z_p(t) = (T_p^{-1} \otimes I_n) \begin{bmatrix} 0 \\ \lambda_2^p\tilde{x}^2(t) \\ \vdots \\ \lambda_N^p\tilde{x}^N(t) \end{bmatrix}$$

$$= (T_p^{-1} \otimes I_n) E_p(t) \begin{bmatrix} 0 \\ \lambda_2^p \tilde{x}^2(\alpha) \\ \vdots \\ \lambda_N^p \tilde{x}^N(\alpha) \end{bmatrix}$$

$$= (T_p^{-1} \otimes I_n) E_p(t) (T_p \otimes I_n) z_p(\alpha),$$

$$t \in [\alpha, \beta) \tag{18}$$

where

$$E_p(t) = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \exp((A + \lambda_2^p BK)(t-\alpha)) & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \exp\left((A + \lambda_N^p BK)(t-\alpha)\right) \end{bmatrix}.$$

Using Corollary 1, we can conclude from (15) the following two facts:

- For any given $\varepsilon > 0$ there exists a K such that

$$\|(T_p^{-1} \otimes I_n) E_p(t) (T_p \otimes I_n)\| < \varepsilon,$$

$$t \geq \alpha + \tau^*, \ \forall\ p \in \Lambda_c. \tag{19}$$

This is due to the fact that the cardinality $|\Lambda_c| < \infty$.

- As long as K is chosen there is a boundary $M(K) < \infty$ for overshoot. That is,

$$\max_{p \in \Lambda_c} \sup_{\alpha \leq t \leq \alpha + \tau^*} \{\|(T_p^{-1} \otimes I_n) E_p(t) (T_p \otimes I_n)\|\}$$

$$< M(K). \tag{20}$$

This is due to the continuity.

Define the synchronization manifold [14]

$$\mathscr{S} := \{x \in \mathbb{R}^{nN} : x^1 = \ldots = x^N\}$$

$$= \{\mathbf{1}_N \otimes s | s \in \mathbb{R}^n\}.$$

Our aim is to show that x will converge to $\mathscr{S}$.

For any $x \in \mathbb{R}^{nN}$, we can decompose it with respect to each $p \in \Lambda$ by

$$x = S + \eta$$

where $S = \mathbf{1}_N \otimes s \in \mathscr{S}$, $\eta \in \mathscr{S}^\perp$, the orthogonal complement of $\mathscr{S}$. Noting that

$$(L_p \otimes I_n) S = (L_p \otimes I_n)(\mathbf{1}_N \otimes s)$$

$$= (L_p \mathbf{1}_N) \otimes (I_n s) = 0$$

we have

$$z_p = (L_p \otimes I_n) x = (L_p \otimes I_n) \eta \tag{21}$$

and the distance of x to $\mathscr{S}$ satisfies

$$d(x, \mathscr{S}) = \|\eta\|.$$

Since the graph $\mathscr{G}_p$ is connected, by Lemma 1, 0 is the algebraically simple eigenvalue of L_p, and is also the eigenvalue of $L_p \otimes I_n$ with multiplicity n. All the other eigenvalues of L_p are positive. The n linearly independent eigenvectors associated with the eigenvalue 0 of $L_p \otimes I_n$ are $\mathbf{1}_N \otimes e_i$, $i = 1, \ldots, n$. Since $\eta \in \mathscr{S}^\perp$, then $\eta \perp (\mathbf{1}_N \otimes e_i)$, $i = 1, \ldots, n$. We have

$$\lambda_m^p \|\eta\| \leq \|(L_p \otimes I_n)\eta\| \leq \lambda_M^p \|\eta\|$$

where λ_m^p and λ_M^p are the second smallest and the largest eigenvalues of L_p, respectively.

Set

$$\lambda_m = \min_{p \in \Lambda_c} \{\lambda_m^p\} > 0, \quad \lambda_M = \max_{p \in \Lambda_c} \{\lambda_M^p\}$$

then we have

$$\lambda_m \|\eta\| \leq \|(L_p \otimes I_n)\eta\| \leq \lambda_M \|\eta\|, \quad \forall\ p \in \Lambda_c. \tag{22}$$

Let $\{t_k\}_{k=1}^\infty$ be the switching moments and assume $t_k \to \infty$ as $k \to \infty$. Assume on $[t_{k-1}, t_k)$, the mode $p_k \in \Lambda$ is active. Choosing

$$\varepsilon = \frac{\lambda_m}{\lambda_M} \delta, \quad \text{where } 0 < \delta < 1$$

equations (18), (19) and Assumption 1 yield that there exists a K such that on each interval $[t_{k-1}, t_k)$

$$\|z_{p_k}(t_k^-)\| \leq \varepsilon \|z_{p_k}(t_{k-1})\|.$$

Using both (21) and (22), we get

$$\begin{aligned}\lambda_m\|\eta(t_k)\| &\le \|(L_{p_k}\otimes I_n)\eta(t_k)\| = \|z_{p_k}(t_k^-)\| \\ &\le \varepsilon\|z_{p_k}(t_{k-1})\| \\ &= \varepsilon\|(L_{p_k}\otimes I_n)\eta(t_{k-1})\| \\ &\le \varepsilon\lambda_M\|\eta(t_{k-1})\|.\end{aligned}$$

It follows that

$$\|\eta(t_k)\|\le\delta\|\eta(t_{k-1})\|, \quad k=1,2,\ldots. \tag{23}$$

Then we have

$$\lim_{k\to\infty}\|\eta(t_k)\|=0.$$

Note that the feedback K is universal. Inequality (20) and the fact that

$$\|\eta(t)\|\le\frac{1}{\lambda_m}\|z(t)\|, \quad \forall t\ge 0$$

imply

$$\|\eta(t)\|\le\frac{M(K)\lambda_M}{\lambda_m}\|\eta(t_{k-1}\|, \quad t_{k-1}<t<t_{k-1}+\tau^*.$$

Hence

$$\lim_{t\to\infty}\|\eta(t)\|=0$$

which means the consensus is achieved. □

IV. GRAPH WITH FREQUENTLY CONNECTED TOPOLOGY

In this section we consider the case when the graph has frequently connected topology.

Definition 2. System (1) is said to have frequently connected topology with time period T, if there exists a $T>0$, for any $t>0$, there exists a $t^*\in[t,t+T)$ such that the graph $\mathcal{G}(t^*)$ is connected.

Through this section we assume

Assumption 2. System (1) has a frequently connected topology with time period T.

Under Assumption 2 we can find an alternating connect-disconnect sequence of time segments. Namely, there is a time sequence $\tau_1<t_1<\tau_2<t_2<\cdots\to\infty$ (refer to Fig. 1) such that

- $t_k-\tau_k\ge\tau^*$ (refer to Assumption 1 for τ^*);
- $\tau^*\le\tau_{k+1}-t_k<T$;

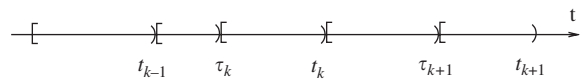

Fig. 1. Frequent connection.

- $\mathcal{G}(t)$ is connected, $\forall\, t\in[\tau_k,t_k)$;
- $\mathcal{G}(t)$ is not connected, $\forall\, t\in[t_{k-1},\tau_k)$.

From the proof of Theorem 1 we have the following:

Lemma 4. Let Assumption 1 hold, then for a given $0<\delta<1$, there exists a set of decentralized controls of the form (3) with an universal K such that

$$\|\eta(t_k)\|\le\delta\|\eta(\tau_k)\|, \quad k=1,2,\ldots. \tag{24}$$

So the problem is to investigate what happens during the time period $[t_{k-1},\tau_k)$ when the graph is not connected. Now consider a non-switching duration $[\alpha,\beta)\subset[t_{k-1},\tau_k)$. Assume $\{\mathcal{G}^\mu(t),\ \mu=1,\ldots,s\}$ are connected components of $\mathcal{G}(t)$, $t\in[\alpha,\beta)$, and the associated vertex sets are $\mathcal{V}^\mu$, $\mu=1,\ldots,s$. Denote the cardinality (size) of the μ-th component by

$$N_\mu=|\mathcal{V}^\mu|;$$

the center of the μ-th component by

$$\bar{x}_\mu=\frac{\sum_{i\in\mathcal{V}^\mu}x^i}{N_\mu}.$$

Similarly, we can define the center of all agents, denoted by $\bar{x}$. Then we have

Lemma 5. Assume $t\in[\alpha,\beta)$, which is a non-switching duration. Then for the closed-loop system with local state error feedback control, the center of each connected component $\mathcal{G}^\mu$ satisfies the following free drift equation:

$$\dot{\bar{x}}_\mu=A\bar{x}_\mu, \quad \mu=1,\ldots,s. \tag{25}$$

Proof. Since for each connected component we have

$$\sum_{i\in\mathcal{V}^\mu}z^i=\sum_{i\in\mathcal{V}^\mu}\sum_{j\in\mathcal{N}_i}(x^i-x^j)=0$$

the conclusion follows immediately. □

Particularly, we have

Corollary 2. The overall center $\bar{x}$ satisfies (25) for all $t \geq 0$.

Now let $\mu_i \in \mathscr{V}^\mu$, $i = 1, \ldots, N_\mu$, where $N_\mu = |\mathscr{V}^\mu|$. Denote $x^\mu = ((x^{\mu_1})^T, \ldots, (x^{\mu_{N_\mu}})^T)^T \in \mathbb{R}^{nN_\mu}$. Similar to the argument in Section III, we split

$$x^\mu = S^\mu + \eta^\mu,$$

where $S^\mu \in \mathscr{S}^\mu$, $\eta^\mu \in [\mathscr{S}^\mu]^\perp$, and $\mathscr{S}^\mu$ is defined as

$$\mathscr{S}^\mu := \{x^{\mu_i} = x^{\mu_j} | \forall \mu_i, \mu_j \in \mathscr{V}^\mu\}.$$

The following lemma gives a precise expression of η^μ.

Lemma 6.

$$\eta^\mu = x^\mu - \mathbf{1}_{N_\mu} \otimes \bar{x}_\mu \tag{26}$$

Proof. First of all, it is easy to see that

$$\mathscr{S}^\mu = \{\mathbf{1}_{N_\mu} \otimes \xi | \xi \in \mathbb{R}^n\}.$$

Then we have

$$\mathbf{1}_{N_\mu} \otimes \bar{x}_\mu \in \mathscr{S}^\mu.$$

Now a straightforward computation shows that

$$\langle x^\mu - \mathbf{1}_{N_\mu} \otimes \bar{x}_\mu, \mathbf{1}_{N_\mu} \otimes \xi \rangle = 0, \quad \forall \xi \in \mathbb{R}^n.$$

The conclusion follows. □

The same argument shows that

$$\eta = x - \mathbf{1}_N \otimes \bar{x}. \tag{27}$$

Let $\alpha \leq t < \beta$. Then we have

$$\begin{aligned} \|x^{\mu_i}(t) - \bar{x}^\mu(t)\| &\leq \|x^\mu(t) - \mathbf{1}_{N_\mu} \otimes \bar{x}^\mu(t)\| \\ &= \|\eta^\mu(t)\|, \quad i = 1, \ldots, N_\mu. \end{aligned} \tag{28}$$

Similarly,

$$\|x^i(t) - \bar{x}(t)\| \leq \|\eta(t)\|. \tag{29}$$

It follows from (29) that

$$\|\bar{x}^\mu(t) - \bar{x}(t)\| \leq \|\eta(t)\|, \quad \forall \mu. \tag{30}$$

Now assume there are two agents, belonging to two different connected components, say, $\mu_i \in \mathscr{V}^\mu$ and $\mu'_j \in \mathscr{V}^{\mu'}$. We are ready to see how much they can diverge.

$$\begin{aligned} \|x^{\mu_i}(t) - x^{\mu'_j}(t)\| &\leq \|x^{\mu_i}(t) - \bar{x}^\mu(t)\| \\ &\quad + \|x^{\mu'_j}(t) - \bar{x}^{\mu'}(t)\| \\ &\quad + \|\bar{x}^\mu(t) - \bar{x}^{\mu'}(t)\| \\ &\leq \|\eta^\mu(t)\| + \|\eta^{\mu'}(t)\| \\ &\quad + \|\bar{x}^\mu(t) - \bar{x}^{\mu'}(t)\|. \end{aligned} \tag{31}$$

From the proof of Theorem 1 and by using the arguments to each connected component, one sees that as long as $t - \alpha \geq \tau^*$ we have

$$\|\eta^\mu(t)\| \leq \|\eta^\mu(\alpha)\|, \quad t \geq \alpha + \tau^*, \quad \forall \mu. \tag{32}$$

Moreover, note that $\mathscr{S}$ is a subspace of $\mathscr{S}^\mu$, and the distance to the whole space is always smaller than or equal to the distance to any of its subspaces. Hence,

$$\|\eta^\mu(t)\| \leq \|\eta(t)\|. \tag{33}$$

We conclude that

$$\|\eta^\mu(t)\| + \|\eta^{\mu'}(t)\| \leq 2\|\eta(\alpha)\|, \quad t \geq \alpha + \tau^*. \tag{34}$$

Using Lemma 5 and equations (29) and (30), we also have

$$\begin{aligned} \|\bar{x}^\mu(t) - \bar{x}^{\mu'}(t)\| &\leq \|e^{A(t-\alpha)}\| \|\bar{x}^\mu(\alpha) - \bar{x}^{\mu'}(\alpha)\| \\ &\leq \|e^{A(t-\alpha)}\| [\|\bar{x}^\mu(\alpha) - \bar{x}(\alpha)\| \\ &\quad + \|\bar{x}^{\mu'}(\alpha) - \bar{x}(\alpha)\|] \\ &\leq 2\|e^{A(t-\alpha)}\| \|\eta(\alpha)\|. \end{aligned} \tag{35}$$

Plugging (34) and (35) into (31), we have

$$\begin{aligned} &\|x^{\mu_i}(t) - x^{\mu'_j}(t)\| \leq 2[1 + \|e^{A(t-\alpha)}\|] \|\eta(\alpha)\|, \\ &\alpha + \tau^* \leq t \leq \beta. \end{aligned} \tag{36}$$

Using (27), we have

$$\|\eta(t)\| \leq \sum_{i=1}^N \|x^i(t) - \bar{x}(t)\|. \tag{37}$$

Note that if $\|x^i - x^j\| \leq \varepsilon$, $\forall i, j$, then

$$\|x^i - \bar{x}\| \leq \varepsilon.$$

Using this fact and equations (36) and (37), we have

$$\|\eta(t)\| \le 2N[1 + \|e^{A(t-\alpha)}\|]\|\eta(\alpha)\|,$$
$$t \ge \alpha + \tau^*. \tag{38}$$

Note that the estimation (38) is independent of the choice of the control, which assures that we can design control to drive the system to the consensus.

Now we are ready to present our main result:

Theorem 2. Assume Assumption 1 and Assumption 2 hold. Then the consensus of system (1) can be achieved by using local state error feedback control (3) with suitably chosen coefficients K.

Proof. Recall that the system has frequently connected topology with time period T. Since $T \ge \tau^*$, there exists an unique $n_0 \in \mathbb{Z}_+$ such that

$$(n_0 - 1)\tau^* < T \le n_0\tau^*.$$

Now in the duration $[t_{k-1}, \tau_k)$ there are at most n_0 times switching. Of course the dwell time for each mode is less than or equal to T. Using (38) we have

$$\|\eta(\tau_k)\| \le [2N(1 + e^{\|AT\|})]^{n_0}\|\eta(t_{k-1})\|. \tag{39}$$

Recall Lemma 4. We can choose a suitable set of coefficients, K, in the feedback control, such that the δ in (24) satisfies

$$\delta \le \frac{\delta_0}{[2N(1 + e^{\|AT\|})]^{n_0}}$$

where $0 < \delta_0 < 1$. It follows that

$$\|\eta(t_k)\| \le \delta_0\|\eta(t_{k-1})\|, \quad k = 1, 2, \ldots.$$

The similar argument for the overshoots between $\{t_k\}$ as in the proof of Theorem 1 completes the proof. □

V. ILLUSTRATIVE EXAMPLES

This section presents two illustrative examples to describe the theoretical result in this paper.

We begin with exploring more about the set of adjacent graphs, which depend on the number of agents only. So they have a universal meaning. In general, we have

$$|\Lambda| = 2^{N(N-1)/2}$$

different graphs. For directed graph, it would be $|\Lambda| = 2^{N(N-1)}$. In this paper, we consider only undirected graphs.

First, we want to order the graphs in Λ. They are one-one corresponding to their Laplacians. So we consider the Laplacians. Note that

$$(l_{12}, \ldots, l_{1N}, l_{23}, \ldots, l_{2N}, \ldots, l_{(N-1)N})$$

are independent elements in a Laplacian, which can be used to describe Laplacians. So we can simply use a binary number

$$\begin{aligned}&|l_{12}|\,|\cdots|\,|l_{1N}|\,|l_{23}|\,|\cdots|\,|l_{2N}|\,|\cdots|\,|l_{(N-1)N}|\\ &= |l_{12}| \times 2^{\frac{N(N-1)}{2}-1} + |l_{13}| \times 2^{\frac{N(N-1)}{2}-2}\\ &\quad + \cdots + |l_{(N-2)N}| \times 2 + |l_{(N-1)N}| := k - 1,\end{aligned}$$

to index L's as $\{L_k\}$, $k = 1, 2, \ldots, 2^{N(N-1)/2}$.

Now let $N = 2$. Then $|\Lambda| = 2$. We have

$$L_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}; \quad L_2 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

and $\mathscr{G}_2$ is connected.

Let $N = 3$. Then $|\Lambda| = 8$. We have

$$L_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad L_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix};$$

$$\cdots; \quad L_8 = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

It is easy to see that $|\Lambda_c| = 4$ and $\Lambda_c = \{\mathscr{G}_4, \mathscr{G}_6, \mathscr{G}_7, \mathscr{G}_8\}$ are connected, and the distinct positive eigenvalues λ's are $\{1, 3\}$.

Let $N = 4$. Then $|\Lambda| = 64$. We have

$$L_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix};$$

$$L_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}; \cdots;$$

$$L_{63} = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{bmatrix};$$

$$L_{64} = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}.$$

Using MatLab, it is easy to calculate that there are $|\Lambda_c| = 38$ connected graphs, which are

$$\Lambda_c = \{\mathscr{G}_p | p \in P\}, \tag{40}$$

where

$$P = \left\{ \begin{matrix} 12 & 14 & 15 & 16 & 20 & 22 & 23 & 24 & 27 & 28 & 29 & 30 & 31 \\ 32 & 36 & 38 & 39 & 40 & 42 & 44 & 45 & 46 & 47 & 48 & 50 & 51 \\ 52 & 54 & 55 & 56 & 57 & 58 & 59 & 60 & 61 & 62 & 63 & 64 & \end{matrix} \right\}.$$

There are 6 different positive λ's, which are

$$\lambda = \{1.0000,\ 4.0000,\ 2.0000,\ 0.5858,\ 3.4142,\ 3.0000\}. \tag{41}$$

When $N = 5$, $|\Lambda| = 1024$, $|\Lambda_c| = 628$. We list the first and last 5 indexes p such that $\mathscr{G}_p \in \Lambda_c$:

$$P = \left\{ \begin{matrix} 76 & 78 & 79 & 80 & 84 \\ \cdots & & & & \\ 1020 & 1021 & 1022 & 1023 & 1024 \end{matrix} \right\}.$$

There are 20 different positive λ's, which are

$$\lambda = \left\{ \begin{matrix} 1.0000 & 5.0000 & 2.3111 & 0.5188 & 3.0000 & 0.3820 & 2.6180 \\ 1.3820 & 4.3028 & 0.6972 & 4.1701 & 3.6180 & 2.0000 & 0.8299 \\ 2.6889 & 4.0000 & 4.4812 & 4.6180 & 4.4142 & 2.3820 & \end{matrix} \right\}.$$

This information is useful in control design.

Example 1. Consider a system with 4 agents, satisfying

$$\dot{x}^i = Ax^i + bu^i, \quad x^i \in \mathbb{R}^2,\ i = 1, 2, 3, 4, \tag{42}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

First, we design controls $K = [k_1^0, k_2^0]$ such that

$$A + \lambda_i bK, \quad i = 1, \ldots, 6 \tag{43}$$

stable, where $\{\lambda_i\}$ are shown in (41).

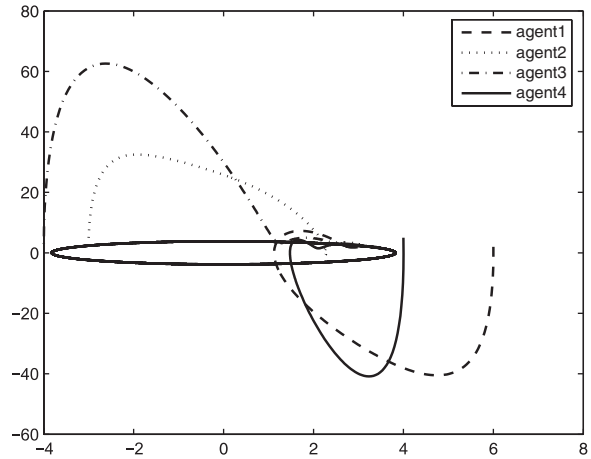

Fig. 2. Consensus with fixed topology.

Choosing

$$K = [\mu^2 k_1^0, \mu k_2^0], \quad \text{with } k_1^0 = -3, k_2^0 = -2, \mu = 10$$

and initial values as

$$x^1(0) = \begin{bmatrix} 6 \\ 2 \end{bmatrix}; \quad x^2(0) = \begin{bmatrix} -3 \\ 5 \end{bmatrix};$$

$$x^3(0) = \begin{bmatrix} -4 \\ 3 \end{bmatrix}; \quad x^4(0) = \begin{bmatrix} 4 \\ 5 \end{bmatrix},$$

three cases are considered. Fig. 2 shows the consensus with fixed topology $\mathscr{G}_{20}$. Fig. 3 shows the consensus with randomly switching connected topology with dwell time $\tau^* = 1$. Fig. 4 shows the consensus with switching frequently connected topology, in which over a time period $3T$, two disconnected modes are active on the first $2T$ duration and then one connected mode is active. The modes are also randomly chosen at switching moments. The four curves in each figure denote the

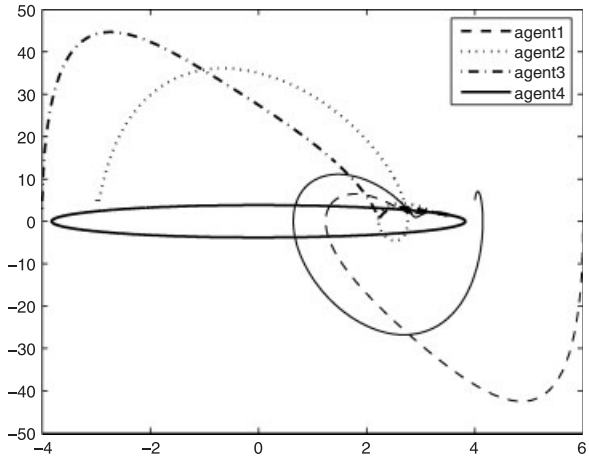

Fig. 3. Consensus with switching topology.

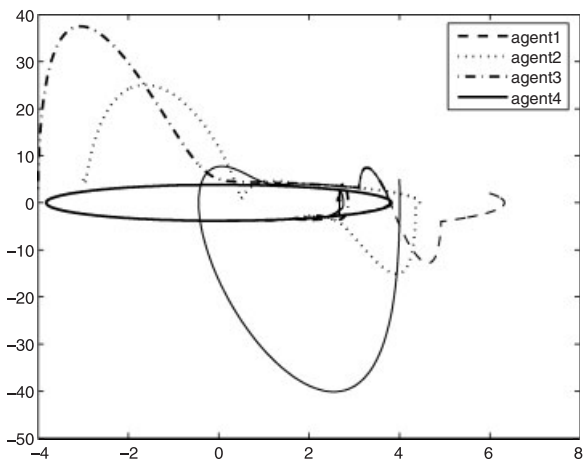

Fig. 4. Consensus with frequently connected topology.

trajectories of four agents. In all three cases, the trajectories of four agents will converge to a common circle which is the trajectory of the center $\bar{x}$. □

Example 2. Consider a system with 4 agents, satisfying

$$\dot{x}^i = Ax^i + bu^i, \quad x^i \in \mathbb{R}^3, \ i = 1, 2, 3, 4, \tag{44}$$

where

$$A = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

First, we design controls $K = [k_1^0, k_2^0, k_3^0]$ such that

$$A + \lambda_i bK, \quad i = 1, \ldots, 6, \tag{45}$$

are stable, where $\{\lambda_i\}$ are given in (41).

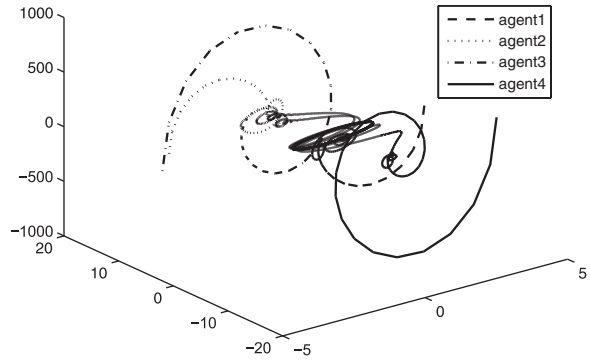

Fig. 5. Consensus with fixed topology.

Then choosing

$$K = [\mu^3 k_1^0, \mu^2 k_2^0, \mu k_3^0],$$

$$\text{with } k_1^0 = -20, \ k_2^0 = -8, \ k_3^0 = -4, \ \mu = 10$$

and initial values as

$$x^1(0) = \begin{bmatrix} 4 \\ 1 \\ -4 \end{bmatrix}; \quad x^2(0) = \begin{bmatrix} -4 \\ 6 \\ 3 \end{bmatrix};$$

$$x^3(0) = \begin{bmatrix} -5 \\ 2 \\ 7 \end{bmatrix}; \quad x^4(0) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix},$$

three cases are considered. Fig. 5 shows the consensus with fixed topology $\mathscr{G}_{20}$. Fig. 6 shows the consensus with randomly switching connected topology with dwell time $\tau^* = 1$. Fig. 7 shows the consensus with randomly chosen switching frequently connected topology, in which over a time period $3T$, two disconnected modes are active on the first $2T$ duration and then one connected mode is active. The modes are also randomly chosen at switching moments. In all three cases, all four trajectories will converge to a circle on a ball surface. □

VI. CONCLUDING REMARKS

In this paper the consensus problem of multi-agent systems was considered. By assuming that every agent shares a common linear dynamic mode that is completely controllable, we showed that the consensus can be achieved via decentralized controls using local

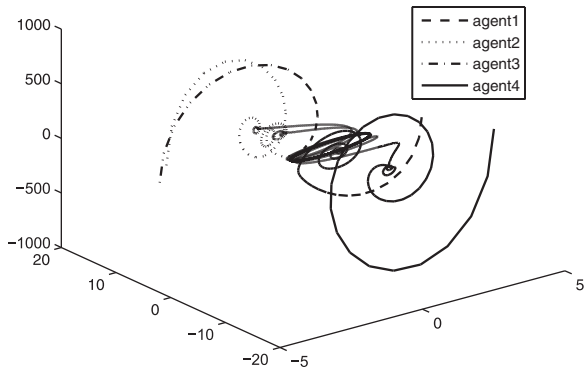

Fig. 6. Consensus with switching topology.

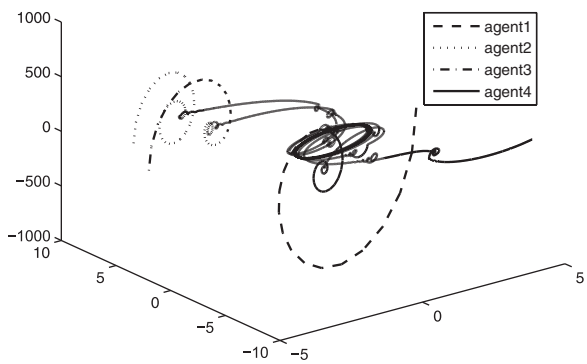

Fig. 7. Consensus with frequently connected topology.

information as long as the adjacent graph is frequently connected.

We give some further remarks as follows:

Remark 1.

1. If the consensus is achieved, where do all the agents go? It is obvious that they will converge to their center $\bar{x}$. According to Lemma 5 we know that the trajectories will converge to

$$\dot{z} = Az, \quad z(0) = \bar{x}(0).$$

2. To design a special target trajectory, a commonly known pre-state-feedback $u^i = K_0 x^i$ can be applied. In this case each agent must know its own precise position.
3. Many interesting problems remain for further study. Among them the joint connection is one of the most challenging problems.

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