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Conservation Laws of a Volterra System and Nonlinear Self-Dual Network Equation

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A systematic method to derive an infinite number of conservation laws for a Volterra system and nonlinear self-dual network equation is presented.

It is well known that an infinite number of conservation laws for certain class of nonlinear evolution equations can be derived from basic equations of the inverse scattering method.[1)] Extending the idea to a discrete problem, we shall present a systematic method to derive an infinite number of conservation laws both for a Volterra equation[2)] and nonlinear self-dual network equation.[3)]

The Volterra equation of our interest is

$$\dot{M}_n=(1+M_n^2)(M_{n+1}-M_{n-1}). \tag{1}$$

We assume the boundary condition $M_n\to 0$ as $|n|\to\infty$. First we derive the infinite number of conservation laws of Eq. (1) from the following basic equations of the inverse scattering method:[4)]

$$v_1(n+1)=zv_1(n)+M_n(t)v_2(n), \tag{2·a}$$

$$v_2(n+1)=\frac{1}{z}v_2(n)-M_n(t)v_1(n), \tag{2·b}$$

$$\dot{v}_1(n)=A_nv_1(n)+B_nv_2(n), \tag{3·a}$$

$$\dot{v}_2(n)=C_nv_1(n)+D_nv_2(n), \tag{3·b}$$

where

$$A_n=z^2+M_{n-1}M_n\,,\quad B_n=zM_n+\frac{1}{z}M_{n-1}\,,$$

$$C_n=-zM_{n-1}-\frac{1}{z}M_n\,,\quad D_n=\frac{1}{z^2}+M_{n-1}M_n\,. \tag{4}$$

Similar to the analysis for the continuous problem,[5)] we set

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$$v_1(n)=z^n \exp\{\sum_{m=-\infty}^{n}\phi_1(z,m)\}, \tag{5·a}$$

$$v_2(n)=z^{-n} \exp\{\sum_{m=n}^{\infty}\phi_2(z,m)\}. \tag{5·b}$$

Substitutions of Eqs. (5) into Eqs. (2) yield

$$(e^{\phi_1(z,n+1)}-1)(e^{-\phi_2(z,n)}-1)=-M_n{}^2, \tag{6·a}$$

$$\frac{e^{\phi_1(z,n+1)}-1}{e^{\phi_1(z,n)}-1}ze^{\phi_1(z,n)}=\frac{M_n}{M_{n-1}}\frac{1}{z}e^{-\phi_2(z,n-1)}. \tag{6·b}$$

Eliminating $\phi_2(z,n)$ from Eqs. (6), we obtain the equation for $\phi_1(z,n)$:

$$z^2\frac{M_{n-1}}{M_n}e^{\phi_1(z,n)}(e^{\phi_1(z,n+1)}-1)=e^{\phi_1(z,n)}-1-M_{n-1}^2. \tag{7}$$

We expand Eq. (7) in power series of $1/z^2$,

$$e^{\phi_1(z,n)}-1\equiv h(z,n)=\sum_{k=1}^{\infty}h^{(k)}(n)z^{-2k}. \tag{8}$$

Recursion formula and explicit form of $h^{(k)}(n)$ are derived from Eqs. (7) and (8):

$$h^{(m+1)}(n+1)=-M_nM_{n-1}\delta_{m,0}+\frac{M_n}{M_{n-1}}h^{(m)}(n)-\sum_{k=1}^{m+1}h^{(k)}(n)h^{(m+1-k)}(n+1), \tag{9}$$

$$\begin{aligned}
&h^{(0)}(n+1)=0,\quad h^{(1)}(n+1)=-M_nM_{n-1},\quad h^{(2)}(n+1)=-M_nM_{n-2}(1+M_{n-1}^2),\\
&h^{(3)}(n+1)=-M_n(1+M_{n-1}^2)\{M_{n-3}(1+M_{n-2}^2)+M_{n-1}M_{n-2}^2\},\\
&h^{(4)}(n+1)=-M_nM_{n-4}(1+M_{n-1}^2)(1+M_{n-2}^2)(1+M_{n-3}^2)\\
&\qquad -M_nM_{n-2}M_{n-3}(1+M_{n-1}^2)(1+M_{n-2}^2)(M_{n-3}+2M_{n-1})\\
&\qquad -M_nM_{n-1}^2M_{n-2}^3(1+M_{n-1}^2),\\
&\cdots\cdots.
\end{aligned} \tag{10}$$

On the other hand, substitution of Eq. (5·a) into Eq. (3·a) gives

$$\frac{\partial\phi_1(z,n+1)}{\partial t}=\left(A_{n+1}+B_{n+1}\frac{v_2(n+1)}{v_1(n+1)}\right)-\left(A_n+B_n\frac{v_2(n)}{v_1(n)}\right) \tag{11}$$

which is the form of the conservation law. We expand Eq. (11) in power series of $1/z^2$ and use Eqs. (2·a), (4), (5·a) and (8) to relate their coefficients with $h^{(k)}(n)$.

$$\phi_1(z,n)\equiv\sum_{k=1}^{\infty}f^{(k)}(n)z^{-2k}=\log\{1+\sum_{k=1}^{\infty}h^{(k)}(n)z^{-2k}\}, \tag{12}$$

$$A_n+B_n\frac{v_2(n)}{v_1(n)}=z^2-\sum_{k=1}^{\infty}g^{(k)}(n)z^{-2k}, \tag{13}$$

$$g^{(k)}(n)=-h^{(k+1)}(n+1)-\frac{M_{n-1}}{M_n}h^{(k)}(n+1). \tag{14}$$

Therefore we obtain the conservation laws for Eq. (1):

$$\frac{\partial f^{(k)}(n)}{\partial t}+g^{(k)}(n+1)-g^{(k)}(n)=0\,. \tag{15}$$

Explicit forms of conserved density $f^{(k)}(n)$ and flux $g^{(k)}(n)$ are

$$\begin{aligned}
f^{(1)}(n)&=h^{(1)}(n)=-M_{n-1}M_{n-2}\,,\\
f^{(2)}(n)&=h^{(2)}(n)-\tfrac{1}{2}\{h^{(1)}(n)\}^2\\
&=-M_{n-1}M_{n-3}(1+M_{n-2}^2)-\tfrac{1}{2}M_{n-1}^2M_{n-2}^2\,,\\
f^{(3)}(n)&=h^{(3)}(n)-h^{(1)}(n)h^{(2)}(n)+\tfrac{1}{3}\{h^{(1)}(n)\}^3\\
&=-M_{n-1}M_{n-4}(1+M_{n-2}^2)(1+M_{n-3}^2)\\
&\quad-M_{n-1}M_{n-2}M_{n-3}(1+M_{n-2}^2)(M_{n-3}+M_{n-1})-\tfrac{1}{3}M_{n-1}^3M_{n-2}^3\,,\\
&\cdots\cdots\cdots\cdots\,.
\end{aligned} \tag{16}$$

$$\begin{aligned}
g^{(1)}(n)&=M_nM_{n-2}(1+M_{n-1}^2)+M_{n-1}^2\,,\\
g^{(2)}(n)&=M_nM_{n-3}(1+M_{n-1}^2)(1+M_{n-2}^2)\\
&\quad+M_{n-1}M_{n-2}(1+M_{n-1}^2)(1+M_nM_{n-2})\,,\\
g^{(3)}(n)&=M_nM_{n-4}(1+M_{n-1}^2)(1+M_{n-2}^2)(1+M_{n-3}^2)\\
&\quad+M_{n-3}(1+M_{n-1}^2)(1+M_{n-2}^2)\{M_nM_{n-2}(M_{n-3}+2M_{n-1})+M_{n-1}\}\\
&\quad+M_{n-1}^2M_{n-2}^2(1+M_{n-1}^2)(1+M_nM_{n-2})\,,\\
&\cdots\cdots\cdots\cdots\,.
\end{aligned} \tag{17}$$

The conservation laws for nonlinear self-dual network equation are derived as follows. The self-dual network equation is obtained by identifying $M_{2n}=V_n$, $M_{2n-1}=-I_n$ in Eq. (1),

$$\begin{aligned}
\frac{1}{1+V_n^2}\frac{dV_n}{dt}&=I_n-I_{n+1}\,,\\
\frac{1}{1+I_n^2}\frac{dI_n}{dt}&=V_{n-1}-V_n\,.
\end{aligned} \tag{18}$$

If we define

$$\begin{aligned}
F^{(k)}(n)&\equiv f^{(k)}(2n)+f^{(k)}(2n-1)\,,\\
G^{(k)}(n)&\equiv g^{(k)}(2n)+g^{(k)}(2n-1)\,,
\end{aligned} \tag{19}$$

it is obvious that Eq. (15) assures that $F^{(k)}(n)$ and $G^{(k)}(n)$ are conserved density and flux, respectively. Explicit forms of $F^{(k)}(n)$ are

$$\begin{aligned}
F^{(1)}(n)&=V_{n-1}(I_n+I_{n-1})\,,\\
F^{(2)}(n)&=-\tfrac{1}{2}V_{n-1}^2(I_n+I_{n-1})^2-I_nI_{n-1}-V_{n-1}V_{n-2}(1+I_{n-1}^2)\,,
\end{aligned}$$

$$F^{(3)}(n)=I_nV_{n-2}(1+V_{n-1}^2)(1+I_{n-1}^2)+I_nV_{n-1}I_{n-1}(1+V_{n-1}^2)(I_n+I_{n-1})$$
$$+V_{n-1}I_{n-2}(1+I_{n-1}^2)(1+V_{n-2}^2)+V_{n-1}I_{n-1}V_{n-2}(1+I_{n-1}^2)(V_{n-2}+V_{n-1})$$
$$+\tfrac{1}{3}V_{n-1}^3(I_n^3+I_{n-1}^3),$$
$$\cdots\cdots\cdots\cdots. \tag{20}$$

$$G^{(1)}(n)=V_nV_{n-1}(1+I_n^2)+I_n^2+I_nI_{n-1}(1+V_{n-1}^2)+V_{n-1}^2,$$
$$G^{(2)}(n)=-V_nI_{n-1}(1+I_n^2)(1+V_{n-1}^2)-I_nV_{n-1}(1+I_n^2)(1+V_nV_{n-1})$$
$$-I_nV_{n-2}(1+V_{n-1}^2)(1+I_{n-1}^2)-V_{n-1}I_{n-1}(1+V_{n-1}^2)(1+I_nI_{n-1}),$$
$$G^{(3)}(n)=(1+V_{n-1}^2)(1+I_{n-1}^2)\{V_nV_{n-2}(1+I_n^2)+I_nI_{n-2}(1+V_{n-2}^2)\}$$
$$+I_{n-1}(1+I_n^2)(1+V_{n-1}^2)\{V_nV_{n-1}(I_{n-1}+2I_n)+I_n\}$$
$$+V_{n-2}(1+V_{n-1}^2)(1+I_{n-1}^2)\{I_nI_{n-1}(V_{n-2}+2V_{n-1})+V_{n-1}\}$$
$$+I_n^2V_{n-1}^2(1+I_n^2)(1+V_nV_{n-1})+V_{n-1}^2I_{n-1}^2(1+V_{n-1}^2)(1+I_nI_{n-1}),$$
$$\cdots\cdots\cdots\cdots. \tag{21}$$

We notice that another identification $M_{2n}=I_{n+1}$ and $M_{2n-1}=-V_n$ also gives Eqs. (18). We also notice that we may define

$$F^{(k)}(n)=f^{(k)}(2n)+f^{(k)}(2n+1),$$
$$G^{(k)}(n)=g^{(k)}(2n)+g^{(k)}(2n+1), \tag{22}$$

instead of Eqs. (19). We find that these ambiguities are related to the fact that Eqs. (18) are invariant under the transformation $V_n\to I_{n+1}$, $I_n\to V_n$, and they do not lead to independent conservation laws.

In this work we have considered a Volterra equation and nonlinear self-dual network equation. However, our method is more general. Further extension and its application will be reported in near future.

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