

CONF 900 4228--1

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ANL-HEP-CP--90-32

DE90 013689

Consistent Multiparameter Quantisation of $GL(n)$

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RECEIVED
APR 11 1990

April 1990

Abstract

We describe a manifold of quantum group structures on the vector space of the universal enveloping algebra of $gl(n)$ and on its dual, the space of polynomials in n^2 variables. The dimension of the manifold is $(n^2 - n + 2)/2$.

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*Talk given at the Informal Spring Workshop on Quantum Groups, Argonne National Laboratory, April 26-May 11, 1990. Work supported by the U.S. Department of Energy, Division of High Energy Physics, Contract W-31-109-ENG-38.

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1. Introduction

Manin [1, 2] has investigated a family of quantum groups containing the algebra of polynomial functions on $GL(n)$ and depending on $N=n(n-1)/2$ parameters. He found, however, that in this family only the usual one-parameter deformations were consistent with functional independence of the generators in the sense of the Poincaré-Birkhoff-Witt theorem, so that the remaining structures were defined on a smaller space than that of the algebra of functions. This paper is devoted to the construction of an $(N+1)$ -parameter family of quantum deformations of $GL(n)$ in which the consistency condition is satisfied for all values of the parameters. The algebra is presented both as an algebra generated by n^2 independent non-commuting matrix elements, with matrix comultiplication, and in the dual form as a deformation G of the universal algebra of $\mathfrak{gl}(n)$. We start with the latter form, in section 2; in section 3 we construct the algebra G^* of non-commuting matrix elements by considering the fundamental representation of G . In section 4 we consider $n=2$ and some other special cases, and comment on the relation of these quantum groups to the Yang-Baxter equation.

2. Multiparameter Deformation of $Uq(n)$

Let A be the polynomial algebra generated by x_1, \dots, x_n with relations

$$x_j x_i = q_{ij} x_i x_j \quad (1)$$

where q_{ij} are c-numbers satisfying $q_{ii}=1$ and $q_{ji}=q_{ij}^{-1}$. It will be convenient to define

$$p_{ij} = \begin{cases} q_{ij} & \text{if } i < j \\ 1 & \text{if } i \geq j \end{cases}$$

and to write (1) as

$$p_{ji} x_j x_i = p_{ij} x_i x_j \quad (2)$$

(no summation convention; throughout this paper all summations will be explicitly marked.)

The algebra A is spanned by the monomials

$$x^r = x_n^{r_n} \dots x_1^{r_1}$$

with $r = (r_1, \dots, r_n) \in \mathbb{N}^n$. We define operators $D_i: A \rightarrow A$ and $X_{ij}: A \rightarrow A$ (for $i \neq j$) by

$$D_i(x^r) = r_i x^r, \quad (3)$$

$$X_{ij}(x^r) = [r_i]_u x^{r - e_i + e_j} \quad (4)$$

where e_i is the elementary vector whose i th component is δ_{ij} , and

$$[x]_u = \frac{u^x - u^{-x}}{u - u^{-1}}, \quad (5)$$

u being a further independent parameter. These operators satisfy

$$D_i(x^r x^s) = (D_i x^r) x^s + x^r (D_i x^s) \quad (6)$$

and

$$X_{ij}(x^r x^s) = (X_{ij} x^r) \frac{u^{s_i} a_j^s}{a_i^s} x^s + \frac{u^{-r_i} b_j^r}{b_i^r} x^r (X_{ij} x^s) \quad (7)$$

where

$$a_i^s = p_{i1}^{s_1} \dots p_{in}^{s_n}, \quad b_i^r = p_{1i}^{r_1} \dots p_{ni}^{r_n}. \quad (8)$$

Thus D_i and X_{ij} are generalised (twisted) derivations of A with the coproducts

$$\Delta(D_i) = D_i \otimes 1 + 1 \otimes D_i \quad (9)$$

and

$$\Delta(X_{ij}) = X_{ij} \otimes u^{D_i} A_i^{-1} A_j + u^{-D_i} B_i^{-1} B_j \otimes X_{ij} \quad (10)$$

where A_i and B_i are the operators (functions of D_1, \dots, D_n) whose eigenvalues are given by (8).

Eq. (7) remains true if u is replaced by u^{-1} ; thus we could also choose the coproduct obtained

from (10) by changing u to u^{-1} .

Let

$$E_i = X_{i,i+1}; \quad F_i = X_{i+1,i};$$

then D_i , E_i and F_i satisfy

$$\left. \begin{aligned} [D_i, D_j] &= 0, \\ [D_i, E_j] &= (-\delta_{ij} + \delta_{i,j+1})E_j, \\ [D_i, F_j] &= (\delta_{ij} - \delta_{i,j+1})F_j, \\ [E_i, F_j] &= [D_{i+1} - D_i]_u; \end{aligned} \right\} \quad (11)$$

$$\left. \begin{aligned} E_{i\pm 1}E_i^2 + E_i^2E_{i\pm 1} &= (u+u^{-1})E_iE_{i\pm 1}E_i, \\ F_{i\pm 1}F_i^2 + F_i^2F_{i\pm 1} &= (u+u^{-1})F_iF_{i\pm 1}F_i; \end{aligned} \right\} \quad (12)$$

and $[E_i, E_j] = [F_i, F_j] = 0$ if $|i-j| \geq 2$. (13)

In view of (9) and (10) and the remark following them, we choose coproducts

$$\left. \begin{aligned} \Delta(D_i) &= D_i \otimes 1 + 1 \otimes D_i, \\ \Delta(E_i) &= E_i \otimes u^{D_i} R_i + u^{-D_i} C_i \otimes E_i, \\ \Delta(F_i) &= F_i \otimes u^{-D_{i+1}} R_i^{-1} + u^{D_{i+1}} C_i^{-1} \otimes F_i, \end{aligned} \right\} \quad (14)$$

where

$$R_i = A_i^{-1} A_{i+1} = \prod_{k=1}^n \left[\frac{p_{i+1,k}}{p_{ik}} \right]^{D_k},$$

$$C_i = B_i^{-1} B_{i+1} = \prod_{k=1}^n \left[\frac{p_{k,i+1}}{p_{ki}} \right]^{D_k}.$$

The commutators (11) give rise to the following relations between D_i , E_i , F_i , R_i and C_i :

$$\left. \begin{aligned} E_i f(D_i) &= f(D_i+1) E_i, & E_i f(D_{i+1}) &= f(D_{i+1}-1) E_i, \\ F_i f(D_i) &= f(D_i-1) F_i, & F_i f(D_{i+1}) &= f(D_{i+1}+1) F_i, \end{aligned} \right\} \quad (15)$$

for any function f ; and

$$\left. \begin{aligned} C_j E_i &= s_{ij} E_i C_j, & C_j F_i &= s_{ij}^{-1} F_i C_j \\ R_j E_i &= s_{ji} E_i R_j, & R_j F_i &= s_{ji}^{-1} F_i R_j \end{aligned} \right\} \quad (16)$$

where

$$s_{ij} = \frac{p_{ij} p_{i+1, j+1}}{p_{i, j+1} p_{i+1, j}}.$$

Using these, it is straightforward to verify that the algebra G generated by D_i , E_i and F_i , with relations (11-13) and comultiplication (14), is a bialgebra [2, 3, 4], i.e. that the coproducts (14) are compatible with the relations (11-13). It becomes a Hopf algebra when furnished with the usual counit and the antipode

$$\left. \begin{aligned} S(D_i) &= -D_i, \\ S(E_i) &= -u^{-1} R_i^{-1} E_i C_i^{-1}, \\ S(F_i) &= -u^{-1} R_i F_i C_i. \end{aligned} \right\} \quad (17)$$

2. The Dual Form

Let V^* be the n -dimensional vector space spanned by x_1, \dots, x_n and V its dual. We think of the elements of V^* as coordinate functions on V , so that the algebra A generated by V^* is a coordinate algebra on V [5]. Eqs. (3) and (4), restricted to V^* , define a representation $\rho: G \otimes V^* \rightarrow V^*$ of our quantum group G in which, with respect to the basis x_1, \dots, x_n , the generators D_i , E_i and F_i are represented by elementary diagonal, subdiagonal and superdiagonal matrices respectively. We denote by a_{ij} the matrix elements of this representation, regarded as functions on G , i.e. elements of the dual G^* ; then

$$\left. \begin{aligned} \langle a_{ij}, D_k \rangle &= \delta_{ik} \delta_{jk} \\ \langle a_{ij}, E_k \rangle &= \delta_{i, k+1} \delta_{jk} \\ \langle a_{ij}, F_k \rangle &= \delta_{ik} \delta_{j, k+1} \end{aligned} \right\} \quad (18)$$

where the angle brackets denote the pairing between G and G^* .

The transposes of these matrices yield $\rho^*: G \otimes V \rightarrow V$ which is an antirepresentation of G , i.e. a representation of the opposite algebra G^{op} . Dualising this gives a map $\delta: V^* \rightarrow G^* \otimes V^*$ given by

$$\delta(x_i) = \sum_k a_{ki} \otimes x_k \quad \text{or, in matrix notation,} \quad \delta(x^T) = x^T \otimes A.$$

The statement that each $\rho(g)$ is a generalised derivation of A with the comultiplication Δ dualises to the statement that δ extends to a homomorphism $\delta: A \rightarrow G^* \otimes A$ (in the terminology of [5], the coordinate algebra A is compatible with the representation ρ^*). Hence (2) is satisfied by $\sum_k a_{ki} \otimes x_k$, and so

$$\sum_{kl} \frac{p_{ij}}{p_{kl}} a_{ki} a_{lj} \otimes y_{kl} = \sum_{kl} \frac{p_{ji}}{p_{lk}} a_{lj} a_{ki} \otimes y_{lk}$$

where $y_{kl} = p_{kl} x_k x_l$. Since these satisfy $y_{kl} = y_{lk}$ (see (2)) but are otherwise independent, it follows that

$$\frac{p_{ij}}{p_{kl}} a_{ki} a_{lj} + \frac{p_{ij}}{p_{lk}} a_{li} a_{kj} = \frac{p_{ji}}{p_{kl}} a_{kj} a_{li} + \frac{p_{ji}}{p_{kl}} a_{kl} a_{ji}. \quad (19)$$

These relations are compatible with matrix comultiplication

$$\Delta(a_{ij}) = \sum_k a_{ik} \otimes a_{kj} \quad (20)$$

(i.e. $\Delta(a_{ij})$ satisfy (19) if a_{ij} do) since this is the comultiplication in G^* , a_{ij} being matrix elements of a representation of G .

Eq. (19) constitutes half of the relations obtained by Manin [1] for the matrices of operators on a quantum superspace in which all the coordinates are even. In order to generate a deformation of the algebra of polynomials in n^2 commuting variables, the a_{ij} must satisfy an equal number of further independent relations. Instead of the remaining relations postulated by Manin, our $a_{ij} \in G^*$ satisfy

$$\frac{p'_{ij}}{p'_{kl}} a_{ik} a_{jl} + \frac{p'_{ij}}{p'_{lk}} a_{il} a_{jk} = \frac{p'_{ji}}{p'_{kl}} a_{jk} a_{il} + \frac{p'_{ji}}{p'_{kl}} a_{jl} a_{ki} \quad (21)$$

where

$$p' = \begin{cases} u^2 q_{ij}^{-1} & \text{if } i < j, \\ 1 & \text{if } i \geq j. \end{cases} \quad (22)$$

These relations could not have been anticipated from the construction of the a_{ij} . To prove them, note that they are the conditions on the matrix $A = (a_{ij})$ for the map $x \rightarrow A \otimes x$ to preserve the relations $p'_{ji} x_j x_i = p'_{ij} x_i x_j$, and therefore they are compatible with the matrix multiplication (20). It follows from this that to prove that they hold in G^* it is sufficient to verify them when bracketed with the generators D_i , E_i and F_i of G . This is readily done, using the brackets (18) and the coproducts (14) together with the definition

$$\langle a_{ij} a_{kl}, X \rangle = \langle a_{ij} \otimes a_{kl}, \Delta(X) \rangle \quad \text{for } X \in G. \quad (23)$$

The relations (19) and (21) can also be related to algebras of q -anticommuting coordinates which are dual to the q -commuting coordinates considered above. The full set of conditions is as follows:

$$\begin{aligned} (19) \quad & \Leftrightarrow x^T \rightarrow x^T A \quad \text{preserves } p_{ji} x_j x_i = p_{ij} x_i x_j \\ & \Leftrightarrow \xi \rightarrow A \xi \quad \text{preserves } p_{ij} \xi_j \xi_i = -p_{ji} \xi_i \xi_j \quad \text{and } \xi_i^2 = 0; \\ (21) \quad & \Leftrightarrow x \rightarrow Ax \quad \text{preserves } p'_{ji} x_j x_i = p'_{ij} x_i x_j \\ & \Leftrightarrow \xi^T \rightarrow \xi^T A \quad \text{preserves } p'_{ij} \xi_j \xi_i = -p'_{ji} \xi_i \xi_j \quad \text{and } \xi_i^2 = 0. \end{aligned}$$

The difference between this algebra and that of ref. [1] is that in the latter p'_{ij} is taken to be equal to p_{ij} . However, it can be shown that they must be related by (22), for some value of u , for the algebra defined by (19) and (21) to have the following consistency property.

The relations (19) and (21) refer to a 2×2 submatrix of A . Suppose $i < j$ and $k < l$, and write

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a_{ik} & a_{il} \\ a_{jk} & a_{jl} \end{bmatrix},$$

$$q = q_{ij}, \quad p = u^2 q_{kl}^{-1}.$$

Then (19) and (21) can be written as

$$\left. \begin{aligned}
 ba &= pab, & db &= qbd, \\
 ca &= qac, & dc &= pcd, \\
 cb &= qp^{-1}bc, \\
 da &= u^{-2}pqad + (1-u^{-2})qbc.
 \end{aligned} \right\} \quad (24)$$

These relations enable any monomial in the a_{ij} to be expressed as a sum of lexicographically ordered monomials. The reordering procedure is not unique, but Manin [1] has shown that the different possible procedures will lead to the same result for all monomials if they do so for cubic monomials, when there are just two possible ways of reordering zyz to xyz :

$$\begin{array}{ccccc}
 & & zxy & \longrightarrow & xzy & & \\
 & \nearrow & & & & \searrow & \\
 zyx & & & & & & xyz \\
 & \searrow & & & & \nearrow & \\
 & & yzx & \longrightarrow & yxz & &
 \end{array} \quad (25)$$

By considering all possible relative positions of the three elements x, y, z in the matrix A , it can be verified that the two ways of reordering zyz , using (24), give the same result in all cases. It follows that the lexicographically ordered monomials are independent and form a basis of G^* .

3. $n = 2$ and Other Special Cases

When $n = 2$ the three parameters p, q, u are related by $u^2 = pq$ and the last relation of (24) becomes

$$da - ad = qbc - q^{-1}cb. \quad (26)$$

The relations (24) can now be expressed in the form

$$\sum_{kl} R_{ij,kl} a_{km} a_{ln} = \sum_{kl} a_{jl} a_{ik} R_{kl,mn} \quad (27)$$

with the R -matrix

$$R = \begin{bmatrix} pq & 0 & 0 & 0 \\ 0 & q & pq-1 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & pq \end{bmatrix} \quad (28)$$

(with rows and columns labelled by ij and kl in the order 11, 12, 21, 22). This satisfies the Yang-Baxter equation.

If $q = p$ this gives the familiar one-parameter deformation [2, 3] of $\mathfrak{gl}(2)$. If $q = p^{-1}$ the R -matrix is in factorised form

$$R = Q \otimes Q^{-1} \quad \text{where} \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & q \end{bmatrix},$$

and the relations in the algebra G^* can be written as

$$a_{kl}a_{ij} = q^{i-j-k+l} a_{ij}a_{kl}.$$

For all values of p and q the algebra defined by (24) and (26) has the power property noted by Corrigan et al [6, 7]: if the matrix elements of A satisfy these relations, then those of A^n satisfy similar relations with (q, p) replaced by (q^n, p^n) .

For general n , the usual one-parameter deformation [2, 3] of $\mathfrak{gl}(n)$ is obtained by taking all the q_{ij} equal to q for $i < j$, and $u = q$. The resulting algebra is quasitriangular, i.e. is described by an R -matrix which satisfies the Yang-Baxter equation. This is also true for general q_{ij} if $u = 1$, when the relations (19) and (21) become

$$q_{ij}a_{ik}a_{jl} = q_{kl}a_{jl}a_{ik}, \quad (29)$$

which is the same as (27) with the R -matrix

$$R_{ij,kl} = q_{ij}\delta_{ik}\delta_{jl}.$$

If q_{ij} factorises as $q_{ij} = q_i q_j^{-1}$ to give an R -matrix $R = Q \otimes Q^{-1}$, then the $n \times n$ matrix A has the power property that the elements of A^n satisfy (29) with q_{ij} replaced by q_{ij}^n .

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