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Maurice A. Biot

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Consolidation Settlement Under a Rectangular Load Distribution

M. A. Biot*
California Institute of Technology, Pasadena, California
(Received December 7, 1940)

The author's general theory is applied to the calculation of the settlement through consolidation of a soil loaded uniformly on an infinite strip of constant width with particular reference to the nature of the settlement at the edge of the loaded area. The solution is obtained by first calculating the settlement produced by a suddenly applied load with sinusoidal distribution. The use of a Dirichlet integral and the principle of superposition leads then directly to the solution for the discontinuous loading.

Introduction

In the calculation of foundations and the prediction of settlement we are not so much interested in the absolute value of the settlement but rather in the differences in settlement which can occur in a loaded area due to differences in load intensity. Then differential settlements are the direct cause of damage in buildings and structures carried by the soil. A typical case of settlement due to differential loading occurs when the load is applied uniformly to an infinite strip of constant width. In particular one may ask what happens at the edge of the loaded area; how much additional settlement is due to the water flowing from the loaded region to the unloaded region; how much restraint does the settlement of the loaded area encounter from the unloaded region; and how much settlement does the unloaded area undergo in the vicinity of the load. The present paper is a quantitative answer to these questions. The problem is essentially one of two-dimensional strain in a plane perpendicular to the axis of the loaded strip.

1. Settlement Under a Load with Sinusoidal Distribution

In the previous paper[1] we have established the following equations for the consolidation of a completely saturated clay:

$$G\nabla^2 u+\frac{G}{1-2\nu}\frac{\partial\epsilon}{\partial x}-\frac{\partial\sigma}{\partial x}=0,$$

$$G\nabla^2 v+\frac{G}{1-2\nu}\frac{\partial\epsilon}{\partial y}-\frac{\partial\sigma}{\partial y}=0, \qquad (1.1)$$

$$G\nabla^2 w+\frac{G}{1-2\nu}\frac{\partial\epsilon}{\partial z}-\frac{\partial\sigma}{\partial z}=0,$$

$$\nabla^2\epsilon=(1/c)(\partial\epsilon/\partial t), \qquad (1.2)$$

where u, v, w are the components of the displacement of the soil and G, ν, are the shear modulus and Poisson ratio for the completely consolidated clay.

$\epsilon=\partial u/\partial x+\partial v/\partial y+\partial w/\partial z.$
$c=k/a$ coefficient of consolidation.
$k=$ coefficient of permeability.
$a=(1-2\nu)/2G(1-\nu)$ final compressibility.
$\sigma=$ water pressure increment in the pores.

We consider an infinitely deep layer of clay and take the xy plane to coincide with the surface while the z axis is oriented positively downward. We propose to find the settlement of the surface of the clay where a uniform load is suddenly applied at the instant $t=0$ at the surface of the clay on a strip infinitely long in the y direction and extending from $x=-l/2$ to $x=l/2$. The water contained in the clay is assumed to escape freely at the surface so that the water pressure at the surface is a constant equal to the atmospheric pressure. This is essentially a two-dimensional problem where $v=0$ and all unknowns are functions only of x and z. We shall solve this problem by first considering a load distributed sinusoidally along the x direction and then using a Fourier expansion and the principle of superposition to find the settlement for a rectangular load distribution.

The settlement due to a sinusoidal load applied suddenly at the instant $t=0$ is found most conveniently by the operational method. In the present two-dimensional problem writing sym-

* On leave of absence from Columbia University.
[1] M. A. Biot, "General theory of three-dimensional consolidation," J. App. Phys. **12**, 155 (1941).

bolically $\partial/\partial t=p$, Eqs. (1.1) and (1.2) become

$$\begin{aligned} G\nabla^2 u+\frac{G}{1-2\nu}\frac{\partial\epsilon}{\partial x}-\frac{\partial\sigma}{\partial x}&=0,\\ G\nabla^2 w+\frac{G}{1-2\nu}\frac{\partial\epsilon}{\partial z}-\frac{\partial\sigma}{\partial z}&=0,\\ \nabla^2\epsilon&=p\epsilon/c. \end{aligned} \tag{1.3}$$

We have to solve these equations with the boundary conditions

(1) that all variables vanish at infinite depth

$z=\infty$,

(2) $\sigma=0$ at $z=0$,

(3) $\sigma_z=2G\left(\frac{\partial w}{\partial z}+\frac{\nu\epsilon}{1-2\nu}\right)=-A\sin\lambda x$ at $z=0$,

(4) $\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}=0$ at $z=0$.

The second condition expresses that the water is free to escape at the surface. The last two equations express that at the surface the normal stress is equal to the load and the shearing stress is zero. These conditions are derived from relations (2.11) in the previous paper by putting $\sigma=0$.

We may verify that a solution of Eqs. (1.3) satisfying the boundary conditions (1) is

$$\begin{aligned} u&=[C_1\lambda e^{-\lambda z}+C_2\lambda e^{-(\lambda^2+p/c)^{\frac{1}{2}}z}-C_3(1-\lambda z)e^{-\lambda z}]\cos\lambda x,\\ w&=[-C_1\lambda e^{-\lambda z}-C_2(\lambda^2+p/c)^{\frac{1}{2}}e^{-(\lambda^2+p/c)^{\frac{1}{2}}z}-C_3\lambda z e^{-\lambda z}]\sin\lambda x,\\ \sigma&=[C_2(p/ac)e^{-(\lambda^2+p/c)^{\frac{1}{2}}z}-2C_3G\lambda e^{-\lambda z}]\sin\lambda x. \end{aligned}\tag{1.4}$$

The arbitrary constants C_1, C_2, C_3, are to be determined by introducing these values of u, v, w, in the three boundary conditions (2), (3), (4). We find

$$\begin{aligned} C_2p/c-2Ga\lambda C_3&=0,\\ C_1\lambda^2+C_2(\lambda^2+p/2Gac)-C_3\lambda&=-A/2G,\\ -C_1\lambda-C_2(\lambda^2+p/c)^{\frac{1}{2}}+C_3&=0. \end{aligned}\tag{1.5}$$

Now we are interested only in the vertical deflection w at the surface $z=0$. This value is

$$w=[-C_1\lambda-C_2(\lambda^2+p/c)^{\frac{1}{2}}]\sin\lambda x=-C_3\sin\lambda x. \tag{1.6}$$

Solving Eqs. (1.5) for C_3 we find

$$C_3=-\frac{A}{2G\lambda}\frac{(1-\nu)\lambda^2+p/\beta c+(1-\nu)\lambda(\lambda^2+p/c)^{\frac{1}{2}}}{\lambda^2+p/\beta c} \tag{1.7}$$

with

$$\beta=(1-2\nu)/(1-\nu)^2.$$

The soil deflection at the surface as a function of time for a sudden sinusoidal load distribution is written in operational form

$$w=\frac{A\sin\lambda x}{2G\lambda}\left[1-\frac{\nu\lambda^2}{\lambda^2+p/\beta c}+\frac{(1-\nu)\lambda(\lambda^2+p/c)^{\frac{1}{2}}}{\lambda^2+p/\beta c}\right]1(t). \tag{1.8}$$

The function represented symbolically by this equation may be calculated from the following operational expressions

$$\frac{\lambda^2}{\lambda^2+p/\beta c}1(t)=1-e^{-\lambda^2\beta ct}, \tag{1.9}$$

$$\frac{\lambda(\lambda^2+p/c)^{\frac{1}{2}}}{\lambda^2+p/\beta c}1(t)=P[\lambda(ct)^{\frac{1}{2}}]-(1-\beta)^{\frac{1}{2}}e^{-\lambda^2\beta ct}P[\lambda((1-\beta)ct)^{\frac{1}{2}}], \tag{1.10}$$

where

$$P(x)=\frac{2}{\pi^{\frac{1}{2}}}\int_0^x e^{-\zeta^2}d\zeta$$

is a tabulated function called the *probability integral.*

The first operator is elementary[2] while the second may be derived by using the shifting formula[3] from the well-known operator.

$$p^{\frac{1}{2}}1(t)=\frac{1}{(\pi t)^{\frac{1}{2}}}. \tag{1.11}$$

With these results the deflection of the soil surface is finally written

$$w=\frac{A\sin\lambda x}{2G\lambda}\{1-\nu(1-e^{-\lambda^2\beta ct})+(1-\nu)P[\lambda(ct)^{\frac{1}{2}}]-\nu e^{-\lambda^2\beta ct}P[\lambda((1-\beta)ct)^{\frac{1}{2}}]\}. \tag{1.12}$$

[2] Th. von Karman and M. A. Biot, *Mathematical Methods in Engineering* (McGraw-Hill, 1940), cf. Chapter X.
[3] V. Bush, *Operational Circuit Analysis* (John Wiley, 1929), pp. 130 and 191.

For $t=0$ the deflection is purely elastic, its value is

$$w_i=\frac{A\sin\lambda x}{2G\lambda}. \tag{1.13}$$

The settlement due to consolidation is $w_s=w-w_i$

$$w_s=\frac{A\sin\lambda x}{2G\lambda}\{-\nu[1-e^{-\lambda^2\beta ct}+e^{-\lambda^2\beta ct}P(\lambda((1-\beta)ct)^{\frac{1}{2}})]+(1-\nu)P[\lambda(ct)^{\frac{1}{2}}]\}. \tag{1.14}$$

It is possible to check the exactness of this expression by considering the case where the distributed load has an infinite wave-length, or what amounts to the same thing when the time interval t after loading is infinitely small. In both cases the product $\lambda^2 t$ is infinitesimal, and expression (1.14) then becomes

$$w_s=A\sin\lambda x 2a(ct/\pi)^{\frac{1}{2}}. \tag{1.15}$$

This checks with the formula (6.11) found in the previous paper for the settlement of a laterally restrained soil column immediately after loading. Note that this settlement is independent of the Poisson ratio.

For convenience we shall write Eq. (1.14) in the form

$$w_s=\frac{aA\sin\lambda x}{\lambda}F[\nu,\lambda(ct)^{\frac{1}{2}}], \tag{1.16}$$

where

$$F(\nu,\lambda(ct)^{\frac{1}{2}})=-\frac{\nu(1-\nu)}{1-2\nu}\{1-e^{-\lambda^2\beta t}+e^{-\lambda^2\beta t}P[\lambda((1-\beta)ct)^{\frac{1}{2}}]\}+\frac{(1-\nu)^2}{1-2\nu}P[\lambda(ct)^{\frac{1}{2}}]. \tag{1.17}$$

In case the Poisson ratio is equal to zero ($\nu=0$) this expression becomes

$$F(0,\lambda(ct)^{\frac{1}{2}}]=P[\lambda(ct)^{\frac{1}{2}}]. \tag{1.18}$$

In order to simplify the numerical work in further calculations we shall restrict ourselves to this case. It must be remembered that the settlement during the initial period as given by expression (1.15) depends only on the compressibility a and is independent of the Poisson ratio. The latter will therefore only have an influence on the later period of the settlement. The order of magnitude of this effect may be found by putting $t=\infty$ in expression (1.17). We find

$$F(\nu,\infty)=1-\nu.$$

This shows that in the later period the actual settlement is smaller than that calculated from expression (1.16) with the assumption $\nu=0$.

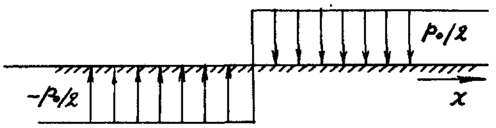

FIG. 1. Discontinuous loading represented by the integral (2.1).

2. Settlement Under a Uniform Load with Discontinuity

For practical purposes we are interested in the *differential* settlements, such as occur when the load distribution has a discontinuity. Such a distribution is represented by the following function of x

$$\varphi(x)=\frac{p_0}{\pi}\int_0^\infty\frac{\sin\lambda x}{\lambda}d\lambda. \tag{2.1}$$

The loading represented by this expression is a constant equal to $-p_0/2$ for $x<0$ and $p_0/2$ for $x>0$ with a discontinuity in the loading at $x=0$ equal to p_0 (Fig. 1).

The settlement due to this load is easily obtained from the solution (1.16) found above for the case of a sinusoidal loading. Using the principle of superposition the expression of the settlement is found to be

$$w_s=\frac{ap_0}{\pi}\int_0^\infty\frac{F[\nu,\lambda(ct)^{\frac{1}{2}}]}{\lambda^2}\sin\lambda x d\lambda. \tag{2.2}$$

This can be written

$$\frac{w_s}{2ap_0}\left(\frac{\pi}{ct}\right)^{\frac{1}{2}}=\frac{1}{2\pi^{\frac{1}{2}}}\int_0^\infty\frac{F(\nu,\gamma)}{\gamma^2}\sin\gamma\xi d\gamma \tag{2.3}$$

with

$$\gamma=\lambda(ct)^{\frac{1}{2}},\quad \xi=x/(ct)^{\frac{1}{2}}.$$

If we now introduce the assumption $\nu=0$, it becomes possible to evaluate this integral numerically by using an approximate analytical expression for the function $F(0,\gamma)$. It may be verified that this function is represented within

1 percent by

$$F(0,\gamma)=1-e^{-2\gamma/\pi^{\frac{1}{2}}}+\gamma^2e^{-1.8\gamma}. \qquad (2.4)$$

The integrations in expression (2.3) can then be performed by means of elementary functions. We find

$$\frac{w_s}{2ap_0}\left(\frac{\pi}{ct}\right)^{\frac{1}{2}}=\frac{1}{4\pi^{\frac{1}{2}}}\xi\log\left(1+\frac{4}{\pi\xi^2}\right)+\frac{1}{\pi}\tan^{-1}\left(\frac{(\pi)^{\frac{1}{2}}\xi}{2}\right)+\frac{1}{2\pi^{\frac{1}{2}}}\frac{\xi}{3.24+\xi^2}. \qquad (2.5)$$

By adding a load equal to $p_0/2$ and extending from $-\infty$ to ∞ we obtain a total load p_0 extending from 0 to ∞ as shown in Fig. 2. The settlement for this case is obtained by superposition. Denoting the right-hand side of Eq. (2.5) by $f(\xi)$ this settlement is

$$w_s=2ap_0\left(\frac{ct}{\pi}\right)^{\frac{1}{2}}\left[\tfrac{1}{2}+f(\xi)\right]. \qquad (2.6)$$

In order to represent the settlement as a function of time it is convenient to introduce for the abscissa a characteristic length l which can be chosen arbitrarily and write

$$w_s=2ap_0\left(\frac{ct}{\pi}\right)^{\frac{1}{2}}\left[\frac{1}{2}+f\left(\frac{x}{l}\frac{l}{(ct)^{\frac{1}{2}}}\right)\right]. \qquad (2.7)$$

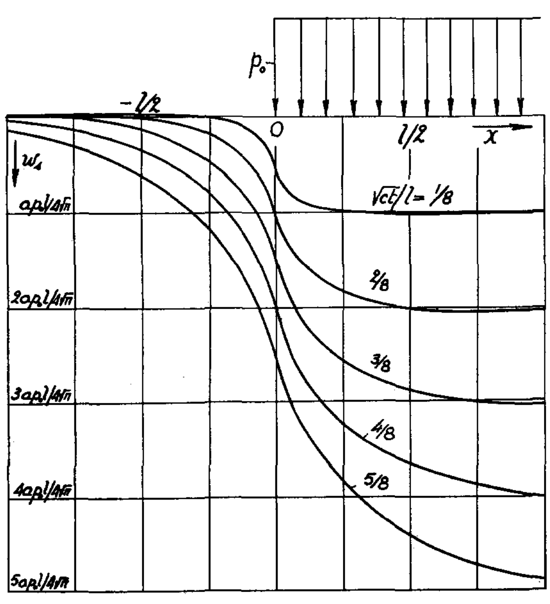

FIG. 2. Settlement of the soil surface at various time intervals for a load extending from $x=0$ to $x=\infty$.

Then settlement curves are plotted as a function of x in Fig. 2 at time intervals corresponding to $(ct^{\frac{1}{2}}/l)=\frac{1}{8},\frac{2}{8},\frac{3}{8},\frac{4}{8},\frac{5}{8}$, and compared directly with the settlement

$$w_{si}=2ap_0(ct/\pi)^{\frac{1}{2}},$$

which would have occurred after the same time intervals if the load extended from $x=-\infty$ to $x=\infty$. These settlements are represented by the horizontal lines of ordinates

$$\frac{2}{\pi^{\frac{1}{2}}}ap_0l\left(\frac{1}{8}\right),\quad \frac{2}{\pi^{\frac{1}{2}}}ap_0l\left(\frac{2}{8}\right)\cdots.$$

The slope of the soil deflection at the edge of the loaded area $(x=0)$ is infinite as may be verified analytically by calculating the derivative $df/d\xi$ for $\xi=0$; it constitutes, therefore, a singular point probably associated with infinite stresses. However, this infinite slope does not show up in the plotted curve because it is a highly localized effect.

It is interesting to follow the settlement at a given point x. Consider first a point located under the load $(x>0)$. The settlements w_s and w_{si} are equal at first, then w_s becomes slightly larger than w_{si} while for large values of the time w_s becomes smaller and smaller compared to w_{si}. This is due to three distinct phases in the settlement. In the first instant the settlement is mostly

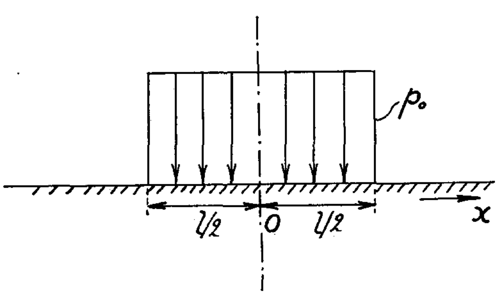

FIG. 3. Rectangular load distribution.

due to water flowing out at the surface directly under the load. In the second phase the settlement is due partly to water flowing from the loaded region to the unloaded region so that this increases the settlement in comparison with the case when the load is applied uniformly from $x=-\infty$ to $x=+\infty$. This effect, however, is very small and is hardly visible in the figure. In the

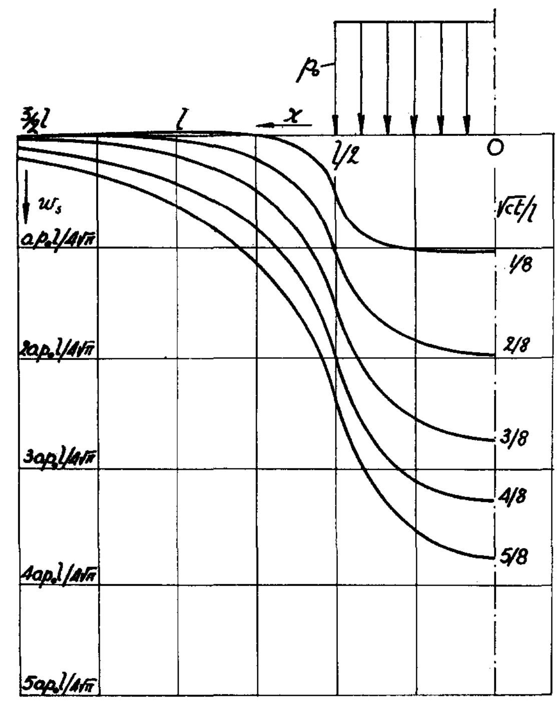

FIG. 4. Settlement of the soil surface at various time intervals for the load distribution represented in Fig. 3.

The settlement curves are represented in Fig. 4 as a function of x at time intervals corresponding to $(ct^{\frac{1}{2}}/l)=\frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \frac{4}{8}, \frac{5}{8}$, and compared directly with the settlement w_{si} which would have occurred after the same time intervals if the load extended from $x=-\infty$ to $x=+\infty$. These settlements are represented by the horizontal lines of ordinates

$$\frac{2}{\pi^{\frac{1}{2}}}ap_0l\left(\frac{1}{8}\right),\quad \frac{2}{\pi^{\frac{1}{2}}}ap_0l\left(\frac{2}{8}\right),\quad \cdots.$$

It will be noted that immediately after loading the settlement is little affected by the unloaded regions on both sides, while in the last phase the settlement is considerably reduced by the restraining effect of the unloaded regions. The settlement at the center of the loaded area is obtained from (3.1) by putting $x=0$

$$w_s=4ap_0(ct/\pi)^{\frac{1}{2}}f[l/2(ct)^{\frac{1}{2}}]. \qquad (3.2)$$

We may also write

$$w_s/w_{si}=2f[l/2(ct)^{\frac{1}{2}}]. \qquad (3.3)$$

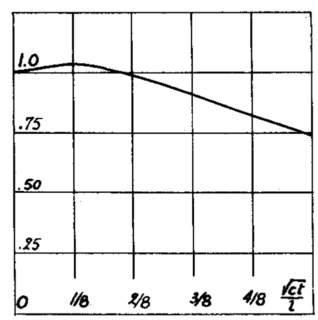

FIG. 5. Ratio of settlement at the middle of the loaded area ($x=0$) to the settlement which would take place if the load extended from $x=-\infty$ to $x=+\infty$.

third phase the unloaded region restrains the settlement of the loaded area because of the elastic stresses originating between the two regions. Similar phases can be distinguished for a point lying outside of the loaded area ($x<0$). At first no motion is observed; then the surface is lifted by a slight amount. This swelling of the soil is due to the water escaping for the loaded region. Finally a settlement is observed because the unloaded region is dragged down elastically by the settlement of the loaded area.

3. Settlement Under a Rectangular Load Distribution

By superposition we may easily derive from the previous solution the settlement due to a constant load extending from $x=-l/2$ to $x=l/2$; as shown in Fig. 3, using the solution (2.6) we may write for the settlement

$$w_s=2ap_0\left(\frac{ct}{\pi}\right)^{\frac{1}{2}}\left[f\left(\frac{x+l/2}{(ct)^{\frac{1}{2}}}\right)-f\left(\frac{x-l/2}{(ct)^{\frac{1}{2}}}\right)\right]. \qquad (3.1)$$

This ratio which represents the restraining effect of the unloaded regions is plotted in Fig. 5 as a function of $(ct)^{\frac{1}{2}}/l$. The larger the size of the loaded area the less quickly this restraining effect comes into play. It will be noted that for small values of t the ratio w_s/w_{si} is slightly larger than unity which means that the settlement is increased by water flowing from the loaded region to the unloaded region. We have assumed the Poisson ratio to be zero. From the remark at the end of paragraph 2 we may deduce that if this is not the case the restraining effect of the unloaded region will be still greater.