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KÔDAI MATH. SEM. REP
25 (1973), 190—201

CONSTANCY OF HOLOMORPHIC SECTIONAL CURVATURE IN ALMOST HERMITIAN MANIFOLDS

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Dedicated to Professor K. Yano on his 60th birthday

§1. Introduction.

Let (M, g, J) be an almost Hermitian manifold with almost complex structure tensor J and almost Hermitian metric tensor g. By R we denote the Riemannian curvature tensor; $R(X, Y)Z=\nabla_{[X,Y]}Z-[\nabla_X, \nabla_Y]Z$. The holomorphic sectional curvature $H(X)$ for a unit tangent vector X is the sectional curvature $K(X, JX)=g(R(X, JX)X, JX)$. Let x be a point of M. If $H(X)$ is constant for every unit tangent vector X at x, (M, g, J) is said to be of constant holomorphic sectional curvature at x. If $H(X)$ is constant for every x and every tangent vector X at x, then (M, g, J) is said to be of constant holomorphic sectional curvature.

One of the main theorems is as follows:

THEOREM A. *Let* $\dim M=m=2n\geq 4$. *Assume that almost Hermitian manifold* (M, g, J) *satisfies*

$$g(R(JX, JY)JX, JZ)=g(R(X, Y)X, Z) \tag{1.1}$$

for every tangent vectors X, Y *and* Z. *Then,* (M, g, J) *is of constant holomorphic sectional curvature at* x, *if and only if*

$$R(X, JX)X \quad \textit{is proportional to} \quad JX \tag{1.2}$$

for every tangent vector X *at* x.

The condition (1.1) is satisfied in every Kählerian manifold or more generally in every K-space (=nearly Kählerian space, almost Tachibana space).

The condition (1.2) itself has a geometric meaning. It is also stated as follows: Let σ be a holomorphic plane and let X, JX be in σ; then $R(X, JX)$ satisfies $R(X, JX)\sigma\subset\sigma$ and $R(X, JX)\sigma^{\perp}\subset\sigma^{\perp}$, where $\sigma^{\perp}$ denotes the orthocomplement of σ in the tangent space.

In §2, as preliminaries we state some Propositions which give conditions for a Riemannian manifold to be of constant curvature.

In §3, we prove Theorem A. Theorem A is concerned with point-wise constant

Received May, 8, 1972.

holomorphic sectional curvature. As for globally-constant holomorphic sectional curvature, we have

THEOREM B. *A K-space* (M, g, J), $m \geq 4$, *is of constant holomorphic sectional curvature, if and only if* (1. 2) *holds at each point.*

Theorem B is a generalization of a result by Kosmanek [9], or a result by Ogiue [15].

In §4, we give a condition for a Sasakian manifold to be of constant ϕ-holomorphic sectional curvature in an analogous way as in Kählerian case.

In §5, we consider K-spaces of constant holomorphic sectional curvature. First we have (cf. Theorem 5. 2)

THEOREM C. *A complete 6-dimensional K-space of constant holomorphic sectional curvature H is one of the following K-spaces*:

$$S^6[H], \qquad CP^3[H], \qquad CE^3[0]/\Gamma_1, \qquad CD^3[H]/\Gamma_2.$$

With respect to the sign of H, we have

THEOREM D. *If a K-space* (M, g, J) *has constant holomorphic sectional curvature* H *and if* (M, g, J) *is not Kählerian, then* $H>0$.

Finally we show a relation between the scalar curvature S and constant holomorphic sectional curvature H.

§2. Riemannian manifolds of constant curvature.

A Riemannian manifold (M, g), $m \geq 3$, is of constant curvature K, if and only if we have

$$R(X, Y)Z = K[g(X, Z)Y - g(Y, Z)X]$$

for every tangent vectors X, Y and Z.

A result of Fialkow [2] is as follows:

PROPOSITION 2. 1. (Fialkow) *Let* $\dim M = m \geq 3$. *Then a Riemannian manifold* (M, g) *is of constant curvature, if and only if*

$$g(R(X, Y)X, Z) = 0 \tag{2. 1}$$

holds for every point x *of* M *and every orthogonal triplet* (X, Y, Z) *in the tangent space* M_x *at* x.

The condition (2. 1) may be stated as follows: For every 2-plane σ, let X, Y be in σ; then

$$R(X, Y)\sigma \subset \sigma \qquad \text{and} \qquad R(X, Y)\sigma^{\perp} \subset \sigma^{\perp}. \tag{2. 1$'$}$$

Or equivalently, for every orthogonal pair (X, Y) in M_x

(2. 1)'' $$R(X, Y)X \text{ is proportional to } Y.$$

Proposition 2. 1 has a consequence:

PROPOSITION 2. 2. (Ogiue [15]) *Let $m \geq 3$. A Riemannian manifold (M, g) is of constant curvature, if and only if $R(X, Y)Z$ is a linear combination of X and Y for every X, Y, and Z.*

In fact, for an orthogonal triplet (X, Y, Z), if $R(X, Y)X$ is a linear combination of X and Y, then it is orthogonal to Z.

§3. Almost Hermitian manifolds of constant holomorphic sectional curvature.

Let (M, g, J) be an almost Hermitian manifold. Then

$$g(JX, JY) = g(X, Y), \qquad JJX = -X$$

holds. If (M, g, J) is of constant holomorphic sectional curvature at x, then we have

$$H(X) = g(R(X, JX)X, JX)/g(X, X)^2 = H = \text{constant}$$

for every non-zero tangent vector X at x. Let $(e_i, i=1, \cdots, m)$ be any basis at x. If we put $R_{ijkl} = g(R(e_k, e_l)e_j, e_i)$ and $g_{jk} = g(e_j, e_k)$, we have

(3. 1) $$[R_{irsl}J^r_jJ^s_k - Hg_{jk}g_{il}]X^iX^jX^kX^l = 0,$$

where $JX = (J^r_jX^j)e_r$ and $X = X^ie_i$. By [symmetrization of $(R_{irsl}J^r_jJ^s_k - Hg_{jk}g_{il})$] $= 0$, we have

LEMMA 3. 1. (Kotō [10]) *An almost Hermitian manifold (M, g, J) is of constant holomorphic sectional curvature at x, if and only if*

(3. 2) $$\begin{aligned} &R_{irsl}(J^r_jJ^s_k + J^r_kJ^s_j) + R_{irsk}(J^r_jJ^s_l + J^r_lJ^s_j) \\ &+ R_{irsj}(J^r_kJ^s_l + J^r_lJ^s_k) + R_{jrsl}(J^r_iJ^s_k + J^r_kJ^s_i) \\ &+ R_{jrsk}(J^r_iJ^s_l + J^r_lJ^s_i) + R_{krsl}(J^r_jJ^s_i + J^r_iJ^s_j) \\ &= 4H(g_{jk}g_{il} + g_{kl}g_{ij} + g_{jl}g_{ik}). \end{aligned}$$

Transvecting (3. 2) with $X^jX^kX^l$ (X: unit), we have

(3. 3) $$(R_{irsl}J^r_jJ^s_k + J^r_iR_{jrsl}J^s_k)X^jX^kX^l = 2HX_i, \text{ i.e.,}$$

(3. 3)' $$R(JX, X)JX + JR(JX, X)X = 2HX.$$

Now we assume that (M, g, J) has a property

$$g(R(JX, JY)JX, JZ)=g(R(X, Y)X, Z) \tag{3.4}$$

for every tangent vectors X, Y, and Z. (3.4) is equivalent to $JR(JX, JY)JX = -R(X, Y)X$. Putting $Y=JX$, we have $JR(JX, X)JX=R(X, JX)X$. Then (3.3)′ implies

$$R(X, JX)X=HJX. \tag{3.5}$$

THEOREM 3.2. *Let $m\geq 4$. Assume that an almost Hermitian manifold (M, g, J) satisfies* (3.4). *Then, (M, g, J) is of constant holomorphic sectional curvature at x, if and only if*

$$R(X, JX)X \text{ is proportional to } JX \tag{3.6}$$

for every tangent vector X at x.

Proof. The necessity follows from (3.5). We prove the converse. Let (X, Y) be an orthonormal pair in M_x such that $g(Y, JX)=0$. Define X^* and Z^* by $X^* =(X+Y)/\sqrt{2}$ and $Z^*=(JX-JY)/\sqrt{2}$. Then Z^* is orthogonal to X^* and JX^*, By property (3.6) and curvature properties, we have

$$0=4g(R(X^*, JX^*)X^*, Z^*)=g(R(X, JX)X, JX)-g(R(Y, JY)Y, JY)$$
$$-g(R(X, JY)X, JY)+g(R(JX, Y)JX, Y).$$

By assumption (3.4) on (M, g, J), we see that the last two terms of the right hand side vanish. Therefore, we get $H(X)=H(Y)$.

(i) First we assume $m\geq 6$. Fix a unit tangent vector X at x. Then the discussion above shows that every holomorphic plane which is orthogonal to the plane (X, JX) has constant holomorphic sectional curvature $H(X)$. Let W be any unit tangent vector at x. Then the orthocomplement of the holomorphic plane (W, JW) and that of (X, JX) have a non-trivial intersection. This proves $H(W) =H(X)$.

(ii) Next we assume $m=4$. For an orthonormal pair (X, Y) such that $g(Y, JX) =0$, we have $H(X)=H(Y)$ as before. Using property (3.6), we get

$$R(X, JX)X=H(X)JX,$$
$$R(X, JX)Y=g(R(X, JX)Y, JY)JY,$$
$$R(X, JY)X=g(R(X, JY)X, Y)Y+g(R(X, JY)X, JY)JY,$$
$$R(X, JY)Y=g(R(X, JY)Y, X)X+g(R(X, JY)Y, JX)JX,$$
$$R(Y, JX)X=g(R(Y, JX)X, Y)Y+g(R(Y, JX)X, JY)JY,$$
$$R(Y, JX)Y=g(R(Y, JX)Y, X)X+g(R(Y, JX)Y, JX)JX,$$
$$R(Y, JY)X=g(R(Y, JY)X, JX)JX,$$
$$R(Y, JY)Y=H(Y)JY=H(X)JY.$$

We define X^θ by $X^\theta=\cos\theta\, X+\sin\theta\, Y$, where θ denotes a real number. Then, using the relations obtained above, we have

$$R(X^\theta, JX^\theta)X^\theta=AX+BY+CJX+DJY, \tag{3.7}$$

where A and B are not necessary for our argument and

$$C=\cos^3\theta\, H(X)+\cos\theta\sin^2\theta\, E,$$

$$D=\sin^3\theta\, H(X)+\cos^2\theta\sin\theta\, E,$$

$$E=g(R(X, JY)Y, JX)+g(R(Y, JX)Y, JX)+g(R(Y, JY)X, JX).$$

On the other hand, we have

$$\begin{aligned} R(X^\theta, JX^\theta)X^\theta&=H(X^\theta)JX^\theta \\ &=H(X^\theta)\cos\theta\, JX+H(X^\theta)\sin\theta\, JY. \end{aligned} \tag{3.8}$$

Comparing (3.7) and (3.8), we have $H(X)=E$. Hence, $H(X^\theta)=\cos^2\theta\, H(X)+\sin^2\theta\, H(X)=H(X)$ for any θ. Changing Y in $\cos\varphi\, Y+\sin\varphi\, JY$, we see that (M, g, J) is of constant holomorphic sectional curvature at x.

COROLLARY 3.3. *Assume that an almost Hermitian manifold (M, g, J), $m\geq 4$, satisfies (3.4) and is of constant holomorphic sectional curvature at each point. If $H(X)$ is not equal to zero for some tangent vector X, then (M, g) is irreducible.*

Proof. Assume the contrary. Let U be an open set in M which is a Riemannian product $(U_1, g_1)\times(U_2, g_2)$. Let X be a vector field on U which is tangent to U_1-part. We decompose JX as $JX=(JX)_1+(JX)_2$ corresponding to the decomposition of the Riemannian product. Then we have $R(X, JX)X=R(X, (JX)_1)X$, and it is tangent to U_1-part. On the other hand, $R(X, JX)X$ is proportional to JX, i.e.,

$$R(X, JX)X=H(X)g(X, X)JX=H(X)g(X, X)[(JX)_1+(JX)_2].$$

That is, $H(X)g(X, X)(JX)_2=0$. Hence, if we choose U so that $H(X)\neq 0$ on U, we have $(JX)_2=0$. Therefore, if X (Y, resp.) is a vector field tangent to U_1-part (U_2-part, resp,), JX (JY, resp.) is tangent to U_1-part (U_2-part, resp.). Assume $g(X+Y, X+Y)=1$. Then

$$R(X+Y, JX+JY)(X+Y)=HJ(X+Y)=HJX+HJY. \tag{3.9}$$

On the other hand, we have

$$\begin{aligned} R(X+Y, JX+JY)(X+Y)&=R(X, JX)X+R(Y, JY)Y \\ &=Hg(X, X)JX+Hg(Y, Y)JY. \end{aligned} \tag{3.10}$$

If $g(X, X)=g(Y, Y)=1/2$, (3.9) and (3.10) are not compatible. Therefore, (M, g) is an irreducible Riemannian manifold. q.e.d.

By ∇ we denote the covariant diffenentiation with respect to g. In an almost Hermitian manifold (M, g, J), if $(\nabla_X J)X=0$ holds for every X, then (M, g, J) is called a K-space (in [17], almost Tachibana space in [25], etc. or nearly Kähler manifold in [6]). In a K-space we have (cf. Gray [6])

$$g(R(JX, JY)JZ, JW)=g(R(X, Y)Z, W). \tag{3.11}$$

THEOREM 3.4. *A K-space (M, g, J), $m\geq 4$, is of constant holomorphic sectional curvature, if and only if*

$$R(X, JX)X \quad \textit{is proportional to} \quad JX$$

for every vector field X on M.

Proof. Nakagawa [13] proved that, in a K-space, if holomorphic sectional curvature is constant at each point then it is constant on M. Therefore, Theorem 3.4 follows from (3.11) and Theorem 3.2. q.e.d.

An almost Hermitian manifold (M, g, J) is called a Kählerian manifold, if J is parallel, i.e., $\nabla J=0$. A Kählerian manifold is a K-space. A K-space is not always Kählerian. In fact, a unit 6-sphere S^6 with J defined by Frölicher [4] is not a Kählerian manifold, but a K-space (cf. Fukami and Ishihara [5]). A Kählerian manifold is of constant holomorphic sectional curvature if and only if (cf. Yano [25])

$$R(X, Y)Z=(H/4)[g(X,Z)Y-g(Y,Z)X+g(Y,JZ)JX-g(X,JZ)JY-2g(X,JY)JZ]. \tag{3.12}$$

It is not known if R can be expressed by g and J in K-spaces of constant holomorphic sectional curvature.

COROLLARY 3.5. (cf. Kosmanek [9]) *A Kählerian manifold, $m\geq 4$, is of constant holomorphic sectional curvature, if and only if*

$$R(X, JX)X \quad \textit{is proportional to} \quad JX$$

for every vector field X on M.

Then we have

COROLLARY 3.6. (Ogiue [15]) *A Kählerian manifold, $m\geq 4$, is of constant holomorphic sectional curvature, if and only if*

$$R(X, Y)Z \quad \textit{is a linear combination of} \quad X, Y, JX, JY, JZ$$

for every vector fields X, Y, and Z.

In fact, assume that $R(X, JX)X$ is a linear combination of X and JX. Since $R(X, JX)X$ is orthogonal to X, Corollary 3.6 follows from (3.12) and Corollary 3.5.

REMARK 1. In the original proof of Ogiue [15], it is not clear, especially for $m=4$, that under the assumption " $R(X, Y)Z$ is a linear combination of X, Y, JX,

JY, JZ for any X, Y, Z" there exist five tensor fields which express $R(X, Y)Z$. Because X, Y, JX, JY, JZ are linearly dependent in almost all cases for $m=4$.

REMARK 2. In an almost Hermitian manifold, an orthonormal basis (X_i) is called a J-basis, if $X_\alpha=X_\alpha$, $X_{\alpha-n}=JX_\alpha$, $\alpha=1, \cdots, n=m/2$. It is stated in [1] (Bishop and Goldberg, p. 532) that a Kählerian manifold ($m\geq 4$) is of constant holomorphic sectional curvature if and only if $g(R(X, Y)X, Z)=0$ for every x and every part (X, Y, Z) of every J-basis at x.

REMARK 3. In a Kählerian manifold, Kosmanek [9] gave a condition for the space to be of constant holomorphic sectional curvature in terms of Jacobi fields along geodesics. Its proof is completed by proving Corollary 3. 5.

§4. Sasakian manifolds of constant ϕ-holomorphic sectional curvature.

Let (M, ϕ, ξ, η, g) be a Sasakian manifold (=normal contact Riemannian manifold, see for example [19], [20], etc.). The structure tensors satisfy

$$\phi\xi=0, \qquad \phi\phi X=-X+g(X, \xi)\xi,$$
$$g(\phi X, \phi Y)=g(X, Y)-g(X, \xi)g(Y, \xi).$$

We denote by R the Riemannian curvature tensor with respect to g. Then R satisfies

$$R(X, Y)\xi=g(X, \xi)Y-g(Y, \xi)X. \tag{4. 1}$$

The ϕ-holomorphic sectional curvature $H(X)$ for a unit tangent vector X orthogonal to ξ is given by $H(X)=g(R(X, \phi X)X, \phi X)$. A Sasakian manifold has constant ϕ-holomorphic sectional curvature $H=H_x$ at x if and only if (cf. Tashiro and Tachibana [23], Ogiue [14])

$$\begin{aligned} 4R(X, Y)Z=&(H+3)[g(X, Z)Y-g(Y, Z)X]\\ &+(H-1)[g(Y, \phi Z)\phi X-g(X, \phi Z)\phi Y-2g(X, \phi Y)\phi Z\\ &+\eta(Y)\eta(Z)X+g(Y, Z)\eta(X)\xi-\eta(X)\eta(Z)Y-g(X, Z)\eta(Y)\xi]. \end{aligned} \tag{4. 2}$$

If $H(X)$ is constant at every point x, then H is constant on M for $m\geq 5$.

Recently, Sekizawa [16] obtained the following

PROPOSITION 4. 1. (Sekizawa) *A Sasakian manifold, $m\geq 5$, is of constant ϕ-holomorphic sectional curvature, if and only if $R(X, Y)Z$ is a linear combination of $X, Y, \phi X, \phi Y, \phi Z$ and ξ, for every vector fields X, Y, and Z.*

As in Remark 1 in §3, in the proof given in [16], it is not clear that there exist six tensor fields of covariant degree 2 and 3, which express $R(X, Y)Z$.

So, we give a generalization and a simple proof.

THEOREM 4. 2. *A Sasakian manifold, $m\geq 5$, is of constant ϕ-holomorphic sectional curvature, if and only if*

$$R(X, \phi X)X \text{ is proportional to } \phi X$$

for every vector field X such that $g(X, \xi)=0$.

Proof. If a Sasakian manifold is of constant ϕ-holomorphic sectional curvature, we see that $R(X, \phi X)X$ is proportional to ϕX by (4. 2). We prove the converse. Let U be a cubical and regular neighborhood of x with respect to ξ. Then we have a local fibering π: $U\rightarrow U/\xi$, where U/ξ is a Kählerian manifold with structure tensore (h, J) such that $g=\pi^{*}h+\eta\otimes\eta$ and $(Ju)^{*}=\phi u^{*}$ for every vector field u on U/ξ, u^{*} denoting the horizontal lift of u with respect to the contact form η (cf. [22], etc.). By $'R$ we denote the Riemannian curvature tensor with respect to h. Then we have (cf. (5. 8) in [19])

$$g(R(u^{*}, z^{*})y^{*}, z^{*})=h('R(u, z)y, z)\cdot\pi-3h(u, Jz)h(y, Jz)\cdot\pi.$$

We put $u=Jz$, and choose y so that $h(y, z)=h(y, Jz)=0$. Then $g(R(z^{*}, \phi z^{*})z^{*}, y^{*})=0$ implies $h('R(z, Jz)z, y)=0$. Hence $(U/\xi, h, J)$ is of constant holomorphic sectional curvature $'H$ by Corollary 3. 5. Consequently, U, and hence M, is of constant ϕ-holomorphic sectional curvature $'H-3$ (cf. for example (3. 6) in [22]). q.e.d.

Proof of Proposition 4. 1. Let X be orthogonal to ξ. Assume that $R(X, JX)X$ is a linear combination of X, JX, ξ. Clearly X-part is vanishing. We show that ξ-part is also vanishing. By (4. 1) we have $R(X, JX)\xi=0$. Hence, ξ-part $=g(R(X, JX)X, \xi)=-g(R(X, JX)\xi, X)=0$. Therefore, Proposition 4. 1 follows from (4. 2) and Theorem 4. 2.

§ 5. *K*-spaces of constant holomorphic sectional curvature.

Let (M, g, J) be a K-space. We pick up the known results which we need later.

(i) In a K-space we have (cf. Takamatsu [18], (3. 2))

$$(R_{ji}-R^{*}{}_{ji})(5R^{*ji}-R^{ji})=0, \tag{5. 1}$$

where R_{ji} denote components of the Ricci curvature tensor and

$$2R^{*}{}_{ji}=J^{ab}R_{abti}J_{j}{}^{t}.$$

(ii) If $m=2n=6$ and (M, g, J) is not Kählerian we have (cf. [18], (4. 3))

$$5S^{*}=S>0, \tag{5. 2}$$

where S denotes the scalar curvature and $S^{*}=g^{ji}R^{*}{}_{ji}$.

(iii) A K-space is Kählerian, if and only if $S=S^{*}$.

(iv) If a K-space is of constant holomorphic sectional curvature H, then we have (Nakagawa [13], p. 272)

$$R_{ji}+3R^*{}_{ji}=2(n+1)Hg_{ji}, \tag{5.3}$$

$$S+3S^*=4n(n+1)H, \tag{5.4}$$

$$S\geq n(n+1)H. \tag{5.5}$$

(v) In a 6-dimensional K-space, J_{ji} is a special Killing tensor and (M, g, J) is an Einstein space (Matsumoto [12]).

(vi) If J_{ji} is a special Killing tensor, (M, g, J) is of point-wise constant type (Yamaguchi, Chuman and Matsumoto [24]), i.e.,

(5. 6) $$||(\nabla_X J)Y||^2=0, \qquad \text{or}$$

(5. 6)′ $$||(\nabla_X J)Y||^2=(S/30)[||X||^2||Y||^2-g(X, Y)^2-g(X, JY)^2].$$

(vii) If a K-space has constant holomorphic sectional curvature H, then for every orthonormal pair (X, Y), we have (Gray [6], p. 288)

$$K(X, Y)=(H/4)[1+3g(X, JY)^2]+(3/4)||(\nabla_X J)Y||^2. \tag{5.7}$$

LEMMA 5. 1. *If a* 6-*dimensional K-space* (M, g, J) *is of constant holomorphic sectional curvature H, then either* (M, g, J) *is Kählerian, or* (M, g, J) *is of constant curvature* $H>0$.

Proof. The case (5. 6) is Kählerian. If (5. 6)′ holds for non-zero S, this is a non-Kählerian case. By (5. 2) and (5. 4), we have

$$S=5n(n+1)H/2=30H>0. \tag{5.8}$$

Then, (5. 6)′ and (5. 7) give $K(X, Y)=H>0$. q.e.d.

By $CP^n[H]$, $CE^n[0]$, and $CD^n[H]$ we denote the simply connected Kählerian space forms of complex n-dimension corresponding to constant holomorphic sectional curvature $H>0$, $H=0$, and $H<0$, respectively. By $S^6[H]$ we denote a 6-sphere of constant curvature $H>0$, which has the natural K-space structure.

THEOREM 5. 2. *Let* (M, g, J) *be a complete* 6-*dimensional K-space. If* $R(X, JX)X$ *is proportional to* JX *for any vector field* X *on* M, *then* (M, g, J) *is one of the following K-spaces*:

$$S^6[H], \qquad CP^3[H], \qquad CE^3[0]/\Gamma_1, \qquad CD^3[H]/\Gamma_2,$$

where Γ_1, Γ_2 *are fixed point free discrete subgroups of the automorphism groups of* $CE^3[0]$, $CD^3[H]$. *The converse is also true.*

Proof. If (M, g, J) is Kählerian, it is one of the last three spaces listed (cf.

Hawley [7], Igusa [8], Lu Qu-Keng [11], etc.). If it is not Kählerian, (M, g, J) is $S^6[H]$. q.e.d.

Transvecting (5. 3) with R^{ji} and R^{*ji}, we have

$$R_{ji}R^{ji}+3R^*_{ji}R^{ji}=2(n+1)HS, \tag{5. 9}$$

$$R^*_{ji}R^{ji}+3R^*_{ji}R^{*ji}=2(n+1)HS^*. \tag{5. 10}$$

Using (5. 9) and (5. 10), by (5. 1) we get

$$32R_{ji}R^{ji}=46(n+1)HS-30(n+1)HS^*.$$

Then using (5. 4) we have

$$4R_{ji}R^{ji}-7(n+1)HS+5n(n+1)^2H^2=0. \tag{5. 11}$$

The equation (5. 11) is also witten as

$$[2R_{ji}-(n+1)Hg_{ji}][2R^{ji}-(n+1)Hg^{ji}]-3(n+1)H[S-n(n+1)H]=0, \tag{5. 11$'$}$$

$$[4R_{ji}-5(n+1)Hg_{ji}][4R^{ji}-5(n+1)Hg^{ji}]+6(n+1)H[2S-5n(n+1)H]=0. \tag{5. 11$''$}$$

THEOREM 5. 3. *If a K-space (M, g, J) has constant holomorphic sectional curvature H, and if it is not Kählerian, then $H>0$.*

Proof. Suppose that $H\leq 0$. If $H=0$, then we have $R_{ji}=0$ by (5. 11) and $S=0$ follows. (5. 4) implies $S=S^*$ and (M, g, J) is Kählerian. If $H<0$, (5. 11)′ gives $S\leq n(n+1)H$. Then (5. 5) implies $S=n(n+1)H$. (5. 4) implies $S=S^*$ and (M, g, J) is Kählerian.

THEOREM 5. 4. *Assume that a K-space (M, g, J) has constant holomorphic sectional curvature H.*

(1) *If (M, g, J) is Kählerian, the scalar curvature $S=n(n+1)H$.*

(2) *If (M, g, J) is not Kählerian, then $H>0$ and*

$$n(n+1)H<S\leq(5/2)n(n+1)H.$$

where equality is attained by and only by (M, g)=Einstein space.

Proof. (1) is well known. (2) follows from Theorem 5. 3 and (5. 11)′ and (5. 11)″. If equality holds, (M, g) is an Einstein space by (5. 11)″. Conversely, if (M, g) is an Einstein space, putting $R_{ji}R^{ji}=S^2/2n$, by (5. 11) we have $[2S-5n(n+1)H][S-n(n+1)H]=0$. Therefore we have $2S=5n(n+1)H$.

REMARK 4. Nakagawa [13] showed that an almost analytic vector X in a compact K-space of constant holomorphic sectional curvature $H\leq 0$ is an infinitesimal automorphism. By Theorem 5. 3, we see that this case is Kählerian and, consequently, (M, g) is an Einstein space. If $H=0$, then X is parallel. If $H<0$, we have no such $X\neq 0$. More generally in the last case, there is no non-trivial isometry which is homotopic to the identity map (cf. Frankel [3]).

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