Constant mean curvature hypersurfaces condensing along a submanifold

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1 Introduction

Let S be an oriented embedded (or possibly immersed) hypersurface in a Riemannian manifold (M^{m+1}, g). The shape operator A_S is the symmetric endomorphism of the tangent bundle TS associated with the second fundamental form of S, b_S, by

$$b_S(X, Y) = g_S(A_S\, X, Y), \quad \forall\, X, Y \in TS; \qquad \text{here} \quad g_S = g|_{TS}\,.$$

The eigenvalues κ_i of the shape operator A_S are the principal curvatures of the hypersurface S. The mean curvature H_S is defined to be the average of these principal curvatures:

$$H(S) := \frac{1}{m}(\kappa_1 + \ldots + \kappa_m).$$

Constant mean curvature (CMC) hypersurfaces in a compact Riemannian manifold (M^{m+1}, g) constitute an important class of submanifolds and have been studied extensively. In this paper we are interested in degenerating families of such submanifolds which 'condense' to a submanifold $K^k \subset M^{m+1}$ of codimension greater than 1. It is not hard to see that the closer a CMC hypersurface is (e.g. in the Hausdorff metric) to such a submanifold, the larger its mean curvature must be; in other words, the mean curvatures of the elements of a condensing family of CMC hypersurfaces must tend to infinity. Less obvious is the fact that under fairly mild geometric assumptions, cf. [10], the existence of such a family implies that K is minimal. Two cases have been studied previously: Ye [14], [15] proved the existence of a local foliation by constant mean curvature hypersurfaces condensing to a point (which is required to be a nondegenerate critical point of the scalar curvature function); more recently, the second and third authors [10] proved the existence of a 'partial foliation' by constant mean curvature hypersurfaces in a neighborhood of a nondegenerate closed geodesic. In this paper we extend the result and methods of [10] to handle the general case, when K is an arbitrary nondegenerate minimal submanifold (no extra curvature hypotheses are required). As we explain below, this more general problem has a number of new analytic and geometric features, and despite the apparent similarities with the case when K is one-dimensional, is considerably more subtle to analyze.

We now describe our result in more detail. Let K^k be a closed (embedded or immersed) submanifold in M^{m+1}, $1 \leq k \leq m-1$; the geodesic tube of radius ρ about K is

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the set

$$\bar{S}_\rho := \{q \in M^{m+1} : \quad \mathrm{dist}_g(q, K) = \rho\}.$$

This is a smooth (immersed) hypersurface provided ρ is smaller than the radius of curvature of K, and henceforth we always tacitly assume that this is the case. An elementary computation shows that the mean curvature at any point of this tube satisfies

$$H(\bar{S}_\rho) = \frac{n-1}{m}\,\rho^{-1} + \mathcal{O}(1), \qquad \text{as} \qquad \rho \searrow 0, \qquad n = m+1-k.$$

At first glance, it seems plausible that we should be able to perturb this tube slightly to obtain a constant mean curvature hypersurface with $H \equiv \frac{n-1}{m}\,\rho^{-1}$. The standard method to do this is to consider all nearby hypersurfaces which can be written as normal graphs over $\bar{S}_\rho$ and to consider the constant mean curvature equation as a nonlinear elliptic equation, to which one can apply familiar PDE methods. More specifically, one expects to be able to find the solution by a contraction mapping argument using the inverse of the linearized mean curvature operator (also known as the Jacobi operator). The complication is that as the mean curvatures of $\bar{S}_\rho$ become more nearly constant, the geometry is degenerating, so this must be treated as a singular perturbation problem.

There are two potential obstacles to carrying this out. First, for general submanifolds K, the difference $H(\bar{S}_\rho) - ((n-1)/m)\rho^{-1}$ does not decay as $\rho \to 0$, but is only bounded, and it turns out that this error term is too large to solve away. When K is minimal, however, one obtains (cf. §4) the finer estimate

$$H(\bar{S}_\rho) = \frac{n-1}{m}\,\rho^{-1} + \mathcal{O}(\rho),$$

and one can then proceed. Second, however, it is necessary that the Jacobi operator on $\bar{S}_\rho$ be invertible. This is not true at all radii; indeed, there is a spectral flow of eigenvalues of this operator across zero as $\rho \to 0$. Fortunately, so long as one assumes that K itself is nondegenerate as a minimal submanifold (i.e. its Jacobi operator is invertible), one can control the rate of this spectral flow as a function of ρ, and hence deduce the existence of infinitely many disjoint intervals in the ρ axis converging to zero and which are disjoint from these 'resonant' values. It is necessary to deduce that these nonresonant intervals are large, so as to control the norm of the inverse of the Jacobi operator. Unfortunately, even then the error term may be too large, so to combat this, we need to obtain much better approximate solutions. These are constructed using a preliminary finite iteration based on the invertibility of the Jacobi operator of K as well as another very crude approximation to the Jacobi operator for $\bar{S}_\rho$.

In the end, we prove the following

Theorem 1.1 *Let $K^k \subset M^{m+1}$ be a closed (embedded or immersed) minimal submanifold, $1 \leq k \leq m-1$, which is nondegenerate in the sense that its Jacobi operator is invertible. There exists an open subset $I \subset (0, \rho_0)$, which is a countable union of disjoint open intervals, such that for all $\rho \in I$, the geodesic tube $\bar{S}_\rho$ may be perturbed to a constant mean curvature hypersurface S_ρ with $H = \frac{n-1}{m}\,\rho^{-1}$. This set I is quite large in the sense that for any $q \geq 2$, there exists a constant $c_q > 0$ such that*

$$|\mathcal{H}^1((0,\rho) \cap I) - \rho| \leq c_q\,\rho^q,$$

where $\mathcal{H}^1$ denotes 1-dimensional Hausdorff measure. Furthermore, the index of the hypersurface S_ρ (for the quadratic form associated to its Jacobi operators) tends to $+\infty$ as $\rho \to 0$, $\rho \in I$.

The nondegeneracy condition on K imposed here is a mild restriction which holds for generic metrics on M [17]. Also, for arbitrary metrics on M it is definitely not possible to obtain a smooth family of hypersurfaces S_ρ for every single radius $\rho > 0$. Indeed, for generic metrics, the moduli space of all CMC hypersurfaces (with mean curvatures assuming any value in $\mathbb{R}$) is a smooth one-dimensional manifold, and the fact that the Morse index attains infinitely many values shows that this moduli space has infinitely many components. For special (nongeneric) metrics, the moduli space may be connected, but singular, and the phenomenon of resonant radii should correspond to other families of CMC hypersurfaces bifurcating from this main 'tubular' family. Explicit examples of this are known when $k = 1$, cf. [10], and in more general cases (also when $k > 1$) can be obtained by general bifurcation-theoretic techniques as in [9]. When $k = 1$ the elements of these bifurcating families have undulations modelled on Delaunay surfaces [5], but the geometric picture when $k > 1$ is unknown. (A similar bifurcation phenomenon appears in [7], and of course in many other settings too.) Finally, we note that an immediate corollary of this theorem is the existence of CMC hypersurfaces with nontrivial topology in any compact Riemannian manifold.

As already noted, our earlier proof of this theorem for the case $k = 1$ [10] follows the same general pattern, but is substantially easier in all the main technical points. Perhaps the biggest difference is that in this one-dimensional case, the spectral gaps are automatically large and so there is no need to estimate these or to find a sequence of improved approximate solutions. Several new ideas were needed in order to extend the result to the general case. These ideas were inspired by recent work of Malchiodi and Montenegro in a somewhat different context [8], [6], cf. also related work by Shatah and Zeng concerning existence of periodic solutions for a penalized Hamiltonian system [12].

Before continuing with a more detailed explanation of the contents of this paper, we make some further remarks about the geometric problem. One interesting question is to determine the extent to which our result has a converse; in other words, one would like to study possible limits of 'condensing families of CMC hypersurfaces. This requires, however, a more general definition of this geometric condensation. One possibility is to consider weak limits of rescaled area and curvature densities. For example, for the family S_ρ constructed in Theorem 1.1, we have

$$\rho^{k-m}\,\mathcal{H}^m \llcorner S_\rho \rightharpoonup \omega_{m-k}\,\mathcal{H}^k \llcorner K, \tag{1.1}$$

and, for all $q \geq 1$,

$$\rho^{k-m+q}\,|A_{S_\rho}|^q\,\mathcal{H}^m \llcorner S_\rho \rightharpoonup (m-k)^{q/2}\,\omega_{m-k}\,\mathcal{H}^k \llcorner K, \tag{1.2}$$

as $\rho \searrow 0$; here $|A_S|^2 := \mathrm{Tr}((A_S)^t\,A_S)$ is the norm squared of the shape operator. In fact, the behaviour of these limits is easy to deduce for the tubes $\bar{S}_\rho$, and the analogues of (1.1) and (1.2) for the family of solutions S_ρ are obtained because we have good quantitative control of the deviation of these surfaces from the $\bar{S}_\rho$. This leads one to think that one should be considering more general families of CMC surfaces Σ_ρ satisfying both (1.1) and (1.2) for some submanifold K. However, even this is not quite general enough, and in [5] families of CMC hypersurfaces are constructed which condense along a nondegenerate geodesic but which do not satisfy (1.1). The point is that, as is well known in geometric measure theory, the limit is a submanifold K along with a density function on K. Furthermore, it is unreasonable to expect the limit always to be supported along a smooth submanifold.

To give some sort of illustration of what might happen, we note that one should be able to construct families of constant mean curvature hypersurfaces which condense along submanifolds with singularities, which are still minimal in an appropriate sense. A simple example of this is when the family Σ_ρ is the homothetic rescaling of a fixed Delaunay triunduloid in $\mathbb{R}^3$. The geometric limit is a union of three rays meeting at a common vertex, and the area density converges to this set with a density computable from the Delaunay necksize on that end of the original triunduloid. Each ray is minimal, of course, and this configuration of rays with density is 'balanced' in the sense that the weighted sum of the vectors along the rays vanishes. One would hope to be able to prove some sort of structure theorem for limiting configurations in greater generality.

As a preliminary step in this direction, Rosenberg has recently proved [11] that if S is an embedded surface with constant mean curvature H in a compact Riemannian 3-manifold (M, g), then if H is sufficiently large, S separates M into two components. Furthermore, there exists a constant c depending only on (M, g) such that the mean convex component of $M \setminus S$ does not contain any geodesic ball of radius c/H. This says intuitively that S is contained in a cylinder of radius c/H around some 1-dimensional set, which one expects in general can be taken as a geodesic network in M.

We now conclude this introduction with an overview of the rest of this paper. Because of the rather technical nature of the proof of our main result, we shall make this overview fairly detailed.

The first step, in §2, is to derive the Taylor expansion of the metric g in Fermi coordinates about K up to second order. This is a fairly routine geometric calculation, which is quite similar to the one in [10] for the case when K is a curve, but we present it anyway since it should help prepare the reader for what comes later. We next define the class of perturbations of the geodesic tubes $\bar{S}_\rho$. It turns out to be most convenient to describe these using a two step process: first deform the minimal submanifold K in the normal direction to a new k-dimensional submanifold, then form the tube of radius ρ around this new submanifold and finally take a normal graph over this tube by some function w. The first of these deformations corresponds to a section Φ of the normal bundle NK. For each $k \in K$, the normal vector $\Phi(k)$ determines a linear (height) function $g(\Phi, \cdot)$ on $N_k K$, or equivalently an element of the first nonzero eigenspace of the Laplacian on the fibre of the spherical normal bundle at that point. Because of this, we can eliminate a redundancy in the parametrization by demanding that w be orthogonal to this eigenspace on each such fibre. We have thus parametrized nearby surfaces by pairs (w, Φ), defined and satisfying the restrictions as written above and and so denote them as $S_\rho(w, \Phi)$. We explain the role of this normal perturbation in more detail below.

The main work in the long technical §3 is to calculate the mean curvature of $S_\rho(w, \Phi)$ as a nonlinear elliptic partial differential operator, depending on ρ, acting on (w, Φ). This requires first computing the metric (§3.3), the unit normal vector (§3.4) and the second fundamental form (the very intricate §3.5). During these calculations it is important to gather together various different types of error terms, some of which depend linearly and some quadratically on (w, Φ), and some of which are inhomogeneous terms vanishing to some order in ρ. To obtain the best forms for these various expressions, we replace w by ρw; more seriously, it turns out to be helpful to rescale the local coordinates in K by ρ, and to consider Φ (but not w) as a function of these rescaled coordinates. (The simultaneous use of 'slow' and 'fast' variables is commonplace in the study of reaction-diffusion equations.) The final expression (4.26) for the mean curvature of $S_\rho(w, \Phi)$ then involves several familiar pieces: the Jacobi operator for K, $\mathcal{J}_K$, the model operator $\mathcal{L}_\rho = -(\rho^2 \Delta_K + \Delta_{S^{n-1}} + (n-1))$ (which can be interpreted as acting on functions on the spherical normal bundle SNK), the contraction of the Hessian of w on K with the second fundamental form, and various error terms. The natural sizes of both w and Φ are $\mathcal{O}(\rho^2)$.

The decomposition of functions on SNK into those which are orthogonal to the linear functions on each fibre and the fibrewise linear functions, expressed as $v = \rho w + g(\Theta, \Phi)$, is also discussed at the end of §4.

Following these preliminaries, the actual construction can now be described. Because the inverse of the Jacobi operator is fairly large, we first improve the approximate solution so that it vanishes to some arbitrarily large order in ρ. This is accomplished in §5 using a finite iteration using the inverse of the operator

$$\rho^{-1}(\Delta_{S^{n-1}} + (n-1)) + g(\mathcal{J}_K \cdot, \Theta)$$

(the two terms here act separately on (w, Φ)). Notice that we are omitting the term Δ_K from $\mathcal{L}_\rho$ in this step; the point is that the two terms are completely decoupled, so we are solving for the first operator acting on the subspace of $L^2(SNK)$ orthogonal to the nullspace of this operator, where the position on K is only a parameter, and the second acting on sections of NK, interpreted as being the complement in L^2 of the previous subspace. Given any positive integer i, we can now arrange that the error term vanishes like ρ^i. Notice, however, that it would be impossible to use this simpler operator for the full infinite iteration because it does not gain regularity for w in the K directions.

Now denote the linearized mean curvature operator about this improved approximate solution (and with all of the preceding normalizations) by $\mathbb{L}_{\rho,i}$. As explained earlier, there is a spectral flow across 0 for this operator as $\rho \to 0$, and hence there exists an infinite sequence of values $\rho_j \to 0$ at which $\mathbb{L}_{\rho,i}$ is not invertible. We are interested in estimating the size of the spectral gaps, which determine the size of the norm of the inverse. This is equivalent to understand the rate of this spectral flow. The usual formula for the variation of eigenvalues with respect to ρ must be interpreted carefully, to allow for multiplicities, but there is a good formalism for this, so in §6 we show that the eigenvalues near 0 are monotone decreasing in ρ (when ρ is small), and hence obtain an estimate for the size of the spectral gaps, and for the Morse index of $\mathbb{L}_{\rho,i}$.

With all of these preliminaries, the final step of solving the precise equation to obtain a CMC perturbation of $\bar{S}_\rho$ is straightforward, and is carried out in §7.

To clarify the need for such an intricate construction, let us recall the steps in Ye's construction [14], corresponding to the case $k = 0$, and contrast them with the ones here. In [14], one is trying to perturb the geodesic sphere of radius ρ around a point $p \in M$ to have constant mean curvature. In order to find an approximate solution to one order better than initially expected, it is necessary to assume that p is a critical point of the scalar curvature function; here, when $k > 0$, this property is guaranteed by the minimality of K. Next, for any value of k, the linearized operator is 'nearly degenerate'. When $k = 0$, the dimension of the 'approximate nullspace', i.e. the number of small (possibly vanishing) eigenvalues is equal to n and the translations of the center of the geodesic sphere provide the correct set of extra parameters to compensate for the corresponding 'approximate cokernel'. When $k > 0$, on the other hand, the number of small eigenvalues tends to infinity as $\rho \searrow 0$, so one needs in practice an infinite number of parameters to compensate. This is precisely the role of the normal perturbation Φ of K; the nondegeneracy of the Jacobi operator of K (which is trivial when $k = 0$) is needed to use these parameters effectively. Note that when $k = 0$ or 1 it is unnecessary to find an improved approximate solution because one has good estimates for the inverse of the linearization, but these fail when $k > 1$. At any rate, the preliminary normal translation of K by Φ when $k \geq 1$ corresponds precisely to the translation parameter of the center of the sphere in Ye's construction.

We emphasize again that despite the many formal similarities of this problem when $k = 1$ and $k > 1$, there is a much better a priori understanding of the geometry and analysis when K is a curve; the turning points of this family of CMC tubes when $k > 1$ remain unclear from a geometric point of view, and as explained above, the details of the analysis needed for the construction are substantially different too.

2 Expansion of the metric in Fermi coordinates near K

2.1 Fermi coordinates

We now introduce Fermi coordinates in a neighborhood of K. For a given $p \in K$, there is a natural splitting

$$T_pM = T_pK \oplus N_pK.$$

Choose orthonormal bases E_a, $a = n+1, \dots, m+1$, for T_pK, and E_i, $i = 1, \dots, n$, of N_pK.

Notation : We shall always use the convention that indices $a, b, c, d, \dots \in \{n+1, \dots, m+1\}$, indices $i, j, k, \ell, \dots \in \{1, \dots, n\}$ and indices $\alpha, \beta, \gamma, \dots \in \{1, \dots, m+1\}$.

Consider, in a neighborhood of p in K, normal geodesic coordinates

$$f(y) := \exp_p^K(y^a\, E_a), \qquad y := (y^{n+1}, \dots, y^{m+1}),$$

where $\exp^K$ is the exponential map on K and summation over repeated indices is understood. This yields the coordinate vector fields $X_a := f_*(\partial_{y^a})$. For any $E \in T_pK$, the curve

$$s \longrightarrow \gamma_E(s) := \exp_p^K(sE),$$

is a geodesic in K, so that

$$\nabla_{X_a} X_b|_p \in N_pK.$$

We define the numbers Γ^i_{ab} by

$$\nabla_{X_a} X_b|_p = \Gamma^i_{ab}\, E_i.$$

Now extend the E_i along each $\gamma_E(s)$ so that they are parallel with respect to the induced connection on the normal bundle NK. This yields an orthonormal frame field X_i for NK in a neighborhood of p in K which satisfies

$$\nabla_{X_a} X_i|_p \in T_pK,$$

and hence defines coefficients Γ^b_{ai} by

$$\nabla_{X_a} X_i|_p = \Gamma^b_{ai}\, E_b.$$

A coordinate system in a neighborhood of p in M is now defined by

$$F(x, y) := \exp^M_{f(y)}(x^i\, X_i), \qquad (x, y) := (x^1, \dots, x^n, y^{n+1}, \dots, y^{m+1}),$$

with corresponding coordinate vector fields

$$X_i := F_*(\partial_{x^i}) \qquad \text{and} \qquad X_a := F_*(\partial_{y^a}).$$

By construction, $X_\alpha\, |_p = E_\alpha$.

2.2 Taylor expansion of the metric

As usual, the Fermi coordinates above are defined so that the metric coefficients

$$g_{\alpha\beta} = g(X_\alpha, X_\beta),$$

equal $\delta_{\alpha\beta}$ at p; furthermore, $g(X_b, X_i) = 0$ in some neighborhood of p in K. This implies that

$$X_a\, g(X_b, X_i) = g(\nabla_{X_a} X_b, X_i) + g(X_b, \nabla_{X_a} X_i) = 0,$$

on K, which yields the identity

$$\Gamma^i_{ab} + \Gamma^b_{ai} = 0. \tag{2.3}$$

at p.

Denote by $\Gamma^b_a : N_pK \longrightarrow \mathbb{R}$ the linear form

$$\Gamma^b_a(\cdot) := g(\nabla_{E_a} E_b, \cdot) = -g(\nabla_{E_a} \cdot, E_b). \tag{2.4}$$

We now compute higher terms in the Taylor expansions of the functions $g_{\alpha\beta}$. The metric coefficients at $q := F(x, 0)$ are given in terms of geometric data at $p := F(0, 0)$ and $|x| = \mathrm{dist}_g(p, q)$.

Notation The symbol $\mathcal{O}(|x|^r)$ indicates a function such that it and its partial derivatives of any order, with respect to the vector fields X_a and $x^i\, X_j$, are bounded by $c\,|x|^r$ in some fixed neighborhood of 0.

We begin with the expansion of the covariant derivative :

Lemma 2.1 *At the point of $q = F(x, 0)$, the following expansions hold*

$$\begin{aligned}
\nabla_{X_i} X_j &= \mathcal{O}(|x|) X_\gamma, \\
\nabla_{X_a} X_b &= \Gamma^b_a(E_i)\, X_i + \mathcal{O}(|x|) X_\gamma, \\
\nabla_{X_a} X_i &= \nabla_{X_i} X_a = -\Gamma^b_a(E_i)\, X_b + \mathcal{O}(|x|)^\gamma\, X_\gamma,
\end{aligned} \tag{2.5}$$

Proof: Observe that, because we are using coordinate vector fields, $\nabla_{X_\alpha} X_\beta = \nabla_{X_\beta} X_\alpha$ for any α, β. We also have $\nabla_{X_i} X_j \;|_p = 0$ since any $X \in N_pK$ is tangent to the geodesic $s \longrightarrow \exp^M_p(sX)$, and hence

$$\nabla_{X_i + X_j}(X_i + X_j)\big|_p = 0.$$

Therefore

$$(\nabla_{X_i} X_j + \nabla_{X_j} X_i)\big|_p = 0,$$

and this completes the proof of the first estimate.

We have by construction

$$\nabla_{X_a} X_b = \Gamma^i_{ab}\, X_i + \mathcal{O}(|x|)\, X_\gamma,$$

and

$$\nabla_{X_a} X_i = \nabla_{X_i} X_a = \Gamma^b_{ai}\, X_b + \mathcal{O}(|x|)\, X_\gamma.$$

The next two estimates follow from the definition of Γ^b_a and (2.3). □

We now give the expansion of the metric coefficients. The expansion of the g_{ij}, $i, j = 1, \ldots, n$, agrees with the well known expansion for the metric in normal coordinates [13], [4], [18], but we briefly recall the proof here for completeness.

Proposition 2.1 *At the point $q = F(x,0)$, the following expansions hold*

$$
\begin{aligned}
g_{ij} &= \delta_{ij} + \tfrac{1}{3}\, g(R(E_k,E_i)\,E_\ell,E_j)\, x^k\, x^\ell + \mathcal{O}(|x|^3),\\
g_{ai} &= \mathcal{O}(|x|^2),\\
g_{ab} &= \delta_{ab} - 2\,\Gamma_a^b(E_i)\, x^i + \left[g(R(E_k,E_a)\,E_\ell,E_b) + \Gamma_a^c(E_k)\,\Gamma_c^b(E_\ell)\right]\, x^k\, x^\ell + \mathcal{O}(|x|^3),
\end{aligned}
\tag{2.6}
$$

where summation over repeated indices is understood.

Proof: By construction, $g_{\alpha\beta} = \delta_{\alpha\beta}$ at p, and so

$$g_{\alpha\beta} = \delta_{\alpha\beta} + \mathcal{O}(|x|).$$

Now, from

$$X_i\, g_{\alpha\beta} = g(\nabla_{X_i} X_\alpha, X_\beta) + g(X_\alpha, \nabla_{X_i} X_\beta),$$

and Lemma 2.1, we get

$$X_i\, g_{aj}\big|_p = 0, \qquad X_i\, g_{jk}\big|_p = 0 \qquad \text{and} \qquad X_i\, g_{ab}\big|_p = \Gamma_{ai}^b + \Gamma_{ib}^a = 2\Gamma_{ai}^b.$$

This yields the first order Taylor expansion

$$g_{aj} = \mathcal{O}(|x|^2), \qquad g_{ij} = \delta_{ij} + \mathcal{O}(|x|^2) \qquad \text{and} \qquad g_{ab} = \delta_{ab} + 2\,\Gamma_{ai}^b\, x^i + \mathcal{O}(|x|^2).$$

To compute the second order terms, it suffices to compute $X_k\, X_k\, g_{\alpha\beta}$ at p and polarize (i.e. replace X_k by $X_i + X_j$, etc.). We compute

$$X_k\, X_k\, g_{\alpha\beta} = g(\nabla^2_{X_k} X_\alpha, X_\beta) + g(X_\alpha, \nabla^2_{X_k} X_\beta) + 2\, g(\nabla_{X_k} X_\alpha, \nabla_{X_k} X_\beta). \tag{2.7}$$

To proceed, first observe that

$$\nabla_X X\big|_{p'} = \nabla^2_X X\big|_{p'} = 0,$$

at $p' \in K$, for any $X \in N_{p'}K$. Indeed, for all $p' \in K$, $X \in N_{p'}K$ is tangent to the geodesic $s \longrightarrow \exp^M_{p'}(sX)$, and so $\nabla_X X = \nabla^2_X X = 0$ at the point p'.

In particular, taking $X = X_k + \varepsilon\, X_j$, we obtain

$$0 = \nabla_{X_k + \varepsilon X_j} \nabla_{X_k + \varepsilon X_j} (X_k + \varepsilon X_j)\ _{|p},$$

equating the coefficient of ε to 0 gives $\nabla_{X_j} \nabla_{X_k} X_k\ _{|p} = -2 \nabla_{X_k} \nabla_{X_k} X_j\ _{|p}$, and hence

$$3\, \nabla^2_{X_k} X_j\big|_p = R(E_k, E_j)\, E_k,$$

So finally, using (2.7) together with the result of Lemma 2.1, we get

$$X_k\, X_k\, g_{ij}\big|_p = \frac{2}{3}\, g(R(E_k, E_i)\, E_k, E_j).$$

The formula for the second order Taylor coefficient for g_{ij} now follows at once.

Recall that, since X_γ are coordinate vector fields, we have from (2.7)

$$\nabla^2_{X_k} X_\gamma = \nabla_{X_k} \nabla_{X_\gamma} X_k = \nabla_{X_\gamma} \nabla_{X_k} X_k + R(X_k, X_\gamma)\, X_k.$$

Using (2.7), this yields

$$\begin{aligned} X_k\, X_k\, g_{ab} &= 2\, g(R(X_k, X_a)X_k, X_b) + 2\, g(\nabla_{X_k} X_a, \nabla_{X_k} X_b) \\ &+ g(\nabla_{X_a} \nabla_{X_k} X_k, X_b) + g(X_a, \nabla_{X_b} \nabla_{X_k} X_k) \end{aligned}$$

Using the result of Lemma 2.1 together with the fact that $\nabla_X X = 0$ at $p' \in K$ for any $X \in N_{p'} K$, we conclude that

$$X_k\, X_k\, g_{ab}|_p = 2\, g(R(E_k, E_a)E_k, E_b) + 2\, \Gamma^c_{ak}\, \Gamma^c_{bk}$$

and using the definition of Γ^b_a given in (2.4) this gives the formula for the second order Taylor expansion for g_{ab}. □

Later on, we will need an expansion of some covariant derivatives which is more accurate than the one given in Lemma 2.1. These are given in the :

Lemma 2.2 *At the point $q = F(x, 0)$, the following expansion holds*

$$\begin{aligned} \nabla_{X_a} X_b &= \Gamma^b_a(E_j)\, X_j - g(R(E_i, E_a)\, E_j, E_b)\, x^i\, X_j \\ &+ \tfrac{1}{2}\, \big(g(R(E_a, E_b)\, E_i, E_j) - \Gamma^c_a(E_i)\, \Gamma^b_c(E_j) - \Gamma^c_a(E_j)\, \Gamma^b_c(E_i)\big)\; x^i\, X_j \qquad (2.8) \\ &+ \mathcal{O}(|x|)^c\, X_c + \mathcal{O}(|x|^2)^j\, X_j, \end{aligned}$$

where summation over repeated indices is understood.

Proof: We compute

$$\begin{aligned} X_i\, g(\nabla_{X_a} X_b, X_j) &= g(\nabla_{X_i} \nabla_{X_a} X_b, X_j) + g(\nabla_{X_a} X_b, \nabla_{X_i} X_j) \\ &= g(R(X_i, X_a)\, X_b, X_j) + g(\nabla_{X_a} \nabla_{X_b} X_i, X_j) + g(\nabla_{X_a} X_b, \nabla_{X_i} X_j). \end{aligned}$$

Observe that, by construction, we have arranged in such a way that

$$\nabla_{X_a + \varepsilon X_b} X_i = (\Gamma^c_{ai} + \varepsilon\, \Gamma^c_{bi})\, X_c,$$

along the geodesic $s \longrightarrow \exp^K_p(s(E_a + \varepsilon E_b))$. Hence, along this geodesic

$$\nabla^2_{X_a + \varepsilon X_b} X_i = ((X_a + \varepsilon\, X_b)(\Gamma^c_{ai} + \varepsilon\, \Gamma^c_{bi}))\; X_c + (\Gamma^c_{ai} + \varepsilon\, \Gamma^c_{bi})\, \nabla_{X_a + \varepsilon\, X_b}\, X_c. \qquad (2.9)$$

Evaluating this at the point p and looking for the coefficient of ε , we obtain

$$(\nabla_{X_a}\, \nabla_{X_b} X_i + \nabla_{X_b}\, \nabla_{X_a} X_i)|_p - (\Gamma^c_{ai}\, \nabla_{X_b}\, X_c + \Gamma^c_{bi}\, \nabla_{X_a}\, X_c)|_p \in T_p K.$$

Hence we get

$$\begin{aligned} g(\nabla_{X_a}\, \nabla_{X_b} X_i, X_j)|_p + g(\nabla_{X_b}\, \nabla_{X_a} X_i, X_j)|_p &= \Gamma^c_{ai}\;\, g(\nabla_{X_b}\, X_c, X_j)|_p \\ &+ \Gamma^c_{bi}\, g(\nabla_{X_a}\, X_c, X_j)|_p \\ &= \Gamma^c_{ai}\, \Gamma^j_{bc} + \Gamma^c_{bi}\, \Gamma^j_{ac} \end{aligned}$$

Finally, we use the fact that

$$g(\nabla_{X_b}\, \nabla_{X_a} X_i, X_j) = g(R(X_b, X_a)\, X_i, X_j) + g(\nabla_{X_a}\, \nabla_{X_b} X_i, X_j)$$

to conclude that, at the point p

$$2\, g(\nabla_{X_a}\, \nabla_{X_b} X_i, X_j)|_p = g(R(E_a, E_b)\, E_i, E_j) + \Gamma^c_{ai}\, \Gamma^j_{bc} + \Gamma^c_{bi}\, \Gamma^j_{ac}$$

Collecting these estimates together with the fact that $\nabla_{X_i} X_j|_p = 0$ we conclude that

$$2\, X_i\, g(\nabla_{X_a} X_b, X_j)|_p = -2 g(R(E_i, E_a)\, E_j, E_b) + g(R(E_a, E_b)\, E_i, E_j) + \Gamma^c_{ai}\, \Gamma^j_{bc} + \Gamma^c_{bi}\, \Gamma^j_{ac}.$$

This, together with the fact that $g_{ij} = \delta_{ij} + \mathcal{O}(|x|)^2$, easily implies (2.8). □

3 Geometry of tubes

We derive expansions as ρ tends to 0 for the metric, second fundamental form and mean curvature of the tubes $\bar{S}_\rho$ and their perturbations. This is an extension of the computation in [10].

3.1 Perturbed tubes

We now describe a suitable class of deformations of the geodesic tubes $\bar{S}_\rho$, depending on a section Φ of NK and a scalar function w on the spherical normal bundle SNK.

Fix $\rho > 0$. It will be convenient to introduce the scaled variable $\bar{y} = y/\rho$; we also use a local parametrization $z \to \Theta(z)$ of S^{n-1}. Now define the map

$$G(z, \bar{y}) := F\left(\rho\,(1 + w(z, \bar{y}))\,\Theta(z) + \Phi(\rho\,\bar{y}), \rho\,\bar{y}\right),$$

and denote its image by $S_\rho(w, \Phi)$, so in particular

$$S_\rho(0, 0) = \bar{S}_\rho.$$

Notation : Because of the definition of these hypersurfaces using the exponential map, various vector fields we shall use may be regarded either as fields along K or along $S_\rho(w, \Phi)$. To help allay this confusion, we write

$$\Phi := \Phi^j\,E_j, \qquad \Phi_a := \partial_{y^a}\,\Phi^j\,E_j, \qquad \Phi_{ab} := \partial_{y^a}\partial_{y^b}\,\Phi^j\,E_j,$$

$$\Theta := \Theta^j\,E_j, \qquad \Theta_i := \partial_{z^i}\Theta^j\,E_j.$$

These are all vectors in the tangent space T_pM at the fixed point $p \in K$. On the other hand, the vectors

$$\Psi := \Phi^j\,X_j, \qquad \Psi_a := \partial_{y^a}\,\Phi^j\,X_j,$$

$$\Upsilon := \Theta^j\,X_j, \qquad \Upsilon_i := \partial_{z^i}\Theta^j\,X_j,$$

lie in the tangent space T_qM, $q = F(z, y)$.

For brevity, we also write

$$w_j := \partial_{z^j} w, \quad w_{\bar{a}} := \partial_{\bar{y}^a} w, \qquad w_{ij} := \partial_{z^i}\,\partial_{z^j} w, \quad w_{\bar{a}\bar{b}} := \partial_{\bar{y}^a}\,\partial_{\bar{y}^b} w, \quad w_{\bar{a}j} := \partial_{\bar{y}^a}\,\partial_{z^j} w.$$

In terms of all this notation, the tangent space to $S_\rho(w, \Phi)$ at any point is spanned by the vectors

$$\begin{aligned} Z_j &= G_*(\partial_{z^j}) = \rho\,((1+w)\,\Upsilon_j + w_j\,\Upsilon), \qquad j = 1, \ldots, n-1, \\ Z_{\bar{a}} &= G_*(\partial_{\bar{y}^a}) = \rho\,(X_a + w_{\bar{a}}\,\Upsilon + \Psi_a), \qquad a = n+1, \ldots, m+1. \end{aligned} \tag{3.10}$$

3.2 Notation for error terms

The formulas for the various geometric quantities of $S_\rho(w, \Phi)$ are potentially very complicated, and so it is important to condense notation as much as possible. Fortunately, we do not need to know the full structure of all of these quantities. Because it is so fundamental, we have isolated the notational conventions we shall use in this separate subsection.

Any expression of the form $L(w, \Phi)$ denotes a linear combination of the functions w together with its derivatives with respect to the vector fields $\rho\, X_a$ and X_i up to order 2, and Φ^j together with their derivatives with respect to the vector fields X_a up to order 2. The coefficients are assumed to be smooth functions on SNK which are bounded by a constant independent of ρ in the $\mathcal{C}^\infty$ topology (i.e. derivatives taken with respect to X_a and X_i).

Similarly, an expression of the form $Q(w, \Phi)$ denotes a nonlinear operator in the functions w together with its derivatives with respect to the vector fields $\rho\, X_a$ and X_i up to order 2, and Φ^j together with their derivatives with respect to the vector fields X_a up to order 2. Again, the coefficients of the Taylor expansion of the corresponding differential operator are smooth functions on SNK which are bounded by a constant independent of ρ in the $\mathcal{C}^\infty$ topology, and Q which vanishes quadratically at $(w, \Phi) = (0, 0)$.

In order to keep notations as simple as possible in the technical proofs, we will use the condensed notations L and Q instead of $L(w, \Phi)$ and $Q(w, \Phi)$.

Finally, any term denoted $\mathcal{O}(\rho^d)$ is a smooth function on SNK which is bounded by a constant by a constant times ρ^d in the $\mathcal{C}^\infty$ topology.

3.3 The first fundamental form

The next step is the computation of the coefficients of the first fundamental form of $S_\rho(w, \Phi)$. We set

$$q := G(z, 0) = F(\rho(1 + w(z, 0))\, \Theta(z) + \Phi(0), 0)$$

and $p := G(0, 0)$. We obtain directly from (2.6) that

$$\begin{array}{rcl} g(X_a, X_b) & = & \delta_{ab} - 2\,\rho\, \Gamma_a^b(\Theta) + \mathcal{O}(\rho^2) - 2\, \Gamma_a^b\,(\Phi) + \rho\, L(w, \Phi) + Q(w, \Phi), \\ g(X_i, X_j) & = & \delta_{ij} + \frac{\rho^2}{3}\, g(R(\Theta, E_i)\, \Theta, E_j) + \mathcal{O}(\rho^3), \\ & + & \frac{\rho}{3}\, (g(R(\Theta, E_i)\, \Phi, E_j) + g(R(\Phi, E_i)\, \Theta, E_j)) + \rho^2\, L(w, \Phi) + Q(w, \Phi) \\ g(X_i, X_a) & = & \mathcal{O}(\rho^2) + \rho\, L(w, \Phi) + Q(w, \Phi). \end{array} \tag{3.11}$$

We now explain a simple argument which will be frequently used throughout the paper. Using the previous expansions, we compute

$$\begin{array}{rcl} g(\Upsilon, \Upsilon_j) & = & g(\Theta, \Theta_j) + \frac{\rho^2}{3}\, g(R(\Theta, \Theta)\, \Theta, \Theta_j) + \mathcal{O}(\rho^3) \\ & + & \frac{\rho}{3}\, (g(R(\Theta, \Theta)\, \Phi, \Theta_j) + g(R(\Phi, \Theta)\, \Theta, \Theta_j)) + \rho^2\, L(w, \Phi) + Q(w, \Phi). \end{array}$$

However, when $w = 0$ and $\Phi = 0$, $g(\Upsilon, \Upsilon_j) = 0$ since Υ is normal and Υ_j is tangent to $S_\rho(0, 0)$ then, so that the sum of the first three terms on the right, which is independent of w and Φ, must also vanish. This, together with the fact that $R(\Theta, \Theta) = 0$ implies that

$$g(\Upsilon, \Upsilon_j) = \frac{\rho}{3}\, g(R(\Phi, \Theta)\, \Theta, \Theta_j) + \rho^2\, L(w, \Phi) + Q(w, \Phi) \tag{3.12}$$

Using similar arguments, we have

$$\begin{array}{rcl} g(\Upsilon, \Upsilon) & = & g(\Theta, \Theta) + \frac{\rho^2}{3}\, g(R(\Theta, \Theta)\, \Theta, \Theta_j) + \mathcal{O}(\rho^3) \\ & + & \frac{\rho}{3}\, (g(R(\Theta, \Theta)\, \Phi, \Theta) + g(R(\Phi, \Theta)\, \Theta, \Theta)) + \rho^2\, L(w, \Phi) + Q(w, \Phi). \end{array}$$

This, together with the fact that $g(\Upsilon, \Upsilon) = 1$ when $w = 0$ and $\Phi = 0$, yields

$$g(\Upsilon, \Upsilon) = 1 + \rho^2 \, L(w, \Phi) + Q(w, \Phi). \tag{3.13}$$

Using these expansions is is easy to obtain the expansion of the first fundamental form of $S_\rho(w, \Phi)$.

Proposition 3.1 *We have*

$$\begin{array}{rcl} \rho^{-2} \, g(Z_{\bar a}, Z_{\bar b}) & = & \delta_{ab} - 2 \, \rho \, \Gamma_a^b(\Theta) + \mathcal{O}(\rho^2) - 2 \, \Gamma_a^b(\Phi) + \rho \, L(w, \Phi) + Q(w, \Phi), \\ \rho^{-2} \, g(Z_{\bar a}, Z_j) & = & \mathcal{O}(\rho^2) + L(w, \Phi) + Q(w, \Phi), \\ \rho^{-2} \, g(Z_i, Z_j) & = & g(\Theta_i, \Theta_j) + \frac{\rho^2}{3} \, g(R(\Theta, \Theta_i) \, \Theta, \Theta_j) + \mathcal{O}(\rho^3) + 2 \, g(\Theta_i, \Theta_j) \, w, \\ & + & \frac{\rho}{3} \, (g(R(\Theta, \Theta_i)\Phi, \Theta_j) + g(R(\Theta, \Theta_j)\Phi, \Theta_i)) + \rho^2 L(w, \Phi) + Q(w, \Phi), \end{array} \tag{3.14}$$

where summation over repeated indices is understood.

3.4 The normal vector field

Our next task is to understand the dependence on (w, Φ) of the unit normal N to $S_\rho(w, \Phi)$. Define the normal (not unitary) vector field

$$\tilde N := - \, \Upsilon + \frac{1}{\rho} \, (\alpha^j \, Z_j + \beta^a \, Z_{\bar a}),$$

where the coefficients α^j and β^a are chosen so that $\tilde N$ is orthogonal to all of the $Z_{\bar b}$ and Z_i. The unit normal vector field $S_\rho(w, \Phi)$ is defined by

$$N := \frac{\tilde N}{|\tilde N|}.$$

We have the following :

Proposition 3.2 *With the above notations, the coefficients α^j are solutions of the system*

$$g(\Theta_i, \Theta_j) \, \alpha^j = w_i + \frac{\rho}{3} \, g(R(\Phi, \Theta) \, \Theta, \Theta_i) + \rho^2 \, L(w, \Phi) + Q(w, \Phi), \qquad i = 1, \ldots, n-1,$$

where summation over j is understood, and the expansion of the coefficients β^a is given by

$$\beta^a = w_{\bar a} + g(\Phi_a, \Theta) + \rho \, L(w, \Phi) + Q(w, \Phi).$$

Finally

$$|\tilde N|^{-1} = 1 + \rho^2 \, L(w, \Phi) + Q(w, \Phi).$$

Proof : We look for coefficients α^j and β^a so that that $\tilde N$ is orthogonal to all of the $Z_{\bar b}$ and Z_i. This leads to a linear system for α^j and β^a.

We have the following expansions

$$\begin{array}{rcl} g(\Upsilon, Z_{\bar a}) & = & \rho \, w_{\bar a} + \rho \, g(\Phi_a, \Theta) + \rho^2 \, L + \rho \, Q, \\ g(\Upsilon, Z_j) & = & \rho \, w_j + \frac{\rho^2}{3} \, g(R(\Phi, \Theta) \, \Theta, \Theta_j) + \rho^3 \, L + \rho \, Q. \end{array} \tag{3.15}$$

These follow from (3.11), (3.12) and (3.13), together with the fact that $g(\Upsilon, Z_{\bar a}) = 0$ and $g(\Upsilon, Z_j) = 0$ when $w = 0$ and $\Phi = 0$.

Using Proposition 3.1, we get with little work the expansions for both β^a and the system α^j satisfy. Collecting these, the estimate for the norm of $\tilde N$ follows at once. □

3.5 The second fundamental form

We now compute the second fundamental form. To simplify the computations below, we henceforth assume that, at the point $\Theta(z) \in S^{n-1}$,

$$g(\Theta_i, \Theta_j) = \delta_{ij}, \qquad \text{and} \qquad \overline{\nabla}_{\Theta_i} \Theta_j = 0,, \quad i, j = 1, \dots, n-1, \tag{3.16}$$

(where $\overline{\nabla}$ is the connection on TS^{n-1}).

Proposition 3.3 *The following expansions hold*

$$\begin{array}{rcl} \rho^{-2}\, g(N, \nabla_{Z_{\bar a}} Z_{\bar a}) & = & -\Gamma_a^a(\Theta) + \rho\, g(R(\Theta, E_a)\,\Theta, E_a) + \rho\, \Gamma_a^c(\Theta)\, \Gamma_c^a(\Theta) + \mathcal{O}(\rho^2) \\ & - & \frac{1}{\rho}\, w_{\bar a \bar a} - g(\Phi_{aa}, \Theta) + g(R(\Phi, E_a)\,\Theta, E_a) + \Gamma_a^c(\Theta)\, \Gamma_c^a(\Phi) \\ & + & w_j\, \Gamma_a^a(\Theta_j) + \rho\, L(w, \Phi) + \frac{1}{\rho}\, Q(w, \Phi), \\ \rho^{-2}\, g(N, \nabla_{Z_j} Z_j) & = & \frac{1}{\rho} + \frac{2}{3}\, \rho\, g(R(\Theta, \Theta_j)\,\Theta, \Theta_j) + \mathcal{O}(\rho^2) \\ & - & \frac{1}{\rho}\, w_{jj} + \frac{1}{\rho}\, w + \frac{2}{3}\, g(R(\Phi, \Theta_j)\,\Theta, \Theta_j) \\ & + & \rho\, L(w, \Phi) + \frac{1}{\rho}\, Q(w, \Phi), \\ \rho^{-2}\, g(N, \nabla_{Z_{\bar a}} Z_{\bar b}) & = & -\Gamma_a^b(\Theta) - \frac{1}{\rho}\, w_{\bar a \bar b} + \mathcal{O}(\rho) + L(w, \Phi) + \frac{1}{\rho}\, Q(w, \Phi), \qquad a \neq b, \\ \rho^{-2}\, g(N, \nabla_{Z_{\bar a}} Z_j) & = & \mathcal{O}(\rho) + \frac{1}{\rho}\, L(w, \Phi) + \frac{1}{\rho}\, Q(w, \Phi), \\ \rho^{-2}\, g(N, \nabla_{Z_i} Z_j) & = & \mathcal{O}(\rho) + \frac{1}{\rho}\, L(w, \Phi) + \frac{1}{\rho}\, Q(w, \Phi), \qquad i \neq j, \end{array} \tag{3.17}$$

where summation over repeated indices is understood.

Proof : Some preliminary computations are needed. First note that by Lemma 2.1, we have

$$\begin{array}{rcl} \nabla_{X_a} X_b & = & \Gamma_a^b(E_i)\, X_i + (\mathcal{O}(\rho) + L + Q)^\gamma\, X_\gamma, \\ \nabla_{X_i} X_j & = & (\mathcal{O}(\rho) + L + Q)^\gamma\, X_\gamma, \\ \nabla_{X_a} X_i & = & -\Gamma_a^b(E_i)\, X_b + (\mathcal{O}(\rho) + L + Q)^\gamma\, X_\gamma. \end{array} \tag{3.18}$$

In particular, this, together with the expression of $Z_{\bar a}$, implies that

$$\begin{array}{rcl} \nabla_{Z_{\bar a}} X_b & = & \rho\, \Gamma_a^b(E_i)\, X_i + (\mathcal{O}(\rho^2) + \rho\, L + \rho\, Q)^\gamma\, X_\gamma, \\ \nabla_{Z_{\bar a}} X_i & = & -\rho\, \Gamma_a^b(E_i)\, X_b + (\mathcal{O}(\rho^2) + \rho\, L + \rho\, Q)^\gamma\, X_\gamma. \end{array} \tag{3.19}$$

We will also need the following expansion which follows from the result of Lemma 2.2

$$\begin{array}{r} \nabla_{X_a} X_b = \Gamma_a^b(E_j)\, X_j - g(R(\rho\,\Theta + \Phi, E_a)\, E_j, E_b)\, X_j \\ + \frac{1}{2} \Big(g(R(E_a, E_b)\, \rho\,\Theta + \Phi, E_j) - \Gamma_a^c(\rho\,\Theta + \Phi)\, \Gamma_c^b(E_j) - \Gamma_c^b(\rho\,\Theta + \Phi)\, \Gamma_a^c(E_j) \Big)\, X_j \\ + (\mathcal{O}(\rho) + L + Q)\, X_c + (\mathcal{O}(\rho^2) + \rho\, L + Q)\, X_j. \end{array} \tag{3.20}$$

Finally, we will need the expansions

$$g(\Upsilon, X_a) = \rho\, L + Q, \qquad \text{and} \qquad g(\Upsilon, \Upsilon_j) = \rho\, L + Q, \tag{3.21}$$

whose proof can be obtained as in §3.2, starting from the estimates (3.11) and using the fact that $g(\Upsilon, X_a) = g(\Upsilon, \Upsilon_j) = 0$ when $w = 0$ and $\Phi = 0$.

Observe that it is enough to get these expansions when N is replaced by $\tilde{N}$ and then multiply the expansion by the expansion of $|\tilde{N}|^{-1}$ which is given in Proposition 3.2.

First estimate : We estimate $g(\tilde{N}, \nabla_{Z_{\bar{a}}} Z_{\bar{b}})$ when $a = b$ since the corresponding estimate, when $a \neq b$ is not as important and follows from the same proof. We must expand

$$\rho^{-2}\, g(\tilde{N}, \nabla_{Z_{\bar{a}}} Z_{\bar{a}}) = \rho^{-1} \, \Big(g(\tilde{N}, \nabla_{Z_{\bar{a}}} X_a) + g(\tilde{N}, \nabla_{Z_{\bar{a}}} (w_{\bar{a}}\, \Upsilon)) + g(\tilde{N}, \nabla_{Z_{\bar{a}}} \Psi_a) \Big) \, .$$

The proof is this estimate is broken into three steps:

Step 1 : From Proposition 3.2, we get

$$g(\tilde{N}, \Upsilon) = -g(\Upsilon, \Upsilon) + \frac{1}{\rho}\, (\alpha^j\, g(Z_j, \Upsilon) + \beta^a\, g(Z_a, \Upsilon)) = -1 + \rho^2\, L + Q.$$

Substituting $\tilde{N} = -\Upsilon + (\tilde{N} + \Upsilon)$ gives

$$g(\tilde{N}, \nabla_{Z_{\bar{a}}} \Upsilon) = -\frac{1}{2}\, \partial_{\bar{y}^a} g(\Upsilon, \Upsilon) + g(\tilde{N} + \Upsilon, \nabla_{Z_{\bar{a}}} \Upsilon).$$

But it follows from (3.13) that

$$\partial_{\bar{y}^a}\, g(\Upsilon, \Upsilon) = \rho^3\, L + \rho\, Q,$$

and (3.19) together with the expression of $\tilde{N}$ implies that

$$g(\tilde{N} + \Upsilon, \nabla_{Z_{\bar{a}}} \Upsilon) = \rho\, L + \rho\, Q.$$

Collecting these estimates we get

$$g(\tilde{N}, \nabla_{Z_{\bar{a}}} \Upsilon) = \rho\, L + Q.$$

Hence we conclude that

$$g(\tilde{N}, \nabla_{Z_{\bar{a}}} (w_{\bar{a}}\, \Upsilon)) = w_{\bar{a}\bar{a}}\, g(\tilde{N}, \Upsilon) + w_{\bar{a}}\, g(\tilde{N}, \nabla_{Z_{\bar{a}}} \Upsilon) = -w_{\bar{a}\bar{a}} + Q.$$

Step 2 : Next,

$$g(\tilde{N}, \nabla_{Z_{\bar{a}}} \Psi_a) = \rho\, g(\tilde{N}, \Psi_{aa}) + \Phi_a^j\, g(\tilde{N}, \nabla_{Z_{\bar{a}}} X_j).$$

From (3.19), we have

$$\Phi_a^j\, g(\tilde{N}, \nabla_{Z_{\bar{a}}} X_j) = \rho^2\, L + \rho\, Q.$$

Also, using the decomposition of $\tilde{N}$ and (3.11), we have

$$g(\tilde{N}, \Psi_{aa}) = -g(\Upsilon, \Psi_{aa}) + g(\tilde{N} + \Upsilon, \Psi_{aa}) = -g(\Theta, \Phi_{aa}) + \rho^2\, L + Q.$$

Collecting these gives

$$g(\tilde{N}, \nabla_{Z_{\bar{a}}} \Psi_a) = -\rho\, g(\Phi_{aa}, \Theta) + \rho^2\, L + \rho\, Q.$$

Step 3 : Expanding $Z_{\bar a}$ gives

$$g(\tilde N, \nabla_{Z_{\bar a}} X_a) = \rho\, (g(\tilde N, \nabla_{X_a} X_a) + w_{\bar a}\, g(\tilde N, \nabla_\Upsilon X_a) + \Phi^j_a\, g(\tilde N, \nabla_{X_j} X_b)). \tag{3.22}$$

With the help of (3.18) and (3.21), we evaluate

$$\begin{aligned} g(\tilde N, \nabla_\Upsilon X_a) &= \mathcal{O}(\rho) + L + Q, \\ g(\tilde N, \nabla_{X_j} X_a) &= \mathcal{O}(\rho) + L + Q, \\ g(\tilde N + \Upsilon, \nabla_{X_a} X_a) &= \alpha^j\, \Gamma^a_a\, (\Theta_j) + \rho\, L + Q, \end{aligned}$$

and plugging these into (3.22) already gives

$$g(\tilde N, \nabla_{Z_{\bar a}} X_a) = -\rho\, g(\Upsilon, \nabla_{X_a} X_a) + \rho\, \alpha_j\, \Gamma^a_a(\Theta_j) + \rho^2\, L + \rho\, Q.$$

Using (3.20) we get the expansion

$$\begin{aligned} \nabla_{X_a} X_a &= \Gamma^a_a(E_j)\, X_j - g(R(\rho\Theta + \Phi, E_a)\, E_j, E_a)\, X_j - \Gamma^c_a(\rho\,\Theta + \Phi)\, \Gamma^a_c(E_j)\, X_j \\ &+ (\mathcal{O}(\rho) + L + Q)^c\, X_c + (\mathcal{O}(\rho^2) + \rho\, L + Q)^j\, X_j. \end{aligned}$$

Finally, using (3.11) again together with the fact that $\alpha_j = w_j + \rho\, L$, we conclude that

$$\begin{aligned} g(\tilde N, \nabla_{Z_{\bar a}} X_a) &= -\rho\, \Gamma^a_a(\Theta) + \rho^2\, g(R(\Theta, E_a)\, \Theta, E_a) + \mathcal{O}(\rho^3) \\ &+ \rho\, g(R(\Phi, E_a)\, \Theta, E_a) + \rho\, \Gamma^c_a(\rho\, \Theta + \Phi)\, \Gamma^a_c(\Theta) + \rho\, w_j\, \Gamma^a_a\, (\Theta_j) \\ &+ \rho^2\, L + \rho\, Q, \end{aligned}$$

which, together with the results of Step 1 and Step 2, completes the proof of the first estimate.

Second estimate : We estimate $g(\tilde N, \nabla_{Z_i} Z_j)$ when $i = j$ since, just as before, the corresponding estimate, when $i \neq j$ is not as important and follows similarly. This part is taken directly from [10]. Recall that

$$\tilde N = -\Upsilon + \frac{1}{\rho}\, (\alpha^j\, Z_j + \beta^a\, Z_a),$$

Now write

$$\begin{aligned} g(\tilde N, \nabla_{Z_j} Z_j) &= -g(\nabla_{Z_j} \tilde N, Z_j) \\ &= g(\nabla_{Z_j} \Upsilon, Z_j) - \tfrac{1}{\rho}\, g(\nabla_{Z_j} (\alpha^i\, Z_i), Z_j) \\ &- \tfrac{1}{\rho}\, \beta^a\, g(Z_{\bar a}, \nabla_{Z_j} Z_j) + \tfrac{1}{\rho}\, \partial_{z^j}\, g(\beta^a\, Z_{\bar a}, Z_j) \end{aligned}$$

Step 1 : We compute

$$g(Z_a, \nabla_{Z_j} Z_j) = \mathcal{O}(\rho^4) + \rho^3\, L + \rho^3\, Q.$$

Hence we already obtain

$$\frac{1}{\rho}\, \beta^a\, g(Z_{\bar a}, \nabla_{Z_j} Z_j) = \rho^3\, L + \rho^2\, Q.$$

Step 2 : Next, using the expansion given in Proposition 3.2 together with (3.14), we find that

$$\frac{1}{\rho}\,\partial_{z_j}\, g(\beta^a\, Z_{\bar a}, Z_j) = \rho^3\, L + \rho\, Q.$$

Step 3 : We now estimate

$$C := 2\, g(\nabla_{Z_j}\Upsilon, Z_j).$$

It is convenient to define

$$C' := \frac{2}{1+w}\, g(\nabla_{Z_j}(1+w)\,\Upsilon, Z_j),$$

It follows from (3.15) that

$$C = C' + \rho\, Q,$$

hence it is enough to focuss on the estimate of C'. To analyze this term, let us revert for the moment and regard w and Φ as functions of the coordinates $(z, \bar y)$ and also consider ρ as a variable instead of just a parameter. Thus we consider

$$\tilde F(\rho, z, \bar y) = F\big(\rho(1 + w(z, \bar y))\Upsilon(z) + \Phi(t\,\bar y), t\,\bar y\big).$$

The coordinate vector fields Z_j are still equal to $\tilde F_*(\partial_{z_j})$, but now we also have $(1+w)\Upsilon = \tilde F_*(\partial_\rho)$, which is the identity we wish to use below. Now, from (3.14), we write

$$C' = \frac{1}{1+w}\, g(\nabla_{\nabla_{(1+w)\Upsilon}} Z_j, Z_j) = \frac{1}{1+w}\,\partial_\rho g(Z_j, Z_j).$$

Therefore, it follows from (3.14) in Proposition 3.1 that

$$\begin{aligned} C \;&=\; \tfrac{1}{1+w}\,\partial_\rho\,[\rho^2\, g(\Theta_j, \Theta_j) + \tfrac{1}{3}\,\rho^4\, g(R(\Theta, \Theta_j)\,\Theta, \Theta_j) + \mathcal{O}(\rho^5) \\ &+\; 2\,\rho^2\, g(\Theta_j, \Theta_j)\, w + \tfrac{2}{3}\,\rho^3\, g(R(\Theta, \Theta_j)\,\Phi, \Theta_j) + \rho^4\, L + \rho^2\, Q] + \rho\, Q \\ &=\; \tfrac{1}{1+w}\,[2\,\rho\, g(\Theta_j, \Theta_j) + \tfrac{4}{3}\,\rho^3\, g(R(\Theta, \Theta_j)\,\Theta, \Theta_j) + \mathcal{O}(\rho^4) \\ &+\; 4\,\rho\, g(\Theta_j, \Theta_j)\, w + 2\,\rho^2\, g(R(\Theta, \Theta_j)\,\Phi, \Theta_j) + \rho^3\, L + \rho\, Q] \\ &=\; 2\,\rho\, g(\Theta_j, \Theta_j) + \tfrac{4}{3}\,\rho^3\, g(R(\Theta, \Theta_j)\,\Theta, \Theta_j) + \mathcal{O}(\rho^4) \\ &+\; 2\, w\, g(\Theta_j, \Theta_j)\,\rho + 2\,\rho^2\, g(R(\Theta, \Theta_j)\,\Phi, \Theta_j) + \rho^3\, L + \rho\, Q. \end{aligned}$$

Step 4 : Finally, we must compute

$$\begin{aligned} D \;&:=\; g(\nabla_{Z_j}(\alpha^i\, Z_i), Z_j) \\ &=\; g(Z_i, Z_j)\,\partial_{z^j}\alpha^i + \alpha^i\, g(\nabla_{Z_i} Z_j, Z_j) \\ &=\; g(Z_i, Z_j)\,\partial_{z^j}\alpha^i + \tfrac{1}{2}\,\alpha^i\,\partial_{z^i}\, g(Z_j, Z_j) \end{aligned}$$

Observe that (3.16) implies

$$\partial_{z^j} g(\Theta_i, \Theta_{j'}) = 0$$

at the point p. Using this together with (3.14) and the expression for the α^i given in Proposition 3.2, we get

$$\alpha^i\,\partial_{z^i}\, g(Z_j, Z_j) = \rho^4\, L + \rho^2\, Q.$$

It follows from (3.14) and the definition of α^i again that

$$g(Z_i, Z_j)\,\partial_{z^j}\alpha^i = \rho^2\, g(\Theta_i, \Theta_j)\,\partial_{z^j}\alpha^i + \rho^4\, L + \rho^2\, Q.$$

Therefore, it remains to estimate $g(\Theta_i, \Theta_j)\,\partial_{z^j}\alpha^i$. By definition, we have

$$g(\Theta_i, \Theta_j)\,\alpha^i = w_j + \frac{\rho}{3}\,g(R(\Phi, \Theta)\,\Theta, \Theta_j) + \rho^2\,L + Q.$$

Differentiating with respect to z^j we get

$$\big(g(\Theta_i, \Theta_j)\,\partial_{z^j}\alpha^i + \alpha^i\,\partial_{z^j} g(\Theta_i, \Theta_j)\big) = w_{jj} + \frac{\rho}{3}\,\partial_{z^j} g(R(\Phi, \Theta)\,\Theta, \Theta_j) + \rho^2\,L + Q. \quad (3.23)$$

Again, it follows from (3.16) that $\partial_{z^j} g(\Theta_i, \Theta_j) = 0$. Moreover this also implies that,

$$\nabla_{\Theta_j}\Theta = \Theta_j, \qquad \text{and} \qquad \nabla_{\Theta_j}\Theta_j = a_j\,\Theta,$$

for some $a_j \in \mathbb{R}$. Therefore, we have

$$g(R(\Phi, \Theta)\,\nabla_{\Theta_j}\Theta, \Theta_j) = g(R(\Phi, \Theta)\,\Theta_j, \Theta_j) = 0,$$

and

$$g(R(\Phi, \Theta)\,\Theta, \nabla_{\Theta_j}\Theta_j) = a_j\,g(R(\Phi, \Theta)\,\Theta, \Theta) = 0.$$

Inserting these information into (3.23) yields

$$g(\Theta_i, \Theta_j)\,\partial_{z^j}\alpha^i \quad = \quad w_{jj} + \tfrac{\rho}{3}\,g(R(\Phi, \Theta_j)\,\Theta, \Theta_j) + \rho^2\,L + Q.$$

Collecting these estimates, we conclude that

$$D = \rho^2\,w_{jj} \; + \frac{\rho^3}{3}\,g(R(\Phi, \Theta_j)\,\Theta, \Theta_j) + \rho^4\,L + \rho^2\,Q,$$

and with the estimates of the previous steps, this finishes the proof of the estimate.

Third estimate: Decompose

$$\frac{1}{\rho}\,g(\tilde{N}, \nabla_{Z_{\bar{a}}} Z_j) = g(\tilde{N}, \Upsilon_j)\,w_{\bar{a}} + g(\tilde{N}, \Upsilon)\,w_{\bar{a}j} + (1+w)\,g(\tilde{N}, \nabla_{Z_{\bar{a}}}\Upsilon_j) + w_j\,g(\tilde{N}, \nabla_{Z_{\bar{a}}}\Upsilon).$$

As above we use the expression of $\tilde{N}$ given in Proposition 3.2 to estimate

$$g(\tilde{N}, \Upsilon_j) = -g(\Upsilon, \Upsilon_j) + g(\tilde{N} + \Upsilon, \Upsilon_j) = L + Q.$$

Similarly

$$g(\tilde{N}, \Upsilon) = -1 + L + Q.$$

But now, by (3.19), we have

$$g(\tilde{N}, \nabla_{Z_{\bar{a}}}\,\Upsilon_j) = \mathcal{O}(\rho^2) + \rho\,L + \rho\,Q,$$

and, as already shown in the first step of the proof of the first estimate

$$g(\tilde{N}, \nabla_{Z_{\bar{a}}}\,\Upsilon) = \rho\,L + Q,$$

and the proof of the estimate follows directly. □

4 The mean curvature of perturbed tubes

Collecting the estimates of the last subsection we obtain the expansion of the mean curvature of the hypersurface $S_\rho(w,\Phi)$. In the coordinate system defined in the previous sections, we get

$$\begin{aligned} &\rho\, mH(w,\Phi) = n-1-\rho\,\Gamma_a^a(\Theta)\\ +&\ \left(g(R(\Theta,\,E_a)\,\Theta,E_a)+\tfrac{1}{3}\,g(R(\Theta,\,E_i)\,\Theta,E_i)-\Gamma_a^c(\Theta)\,\Gamma_c^a(\Theta)\right)\,\rho^2+\mathcal{O}(\rho^3)\\ -&\ (w_{\bar a\bar a}+\Delta_{S^{n-1}}w+(n-1)\,w)-2\,\rho\,\Gamma_a^b(\Theta)\,w_{\bar a\bar b}+\rho\,\Gamma_a^a(\Theta_j)\,w_j\\ -&\ \rho\,g(\Phi_{aa},\Theta)+\rho\,g(R(E_a,\Phi)\,E_a,\Theta)-\rho\,\Gamma_a^c(\Phi)\,\Gamma_c^a(\Theta)\\ +&\ \rho^2\,L(w,\Phi)+Q(w,\Phi), \end{aligned}$$

where summation over repeated indices is understood. We can simplify this rather complicated expression as follows. First, note that

$$K \text{ minimal} \qquad \Longleftrightarrow \qquad \Gamma_a^a=0,$$

where summation over a is understood. Next, define

$$\mathcal{L}_\rho := -\left(\rho^2\,\Delta_K+\Delta_{S^{n-1}}+(n-1)\right), \tag{4.24}$$

as an operator on the spherical normal bundle SNK with the expression (4.24) in any local coordinates. Also, the Jacobi (linearized mean curvature) operator, for K is defined by

$$\mathfrak{J} := -\Delta^N+\mathcal{R}^N-\mathcal{B}^N, \tag{4.25}$$

cf. [3]. To explain the terms here, recall that the Levi-Civita connection for g induces not only the Levi-Civita connection on K, but also a connection ∇^N on the normal bundle NK. The first term here is simply the rough Laplacian for this connection, i.e.

$$\Delta^N := (\nabla^N)^*\nabla^N = \nabla^N_{E_a}\,\nabla^N_{E_a}-\nabla^N_{(\nabla_{E_a}E_a)^T}.$$

in the coordinates we have chosen. The second term is the contraction (in normal directions) of the curvature operator for this connection:

$$\mathcal{R}^N := (R(E_a,\cdot)\,E_a)^N,$$

where E_a is (any) orthonormal frame for TK. Finally, the second fundamental form

$$B: T_pK\times T_pK \longrightarrow N_pK, \qquad B(X,Y) := (\nabla_X Y)^N, \quad X,Y\in T_pK,$$

defines a symmetric operator

$$\mathcal{B}^N := B^t\cdot B;$$

in terms of the coefficients $\Gamma_a^b := B(E_a,E_b)$,

$$g(\mathcal{B}^N X,Y) = \Gamma_a^b(X)\,\Gamma_b^a(Y).$$

where summation over repeated indices is understood. We also use the Ricci tensor

$$\mathrm{Ric}(X,Y) = g(R(X,E_\gamma)\,E_\gamma,Y), \qquad X,Y\in T_pM.$$

Finally, we introduce the operator

$$g(\cdot, B) \circ \nabla_K^2 = g(\cdot, B(E_a, E_b))\,(\nabla_{E_a} \nabla_{E_b} - \nabla_{(\nabla_{E_a} E_b)^T})$$

in the coordinates we have chosen and the quadratic form

$$\Omega\,(\cdot, \cdot) := -\frac{2}{3}\, g(\mathcal{R}^N \cdot, \cdot) + \frac{1}{3}\,\mathrm{Ric}(\cdot, \cdot) + g(\mathcal{B}^N \cdot, \cdot)$$

acting on N_pK. In terms of all of these notations, we have the

Proposition 4.1 *Let K be a minimal submanifold. Then the mean curvature of $\mathcal{T}_\rho(w, \Phi)$ can be expanded as*

$$\begin{aligned} \rho\, m\, H(w, \Phi) &= (n-1) - \Omega(\Theta, \Theta)\,\rho^2 + \mathcal{O}(\rho^3) \\ &+ \mathcal{L}_\rho\, w + \rho\, g(\mathfrak{J}\,\Phi, \Theta) - 2\,\rho^3\, g(\Theta, B) \circ \nabla_K^2 w \\ &+ \rho^2\, L(w, \Phi) + Q(w, \Phi). \end{aligned} \tag{4.26}$$

The equation $\rho\, m\, H = n - 1$ can now be written as

$$\begin{aligned} \mathcal{L}_\rho\, w + \rho\, g(\mathfrak{J}\,\Phi, \Theta) &= \Omega(\Theta, \Theta)\,\rho^2 + \mathcal{O}(\rho^3) \\ &+ 2\,\rho^3\, g(\Theta, B) \circ \nabla_K^2\, w + \rho^2\, L(w, \Phi) + Q(w, \Phi). \end{aligned} \tag{4.27}$$

4.1 Decomposition of functions on SNK

Before proceeding, we now state more clearly our notation for functions on SNK.

Let (φ_j, λ_j) be the eigendata of $\Delta_{S^{n-1}}$, with eigenfunctions orthonormal and counted with multiplicity. These individual eigenfunctions do not make sense on all of SNK, but their span is a well-defined subspace $\mathcal{S} \subset L^2(SNK)$; thus $v \in \mathcal{S}$ if its restriction to each fibre of SNK lies in the span of $\{\varphi_1, \ldots, \varphi_n\}$. We denote by Π and $\Pi^\perp$ the L^2 orthogonal projections of $L^2(SNK)$ onto $\mathcal{S}$ and $\mathcal{S}^\perp$, respectively.

Now, given any function $v \in L^2(SNK)$, we write

$$\Pi\, v = g(\Phi, \Theta), \qquad \Pi^\perp v = \rho\, w,$$

so that

$$v = \rho\, w + g(\Phi, \Theta),$$

here Φ is a section of the normal bundle NK, and the somewhat elaborate notation in the second summand here reflects the fact that any element of $\mathcal{S}$ can be written (locally) as the inner product of a section of NK and the vector Θ, whose components are the linear coordinate functions on each S^{n-1}. We shall often identify this summand with Φ, and thus, in the following, w and Φ will always represent the components of v in $\mathcal{S}^\perp$ and $\mathcal{S}$, respectively.

Later on we shall further decompose

$$w = w_0 + w_1, \tag{4.28}$$

where w_0 is a function on K and the integral of w_1 over each fibre of SNK vanishes.

Note that the operator

$$J : v \longrightarrow g(\mathfrak{J}\,\Phi, \Theta),$$

defined for $v = g(\Phi, \Theta)$, preserves $\mathcal{S}$ and is invertible since K is a nondegerate minimal submanifold.

5 Improvement of the approximate solution

The first important step in solving (4.27) is to use an iteration scheme to find a sequence of approximate solutions $(w^{(i)}, \Phi^{(i)})$ for which the estimates for the error term are increasingly small. Namely

$$\rho\, m\, H(w^{(i)}, \Phi^{(i)}) = n - 1 + \mathcal{O}(\rho^{i+3}),$$

for all $i \geq 1$.

Letting $(w^{(0)}, \Phi^{(0)}) = (0, 0)$, we define the sequence $(w^{(i+1)}, \Phi^{(i+1)}) \in \mathcal{S}^\perp \oplus \mathcal{S}$ inductively as the unique solution to

$$\begin{aligned} \mathcal{L}_0\, w^{(i+1)} &+ \rho\, g(\mathfrak{J}\,\Phi^{(i+1)}, \Theta) = \Omega(\Theta, \Theta)\, \rho^2 + \mathcal{O}(\rho^3) - \rho^2\, \Delta_K w^{(i)} \\ &+ 2\,\rho^3\, g(\Theta, B) \circ \nabla^2_K w^{(i)} + \rho^2\, L(w^{(i)}, \Phi^{(i)}) + Q(w^{(i)}, \Phi^{(i)}). \end{aligned} \tag{5.29}$$

here

$$\mathcal{L}_0 := -\left(\Delta_{S^{n-1}} + (n-1)\right).$$

Observe, and this is the key point, that the operator Δ_K acting on functions has been moved to the right hand side and hence, the operator on the left hand side is not elliptic anymore. This equation becomes simpler when divided into its $\mathcal{S}^\perp$ and $\mathcal{S}$ components. Thus using that $\mathcal{L}_0$ annihilates $\mathcal{S}$ and

$$\Omega(\Theta, \Theta) \in \mathcal{S}^\perp,$$

since it is quadratic in Θ, (5.29) can be rewritten as the two separate equations:

$$\begin{aligned} \mathcal{L}_0\, w^{(i+1)} &= \Pi^\perp \big(\Omega(\Theta, \Theta)\rho^2 + \mathcal{O}(\rho^3) - \rho^2\, \Delta_K w^{(i)} + 2\,\rho^3\, g(\Theta, B) \circ \nabla^2_K w^{(i)} \\ &+ \rho^2\, L(w^{(i)}, \Phi^{(i)}) + Q(w^{(i)}, \Phi^{(i)})\big)\,, \end{aligned} \tag{5.30}$$

and

$$\begin{aligned} g(\mathfrak{J}\,\Phi^{(i+1)}, \Theta) &= \Pi\, \big(\mathcal{O}(\rho^2) + 2\rho^2\, g(\Theta, B) \circ \nabla^2_K w^{(i)} \\ &+ \rho\, L(w^{(i)}, \Phi^{(i)}) + \rho^{-1}\, Q(w^{(i)}, \Phi^{(i)})\big)\,, \end{aligned}$$

since $\Pi\,(\Delta_K w) = 0$ for all $w \in \mathcal{S}$.

That there is a unique solution now follows directly from the invertibility of the operators J on $\mathcal{S}$ and $\mathcal{L}_0$ on $\mathcal{S}^\perp$, so the only issue is to obtain estimates.

Lemma 5.1 *For this sequence $(w^{(i)}, \Phi^{(i)})$, we have the estimates*

$$w^{(i)} = \mathcal{O}(\rho^2), \qquad \Phi^{(i)} = \mathcal{O}(\rho^2),$$
$$w^{(i+1)} - w^{(i)} = \mathcal{O}(\rho^{i+3}), \qquad \Phi^{(i+1)} - \Phi^{(i)} = \mathcal{O}(\rho^{i+2}),$$

for all $i \geq 1$.

Proof: The estimates for $(w^{(1)}, \Phi^{(1)})$ are immediate, and the result for $i > 1$ is proved by a standard induction using the general structure of the operators L and Q. □

As already mentioned, the operator in the right hand side of (5.30) is not elliptic since $\mathcal{L}_0$ acts on functions defined on SNK and $\mathcal{L}_0$ does not involve any derivatives with respect to y^a. Nevertheless, since we are working with functions in $\mathcal{S}$, the equation

$$\mathcal{L}_0\, w = f,$$

can always be solved for any $f \in \mathcal{S}$ (we have in mind that this equation is solved on each fiber of NK with the base point as a parameter), but without any gain of regularity in the y^a variables and in fact there is a "loss" of two derivatives in the y^a variables at each iteration. At first glance, it would have been more natural to work with the operator $\mathcal{L}_\rho$, which is elliptic, and solve the equation

$$\mathcal{L}_\rho \, w = f,$$

but the operator $\mathcal{L}_\rho$ has the disadvantage to have a nontrivial kernel in $\mathcal{S}$ each time $\frac{n-1}{\rho^2}$ belongs to the spectrum of $-\Delta_K$. This implies that the corresponding iteration scheme, using the operator $\mathcal{L}_\rho$ instead of $\mathcal{L}_0$ does not work for any value of ρ. In addition, even if $\frac{n-1}{\rho^2}$ is chosen not to belong to the spectrum of $-\Delta_K$, the norm of the inverse of $\mathcal{L}_\rho$ will blow up as ρ tends to 0 and hence the estimates for w_i and Φ_i will not be as good as the one stated in Lemma 5.1.

To conclude, the use of the iteration scheme (5.29) allows one to improve the approximate solution to any finite order. Observe that the error $\Omega(\Theta, \Theta)\, \rho^2 + \mathcal{O}(\rho^3)$ in (5.29) is smooth in the y^a variables and hence loosing finitely regularity in these variables is not a real issue.

Finally, replacing (w, Φ) by $(w^{(i)} + w, \Phi^{(i)} + \Phi)$ in (4.27), the equation we must solve becomes

$$\tfrac{1}{\rho}\, \mathcal{L}_\rho \, w + g(\mathfrak{J}\, \Phi, \Theta) - 2\, \rho^2 \, g(\Theta, B) \circ \nabla^2_K w + \rho \, L_i(w, \Phi) = \mathcal{O}_i(\rho^{i+2}) + \tfrac{1}{\rho}\, Q_i(w, \Phi). \tag{5.31}$$

This is of course simply the expansion of the equation

$$m \, H(w^{(i)} + w, \Phi^{(i)} + \Phi) = \frac{n-1}{\rho}.$$

The linear and nonlinear operators L_i and Q_i appearing in this equation are different from the ones before, but enjoy similar properties, uniformly in i. The indices i are here to remind the reader that these quantities depend on i.

6 Estimating the spectrum of the linearized operators

We now examine the mapping properties of the linear operator

$$(w, \Phi) \longmapsto \frac{1}{\rho}\, \mathcal{L}_\rho \, w + g(\mathfrak{J}\, \Phi, \Theta) - 2\, \rho^2 \, g(\Theta, B) \circ \nabla^2_K \, w + \rho \, L_i(w, \Phi) \tag{6.32}$$

which appears in (5.31). This is not precisely the usual Jacobi operator (applied to the function $\rho\, w + g(\Phi, \Theta)$), because we are parametrizing this hypersurface as a graph over $S_\rho(w^{(i)}, \Phi^{(i)})$ using the vector field $-\Upsilon$ rather than the unit normal.

To understand the difference between (6.32) and the Jacobi operator, recall that if N is the unit normal to a hypersurface Σ and $\tilde{N}$ is any other transverse vector field, then hypersurfaces which are $\mathcal{C}^2$ close to Σ can be parameterized as either

$$\Sigma \ni q \mapsto \exp^M_q(vN) \qquad \text{or} \qquad \Sigma \ni q \mapsto \exp^M_q(\tilde{v}\tilde{N}).$$

The corresponding linearized mean curvature operators $\mathbb{L}_{\Sigma,N}$ and $\mathbb{L}_{\Sigma,\tilde{N}}$ are related by

$$\mathbb{L}_{\Sigma,\tilde{N}} \, \tilde{v} = \mathbb{L}_{\Sigma,N}(g(N, \tilde{N}) \, \tilde{v}) + m \, (\tilde{N}^T H_\Sigma) \, \tilde{v},$$

here $\tilde{N}^T$ is the orthogonal projection of $\tilde{N}$ onto $T\Sigma$. Since $\mathbb{L}_{\Sigma,N}$ is self-adjoint with respect to the usual inner product, we conclude that $L_{\Sigma,\tilde{N}}$ is self-adjoint with respect to the inner product

$$\langle v, v' \rangle := \int_\Sigma v\, v'\, g(N, \tilde{N})\, dvol_\Sigma.$$

Now suppose that $\Sigma = S_\rho(w^{(i)}, \Phi^{(i)})$ and $\tilde{N} = -\Upsilon$. From Lemma 5.1 and Proposition 3.2 we have

$$g(N, -\Upsilon) = 1 + \mathcal{O}(\rho^4).$$

Furthermore, from Proposition 3.1 and Lemma 5.1, and the fact that K is minimal, the volume forms of the tubes $S_\rho(w^{(i)}, \Phi^{(i)})$ and SNK are related by

$$dvol_{S_\rho(w^{(i)},\Phi^{(i)})} = \rho^{(n-1)/2}\,(1 + \mathcal{O}(\rho^2))\, dvol_{SNK}.$$

We define $c_{\rho,i} > 0$ by

$$g(N, -\Upsilon)\, dvol_{S_\rho(w^{(i)},\Phi^{(i)})} = \rho^{(n-1)/2}\, c_{\rho,i}\, dvol_{SNK}. \tag{6.33}$$

and the operator

$$\mathbb{L}_{\rho,i}\, v := c_{\rho,i} \left(\frac{1}{\rho}\, \mathcal{L}_\rho\, w + g(\mathfrak{J}\,\Phi, \Theta) - 2\,\rho^2\, g(\Theta, B) \circ \nabla_K^2 w + \rho\, L_i(w, \Phi) \right),$$

where we have decomposed as usual $v = \rho\, w + g(\Phi, \Theta)$. Thanks to (6.33), we can write

$$\mathbb{L}_{\rho,i}\, v = \frac{1}{\rho}\, \mathcal{L}_\rho\, w + g(\mathfrak{J}\,\Phi, \Theta) - 2\,\rho^2\, g(\Theta, B) \circ \nabla_K^2 w + \rho\, \bar{L}_i(w, \Phi), \tag{6.34}$$

where $\bar{L}_i$ enjoys properties similar to the one enjoyed by L_i.

Finally, multiplying (5.31) by $c_{\rho,i}$ gives one further equivalent form of this equation,

$$\mathbb{L}_{\rho,i}\, v = \mathcal{O}_i(\rho^{2+i}) + \frac{1}{\rho}\, \bar{Q}_i \left(\frac{1}{\rho} \Pi^\perp v, \Pi v \right), \tag{6.35}$$

where the nonlinear operator on the right has the same properties as before.

Associated to $\mathbb{L}_{\rho,i}$ is the symmetric bilinear form form

$$\mathcal{C}_{\rho,i}(v, v') := \int_{SNK} v\, \mathbb{L}_{\rho,i}\, v'\, dvol_{SNK},$$

and its associated quadratic form $\mathcal{Q}_{\rho,i}(v) := \mathcal{C}_{\rho,i}(v, v)$.

We shall study these forms as perturbations of the model forms

$$\begin{aligned} \mathcal{C}_0(v, v') \;&:=\; -\int_{SNK} w'\,(\rho^2\,\Delta_K w + \Delta_{S^{n-1}} w + (n-1)\, w)\, dvol_{SNK} \\ &+\; \frac{\omega_{n-1}}{n} \int_K g(\mathfrak{J}\Phi, \Phi')\, dvol_K, \end{aligned}$$

and associated quadratic form $\mathcal{Q}_0(v) := \mathcal{C}_0(v, v)$, where $\omega_{n-1} = |S^{n-1}|$ is the volume of S^{n-1}. Observe that

$$\int_{SNK} g(\Phi, \Theta)^2 dvol_{SNK} = \frac{\omega_{n-1}}{n} \int_K |\Phi|^2\, dvol_K.$$

To make precise the sense in which $\mathcal{Q}_0$ and $\mathcal{Q}_{\rho,i}$ are close, define the weighted norm

$$\|v\|^2_{H^1_\rho} := \int_{SNK} (\rho^2 \, |\nabla_K w|^2 + |\nabla_{S^{n-1}} w|^2 + |w|^2) \, dvol_{SNK} + \int_K (|\nabla_K \Phi|^2 + |\Phi|^2) \, dvol_K,$$

and also

$$\|v\|^2_{L^2_\rho} := \int_{SNK} |w|^2 \, dvol_{SNK} + \int_K |\Phi|^2 \, dvol_K.$$

Using (6.33) and the properties of $\bar{L}_i$, we have the important :

Proposition 6.1 *There exists a constant $c > 0$ (independent of i) such that*

$$\left| \mathcal{C}_{\rho,i}(v, v') - \mathcal{C}_0(v, v') \right| \leq c \, \rho \, \|v\|_{H^1_\rho} \, \|v'\|_{H^1_\rho}. \tag{6.36}$$

Proof : This estimate arises from the fact that, $-2 \, \rho \, g(\Theta, B) \circ \nabla^2_K w + \bar{L}_i(w, \Phi)$ certainly involves terms of the form w, $\rho \, \partial_{y^a} w$, $\rho \, \partial_{y^a} \partial_{y^b} w$, $\partial_{z^j} w$, $\partial_{z^j} \partial_{z^{j'}} w$ and also Φ^j, $\partial_{y^a} \Phi^j$ and $\partial_{y^a} \partial_{y^b} \Phi^j$. Hence, after integration by parts,

$$\int_{SNK} (-2 \, \rho \, g(\Theta, B) \circ \nabla^2_K w + \bar{L}_i(w, \Phi)) \, (\rho \, w' + g(\Phi', \Theta)) \, dvol_{SNK},$$

can be bounded by a constant times $\|v\|_{H^1_\rho} \, \|v'\|_{H^1_\rho}$. □

6.1 Estimates for eigenfunctions with small eigenvalues

We prove that eigenfunctions of $\mathbb{L}_{\rho,i}$ corresponding to small eigenvalues are localized in the sense that they are essentially functions defined on K.

Lemma 6.1 *Let σ be an eigenvalue of $\mathbb{L}_{\rho,i}$ and $v = \rho \, w + g(\Phi, \Theta)$ a corresponding eigenfunction. There exist constants $c, c_0 > 0$ such that if $|\sigma| \leq c_0$, then*

$$\|v - \rho \, w_0\|^2_{H^1_\rho} \leq c \, \rho \, \|v\|^2_{H^1_\rho},$$

for all $\rho \in (0, 1)$, where $w = w_0 + w_1$ is the decomposition from (4.28).

Proof: For any $v' = \rho \, w' + g(\Phi', \Theta)$, we have

$$\begin{aligned} \mathcal{C}_{\rho,i}(v, v') &= \sigma \int_{SNK} (\rho^2 w \, w' + g(\Phi, \Theta) g(\Phi', \Theta)) \, dvol_{SNK} \\ &= \sigma \int_{SNK} \rho^2 w \, w' \, dvol_{SNK} + \sigma \, \frac{\omega_{n-1}}{n} \int_K g(\Phi, \Phi') \, dvol_K. \end{aligned}$$

In addition, (6.36) gives

$$\begin{aligned} \Bigg| \int_{SNK} (\rho^2 \, \nabla_K w \, \nabla_K w' + \nabla_{S^{n-1}} w \nabla_{S^{n-1}} w' - (n - 1 + \sigma \, \rho^2) \, w \, w') \, dvol_{SNK} \\ - \frac{\omega_{n-1}}{n} \int_K (g(\mathfrak{J}\Phi, \Phi') - \sigma \, g(\Phi, \Phi')) \, dvol_K \Bigg| \leq c \, \rho \, \|v\|_{H^1_\rho} \, \|v'\|_{H^1_\rho}. \end{aligned} \tag{6.37}$$

Step 1 : Take $w' = 0$ and $\Phi' = \Phi^+$ (resp. $\Phi' = \Phi^-$) in (6.37), where Φ^+ (resp. Φ^-) is the L^2 projection of Φ over the space of eigenfunctions of $\mathfrak{J}$ associated to positive (resp. negative) eigenvalues. This yields

$$\left| \int_K (g(\mathfrak{J}\Phi, \Phi^\pm) - \sigma \, g(\Phi, \Phi^+)) \, dvol_K \right| \leq c \, \rho \, \|v\|_{H^1_\rho} \, \|g(\Phi^\pm, \Theta)\|_{H^1_\rho}.$$

Since $\mathfrak{J}$ is invertible, there exists $c_1 > 0$ such that

$$2\,c_1\,\|g(\Phi^\pm,\Theta)\|^2_{H^1_\rho} \leq \left|\int_K g(\mathfrak{J}\Phi,\Phi^\pm)\,dvol_K\right|,$$

hence

$$(2\,c_1 - |\sigma|)\,\|g(\Phi^\pm,\Theta)\|^2_{H^1_\rho} \leq c\,\rho\,\|v\|^2_{H^1_\rho}.$$

Assuming $c_1 \geq |\sigma|$, we conclude that

$$\|g(\Phi^\pm,\Theta)\|^2_{H^1_\rho} \leq c\,\rho\,\|v\|^2_{H^1_\rho}.$$

Step 2 : Now use (6.37) with $\Phi' = 0$ and $w' = w_1$ to get

$$\left|\int_{SNK}(\rho^2\,|\nabla_K w_1|^2 + |\nabla_{S^{n-1}} w_1|^2 - (n-1-\sigma\,\rho^2)\,|w_1|^2)\,dvol_{SNK}\right| \leq c\,\rho\,\|v\|_{H^1_\rho}\,\|\rho\,w_1\|_{H^1_\rho}.$$

However, since

$$\Pi\,w_1 = 0 \qquad \text{and} \qquad \int_{S^{n-1}} w_1\,dvol_{S^{n-1}} = 0,$$

we have

$$\int_{S^{n-1}} |\nabla_{S^{n-1}} w_1|^2\,dvol_{S^{n-1}} \geq 2\,n\int_{S^{n-1}} |w_1|^2\,dvol_{S^{n-1}},$$

hence

$$\left|\int_{SNK}(\rho^2\,|\nabla_K w_1|^2 + \frac{1}{2}\,|\nabla_{S^{n-1}} w_1|^2 + (1-|\sigma|\,\rho^2)\,|w_1|^2)\,dvol_{SNK}\right| \leq c\,\rho\,\|v\|_{H^1_\rho}\,\|\rho\,w_1\|_{H^1_\rho}.$$

This implies that

$$\|\rho\,w_1\|^2_{H^1_\rho} \leq c\,\rho\,\|v\|^2_{H^1_\rho},$$

for all $\rho \in (0,1)$, provided $|\sigma| \leq 1/2$. This completes the proof if $c_0 = \min(c_1, 1/2)$, since $v - \rho\,w_0 = \rho\,w_1 + g(\Phi,\Theta)$. □

6.2 Variation of small eigenvalues with respect to ρ

We shall need to obtain some information about the spectral gaps of $\mathbb{L}_{\rho,i}$ when ρ is small, and to do this, it is necessary to understand the rate of variation of the small eigenvalues of this operator.

Lemma 6.2 *There exist constants $c_0, c > 0$ such that, if σ is an eigenvalue of $\mathbb{L}_{\rho,i}$ with $|\sigma| < c_0$, then*

$$\rho\,\partial_\rho\sigma \geq 2\,(n-1) - c\,\rho,$$

provided ρ is small enough.

Proof: There is a well-known formula for the variation of a simple eigenvalue

$$\partial_\rho\sigma = \int_{SNK} v\,(\partial_\rho\mathbb{L}_{\rho,i})\,v\,dvol_{SNK},$$

where $\mathbb{L}_{\rho,i} v = \sigma\,v$, is normalized by $\|v\|_{L^2} = 1$. Here, by definition,

$$\|v\|^2_{L^2} := \int_{SNK} v^2\,dvol_{SNK}.$$

Complications arise in the presence of multiplicities, but a result of Kato [2] shows that if one considers the derivative of the eigenvalue as a multi-valued function, then an analogue of this same formula holds for self adjoint operators :

$$\partial_\rho \sigma \in \left\{ \int_{SNK} v\, (\partial_\rho \mathbb{L}_{\rho,i})\, v \, dvol_{SNK} \qquad : \qquad \mathbb{L}_{\rho,i} v = \sigma\, v, \qquad \|v\|_{L^2} = 1 \right\}.$$

Hence we must provide bounds for the set on the right. We do this by comparing to the model case and using the bounds for eigenfunctions obtained in the last subsection.

Assume that $\mathbb{L}_{\rho,i} v = \sigma\, v$, but rather than normalizing the function v by $\|v\|_{L^2} = 1$, assume instead that $\|v\|_{L^2_\rho} = 1$. In order to compute $\partial_\rho \mathbb{L}_{\rho,i}$, recall that

$$w = \rho^{-1}\, \Pi^\perp v \qquad \text{and that} \qquad g(\mathfrak{J}\Phi, \Theta) = \Pi\, v,$$

so we can write

$$\mathbb{L}_{\rho,i}\, v = -\Delta_K\, (\Pi^\perp v) + \frac{1}{\rho^2}\, \mathcal{L}_0\, (\Pi^\perp v) + \Pi\, v - 2\,\rho\, g(\Theta, B) \circ \nabla^2_K\, (\Pi^\perp v) + \rho\, \bar{L}_i(\rho^{-1}\, \Pi^\perp v, \Pi\, v).$$

Since Π and $\Pi^\perp$ are independent of ρ, we have

$$\partial_\rho \mathbb{L}_{\rho,i} v = -\frac{2}{\rho^3}\, \mathcal{L}_0\, (\Pi^\perp v) - 2\, g(\Theta, B) \circ \nabla^2_K\, (\Pi^\perp v) + \tilde{L}_i(\rho^{-1}\, \Pi^\perp v, \Pi v),$$

where the operator $\tilde{L}_i$ varies from line to line but satisfies the usual assumptions. This now gives

$$\left| \int_{SNK} v\, (\partial_\rho \mathbb{L}_{\rho,i})\, v\, dvol_{SNK} - \frac{2}{\rho} \int_{SNK} (|\nabla_{S^{n-1}} w|^2 - (n-1)\, |w|^2)\, dvol_{SNK} \right| \leq c\, \|v\|^2_{H^1_\rho}. \tag{6.38}$$

Now, if v is an eigenfunction of $\mathbb{L}_{\rho,i}$, we have

$$Q_{\rho,i}(v) = \sigma\, \|v\|^2_{L^2} = \sigma \int_{SNK} \rho^2\, |w|^2\, dvol_{SNK} + \sigma\, \frac{\omega_{n-1}}{n} \int_K |\Phi|^2\, dvol_K,$$

and hence by (6.36),

$$\begin{aligned} \Bigg| \int_{SNK} (\rho^2\, |\nabla_K w|^2 + |\nabla_{S^{n-1}} w|^2 - (n - 1 + \sigma\, \rho^2)\, |w|^2)\, dvol_{SNK} \\ - \frac{\omega_{n-1}}{n} \int_K (g(\mathfrak{J}\Phi, \Phi) + \sigma\, g(\Phi, \Phi))\, dvol_K \Bigg| \leq c\, \rho\, \|v\|^2_{H^1_\rho}, \end{aligned} \tag{6.39}$$

By Lemma 6.1, if we assume that $|\sigma| \leq c_0$ and if, as usual we decompose $v = \rho\, w + g(\Phi, \Theta)$, we get

$$\int_{SNK} |\nabla_{S^{n-1}} w|^2\, dvol_{SNK} + \int_K (|\nabla_K \Phi|^2 + |\Phi|^2)\, dvol_K \leq c\, \rho\, \|v\|^2_{H^1_\rho}, \tag{6.40}$$

(observe that $\nabla_{S^{n-1}} w = \nabla_{S^{n-1}} w_1$ if w is decomposed as $w = w_0 + w_1$ as usual) and inserting this in (6.39) gives

$$\left| \int_{SNK} (\rho^2\, |\nabla_K w|^2 - (n - 1 + \sigma)\, |w|^2)\, dvol_{SNK} \right| \leq c\, \rho\, \|v\|^2_{H^1_\rho}. \tag{6.41}$$

Adding these last two estimates now implies that

$$\|v\|^2_{H^1_\rho} \leq c\,\rho\,\|v\|^2_{H^1_\rho} + c \int_{SNK} |w|^2\, dvol_{SNK};$$

Thus, when ρ is small enough,

$$\|v\|^2_{H^1_\rho} \leq c \int_{SNK} w^2\, dvol_{SNK} \leq c\,\|v\|^2_{L^2_\rho} \leq c,$$

if we normalize v by $\|v\|_{L^2_\rho} = 1$. From (6.40) again

$$\int_{SNK} |\nabla_{S^{n-1}} w|^2\, dvol_{SNK} + \int_K (|\nabla_K \Phi|^2 + |\Phi|^2)\, dvol_K \leq c\,\rho.$$

Inserting this into (6.38), and using again that $\|v\|_{L^2_\rho} = 1$, we get

$$\left| \int_{SNK} v\,(\partial_\rho \mathbb{L}_{\rho,i})\, v\, dvol_{SNK} - \frac{2}{\rho}\,(n-1) \right| \leq c \tag{6.42}$$

for all eigenfunction v such that $\mathbb{L}_{\rho,i} v = \sigma\, v$ which is normalized by $\|v\|_{L^2_\rho} = 1$.

This already implies that $\partial_\rho \sigma > 0$ for ρ small enough. But observing that we always have $||v||_{L^2} \leq \|v\|_{L^2_\rho}$, we conclude that

$$\inf_{\substack{\mathbb{L}_\rho v = \sigma\, v \\ \|v\|_{L^2} = 1}} \int_{SNK} v\,(\partial_\rho \mathbb{L}_\rho)\, v\, dvol_{SNK} \quad \geq \inf_{\substack{\mathbb{L}_\rho v = \sigma\, v \\ \|v\|_{L^2_\rho} = 1}} \int_{SNK} v\,(\partial_\rho \mathbb{L}_\rho)\, v\, dvol_{SNK},$$

and (6.42) implies that

$$\partial_\rho \sigma \geq \frac{2}{\rho}\,(n-1) - c.$$

This completes the proof of the result. □

6.3 The spectral gap at 0 of $\mathbb{L}_{\rho,i}$

We can now prove a quantitative statement about the clustering of the spectrum at 0 of $\mathbb{L}_{\rho,i}$ as $\rho \searrow 0$. The ultimate goal is to estimate the norm of the inverse of this operator, but by self-adjointness, this is equivalent to an estimate on the size of the spectral gap at 0.

Lemma 6.3 *Fix any $q \geq 2$. Then there exists a sequence of disjoint nonempty open intervals $I_\ell = (\rho_\ell^-, \rho_\ell^+)$, $\rho_\ell^\pm \to 0$ and a constant $c_q > 0$ such that when $\rho \in I^q := \cup_\ell I_\ell$, the operator $\mathbb{L}_{\rho,i}$ is invertible and*

$$(\mathbb{L}_{\rho,i})^{-1} : L^2(SNK) \longrightarrow L^2(SNK),$$

has norm bounded by $c_q\,\rho^{-k-q+1}$, uniformly in $\rho \in I$. Furthermore, $I^q := \cup_\ell I_\ell$ satisfies

$$\left| \mathcal{H}^1((0,\rho) \cap I^q) - \rho \right| \leq c\,\rho^q, \qquad \rho \searrow 0.$$

Proof: An estimate for the size of the spectral gap at 0 is related to the spectral flow of $\mathbb{L}_{\rho,i}$, and so it suffices to find an asymptotic estimate for the number of negative eigenvalues of $\mathbb{L}_{\rho,i}$. Define the two quadratic forms

$$\mathcal{Q}^{\pm}(v) := \mathcal{Q}_0(v) \pm \gamma\,\rho\,\|v\|^2_{H^1_\rho}.$$

¿From (6.36), if $\gamma > 0$ is sufficiently large, then

$$\mathcal{Q}^{-} \leq \mathcal{Q}_{\rho,i} \leq \mathcal{Q}^{+},$$

and this will give a two-sided bound for the index of $\mathcal{Q}_{\rho,i}$, i.e. the dimension of the largest space where $\mathcal{Q}_{\rho,i}$ is negative.

Given any function w defined on SNK, we write

$$D_0^{\pm}(w) := (1 \pm \gamma\,\rho) \int_K \rho^2\,|\nabla_K w|^2\,dvol_{SNK} - (n-1 \mp \gamma\,\rho) \int_K |w|^2\,dvol_K,$$

$$D_1^{\pm}(w) := (1 \pm \gamma\,\rho) \int_{SNK} (\rho^2\,|\nabla_K w|^2 + |\nabla_{S^{n-1}} w|^2)\,dvol_{SNK} - (n-1 \mp \gamma\,\rho) \int_{SNK} |w|^2\,dvol_K,$$

and finally, we define

$$D^{\pm}(\Phi) := -(1 \pm \gamma\,\rho) \int_K g(\mathfrak{J}\,\Phi, \Phi)\,dvol_K.$$

With these definitions in mind, we have

$$\mathcal{Q}^{\pm}(v) = \omega_{n-1}\,D_0^{\pm}(w_0) + D_1^{\pm}(w_1) + \frac{\omega_{n-1}}{n}\,D^{\pm}(\Phi),$$

if we decompose $v = \rho\,w + g(\Phi, \Theta)$ and further decompose $w = w_0 + w_1$ as usual.

If $1 - \gamma\,\rho > 0$, then the index of $D^{\pm}$ is equal to the index of the minimal submanifold K, and hence does not depend on ρ. Next, if $2\,n\,(1-\gamma\,\rho) - (n-1+\gamma\,\rho) > 0$, then the index of $D_1^{\pm}$ equals 0 since we have

$$\int_{S^{n-1}} |\nabla_{S^{n-1}} w_1|^2\,dvol_{S^{n-1}} \geq 2\,n \int_{S^{n-1}} |w_1|^2\,dvol_{S^{n-1}}.$$

So it remains only to study the index of $D_0^{\pm}$. We denote by

$$\mu_0 < \mu_1 \leq \ldots \leq \mu_j \leq \ldots$$

the eigenvalues of $-\Delta_K$ which are counted with multiplicity. Weyl's asymptotic formula states that

$$\#\{j \in \mathbb{N} \quad : \quad \mu_j \leq \mu\} \sim c_K\,\mu^{\frac{k}{2}}.$$

where $c_K > 0$ only depends on the dimension and the volume of K. Now, the index of $D_0^{\pm}$ is equal to the largest $j \in \mathbb{N}$ such that

$$(1 \pm \gamma\,\rho)\,\rho^2\,\mu_j < (n-1 \mp c\,\rho).$$

Using Weyl's asymptotic formula, we conclude that

$$\mathrm{Ind}\,D_0^{\pm} \sim c_K \left(\frac{n-1}{\rho^2}\right)^{\frac{k}{2}},$$

and hence we have proved that the index $\mathcal{Q}_{\rho,i}$ is asymptotic to $c\,\rho^{-k}$, where c only depends on K and m.

Let $\rho_\ell \searrow 0$ be the decreasing sequence corresponding to the values at which the index of $\mathcal{Q}_{\rho,i}$ changes, counted according to the dimension of the nullspace of $\mathbb{L}_{\rho_\ell,i}$, i.e.

$$\rho_{\ell-1} < \rho_\ell = \ldots = \rho_{\ell'} < \rho_{\ell'+1},$$

if $\dim \operatorname{Ker} \mathbb{L}_{\rho_\ell,i} = \ell' - \ell + 1$. This is well-defined since, by Lemma 6.2 the small eigenvalues of $\mathbb{L}_{\rho,\ell}$ are monotone increasing for ρ small enough and hence, the function

$$\rho \longrightarrow \operatorname{Ind} \mathcal{Q}_{\rho,i},$$

is monotone decreasing for ρ small.

The asymptotic estimates for $\operatorname{Ind} Q_{2\rho,i}$ and $\operatorname{Ind} Q_{\rho,i}$ imply that

$$r_\rho := \#\{\ell \quad : \quad \rho_\ell \in (\rho, 2\rho)\} \sim c\,\rho^{-k}.$$

Letting λ_ρ denote the sum of lengths of intervals $(\rho_{\ell+1}, \rho_\ell)$ for which

$$\rho_{\ell+1} \in (\rho, 2\rho) \qquad \text{and} \qquad (\rho_\ell - \rho_{\ell+1}) \leq \rho^{k+q},$$

then we have $\lambda_\rho \leq c\,\rho^q$. From this we conclude that $\tilde\lambda_\rho$, the sum of lengths of all intervals $(\rho_{\ell+1}, \rho_\ell)$ where

$$\rho_{\ell+1} < \rho \qquad \text{and} \qquad (\rho_\ell - \rho_{\ell+1}) \leq \rho^{k+q},$$

is also bounded by $c\,\rho^q$, where the constant $c > 0$ depends on q.

We set

$$\tilde I^q := \bigcup_{\ell \in J_q} (\rho_{\ell+1}, \rho_\ell), \qquad \text{where} \qquad \ell \in J_q \qquad \Leftrightarrow \qquad \rho_\ell - \rho_{\ell+1} > \rho_\ell^{k+q}.$$

Then by the above, we have

$$\left| \mathcal{H}^1((0,\rho) \cap \tilde I^q) - \rho \right| \leq c_q\,\rho^q.$$

Finally, consider for any $\ell \in J_q$ and $\rho \in (\rho_{\ell+1}, \rho_\ell)$. We denote by

$$\sigma^-(\rho) < 0 < \sigma^+(\rho)$$

the eigenvalues of $\mathbb{L}_{\rho,i}$ which are closest to 0. By construction,

$$\lim_{\rho \searrow \rho_{\ell+1}} \sigma^+(\rho) = \lim_{\rho \nearrow \rho_\ell} \sigma^-(\rho) = 0.$$

By Lemma 6.2,

$$\begin{aligned} \sigma^-(\rho) &\leq 2\,(n-1)\,\log(\rho/\rho_\ell) + c\,(\rho_\ell - \rho), \\ \sigma^+(\rho) &\geq 2\,(n-1)\,\log(\rho/\rho_{\ell+1}) - c\,(\rho - \rho_{\ell+1}), \end{aligned} \tag{6.43}$$

for all $\rho \in (\rho_{\ell+1}, \rho_\ell)$. Hence by the monotonicity of small eigenvalues,

$$\sigma^-(\rho) \leq \sigma^-(\rho_\ell - \rho_\ell^{k+q}/4) < 0 < \sigma^+(\rho_{\ell+1} + \rho_\ell^{k+q}/4) \leq \sigma^+(\rho),$$

if

$$\rho \in I^q := \bigcup_\ell (\rho_{\ell+1} + \rho_\ell^{k+q}/4, \rho_\ell - \rho_\ell^{k+q}/4),$$

and, using (6.43) we conclude that the infimum of the absolute value of the eigenvalues of $\mathbb{L}_{\rho,i}$ is bounded from below by a constant (only depending on K and m) times ρ_ℓ^{k+q-1}, provided ρ_ℓ is small enough. Moreover, as above we have

$$\left|\mathcal{H}^1((0,\rho)\cap I^q) - \rho\right| \leq c_q\, \rho^q.$$

The result then follows at once. □

7 Existence of constant mean curvature hypersurfaces

We now use the results of the previous sections in order to solve the equation (6.35) which reduces to find a fixed point

$$v = (\mathbb{L}_{\rho,i})^{-1}\left(\mathcal{O}_i(\rho^{2+i}) + \frac{1}{\rho}\,\bar{Q}_i\left(\frac{1}{\rho}\,\Pi^\perp v, \Pi v\right)\right).$$

We start with the following elementary observation

Lemma 7.1 *There exists a constant $c > 0$ such that*

$$\rho^{2+\alpha}\,\|v\|_{\mathcal{C}^{2,\alpha}(SNK)} \leq c\,\rho^2\,\|\mathbb{L}_{\rho,i}\,v\|_{\mathcal{C}^{0,\alpha}(SNK)} + c\,\rho^{-\frac{k}{2}}\,\|v\|_{L^2(SNK)}$$

Proof : This is a simple application of (rescaled) standard elliptic estimates. We set $f := \mathbb{L}_{\rho,i}\,v$ and, as in §3.1, we use local normal coordinates $\bar{y} = y/\rho$ to parameterize a ball of radius $2\,\rho\,R$ in K, for some fixed small constant $R > 0$, and local coordinates z to parameterize S^{n-1}. Define the functions

$$\bar{v}(z,\bar{y}) := v(z,\rho\,\bar{y}) \qquad \text{and} \qquad \bar{f}(z,\bar{y}) := \rho^2\,f(z,\rho\,\bar{y})$$

It is easy to check that $f := \mathbb{L}_{\rho,i}\,v$ translates into $\bar{\mathbb{L}}_{\rho,i}\bar{v} = \bar{f}$, where $\bar{\mathbb{L}}_{\rho,i}$ is a second order elliptic operator whose coefficients are bounded uniformly as ρ tends to 0. Moreover, the principal part of $\bar{\mathbb{L}}_{\rho,i}$ is the Laplace operator on SNK. Standard elliptic estimates yield

$$\|\bar{v}\|_{\bar{\mathcal{C}}^{2,\alpha}(B_R\times S^{n-1})} \leq c\,\|\bar{f}\|_{\bar{\mathcal{C}}^{0,\alpha}(B_R\times S^{n-1})} + c\left(\int_{S^{n-1}}\left(\int_{B_{2R}}|\bar{v}|^2\,d\bar{y}\right)\,dvol_{S^{n-1}}\right)^{1/2}$$

where, to evaluate the Hölder norms in $\bar{\mathcal{C}}^{p,\alpha}$ one takes derivatives with respect to $\bar{y}$ and z. Going back to the functions v and f we have

$$\rho^{2+\alpha}\,\|v\|_{\mathcal{C}^{2,\alpha}(B_{\rho R}\times S^{n-1})} \leq c\,\|\bar{v}\|_{\bar{\mathcal{C}}^{2,\alpha}(B_R\times S^{n-1})}, \quad \|\bar{f}\|_{\bar{\mathcal{C}}^{0,\alpha}(B_R\times S^{n-1})} \leq c\,\rho^2\,\|f\|_{\mathcal{C}^{0,\alpha}(B_{\rho R}\times S^{n-1})}$$

and

$$\left(\int_{S^{n-1}}\left(\int_{B_{2R}}|\bar{v}|^2\,d\bar{y}\right)\,dvol_{S^{n-1}}\right)^{1/2} \leq c\,\rho^{-\frac{k}{2}}\left(\int_{S^{n-1}}\left(\int_{B_{2\rho R}}|v|^2\,dy\right)\,dvol_{S^{n-1}}\right)^{1/2}$$

the result then follows at once. □

We fix $q \geq 2$ and $\alpha \in (0,1)$ and define

$$D := \frac{3}{2}\,k + q + \alpha + 1, \qquad \text{and} \qquad i = 3\,k + 2q + 4 > 2D + 1.$$

Collecting the result of Lemma 6.3 and the result of the previous Lemma, we conclude that, if $\rho \in I^q$, then

$$\|v\|_{\mathcal{C}^{2,\alpha}(SNK)} \leq c\,\rho^{-D}\,\|\mathbb{L}_{\rho,i}\,v\|_{\mathcal{C}^{0,\alpha}(SNK)}, \tag{7.44}$$

where the constant $c > 0$ does not depend on ρ (but depends on i, hence on q).

We define the nonlinear mapping

$$\mathcal{N}_\rho(v) := (\mathbb{L}_{\rho,i})^{-1}\left(\mathcal{O}_i(\rho^{2+i}) + \frac{1}{\rho}\,\bar{Q}_i\left(\frac{1}{\rho}\,\Pi^\perp v, \Pi v\right)\right).$$

It follows from (7.44) that we have

$$\|\mathcal{N}_\rho(0)\|_{\mathcal{C}^{2,\alpha}} \leq \frac{c_q}{2}\,\rho^{2+i-D},$$

for some constant $c_q > 0$ depending on q but independent of $\rho \in I^q$.

Given $\rho > 0$, we set

$$B_\rho := \{v \in \mathcal{C}^{2,\alpha}(SNK) \qquad : \qquad \|v\|_{\mathcal{C}^{2,\alpha}} \leq c_q\,\rho^{2+i-D}\}.$$

Using the properties of the operator $\bar{Q}_i$, it is easy to check that there exists $\rho_q > 0$, only depending on q, such that, for all $\rho \in (0, \rho_q) \cap I^q$,

$$\|\mathcal{N}_\rho(v)\|_{\mathcal{C}^{2,\alpha}(SNK)} \leq c_0\,\rho^{2+i-D},$$

and

$$\|\mathcal{N}_\rho(v) - \mathcal{N}_\rho(v')\|_{\mathcal{C}^{2,\alpha}} \leq c\,\rho^{i-1-2D}\,\|v - v'\|_{\mathcal{C}^{2,\alpha}},$$

for all $v, v' \in B_\rho$. In particular, the mapping $\mathcal{N}_\rho$ admits a (unique) fixed point

$$v_\rho = \rho\,w_\rho + g(\Phi_\rho, \Theta),$$

in B_ρ. This yields the existence of $S_\rho(w^{(i)} + w_\rho, \Phi^{(i)} + \Phi_\rho)$, a constant mean curvature perturbation of the tube $S_\rho(w^{(i)}, \Phi^{(i)})$ for all $\rho \in (0, \rho_q) \cap I^q$. The proof of the Theorem is therefore complete with

$$I := \cup_{q\geq 2}((0, \rho_q) \cap I^q).$$

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