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# CONSTANT MEAN CURVATURE HYPERSURFACES IN WARPED PRODUCT SPACES

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*Abstract* We study hypersurfaces of constant mean curvature immersed into warped product spaces of the form $\mathbb{R} \times_{\varrho} \mathbb{P}^n$, where $\mathbb{P}^n$ is a complete Riemannian manifold. In particular, our study includes that of constant mean curvature hypersurfaces in product ambient spaces, which have recently been extensively studied. It also includes constant mean curvature hypersurfaces in the so-called *pseudo-hyperbolic* spaces. If the hypersurface is compact, we show that the immersion must be a leaf of the trivial totally umbilical foliation $t \in \mathbb{R} \mapsto \{t\} \times \mathbb{P}^n$, generalizing previous results by Montiel. We also extend a result of Guan and Spruck from hyperbolic ambient space to the general situation of warped products. This extension allows us to give a slightly more general version of a result by Montiel and to derive height estimates for compact constant mean curvature hypersurfaces with boundary in a leaf.

*Keywords:* mean curvature; warped product spaces; totally umbilical foliation; maximum principle

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## 1. Introduction

It is a classical result that a compact hypersurface embedded in Euclidean space with constant mean curvature must be a round sphere. Alexandrov [**1**] gave a proof of this fact by making a clever use of the maximum principle for elliptic partial differential equations. The so-called Alexandrov reflection method also works for hypersurfaces in Euclidean sphere and hyperbolic space, since its main requirement of having a large number of isometric reflections is satisfied in such ambient spaces.

An attempt to extend the above result from constant sectional curvature manifolds to a larger class of Riemannian spaces should consider manifolds with an abundance of complete embedded constant mean curvature hypersurfaces. Such hypersurfaces play the role of the umbilical hypersurfaces in spaces of constant sectional curvature. Then, one looks for geometric conditions on an immersed complete constant mean curvature hypersurface that force it to be one of those already classified. In space forms, one proves such classification results by using the abundance of isometries of the space. Since here we consider more general ambient manifolds, we need to develop an appropriate method of proof.

Montiel [**12**] observed that a natural class of manifolds to consider is that of warped products $M^{n+1} = \mathbb{R} \times_\varrho \mathbb{P}^n$, where $\mathbb{P}^n$ is a complete $n$-dimensional Riemannian manifold, $\varrho : \mathbb{R} \to \mathbb{R}_+$ is a smooth function and the product manifold $\mathbb{R} \times \mathbb{P}^n$ is endowed with the complete Riemannian metric

$$\langle \cdot\,, \cdot \rangle = \pi_{\mathbb{R}}^*(dt^2) + \varrho^2(\pi_{\mathbb{R}})\pi_{\mathbb{P}}^*(\langle \cdot\,, \cdot \rangle_{\mathbb{P}}).$$

Here $\pi_{\mathbb{R}}$ and $\pi_{\mathbb{P}}$ denote the projections onto the corresponding factor and $\langle \cdot\,, \cdot \rangle_{\mathbb{P}}$ is the Riemannian metric on $\mathbb{P}^n$. Each leaf $\mathbb{P}_t = \{t\} \times \mathbb{P}^n$ (called here a *slice*) of the foliation $t \in \mathbb{R} \mapsto \mathbb{P}_t$ of $M^{n+1}$ by complete hypersurfaces has constant mean curvature. Its mean curvature vector field is

$$\boldsymbol{H}_t = -\mathcal{H}(t)T,$$

where $\mathcal{H}(t) = \varrho'(t)/\varrho(t)$ and $T = \partial/\partial t \in TM$. For further geometric interpretation, observe that $\mathcal{T} = \varrho T$ is a *closed conformal vector field* on $M^{n+1}$, that is, it satisfies

$$\bar{\nabla}_V \mathcal{T} = \varrho' V \quad \text{for any } V \in TM. \tag{1.1}$$

Here and elsewhere $\bar{\nabla}$ denotes the Levi-Civita connection in $M^{n+1}$ and, by abuse of notation, we denote in the same way functions on $\mathbb{R}$ and their lift to $M^{n+1}$. In [**12**, § 3] it is carefully shown that any Riemannian manifold $M^{n+1}$ with a closed conformal vector field is locally isometric to a warped product manifold with one-dimensional factor. Furthermore, the isometry is global if $M^{n+1}$ is complete and simply connected.

Extending the well-known *Mercator projection*, used in cartography to conformally project the two-dimensional sphere into the Euclidean plane [**17**, p. 173] (see [**14**] for the hyperbolic case), we conformally transform the warped product space $\mathbb{R} \times_\varrho \mathbb{P}^n$ into a product space with factor $\mathbb{P}^n$. In fact, let $\tau : \mathbb{R}\times\mathbb{P}^n \to \mathbb{J}\times\mathbb{P}^n$ be given by $\tau(t, x) = (s(t), x)$, where $\mathbb{J} = s(\mathbb{R})$ and

$$s(t) = s_0 - \int_0^t \frac{1}{\varrho(u)}\,\mathrm{d}u.$$

Then $\tau$ is a reversing orientation isometry between $M^{n+1}$ and $\mathbb{J} \times \mathbb{P}^n$ endowed with the conformal metric

$$\langle \cdot\,, \cdot \rangle = \lambda^2(s)(ds^2 + \langle \cdot\,, \cdot \rangle_{\mathbb{P}^n}), \tag{1.2}$$

where the conformal factor is $\lambda(s) = \varrho(t(s))$. Suppose that $\varrho(t)$ satisfies

$$\int_0^{+\infty} \frac{1}{\varrho} < +\infty \quad \text{and} \quad \int_{-\infty}^0 \frac{1}{\varrho} = +\infty, \tag{1.3}$$

and take $s_0 = \int_0^{+\infty} 1/\varrho$. Then, we have that $\mathbb{J} = \mathbb{R}_+$ and, therefore, $\mathbb{P}^n$ acts as a *boundary at infinity* of $\mathbb{R} \times_\varrho \mathbb{P}^n$, as does $\{0\} \times \mathbb{R}^n$ in $\mathbb{H}^{n+1}$, and the leaves $\mathbb{P}_t$ can be thought of as horospheres in a fixed direction of $\mathbb{H}^{n+1}$.

There are two cases (after normalization) in which *all* slices have the *same* constant mean curvature $\mathcal{H}$. The first one is when $\mathcal{H}(t) = 0$ ($\varrho(t) = 1$), and the ambient space is just

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a Riemannian product $M^{n+1} = \mathbb{R} \times \mathbb{P}^n$. Constant mean curvature hypersurfaces in these spaces have been extensively studied in recent years. The second case is when $\mathcal{H}(t) = 1$ ($\varrho(t) = \mathrm{e}^t$), and then $M^{n+1}$ belongs to the class of *pseudo-hyperbolic* manifolds defined in [**18**]. In this case, the conformal factor in (1.2) is $\lambda(s) = 1/s$ and (1.3) is satisfied. Moreover, if $\mathbb{P}^n$ is Ricci flat, then $M^{n+1}$ is Einstein with negative Ricci curvature, and if $\mathbb{P}^n$ is flat then $M^{n+1}$ is a negatively curved space form. Thus, for $\varrho(t) = \mathrm{e}^t$ we deal with ambient spaces that have many resemblances with hyperbolic space $\mathbb{H}^{n+1}$.

Montiel's method of proof in [**12**] combines the use of two Minkowski-type formulae. In his Corollary 7 he gives the following.

**Theorem 1.1 (Montiel's first result).** *Let $\mathbb{P}^n$ be a compact manifold satisfying $\mathrm{Ric}_{\mathbb{P}} > \sup_{\mathbb{R}}\{-\varrho^2\mathcal{H}'(t)\}$. Then any compact orientable immersed constant mean curvature hypersurface in $\mathbb{R} \times_\varrho \mathbb{P}^n$ that is locally a graph over $\mathbb{P}^n$ must be a slice.*

This result has the following consequences (see [**12**, Corollary 8]) for the class of pseudo-hyperbolic ambient spaces.

**Theorem 1.2 (Montiel's second result).**

(a) *Let $\mathbb{P}^n$ be compact with non-negative Ricci curvature. Then any compact constant mean curvature hypersurface in $\mathbb{R} \times_{\mathrm{e}^t} \mathbb{P}^n$ that is locally a graph on $\mathbb{P}^n$ must be a slice.*

(b) *Let $\mathbb{P}^n$ be compact with Ricci curvature satisfying $\mathrm{Ric}_{\mathbb{P}} \geqslant -1$. Then any compact constant mean curvature hypersurface in $\mathbb{R} \times_{\cosh t} \mathbb{P}^n$ that is locally a graph on $\mathbb{P}^n$ must be a slice.*

In § 2 we compute the Laplacian of $\sigma \circ h \in \mathcal{C}^\infty(\Sigma)$, where $h$ is the height function of an immersed hypersurface $\Sigma^n \mapsto \mathbb{R} \times_\varrho \mathbb{P}^n$ and $\sigma \in \mathcal{C}^\infty(\mathbb{R})$ satisfies $\sigma'(t) = \varrho(t)$. This yields a rather simple differential equation that has several applications for compact hypersurfaces. In particular, it allows a generalization of Montiel's second result, for instance, by removing the assumption about the Ricci curvature. We also consider the case of complete hypersurfaces via the Omori–Yau maximum principle.

In § 3 we extend a result of Guan and Spruck [**7**] from hyperbolic ambient space to the general situation studied in this paper. Such an extension allows a slight generalization of Montiel's first result. Then we use our result to provide height estimates for compact constant mean curvature hypersurfaces with boundary contained in a slice of either a product or a pseudo-hyperbolic ambient space, thus extending results in [**7, 10**]. Further applications for graphs with boundary are given in [**2**].

## Note added in proof

Reference [**2**] uses some of the results proved in this paper. As the systems used to number the results in this paper and [**2**] are different, for each result below, the numbering system used in [**2**] is also given in parentheses.

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## 2. The first equation

In this section we compute a basic partial differential equation whose strength is based on its independence of the curvature tensor of the ambient space. We then derive several consequences, in particular a generalization of Montiel's first result.

Let $f : \Sigma^n \to \mathbb{R} \times_\varrho \mathbb{P}^n$ be an isometric immersion of an $n$-dimensional Riemannian manifold $\Sigma^n$; its *height function* $h \in \mathcal{C}^\infty(\Sigma)$ is defined as $h = \pi_{\mathbb{R}} \circ f$, where $\pi_{\mathbb{R}}$ denotes the projections onto the first factor.

**Proposition 2.1.** *Let $f : \Sigma^n \to M^{n+1} = \mathbb{R} \times_\varrho \mathbb{P}^n$ be an isometric immersion with mean curvature vector field $\boldsymbol{H}$. If*

$$\sigma(t) = \int_{t_0}^{t} \varrho(r)\,\mathrm{d}r,$$

*then*

$$\Delta\sigma(h) = n\varrho(h)(\mathcal{H}(h) + \langle \boldsymbol{H}, T\rangle), \tag{2.1}$$

*where $\mathcal{H}(t) = \varrho'(t)/\varrho(t)$ and $T = \partial/\partial t \in TM$.*

**Proof.** The gradient of $\pi_{\mathbb{R}} \in \mathcal{C}^\infty(M)$ is $\bar{\nabla}\pi_{\mathbb{R}} = T$, and thus the gradient of $h$ is

$$\nabla h = (\bar{\nabla}\pi_{\mathbb{R}})^\top = T - \langle T, N\rangle N, \tag{2.2}$$

where by $(\cdot)^\top$ denotes the tangential component of a vector field along $f$ and $N$ is a (local) smooth unit normal vector field. It is a standard fact that the Levi-Civita connection of a warped product satisfies

$$\bar{\nabla}_V T = \mathcal{H}(V - \langle V, T\rangle T) \quad \text{for any } V \in TM. \tag{2.3}$$

It follows from (2.2) and (2.3) that

$$\bar{\nabla}_X \nabla h = \mathcal{H}(h)(X - \langle X, T\rangle T) - X(\langle T, N\rangle)N + \langle T, N\rangle AX \tag{2.4}$$

for any $X \in T\Sigma$. Here $AX = -\bar{\nabla}_X N$ denotes the second fundamental form of $f$ with respect to $N$. Then, we get

$$\nabla_X \nabla h = (\bar{\nabla}_X \nabla h)^\top = \mathcal{H}(h)(X - \langle X, \nabla h\rangle \nabla h) + \langle T, N\rangle AX, \tag{2.5}$$

where $\nabla$ denotes the Levi-Civita connection in $\Sigma^n$. It follows from here that the Laplacian of $h$ is given by

$$\Delta h = \mathcal{H}(h)(n - \|\nabla h\|^2) + n\langle \boldsymbol{H}, T\rangle. \tag{2.6}$$

Since $\nabla\sigma(h) = \varrho(h)\nabla h$, we have

$$\Delta\sigma(h) = \varrho(h)\Delta h + \varrho'(h)\|\nabla h\|^2 = n\varrho(h)(\mathcal{H}(h) + \langle \boldsymbol{H}, T\rangle),$$

and this concludes the proof. □

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We first analyse the case of compact hypersurfaces (without boundary). Our first result is mostly technical because of the assumption on the immersion itself.

**Proposition 2.2.** *Let $f : \Sigma^n \to \mathbb{R} \times_\varrho \mathbb{P}^n$ be a compact hypersurface such that either*

$$\|\boldsymbol{H}\| \leqslant \mathcal{H} \circ h \quad \text{or} \quad \|\boldsymbol{H}\| \leqslant -\mathcal{H} \circ h \tag{2.7}$$

*holds along $\Sigma^n$. Then $\mathbb{P}^n$ is compact and $f(\Sigma^n)$ is a slice.*

**Proof.** At any point of $\Sigma^n$ we have by the Cauchy–Schwarz inequality that

$$\mathcal{H}(h) - \|\boldsymbol{H}\| \leqslant \mathcal{H}(h) + \langle \boldsymbol{H}, T\rangle \leqslant \mathcal{H}(h) + \|\boldsymbol{H}\|.$$

By assumption the function $\mathcal{H}(h)+\langle \boldsymbol{H}, T\rangle$ does not change sign. It follows from (2.1) that $\Delta\sigma(h)$ does not change sign either. But $\Sigma^n$ being compact, the divergence theorem gives $\Delta\sigma(h) = 0$, and hence $\sigma(h)$ must be constant (that is, any subharmonic or superharmonic function on a compact Riemannian manifold without boundary must be constant; this property is used several times in the paper). Since $\sigma'(t) = \varrho(t) > 0$, we conclude that $h$ itself must be constant. □

Notice that (2.7) implies that the function $\mathcal{H} \circ h \in \mathcal{C}^\infty(\Sigma^n)$ does not change sign, and means just that in the minimal case. It is thus natural (and convenient) to assume that $\mathcal{H} \in \mathcal{C}^\infty(\mathbb{R})$ does not change sign, instead of involving the immersion $f$ in the hypothesis. Geometrically, the fact that $\mathcal{H}$ does not change sign means that the mean curvature vectors of *all* slices $\mathbb{P}_t$ point in the same direction.

The next corollary of Proposition 2.2 states the analogue in $\mathbb{R} \times_\varrho \mathbb{P}^n$ of the non-existence of compact hypersurfaces either that are minimal in $\mathbb{R}^{n+1}$ and $\mathbb{R} \times \mathbb{H}^n$ or with mean curvature function $0 \leqslant H \leqslant 1$ in $\mathbb{H}^{n+1}$. The case when $\mathcal{H}(t) \leqslant 0$ can be reduced to that when $\mathcal{H}(t) \geqslant 0$ by changing the orientation of the factor $\mathbb{R}$.

**Proposition 2.3.** *Assume that $\mathcal{H}(t) \geqslant 0$ and set $\mathcal{H}_0 = \inf_{\mathbb{R}} \mathcal{H}(t)$. A compact hypersurface in $\mathbb{R} \times_\varrho \mathbb{P}^n$ with mean curvature function $0 \leqslant H \leqslant \mathcal{H}_0$ occurs only if $\mathbb{P}^n$ is compact and, then, it is any slice $\mathbb{P}_{t_0}$ in which $\mathcal{H}(t_0) = \mathcal{H}_0$.*

**Proof.** Proposition 2.2 yields that $\mathbb{P}^n$ must be compact and that $H = \mathcal{H}(t_0)$ for some $t_0 \in \mathbb{R}$. Thus, $H = \mathcal{H}_0$ by assumption. □

A submanifold $f : \Sigma^n \to \mathbb{R} \times_\varrho \mathbb{P}^n$ is called *two sided* if its normal bundle is trivial, i.e. there is a globally defined unit normal vector field. For instance, every hypersurface with non-zero constant mean curvature is trivially two sided. We then define the smooth *angle function* $\Theta : \Sigma^n \to [-1, 1]$ by

$$\Theta(p) = \langle N(p), T\rangle,$$

where $N$ denotes the global normal field.

If $f$ is locally a graph over $\mathbb{P}^n$ (i.e. transversal to $T$), then either $\Theta < 0$ or $\Theta > 0$ along $\Sigma^n$. Thus, requiring $\Theta$ not to change sign is a weaker assumption than requiring it to be a local graph. Notice that $\Theta^2 = 1$ if and only if $\Sigma^n$ is a slice (see (2.8), below).

From now on, every time the angle function of a two-sided hypersurface does not change sign, the orientation $N$ is chosen so that $\Theta \leqslant 0$, and then the mean curvature function is $H = \langle \boldsymbol{H}, N \rangle$.

Montiel observed that if $\mathbb{P}^n$ is compact and if the mean curvature of the slices is non-decreasing ($\mathcal{H}'(t) \geqslant 0$), then any compact constant mean curvature graph over $\mathbb{P}^n$ must be a slice (see [**12**, Remark 6]). To see this, he compares the hypersurface with slices and then invokes the maximum principle. The following theorem generalizes such results as well as [**12**, Corollary 8] (Montiel's second result).

**Theorem 2.4.** *Let $f : \Sigma^n \to \mathbb{R} \times_\varrho \mathbb{P}^n$ be a compact two-sided hypersurface of constant mean curvature $H$. Assume that $\mathcal{H}'(t) \geqslant 0$ and that the angle function $\Theta$ does not change sign. Then $\mathbb{P}^n$ is compact and $f(\Sigma^n)$ is a slice.*

**Proof.** Let $p_{\min}, p_{\max} \in \Sigma^n$ be such that

$$h(p_{\min}) = \underline{h} := \min_{\Sigma} h \quad \text{and} \quad h(p_{\max}) = \bar{h} := \max_{\Sigma} h.$$

Therefore, $\nabla h(p_{\min}) = 0$ and $\nabla h(p_{\max}) = 0$. From (2.2) we have that

$$\|\nabla h\|^2 = 1 - \Theta^2, \tag{2.8}$$

and therefore

$$\Theta(p_{\min}) = \pm 1 \quad \text{and} \quad \Theta(p_{\max}) = \pm 1. \tag{2.9}$$

Moreover, (2.6) gives

$$\begin{aligned} \Delta h(p_{\min}) &= n(\mathcal{H}(\underline{h}) + \langle \boldsymbol{H}(p_{\min}), T \rangle) \geqslant 0, \\ \Delta h(p_{\max}) &= n(\mathcal{H}(\bar{h}) + \langle \boldsymbol{H}(p_{\max}), T \rangle) \leqslant 0. \end{aligned}$$

Hence,

$$-\langle \boldsymbol{H}(p_{\min}), T \rangle \leqslant \mathcal{H}(\underline{h}) \quad \text{and} \quad \mathcal{H}(\bar{h}) \leqslant -\langle \boldsymbol{H}(p_{\max}), T \rangle. \tag{2.10}$$

Before we proceed, for later use note that the proof of (2.10) uses only the fact that $\Sigma^n$ is compact. From (2.9), (2.10) and $\mathcal{H}' \geqslant 0$, we obtain

$$-\Theta(p_{\min}) H(p_{\min}) \leqslant \mathcal{H}(\underline{h}) \leqslant \mathcal{H}(\bar{h}) \leqslant -\Theta(p_{\max}) H(p_{\max}).$$

By assumption $\Theta(p_{\min}) = \Theta(p_{\max}) = \operatorname{sgn} \Theta$, and hence

$$-H \operatorname{sgn} \Theta \leqslant \mathcal{H}(\underline{h}) \leqslant \mathcal{H}(\bar{h}) \leqslant -H \operatorname{sgn} \Theta.$$

It follows that $\mathcal{H} \circ h = -H \operatorname{sgn} \Theta$. We obtain from (2.1) that

$$\Delta \sigma(h) = n \varrho(h) H (\Theta - \operatorname{sgn} \Theta),$$

and thus $\Delta(\sigma \circ h)$ does not change sign. Therefore, $\sigma \circ h$ and hence $h$ itself must be constant. $\square$

We have two useful corollaries of the proof of Theorem 2.4. For instance, dropping the assumption that $\mathcal{H}' \geqslant 0$, we still have the following result.

**Proposition 2.5.** *Let $f : \Sigma^n \to \mathbb{R} \times_\varrho \mathbb{P}^n$ be a compact two-sided hypersurface such that $\Theta$ does not change sign. We then have*

$$\min_{\Sigma} H \leqslant \mathcal{H}\Big(\min_{\Sigma} h\Big) \quad \textit{and} \quad \max_{\Sigma} H \geqslant \mathcal{H}\Big(\max_{\Sigma} h\Big).$$

**Proof.** We find that $\Theta(p_{\min}) = \Theta(p_{\max}) = -1$, since we have agreed always to choose $\Theta \leqslant 0$. It follows from (2.10) that

$$\min_{\Sigma} H \leqslant H(p_{\min}) \leqslant \mathcal{H}\Big(\min_{\Sigma} h\Big) \quad \text{and} \quad \mathcal{H}\Big(\max_{\Sigma} h\Big) \leqslant H(p_{\max}) \leqslant \max_{\Sigma} H,$$

and this concludes the proof. □

Our second result is for minimal immersions.

**Proposition 2.6.** *Assume that $\varrho''(t) \geqslant 0$. A compact minimal hypersurface in $\mathbb{R} \times_\varrho \mathbb{P}^n$ occurs only if $\mathbb{P}^n$ is compact and then it is any slice $\mathbb{P}_{t_0}$ where $\mathcal{H}(t_0) = 0$.*

**Proof.** From (2.10) we have $\varrho'(\bar{h}) \leqslant 0 \leqslant \varrho'(\underline{h})$. Then $\varrho''(t) \geqslant 0$ yields $\varrho' \circ h = 0$, and hence $\mathcal{H} \circ h = 0$. The proof follows from Proposition 2.2. □

To extend the preceding results from compact to complete submanifolds we use the following well-known Omori–Yau maximum principle [**19**].

**Lemma 2.7.** *Let $M$ be a complete Riemannian manifold with Ricci curvature bounded from below. If $u \in \mathcal{C}^\infty(M)$ is bounded from below, then there exists a sequence of points $\{p_j\} \in M$ such that*

$$\lim_{j\to\infty} u(p_j) = \inf_M u, \quad \|\nabla u(p_j)\| < \frac{1}{j} \quad \textit{and} \quad \Delta u(p_j) > -\frac{1}{j}.$$

**Remark 2.8.** The Omori–Yau maximum principle (and thus our next result) holds under the weaker assumption [**3**]

$$\mathrm{Ric}_M \geqslant -C(1 + r^2 \log^2(r+2)),$$

where $r$ is the distance function in $M$ to a fixed point and $C$ is a positive constant.

The next theorem is analogous to Theorem 2.4 for complete hypersurfaces contained in a slab.

**Theorem 2.9.** *Let $f : \Sigma^n \to \mathbb{R} \times_\varrho \mathbb{P}^n$ be a two-sided complete hypersurface of constant mean curvature $H$, with Ricci curvature bounded from below and*

$$f(\Sigma^n) \subset [t_1, t_2] \times \mathbb{P}^n,$$

*where $t_1, t_2 \in \mathbb{R}$ are finite. Assume that $\mathcal{H}'(t) > 0$ almost everywhere and that the angle function $\Theta$ does not change sign. Then $f(\Sigma^n)$ is a slice.*

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**Proof.** By Lemma 2.7 using (2.6) and (2.8), there exists a sequence $\{p_j\} \in \varSigma^n$ such that

$$\lim_{j\to\infty} h(p_j) = \underline{h} := \inf h > -\infty, \tag{2.11}$$

$$\|\nabla h(p_j)\|^2 = 1 - \varTheta^2(p_j) < \left(\frac{1}{j}\right)^2 \tag{2.12}$$

and

$$\Delta h(p_j) = \mathcal{H}(h(p_j))(n - \|\nabla h(p_j)\|^2) + nH(p_j)\varTheta(p_j) > -\frac{1}{j}. \tag{2.13}$$

Equation (2.13) gives

$$-nH(p_j)\varTheta(p_j) < \frac{1}{j} + \mathcal{H}(h(p_j))(n - \|\nabla h(p_j)\|^2).$$

Since $\lim_{j\to\infty} \varTheta(p_j) = \operatorname{sgn}\varTheta$ by (2.12), it follows that

$$-\operatorname{sgn}\varTheta \lim_{j\to+\infty} H(p_j) \leqslant \mathcal{H}(\underline{h}). \tag{2.14}$$

Similarly, applying Lemma 2.7 to $-h$ yields a sequence $\{q_j\} \in \varSigma^n$ such that

$$\mathcal{H}(\bar{h}) \leqslant -\operatorname{sgn}\varTheta \lim_{j\to+\infty} H(q_j), \tag{2.15}$$

where $\bar{h} := \sup h < \infty$. We obtain from (2.14), (2.15) and our assumptions that

$$-\operatorname{sgn}\varTheta H \leqslant \mathcal{H}(\underline{h}) \leqslant \mathcal{H}(\bar{h}) \leqslant -\operatorname{sgn}\varTheta H,$$

and, since $\mathcal{H}'(t) > 0$ almost everywhere, we conclude that $\underline{h} = \bar{h}$. □

For complete hypersurfaces we have the following version of Proposition 2.5.

**Proposition 2.10.** *Let $f : \varSigma^n \to \mathbb{R} \times_\varrho \mathbb{P}^n$ be a two-sided complete hypersurface with Ricci curvature bounded from below and contained in a slab. Assume that the angle function $\varTheta$ does not change sign. Then*

$$\inf_{\varSigma} H \leqslant \mathcal{H}\Big(\inf_{\varSigma} h\Big) \quad \textit{and} \quad \sup_{\varSigma} H \geqslant \mathcal{H}\Big(\sup_{\varSigma} h\Big).$$

**Proof.** Since $\operatorname{sgn}\varTheta = -1$, from (2.14) and (2.15) we obtain

$$\inf_{\varSigma} H \leqslant \lim_{j\to+\infty} H(p_j) \leqslant \mathcal{H}\Big(\inf_{\varSigma} h\Big) \quad \text{and} \quad \mathcal{H}\Big(\sup_{\varSigma} h\Big) \leqslant \lim_{j\to+\infty} H(q_j) \leqslant \sup_{\varSigma} H,$$

and this concludes the proof. □

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Any function $u \in \mathcal{C}^\infty(\mathbb{P})$ determines an entire graph $\Gamma(u)$ over $\mathbb{P}^n$ by the map $f_u : \mathbb{P}^n \to \mathbb{R} \times \mathbb{P}^n$ defined as $f_u(q) = (u(q), q)$. A straightforward computation shows that the equation for the mean curvature function $H$ of $\Gamma(u)$ is

$$\operatorname{div} \frac{\mathrm{D}u}{\sqrt{1 + \|\mathrm{D}u\|^2}} = -nH, \tag{2.16}$$

where $\mathrm{D}u$ denotes the gradient of $u \in \mathcal{C}^\infty(\mathbb{P})$ and 'div' the divergence on $\mathbb{P}^n$. If $\mathbb{P}^n$ is compact, it follows easily from (2.16) that any entire graph in $\mathbb{R} \times \mathbb{P}^n$ whose mean curvature $H$ does not change sign is necessarily minimal. As $\mathcal{H} = 0$, from (2.6) it follows that the height function $u$ is harmonic on the compact $\Gamma(u)$, and thus the graph must be a slice.

Extending a result due to Heinz [**9**] ($n = 2$), it was proved independently by Chern [**4**] and Flanders [**5**] that any entire graph in Euclidean space $\mathbb{R}^{n+1}$ with constant mean curvature must be minimal. A beautiful argument by Salavessa [**16**] shows that, for a complete non-compact $\mathbb{P}^n$, an entire graph in $\mathbb{R} \times \mathbb{P}^n$ with constant mean curvature $H$ is minimal, provided that the Cheeger constant $\mathfrak{h}(\mathbb{P})$ of $\mathbb{P}^n$ vanishes. To see this, recall that

$$\mathfrak{h}(\mathbb{P}) = \inf_D \frac{\operatorname{area}(\partial D)}{\operatorname{area}(D)},$$

where $D \subset \mathbb{P}^n$ is any compact domain with smooth boundary. Integrating (2.16) over $D$ and using the divergence theorem, we obtain

$$n \operatorname{area}(D) \min_D H \leqslant n \int_D H \,\mathrm{d}A_{\mathbb{P}} = \oint_{\partial D} \frac{\langle \mathrm{D}u, \nu \rangle}{\sqrt{1 + \|\mathrm{D}u\|^2}} \,\mathrm{d}s \leqslant \operatorname{area}(\partial D)$$

and, similarly, $n \operatorname{area}(D) \max_D \geqslant -\operatorname{area}(\partial D)$. We thus have

$$\inf_{\mathbb{P}} H \leqslant \frac{1}{n} \frac{\operatorname{area}(\partial D)}{\operatorname{area}(D)} \quad \text{and} \quad \sup_{\mathbb{P}} H \geqslant -\frac{1}{n} \frac{\operatorname{area}(\partial D)}{\operatorname{area}(D)},$$

and hence

$$\inf_{\mathbb{P}} H \leqslant \frac{1}{n} \mathfrak{h}(\mathbb{P}) \quad \text{and} \quad \sup_{\mathbb{P}} H \geqslant -\frac{1}{n} \mathfrak{h}(\mathbb{P}).$$

In particular, when $\mathfrak{h}(\mathbb{P}) = 0$ we obtain $\inf_{\mathbb{P}} H \leqslant 0 \leqslant \sup_{\mathbb{P}} H$. Then, if $H$ is constant, it must vanish.

As a consequence of Proposition 2.10 we have the following result for graphs in Riemannian products.

**Corollary 2.11.** *Let $\Gamma(u)$ be an entire graph over $\mathbb{P}^n$ determined by $u \in \mathcal{C}^\infty(\mathbb{P})$. If $u$ is bounded and if the Ricci curvature of $\Gamma(u)$ is bounded from below, then the mean curvature function of the graph satisfies*

$$\inf_{\Gamma} H \leqslant 0 \leqslant \sup_{\Gamma} H.$$

*In particular, if $H$ is constant then the graph must be minimal.*

For other results of this type see the corollary in [**15**, p. 445] and [**8**, Theorem 2]. Examples of entire graphs in the product space $\mathbb{R}\times\mathbb{H}^2$ of constant mean curvature $H\in(0,\frac{1}{2}]$ with $u$ bounded only on one side where given in [**13**].

To conclude this section, we consider the case of parabolic submanifold, where by *parabolic* we mean that any subharmonic function on the submanifold, bounded from above, must be constant.

**Proposition 2.12.** *Let $f:\Sigma^n\to\mathbb{R}\times_\varrho\mathbb{P}^n$ be an isometric immersion. Assume that $\Sigma^n$ is parabolic and that either*

(i) *$h\leqslant\bar{h}<+\infty$ and $\|\boldsymbol{H}\|\leqslant\mathcal{H}\circ h$, or*

(ii) *$h\geqslant\underline{h}>-\infty$ and $\|\boldsymbol{H}\|\leqslant-\mathcal{H}\circ h$.*

*Then $f(\Sigma^n)$ is a slice.*

**Proof.** For case (i), using (2.1), we have $\Delta\sigma(h)\geqslant 0$, and the proof follows since $\Sigma^n$ is parabolic and $\sigma(h)\leqslant\sigma(\bar{h})$. Case (ii) is analogous using $-\sigma$. □

## 3. The second equation

We have already reached several conclusions for hypersurfaces whose mean curvature is smaller than that of slices. To remove this restriction, we assume a bound on the normalized Ricci curvature of $\mathbb{P}^n$ and introduce a partial differential equation coming from the Codazzi equation.

The following result extends Theorems 1.2 and 2.2 in [**7**], which were proved for hypersurfaces in hyperbolic space (we will explain in which sense in Remark 3.3).

Given a two-sided hypersurface $f:\Sigma^n\to\mathbb{R}\times_\varrho\mathbb{P}^n$, we fix an orientation $N$ and define $\phi\in\mathcal{C}^\infty(\Sigma^n)$ by

$$\phi=\sigma(h)H+\varrho(h)\Theta. \tag{3.1}$$

**Theorem 3.1 (cited as Theorem 13 in [2]).** *Let $f:\Sigma^n\to\mathbb{R}\times_\varrho\mathbb{P}^n$ be a two-sided hypersurface of constant mean curvature. If the angle function $\Theta$ does not change sign and the Ricci curvature of $\mathbb{P}^n$ satisfies*

$$\mathrm{Ric}_{\mathbb{P}}\geqslant\sup_{\mathbb{R}}\{-\varrho^2\mathcal{H}'(t)\}, \tag{3.2}$$

*then $\phi$ is subharmonic.*

**Proof.** The Codazzi equation of $f:\Sigma^n\to\mathbb{R}\times_\varrho\mathbb{P}^n$ is

$$(\bar{R}(X,Y)N)^\top=(\nabla_Y A)X-(\nabla_X A)Y, \tag{3.3}$$

where $\bar{R}$ denotes the curvature tensor of $\mathbb{R}\times_\varrho\mathbb{P}^n$. It follows from (1.1) that

$$\nabla\langle N,\mathcal{T}\rangle=-A(\mathcal{T}^\top)=-\varrho(h)A(\nabla h), \tag{3.4}$$

where $\mathcal{T} = \varrho T$. Therefore, using (2.5) and (3.3), we conclude from (3.4) that

$$\begin{aligned}\nabla_X \nabla \langle N, \mathcal{T} \rangle &= -\varrho'(h)\langle X, \nabla h\rangle A\nabla h - \varrho(h)(\nabla_X A)(\nabla h) - \varrho(h)A(\nabla_X \nabla h) \\ &= -\varrho(h)(\nabla_{\nabla h} A)X - \varrho(h)(\bar{R}(\nabla h, X)N)^{\top} - \varrho'(h)AX - \langle N, \mathcal{T}\rangle A^2 X.\end{aligned}$$

Let $\overline{\mathrm{Ric}}$ be the Ricci tensor of $M^{n+1} = \mathbb{R} \times_\varrho \mathbb{P}^n$. Using $\operatorname{tr}(\nabla_Z A) = \langle \nabla \operatorname{tr} A, Z\rangle$, we obtain

$$\Delta \langle N, \mathcal{T}\rangle = -n\varrho(h)\langle \nabla H, \nabla h\rangle + \varrho(h)\overline{\mathrm{Ric}}(N, \nabla h) - n\varrho'(h)H - \langle N, \mathcal{T}\rangle \|A\|^2. \tag{3.5}$$

The curvature tensor of $M^{n+1}$ expressed in terms of the curvature tensor of $\mathbb{P}^n$ is

$$\begin{aligned}\bar{R}(U, V)W &= R_{\mathbb{P}}(\hat{U}, \hat{V})\hat{W} - \mathcal{H}^2(\langle V, W\rangle U - \langle U, W\rangle V) \\ &\quad + \mathcal{H}'\langle W, T\rangle(\langle U, T\rangle V - \langle V, T\rangle U) - \mathcal{H}'(\langle V, W\rangle\langle U, T\rangle - \langle U, W\rangle\langle V, T\rangle)T,\end{aligned}$$

where $\hat{U} = \pi_{\mathbb{P}*}U$. Then, the Ricci tensor of $M^{n+1}$ can be given in terms of the Ricci tensor of $\mathbb{P}^n$, namely,

$$\overline{\mathrm{Ric}}(V, W) = \mathrm{Ric}_{\mathbb{P}}(\hat{V}, \hat{W}) - (n\mathcal{H}^2 + \mathcal{H}')\langle V, W\rangle - (n-1)\mathcal{H}'\langle V, T\rangle\langle W, T\rangle. \tag{3.6}$$

Thus,

$$\overline{\mathrm{Ric}}(N, X) = \mathrm{Ric}_{\mathbb{P}}(\hat{N}, \hat{X}) - (n-1)\mathcal{H}'(h)\Theta\langle X, \nabla h\rangle$$

for any $X \in T\Sigma$. Since $T = \nabla h + \Theta N$, we get $(\nabla h)^* = -\Theta N^*$, where $(\cdot)^*$ means taking the $\mathbb{P}^n$-component of a vector field in $TM$. Thus,

$$\overline{\mathrm{Ric}}(N, \nabla h) = -(n-1)\Theta(\mathrm{Ric}_{\mathbb{P}}(\hat{N}) + \mathcal{H}'(h)\|\nabla h\|^2), \tag{3.7}$$

where, as usual, $(n-1)\,\mathrm{Ric}_{\mathbb{P}}(\cdot) = \mathrm{Ric}_{\mathbb{P}}(\cdot\,, \cdot)$. Since the mean curvature $H$ is constant, we conclude from (2.1), (3.5), (3.7) and $\varrho(h)\Theta = \langle N, \mathcal{T}\rangle$ that

$$\Delta\phi = -\varrho(h)\Theta\{\|A\|^2 - nH^2 + (n-1)(\mathrm{Ric}_{\mathbb{P}}(\hat{N}) + \mathcal{H}'(h)\|\nabla h\|^2)\}. \tag{3.8}$$

From (3.2) and $\|\hat{N}\|^2_{\mathbb{P}} = \varrho^{-2}(h)\|\nabla h\|^2$ we obtain

$$\mathrm{Ric}_{\mathbb{P}}(\hat{N}) + \mathcal{H}'(h)\|\nabla h\|^2 \geqslant 0, \tag{3.9}$$

and the proof follows by using $\|A\|^2 \geqslant nH^2$. □

**Remark 3.2.** It follows easily from (3.6) that (3.2) is equivalent to

$$\overline{\mathrm{Ric}}(X) \geqslant \overline{\mathrm{Ric}}(T) \quad \text{for all } X \in TM.$$

In other words, the direction $T$ must be of least Ricci curvature.

**Remark 3.3.** Given a vertical graph over $\mathbb{R}^n$ in $\mathbb{H}^{n+1}$ with constant mean curvature, the result in [**7**, § 2] asserts that the mean curvature function computed with respect to the underlying Euclidean metric is subharmonic. To see that the preceding result extends the one in [**7**], we consider the case of pseudo-hyperbolic ambient spaces $\mathbb{R} \times_{\mathrm{e}^t} \mathbb{P}^n$. Then

(3.2) reduces to $\mathrm{Ric}_{\mathbb{P}} \geqslant 0$ (this holds for $\mathbb{H}^{n+1}$ since $\mathbb{P}^n = \mathbb{R}^n$) and the subharmonic function $\phi$ takes the simple form

$$\phi = \mathrm{e}^h (H + \Theta). \tag{3.10}$$

It turns out that $\phi$ is also the mean curvature of the hypersurface when computed in the product metric of $\mathbb{R}_+ \times \mathbb{P}^n$. In fact, a straightforward computation yields that the mean curvature function $\hat{H}$ of $\hat{f} = \tau \circ f : \Sigma^n \to \mathbb{J} \times \mathbb{P}^n$ is $\hat{H} = \varrho(h)H + \varrho'(h)\Theta$. Then observe that $\hat{H} = \phi$ if and only if $\varrho(t) = \mathrm{e}^t$.

Montiel's first result [**12**, Corollary 7] in our case is an easy consequence of Theorem 3.1.

**Theorem 3.4.** *Let $f : \Sigma^n \to M^{n+1} = \mathbb{R} \times_\varrho \mathbb{P}^n$ be a compact two-sided hypersurface of constant mean curvature. Assume that (3.2) holds and that the angle function $\Theta$ does not change sign. Then either $f(\Sigma^n)$ is a slice over a compact $\mathbb{P}^n$ or $M^{n+1}$ has constant sectional curvature and $\Sigma^n$ is a geodesic hypersphere. The latter case cannot occur if we assume that the inequality in (3.2) is strict.*

**Proof.** We know by Theorem 3.1 that $\phi$ is a subharmonic function on $\Sigma^n$, but being $\Sigma^n$ compact implies that $\phi$ is constant. Then (3.8) gives

$$\Theta(\|A\|^2 - nH^2 + (n-1)(\mathrm{Ric}_{\mathbb{P}}(\hat{N}) + \mathcal{H}'(h)\|\nabla h\|^2)) = 0. \tag{3.11}$$

We claim that $\mathcal{U} = \{p \in \Sigma^n : \Theta(p) = 0\}$ has an empty interior. To see this, assume on the contrary that $\mathcal{U}$ contains a non-empty open subset $\mathcal{V}$ of $\Sigma^n$. On $\mathcal{V}$ the function $\sigma(h)H = \phi$ is constant and, if $H \neq 0$, then $\sigma(h)$ and, equivalently, $h$ is constant. But this is not possible, since $\|\nabla h\|^2 = 1 - \Theta^2 = 1$ on $\mathcal{V}$. Therefore, we must have $H = 0$, and then $\varrho(h)\Theta = \phi$ is constant on $\Sigma^n$. Since $\phi = \varrho(h)\Theta$ vanishes on $\mathcal{V}$, it must vanish on all of $\Sigma^n$. Hence, $\mathcal{U} = \Sigma^n$, but this is not possible because $\Theta^2 = 1$ at least where $h$ attains its extrema. Summing up, $\mathcal{U}$ has an empty interior. Then (3.11) implies that

$$\|A\|^2 - nH^2 + (n-1)(\mathrm{Ric}_{\mathbb{P}}(\hat{N}) + \mathcal{H}'(h)\|\nabla h\|^2) = 0,$$

that is,

$$\|A\|^2 - nH^2 = 0 \tag{3.12}$$

and

$$\mathrm{Ric}_{\mathbb{P}}(\hat{N}) + \mathcal{H}'(h)\|\nabla h\|^2 = 0. \tag{3.13}$$

Equality (3.12) means that $f$ is totally umbilical. Moreover, we observe that Montiel's reasoning in his proof of [**12**, Corollary 7] also applies here, and allows us to conclude that the case in which $f$ is totally umbilical (but not a slice) can only occur if $M^{n+1}$ has constant sectional curvature and $\Sigma^n$ is a geodesic hypersphere.

Finally, when inequality in (3.2) is strict, (3.13) is equivalent to $\hat{N}(p) = 0$ at any $p \in \Sigma^n$, that is, $\nabla h = 0$ on $\Sigma^n$, and hence $f(\Sigma^n)$ is a slice over a compact $\mathbb{P}^n$. $\square$

One can use Theorem 3.1 to obtain height estimates for constant mean curvature hypersurfaces with (non-empty) boundary contained in a slice. The next result is an extension of the height estimates for vertical graphs in $\mathbb{R} \times \mathbb{P}^2$ in [**10**].

**Theorem 3.5.** *Let $f : \Sigma^n \to \mathbb{R} \times \mathbb{P}^n$ be a compact hypersurface of constant mean curvature $H > 0$ and non-empty boundary $\partial\Sigma^n \subset \mathbb{P}_0$. Assume that*

$$\mathrm{Ric}_{\mathbb{P}} \geqslant \frac{n}{n-1}\alpha \quad \textit{for some } \alpha \leqslant 0,$$

*$\Theta \leqslant 0$ and $H^2 \geqslant |\alpha|$. Then, we have*

$$f(\Sigma^n) \subset \left[0, \frac{H}{H^2 - |\alpha|}\right] \times \mathbb{P}^n.$$

**Proof.** By assumption, $(n-1)\,\mathrm{Ric}_{\mathbb{P}}(\hat{N}) \geqslant n\alpha\|\hat{N}\|^2_{\mathbb{P}} \geqslant n\alpha$ because of $\alpha \leqslant 0$ and $\|\hat{N}\|^2_{\mathbb{P}} = \|\nabla h\|^2 \leqslant 1$. Consider $\psi \in \mathcal{C}^\infty(\Sigma^n)$ defined as

$$\psi = \phi + \frac{\alpha}{H}h = \frac{H^2 - |\alpha|}{H}h + \Theta.$$

Using (2.6) and (3.8), we have

$$\begin{aligned}\Delta\psi &= -\Theta(\|A\|^2 - nH^2 + (n-1)\,\mathrm{Ric}_{\mathbb{P}}(\hat{N}) - n\alpha)\\ &\geqslant -\Theta((n-1)\,\mathrm{Ric}_{\mathbb{P}}(\hat{N}) - n\alpha),\end{aligned}$$

and thus $\psi$ is subharmonic on $\Sigma^n$. The maximum principle yields

$$\frac{H^2 - |\alpha|}{H}h - 1 \leqslant \frac{H^2 - |\alpha|}{H}h + \Theta = \psi \leqslant \max_{\partial\Sigma}\psi = \max_{\partial\Sigma}\Theta \leqslant 0,$$

and hence $0 \leqslant h \leqslant H/(H^2 - |\alpha|)$. □

For our next result we first recall a well-known tangency principle. Let $\Sigma^n_1$ and $\Sigma^n_2$ be two hypersurfaces in an arbitrary Riemannian manifold $N^{n+1}$ that are tangent at a common point $p_0$. Fix a normal vector $\eta_0$ at $p_0$ and locally parametrize both hypersurfaces in a neighbourhood $U$ of zero in $T_{p_0}\Sigma_1 = T_{p_0}\Sigma_2$ by means of the exponential map of $N^{n+1}$ as follows:

$$\varphi_j(x) = \exp_{p_0}(x + \mu_j(x)\eta_0), \quad j = 1, 2,$$

where $\mu_j \in \mathcal{C}^\infty(U)$ are well-determined functions satisfying $\mu_j(0) = 0$. One says that $\Sigma^n_1$ lies above $\Sigma^n_2$ in a neighbourhood of $p_0$ if $\mu_1(x) \geqslant \mu_2(x)$ in a neighbourhood of zero. This is equivalent to requiring that the geodesics of $N^{n+1}$ normal to the hypersurface $\exp_{p_0}(U)$ in a neighbourhood of $p_0$ in the orientation determined by $\eta_0$ intercept $\Sigma^n_2$ before $\Sigma^n_1$. The following fact is well known [**6**].

**Theorem 3.6 (Fontenele and Silva [6]).** *Let $\Sigma^n_1$ and $\Sigma^n_2$ be hypersurfaces as above with constant mean curvature satisfying $H_{\Sigma_1} \leqslant H_{\Sigma_2}$ with respect to $\eta_0$. Then $\Sigma^n_1$ and $\Sigma^n_2$ coincide in a neighbourhood of $p_0$.*

The following general result involves no assumption on the curvature of $\mathbb{P}^n$ and is of independent interest.

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**Proposition 3.7 (cited as Proposition 18 in [2]).** *Let $f : \Sigma^n \to \mathbb{R} \times_\varrho \mathbb{P}^n$ be a compact two-sided constant mean curvature hypersurface with non-empty boundary $f(\partial\Sigma) \subset \mathbb{P}_\tau$, and whose angle function $\Theta$ does not change sign. We then have the following conditions.*

(i) *If $H \leqslant \inf_{[\tau,+\infty)} \mathcal{H}$, then $f(\Sigma^n) \subset (-\infty, \tau] \times \mathbb{P}^n$.*

(ii) *If $H \geqslant \sup_{(-\infty,\tau]} \mathcal{H}$, then $f(\Sigma^n) \subset [\tau, +\infty) \times \mathbb{P}^n$.*

*In particular, if $\mathcal{H}'(t) \geqslant 0$ and $H = \mathcal{H}(\tau)$ then $f(\Sigma^n) \subset \mathbb{P}_\tau$.*

**Proof.** Assume that $H \leqslant \inf_{[\tau,+\infty)} \mathcal{H}$ but that $h \leqslant \tau$ does not hold. Hence, we obtain

$$\max_{\Sigma} h = h(p_0) = \tau_0 > \tau$$

at some interior point $p_0$ of $\Sigma^n$. Take $\Sigma_1 = \Sigma^n$, $\Sigma_2 = \mathbb{P}_{\tau_0}$, and hence $\Sigma_1 \neq \Sigma_2$. Observe that $\Sigma_1$ and $\Sigma_2$ are tangent at the common point $p_0$, and that $\Sigma_1$ lies above $\Sigma_2$ with respect to the common normal $\eta_0 = -T$ at $p_0$. Since

$$H_{\Sigma_1} = H \leqslant \inf_{[\tau,+\infty)} \mathcal{H} \leqslant \mathcal{H}(\tau_0) = H_{\Sigma_2},$$

by the tangency principle we may find that $\Sigma_1$ and $\Sigma_2$ coincide in some open neighbourhood of $p_0$. This is in contradiction to $\Sigma_1 \neq \Sigma_2$. The proof for the case $H \geqslant \sup_{(-\infty,\tau]} \mathcal{H}$ is similar. $\square$

The following consequence of Proposition 3.7 extends [**11**, Proposition 2.3].

**Corollary 3.8.** *Let $f : \Sigma^n \to \mathbb{R} \times_{\mathrm{e}^t} \mathbb{P}^n$ be a compact two-sided hypersurface of constant mean curvature with non-empty boundary $f(\partial\Sigma) \subset \mathbb{P}_\tau$, and whose angle function $\Theta$ does not change sign. Then, we have*

(1) *$H \leqslant 1$ if and only if $h \leqslant \tau$,*

(2) *$H \geqslant 1$ if and only if $h \geqslant \tau$ on $\Sigma^n$.*

*In particular, $H = 1$ if and only if $f(\Sigma^n) \subset \mathbb{P}_\tau$.*

To conclude we extend [**11**, Theorem 3.3], which holds for graphs in hyperbolic space $\mathbb{H}^{n+1}$, to the following results, which hold for graphs in pseudo-hyperbolic manifolds. For the case of pseudo-hyperbolic space with $\varrho(t) = \mathrm{e}^t$ we have the following.

**Theorem 3.9.** *Let $f : \Sigma^n \to \mathbb{R} \times_{\mathrm{e}^t} \mathbb{P}^n$ be a compact hypersurface of constant mean curvature $H \notin [0, 1)$ and non-empty boundary $f(\partial\Sigma^n) \subset \mathbb{P}_\tau$. Assume that $\mathrm{Ric}_{\mathbb{P}} \geqslant 0$ and that the angle function $\Theta$ does not change sign. Set $C = \log(H/H - 1)$. Then*

(i) *if $H < 0$, then $f(\Sigma^n) \subset [\tau + C, \tau] \times \mathbb{P}^n$,*

(ii) *if $H > 1$, then $f(\Sigma^n) \subset [\tau, \tau + C] \times \mathbb{P}^n$,*

(iii) *if $H = 1$ then $f(\Sigma^n) \subset \mathbb{P}_\tau$.*

**Proof.** By the preceding result, we observe that $f(\Sigma^n) \subset (-\infty, \tau] \times \mathbb{P}^n$ if $H < 0$, that $f(\Sigma^n) \subset [\tau, +\infty) \times \mathbb{P}^n$ if $H > 1$, and that $f(\Sigma^n) \subset \mathbb{P}_\tau$ if $H = 1$. From the maximum principle applied to the subharmonic function $\phi$ given by (3.10), we obtain

$$\mathrm{e}^h(H-1) \leqslant \mathrm{e}^h(H+\Theta) \leqslant \max_{\partial\Sigma} \mathrm{e}^h(H+\Theta) = \mathrm{e}^\tau\Big(H + \max_{\partial\Sigma}\Theta\Big) \leqslant \mathrm{e}^\tau H,$$

and the proof follows easily. □

Finally, for the case of pseudo-hyperbolic space with $\varrho(t) = \cosh t$, we obtain the following.

**Theorem 3.10.** *Let $f : \Sigma^n \to \mathbb{R} \times_{\cosh t} \mathbb{P}^n$ be a compact hypersurface of constant mean curvature $H$ and non-empty boundary $f(\partial\Sigma^n) \subset \mathbb{P}_0$. Assume that $\mathrm{Ric}_{\mathbb{P}} \geqslant -1$ and that the angle function $\Theta$ does not change sign. Set $\tanh C = 1/H$. Then*

(i) *if $H < -1$, then $f(\Sigma^n) \subset [C, 0] \times \mathbb{P}^n$,*

(ii) *if $H > 1$, then $f(\Sigma^n) \subset [0, C] \times \mathbb{P}^n$,*

(iii) *if $H = 0$, then $f(\Sigma^n) \subset \mathbb{P}_0$.*

**Proof.** By Proposition 3.7, we observe that $f(\Sigma^n) \subset (-\infty, 0] \times \mathbb{P}^n$ if $H < 0$, that $f(\Sigma^n) \subset [0, +\infty) \times \mathbb{P}^n$ if $H > 0$ and that $f(\Sigma^n) \subset \mathbb{P}_0$ if $H = 0$. Now $\sigma(t) = \sinh t$, and from the maximum principle applied to the subharmonic function $\phi$ given by (3.1), we obtain

$$H \sinh h - \cosh h \leqslant \phi \leqslant \max_{\partial\Sigma} \phi = \max_{\partial\Sigma} \Theta \leqslant 0,$$

that is, $H \tanh h \leqslant 1$. Then, when $H < -1$, this gives $\tanh h \geqslant 1/H$, and when $H > 1$ this yields $\tanh h \leqslant 1/H$. □

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