Constant mean curvature spheres in Riemannian manifolds

F. Pacard *and X. Xu †

Abstract

We prove the existence of embedded spheres with large constant mean curvature in any compact Riemannian manifold (M, g). This result partially generalizes a result of R. Ye which handles the case where the scalar curvature function of the ambient manifold (M, g) has non-degenerate critical points.

1 Introduction

Assume that S is an oriented embedded (or possibly immersed) hyper-surface in a compact Riemannian manifold (M, g) of dimension $m + 1$. The mean curvature of S is defined to be the sum of the principal curvatures κ_j, i.e.

$$H(S) := \sum_j \kappa_j.$$

We are interested in the existence of compact embedded spheres $S^m \hookrightarrow M$ that have constant mean curvature. In this direction, a first general result was obtained by Ye in [13] in the case where $\mathbf{s}$, the scalar curvature of (M, g), has non-degenerate critical points.

Given $p \in M$ and $\rho > 0$ small enough, we denote by S^m the unit sphere in T_pM. Let $S_{p,\rho}$ be the geodesic sphere of radius ρ, centered at the point p. This hyper-surface can be parameterized by

$$\Theta \in S^m \subset T_pM \longrightarrow \mathrm{Exp}_p(\rho\,\Theta) \in S_{p,\rho} \subset M$$

*Laboratoire d'Analyse et de Mathématiques Appliquées, Université Paris 12 and Institut Universitaire de France. email : pacard@univ-paris12.fr

†Department of Mathematics, National University of Singapore, 2 Science drive 2, S117543. email: matxuxw@nus.edu.sg

and the mean curvature of $S_{p,\rho}$ can be expanded in powers of ρ according to

$$H(S_{p,\rho}) = \frac{m}{\rho} - \frac{1}{3}\,\mathrm{Ric}_p(\Theta,\Theta)\,\rho + \mathcal{O}(\rho^2),$$

as ρ tends to 0. Here Ric_p is the Ricci tensor computed at p. When p is a non-degenerate critical point of the scalar curvature function $\mathbf{s}$ of (M,g), Ye proves that it is possible to perturb the geodesic sphere $S_{p,\rho}$ into a hyper-surface $S^\natural_{p,\rho}$ whose mean curvature is constant equal to $\frac{m}{\rho}$, provided $\rho > 0$ is chosen small enough. In addition, the construction shows that $S^\natural_{p,\rho}$ is a normal geodesic graph over $S_{p,\rho}$ for some (smooth) function bounded by a constant times ρ^2 in $\mathcal{C}^{2,\alpha}(S^m)$ topology and, as ρ varies, these hyper-surfaces constitute a local foliation of a neighborhood of p.

Ye's construction yields the existence of branches of constant mean curvature hyper-surfaces each of which is associated to non-degenerate critical points of the scalar curvature. Furthermore, Ye shows that the elements of these branches form a local foliation of a neighborhood of p by constant mean curvature hyper-surfaces.

In this short note, we are interested in the cases that are not covered by the result of Ye, namely the cases where the scalar curvature $\mathbf{s}$ has degenerate critical points. This is for example the case when (M,g) is an Einstein manifold or more generally when the scalar curvature of g is constant ! The later manifolds are known to be abundant thanks to the solution of the Yamabe problem given by Trudinger, Aubin and Schoen (see [1] and [11] for references) and extending Ye's result in this setting is a natural problem.

Recall that constant mean curvature hyper-surfaces can be obtained by solving the isoperimetric problem : Minimize the m-dimensional volume of a hyper-surface S among all hyper-surfaces that enclose a domain whose $(m+1)$-dimensional volume is fixed. Solutions of the isoperimetric problem are constant mean curvature hyper-surfaces (when they are smooth) and hence, varying the volume constraint one obtains many interesting constant mean curvature hyper-surfaces. We refer to [10] for further references and recent advances on this problem. Beside the fact that, in dimension $m+1 \geq 7$, the solutions of the isoperimetric problem might not be smooth hyper-surfaces, the main drawback of this approach is that very little information is available on the solution itself and the control of the mean curvature in terms of the volume constraint seems a difficult task. On the other hand, more information are available for solutions of the isoperimetric problem with small volume constraint. In this case it is known that the solutions of the isoperimetric problem are close to geodesic spheres with small radius [10], [2] and Druet [3] has shown that in fact the solutions of the isoperimetric problem concentrate at critical points of the scalar curvature as the volume constraint tends to 0, unearthing once more the crucial role of critical points of the scalar curvature function in this context.

We have already mentioned that $\mathbf{s}$ denotes the scalar curvature on (M, g). We define the function

$$\begin{array}{rcl} \mathbf{r}(p) & = & \frac{1}{36(m+5)} \left(5\, \mathbf{s}^2(p) + 8\, ||\mathrm{Ric}_p||^2 - 3\, ||R_p||^2 - 18\, \Delta_g \mathbf{s}(p)\right) \\ & + & \frac{1}{9(m+1)(m+2)} \left(\frac{m+6}{m}\, \mathbf{s}^2(p) - 2||\mathrm{Ric}_p||^2\right), \end{array}$$

where Ric_p denotes the Ricci tensor and R_p is the Riemannian tensor at the point p. Our main result reads :

Theorem 1.1. *There exists* $\rho_0 > 0$ *and a smooth function*

$$\phi : M \times (0, \rho_0) \longrightarrow \mathbb{R},$$

such that :

(i) For all $\rho \in (0, \rho_0)$*, if* p *is a critical point of the function* $\phi(\cdot, \rho)$ *then, there exists an embedded hyper-surface* $S^\flat_{p,\rho}$ *whose mean curvature is constant equal to* $\frac{m}{\rho}$ *and that is a normal graph over* $S_{p,\rho}$ *for some function which is bounded by a constant times* ρ^2 *in* $\mathcal{C}^{2,\alpha}$ *topology.*

(ii) For all $k \geq 0$*, there exists* $c_k > 0$ *which does not depend on* $\rho \in (0, \rho_0)$ *such that*

$$\|\phi(\cdot, \rho) - \mathbf{s} + \rho^2\, \mathbf{r}\|_{\mathcal{C}^k(M)} \leq c_k\, \rho^3.$$

Some remarks are due. If the scalar curvature function $\mathbf{s}$ on M has a non-degenerate critical point p_0, then it follows from (ii) that, for all ρ small enough there exists $p = p(\rho)$, a critical point of $\phi(\cdot, \rho)$ close to p_0 in the sense that

$$\mathrm{dist}(p, p_0) \leq c\, \rho^2.$$

Then (i) shows that the geodesic sphere $S_{p_0,\rho}$ can be perturbed into a constant mean curvature hyper-surface (provided ρ is chosen small enough). In fact, we only recover Ye's existence result (loosing the information that the hyper-surfaces constitute a local foliation).

In the case where $\mathbf{s}$ is a constant function, one gets the existence of constant mean curvature hyper-surfaces close to any non-degenerate critical point of the function $\mathbf{r}$. In the particular case where the metric g is

Einstein, we obtain constant mean curvature hyper-surfaces close to any non-degenerate critical point of the function

$$p \longmapsto \|R_p\|^2.$$

Observe that Theorem 1.1 allows one to weaken the non-degeneracy condition imposed in [13]. For example, using (ii) and arguments developed in [7] we see that, provided ρ is small enough, one can find critical points of $\phi(\cdot, \rho)$ near : (a) any local strict maximum (or minimum) of $\mathbf{s}$, (b) any critical point of $\mathbf{s}$ for which the Browder degree of $\nabla \mathbf{s}$ at p is not zero,

In the same vain, given any smooth function ψ defined on M, we denote by $\lambda_M(\psi)$ the number of critical points of ψ. Recall that Λ_M, the Ljusternik-Shnirelman category of M, is defined to be the minimal value of $\lambda_M(\psi)$ as $\psi \in \mathcal{C}^\infty(M)$ varies (for example, the Lusternik-Shnirelman category of a n-dimensional torus is equal to $n+1$ [8], the Lusternik-Shnirelman category of $\mathbb{R}P^n$ is equal to n [8], the Lusternik-Shnirelman category of a surface of genus $k \geq 2$ is equal to 3 [12]). As a simple byproduct of our analysis, we find that there exists at least Λ_M embedded hyper-surfaces in (M, g) whose mean curvature is constant equal to $\frac{m}{\rho}$, provided ρ is chosen small enough.

Together with the existence of constant mean curvature spheres, we also get precise expansions of their volume as well as the volume of the domain they enclose.

Corollary 1.1. *Assume that* $\rho \in (0, \rho_0)$ *and that* p *is a critical point of* $\phi(\cdot, \rho)$*, then the* m*-dimensional volume of* $S^\flat_{p,\rho}$ *can be expanded as*

$$Vol(S^\flat_{p,\rho}) = \rho^m \; Vol(S^m) \, \left(1 - \tfrac{1}{2(m+1)} \, \rho^2 \, \mathbf{s}(p) + \mathcal{O}_p(\rho^4)\right),$$

while the $(m+1)$ *dimensional volume of the domain* $B^\flat_{p,\rho}$ *enclosed by* $S^\flat_{p,\rho}$ *and containing* p *can be expanded as*

$$Vol(B^\flat_{p,\rho}) = \rho^{m+1} \, \tfrac{1}{m+1} \; Vol(S^m) \, \left(1 - \tfrac{3(m+2)}{2m(m+3)} \, \rho^2 \, \mathbf{s}(p) + \mathcal{O}_p(\rho^4)\right).$$

These formula should be compared to the corresponding expansions for geodesic spheres [4], [14] (which are recalled in the Appendix). Expansions up to order 5 can be obtained but the formula being rather involved and not particularly interesting, we have chosen not to state them.

The result of Ye, the result of Druet [3] and the result of the present paper single out the relation between critical points of the scalar curvature function and the existence of constant mean curvature surfaces with high mean curvature. In particular, these results raise the following question :

Question : *Assume that for all $\rho > 0$ small enough there exists a constant mean curvature (not necessarily embedded) hyper-surface in $B_\rho(p)$, the geodesic ball of radius ρ centered at p. Is it true that p is a critical point of the scalar curvature function ?*

Thanks to the result of [3], the answer to this question is known to be positive under the additional assumption that the constant mean curvature hyper-surfaces are solutions of the isoperimetric problem. Let us mention that Nardulli [9] has recently shown that the solutions of the isoperimetric problem belong to the families constructed by Ye when the volume constraint tends to 0. In our construction, we also obtain the precise expansion of the volume of the constant mean curvature surfaces we construct and also the volume of the body enclosed by these hyper-surfaces (see Appendix). These expansions, together with the result of Nardulli, allows one to give a precise expansion of the isoperimetric profile as the volume constrain tends to 0.

The proof of Theorem 1.1 is based on the following ingredients : First we consider a geodesic sphere $S_{p,\rho}$ of radius ρ centered at p which we try to perturb in the normal direction to obtain a constant mean curvature hyper-surface. It turns out that this will not be always possible since the corresponding Jacobi operator about $S_{p,\rho}$ has small (eventually zero) eigenvalues. Therefore, we need to perform some Liapunov-Schmidt reduction argument and instead of trying to solve the equation $H = \frac{m}{\rho}$, we prefer to solve the equation $H = \frac{m}{\rho}$ modulo some linear combination of eigenfunctions of the Jacobi operator about $S_{p,\rho}$ associated to small eigenvalues. This idea was already used by Ye even though the analysis he uses is somehow different from ours. This yields a hyper-surface $S^\flat_{p,\rho}$ which is close to $S_{p,\rho}$. Next, we use the variational characterization of $H = \frac{m}{\rho}$ constant mean curvature hyper-surfaces as critical points of the functional

$$\mathcal{E}(S) := \mathrm{Vol}_m(S) - \frac{m}{\rho}\,\mathrm{Vol}_{m+1}(B_S),$$

where B_S is the domain enclosed by S. We prove that, provided ρ is small enough, any critical point of the function $\phi(p,\rho) := \mathcal{E}(S^\flat_{p,\rho})$ gives rise to a constant mean curvature surface. This last part of the argument borrows an idea already used by Kapouleas in [5].

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2 Perturbed geodesic spheres

2.1 Expansion of the metric in geodesic normal coordinates

First we make the following convention : the Greek index letters, such as $\mu, \nu, \iota, \cdots$, range from 1 to $m+1$ while the Latin index letters, such as $i, j, k, \cdots$, will run from 1 to m.

We introduce geodesic normal coordinates in a neighborhood of a point $p \in M$. To this aim, we choose an orthonormal basis E_μ, $\mu = 1, \ldots, m+1$, of T_pM and introduce coordinates $x := (x^1, \ldots, x^{m+1})$ in $\mathbb{R}^{m+1}$. We set

$$F(x) := \mathrm{Exp}_p(x^\mu \, E_\mu),$$

where Exp is the exponential map on M and summation over repeated indices is understood. This choice of coordinates induces coordinate vector fields $X_\mu := F_*(\partial_{x^\mu})$. As usual, geodesic normal coordinates are defined so that the metric coefficients

$$g_{\mu\nu} = g(X_\mu, X_\nu)$$

equal $\delta_{\mu\nu}$ at p. We now recall the Taylor expansion of the metric coefficients at $q := F(x)$ in terms of geometric data at $p := F(0)$ and $|x| := ((x^1)^2 + \ldots + (x^{m+1})^2)^{1/2}$.

Proposition 2.1. *At the point $q = F(x)$, the following expansion holds for all $\mu, \nu = 1, \ldots, m+1$*

$$\begin{aligned} g_{\mu\nu} &= \delta_{\mu\nu} + \tfrac{1}{3}\, g(R_p(\Xi, E_\mu)\,\Xi, E_\nu) + \tfrac{1}{6}\, g(\nabla_\Xi R_p(\Xi, E_\mu)\,\Xi, E_\nu) \\ &+ \tfrac{1}{20}\, g(\nabla_\Xi\, \nabla_\Xi R_p(\Xi, E_\mu)\,\Xi, E_\nu) \\ &+ \tfrac{2}{45}\, g(R_p(\Xi, E_\mu)\,\Xi, E_\iota)\, g(R_p(\Xi, E_\nu)\,\Xi, E_\iota) + \mathcal{O}_p(|x|^5), \end{aligned} \tag{2.1}$$

where R_p is the curvature tensor at the point p and $\Xi := \sum x^\mu \, E_\mu \in T_pM$.

The proof of this result can be found for example in [11], [6] or in [14].

Notation The symbol $\mathcal{O}_p(|x|^r)$ indicates a smooth function (depending on p) such that it and its partial derivatives of any order, with respect to the vector fields $x^\mu \, X_\mu$, are bounded by a constant times $|x|^r$ in some fixed neighborhood of 0, uniformly in p.

2.2 Geometry of perturbed geodesic spheres

We derive expansions, as ρ tends to 0, for the metric, the volume, the enclosed volume, the second fundamental form and the mean curvature of normal perturbations of geodesic spheres.

We fix $\rho > 0$, a small $C^{2,\alpha}$ function w on S^m and we use a local parametrization $z \longmapsto \Theta(z)$ of $S^m \subset T_pM$. We define the map

$$G(z) := \mathrm{Exp}_p \left(\rho \, (1 - w(z)) \, \Theta(z) \right)$$

and denote its image by $S_{p,\rho}(w)$. Observe that $S_{p,\rho}(0) = S_{p,\rho}$. Various vector fields we shall use may be regarded either as vector fields along $S_{p,\rho}(w)$ or as vectors fields along $S^m \subset T_pM$. We agree that the coordinates of Θ in $\{E_1, \ldots, E_{m+1}\}$ are given by $\Theta^1, \ldots, \Theta^{m+1}$ so that

$$\Theta = \Theta^\mu \, E_\mu .$$

We define

$$\Theta_i := \partial_{z^i} \Theta^\mu \, E_\mu$$

which are vector fields along $S^m \subset T_pM$ while

$$\Upsilon := \Theta^\mu \, X_\mu \qquad \text{and} \qquad \Upsilon_i := \partial_{z^i} \Theta^\mu \, X_\mu$$

are vector fields along $S_{p,\rho}(w)$. For brevity, we also write

$$w_j := \partial_{z^j} w, \qquad w_{ij} := \partial_{z^i} \, \partial_{z^j} w.$$

In terms of all these notation, the tangent space to $S_{p,\rho}(w)$ at any point is spanned by the vectors

$$Z_j = G_*(\partial_{z^j}) = \rho \, ((1 - w) \, \Upsilon_j - w_j \, \Upsilon), \tag{2.2}$$

for $j = 1, \ldots, m$.

The formulas for the various geometric quantities of $S_{p,\rho}(w)$ are potentially very complicated, and to keep notations short, we agree on the following :

Notation Any expression of the form $L_p(w)$ denotes a linear combination of the function w together with its derivatives with respect to the vector fields Θ_i up to order 2. The coefficients of L_p might depend on ρ and p but, for all $k \in \mathbb{N}$, there exists a constant $c > 0$ independent of $\rho \in (0, 1)$ and $p \in M$ such that

$$\| L_p \, (w) \|_{\mathcal{C}^{k,\alpha}(S^m)} \leq c \, \| w \|_{\mathcal{C}^{k+2,\alpha}(S^m)} .$$

Similarly, given $a \in \mathbb{N}$, any expression of the form $Q_p^{(a)}(w)$ denotes a nonlinear operator in the function w together with its derivatives with respect to the vector fields Θ_i up to order 2. The coefficients of the Taylor expansion of $Q_p^{(a)}(w)$ in powers of w and its partial derivatives might depend on ρ and p and, given $k \in \mathbb{N}$, there exists a constant $c > 0$ independent of $\rho \in (0,1)$ and $p \in M$ such that $Q_p^{(a)}(0) = 0$ and

$$\begin{aligned} \|Q_p^{(a)}(w_2) - Q_p^{(a)}(w_1)\|_{\mathcal{C}^{k,\alpha}(S^m)} &\leq c\,(\|w_2\|_{\mathcal{C}^{k+2,\alpha}(S^m)} + \|w_1\|_{\mathcal{C}^{k+2,\alpha}(S^m)})^{a-1} \\ &\quad \times \|w_2 - w_1\|_{\mathcal{C}^{k+2,\alpha}(S^m)}, \end{aligned}$$

provided $\|w_l\|_{\mathcal{C}^1(S^m)} \leq 1, \;\; l = 1, 2.$

We also agree that any term denoted by $\mathcal{O}_p(\rho^d)$ is a smooth function on S^m that might depend on p but which is bounded by a constant (independent of p) times ρ^d in $\mathcal{C}^k$ topology, for all $k \in \mathbb{N}$.

The next step is the computation of the coefficients of the first fundamental form of $S_{p,\rho}(w)$. We fix p and set $q := G(z)$. We obtain directly from (2.1), taking $\Xi = \rho\,(1-w)\,\Theta$, that

$$\begin{aligned} g(X_\mu, X_\nu) &= \delta_{\mu\nu} + \tfrac{1}{3}\, g(R_p(\Theta, E_\mu)\,\Theta, E_\nu)\,\rho^2\,(1-w)^2 \\ &+ \tfrac{1}{6}\, g(\nabla_\Theta R_p(\Theta, E_\mu)\,\Theta, E_\nu)\,\rho^3\,(1-w)^3 \\ &+ \tfrac{1}{20}\, g(\nabla_\Theta\,\nabla_\Theta R_p(\Theta, E_\mu)\,\Theta, E_\nu)\,\rho^4\,(1-w)^4 \\ &+ \tfrac{2}{45}\, g(R_p(\Theta, E_\mu)\,\Theta, E_\iota)\, g(R_p(\Theta, E_\nu)\,\Theta, E_\iota)\,\rho^4\,(1-w)^4 \\ &+ \mathcal{O}_p(\rho^5) + \rho^5\, L_p(w) + \rho^5\, Q_p^{(2)}(w), \end{aligned} \tag{2.3}$$

where all the curvature terms are evaluated at p. Observe that we have

$$g(\Upsilon, \Upsilon) \equiv 1 \qquad \text{and} \qquad g(\Upsilon, \Upsilon_j) \equiv 0, \quad j = 1, 2, \cdots, m. \tag{2.4}$$

Let $\mathring{g}_{ij} := g(Z_i, Z_j)$ be the coefficients of the first fundamental form of $S_{p,\rho}(w)$. Using these two equalities as well as (2.3), it is easy to obtain the expansion of $\mathring{g}_{ij}$ in powers of ρ and w.

Lemma 2.1. *The following expansion holds :*

$$\begin{aligned}(1-w)^{-2}\,\rho^{-2}\,\mathring{g}_{ij} &= g(\Theta_i,\Theta_j) + (1-w)^{-2}\,w_i\,w_j \\ &+ \tfrac{1}{3}\,g(R_p(\Theta,\Theta_i)\,\Theta,\Theta_j)\,\rho^2\,(1-w)^2 \\ &+ \tfrac{1}{6}\,g(\nabla_\Theta R_p(\Theta,\Theta_i)\,\Theta,\Theta_j)\,\rho^3\,(1-w)^3 \\ &+ \tfrac{1}{20}\,g(\nabla_\Theta\,\nabla_\Theta R_p(\Theta,\Theta_i)\,\Theta,\Theta_j)\,\rho^4\,(1-w)^4 \\ &+ \tfrac{2}{45}\,g(R_p(\Theta,\Theta_i)\,\Theta,E_\mu)\,g(R_p(\Theta,\Theta_j)\,\Theta,E_\mu)\,\rho^4\,(1-w)^4 \\ &+ \mathcal{O}_p(\rho^5) + \rho^5\,L_p(w) + \rho^5\,Q_p^{(2)}(w),\end{aligned}$$

where all curvature terms are evaluated at p.

For any (oriented) hyper-surface S that encloses a domain B_S, we agree that the orientation of S is chosen such that the normal vector points towards B_S and we define the functional

$$\Psi(S) := \mathrm{Vol}_m(S) - \tfrac{m}{\rho}\,\mathrm{Vol}_{m+1}(B_S).$$

When $w=0$, $S_{p,\rho}(0) = S_{p,\rho}$ is the geodesic sphere of radius ρ centered at p and the expansion of the volume of $S_{p,\rho}$ as well as the volume of the enclosed domain $B_{S_{p,\rho}}$ have been derived in [4] or [14] therefore, we have

$$\begin{aligned}&\rho^{-m}\,\Psi(S_{p,\rho}(w)) = \tfrac{1}{m+1}\,\mathrm{Vol}_m(S^m)\,\Big(1 - \tfrac{1}{2(m+3)}\,\rho^2\,\mathbf{s}(p) \\ &+ \tfrac{1}{72}\,\tfrac{1}{(m+3)(m+5)}\rho^4\,\big(5\mathbf{s}^2(p) + 8\,||\mathrm{Ric}_p||^2 - 3\,||R_p||^2 - 18\,\Delta_g\mathbf{s}(p)\big) + \mathcal{O}_p(\rho^5)\Big).\end{aligned}$$

In the following Lemma, we obtain the expansion of $\Psi(S_{p,\rho}(w))$ in powers of ρ and w.

Lemma 2.2. *The function* $\Psi(S_{p,\rho}(w))$ *can be expanded as*

$$\begin{aligned}&\rho^{-m}\,\Psi(S_{p,\rho}(w)) = \tfrac{1}{m+1}\,\mathit{Vol}_m(S^m)\,\Big(1 - \tfrac{1}{2(m+3)}\,\rho^2\,\mathbf{s}(p) \\ &+ \tfrac{1}{72}\,\tfrac{1}{(m+3)(m+5)}\rho^4\,\big(5\mathbf{s}^2(p) + 8\,||Ric_p||^2 - 3\,||R_p||^2 - 18\,\Delta_g\mathbf{s}(p)\big) + \mathcal{O}_p(\rho^5)\Big) \\ &+ \tfrac{1}{3}\,\rho^2 \int_{S^m} Ric_p(\Theta,\Theta)\,w\,d\sigma - \tfrac{m}{2}\int_{S^m} w^2\,d\sigma + \tfrac{1}{2}\int_{S^m} |\nabla w|^2\,d\sigma \\ &+ \int_{S^m}\Big(\rho^3\,L_p(w) + \rho^2\,Q_p^{(2)}(w) + Q_p^{(3)}(w)\Big)\,d\sigma.\end{aligned}$$

The proof of this Lemma is not very enlightening and is postponed to the Appendix.

Hopefully, in most of the paper, we do not need to keep such precise expansions. For example, the expansion of the first fundamental form of $S_{p,\rho}(w)$ that will be needed from now on is just

$$\begin{array}{rcl}\rho^{-2}\,(1-w)^{-2}\,\mathring{g}_{ij} & = & g(\Theta_i,\Theta_j)+\frac{1}{3}\,g(R_p(\Theta,\Theta_i)\,\Theta,\Theta_j)\,\rho^2\,(1-w)^2 \\ & + & \mathcal{O}_p(\rho^3)+\rho^3\,L_p(w)+Q_p^{(2)}(w).\end{array}$$

Our next task is to understand the dependence on w and ρ of the unit normal N to $S_{p,\rho}(w)$. Define the vector field

$$\mathring{\tilde{N}} := -\,\Upsilon + a^j\,Z_j,$$

and choose the coefficients a^j so that that $\tilde{N}$ is orthogonal to all the tangent vectors Z_i, for $i=1,\ldots,m$. Using (2.4), we obtain a linear system for the coefficients a^j

$$\mathring{g}_{ij}\,a^j = -\rho\,w_i.$$

Observe that

$$g(\mathring{\tilde{N}},\mathring{\tilde{N}}) = 1+\rho\,a^j\,w_j = 1-\rho^2\,\mathring{g}^{ij}w_i\,w_j.$$

The unit normal vector field $\mathring{N}$ about $S_{p,\rho}(w)$ is defined to be

$$\mathring{N} := g(\mathring{\tilde{N}},\mathring{\tilde{N}})^{-1/2}\,\mathring{\tilde{N}}.$$

We can now compute the second fundamental form $\mathring{h}_{ij} = -g(\nabla_{Z_i}\mathring{N},Z_j)$.

Lemma 2.3. *The following expansion holds for $i,j=1,\ldots,m$*

$$\begin{array}{rcl}\mathring{h}_{ij} & = & \rho\,(1-w)\,g(\Theta_i,\Theta_j)+\rho\,(Hess_{g_{S^m}}\,w)_{ij} \\ & + & \frac{2}{3}\,g(R_p(\Theta,\Theta_i)\,\Theta,\Theta_j)\,\rho^3\,(1-w)^3 \\ & + & \mathcal{O}_p(\rho^4)+\rho^3\,L_p(w)+\rho\,Q_p^{(2)}(w),\end{array}$$

where as usual, all curvature terms are computed at the point p.

Proof : We will first obtain the expansion of $\mathring{\tilde{h}}_{ij} = g(\nabla_{Z_i}\mathring{\tilde{N}},Z_j)$. To this aim, using (2.2) together with the definition of $\mathring{\tilde{N}}$, we compute,

$$\begin{array}{rcl}\mathring{\tilde{h}}_{ij} & = & g(\nabla_{Z_i}\Upsilon,Z_j)-g(\nabla_{Z_i}(a^k\,Z_k),Z_j) \\ & = & \frac{1}{1-w}g(\nabla_{Z_i}((1-w)\,\Upsilon),Z_j)+\frac{1}{1-w}\,w_i\,g(\Upsilon,Z_j)-g(\nabla_{Z_i}(a^k\,Z_k),Z_j) \\ & = & \frac{1}{1-w}g(\nabla_{Z_i}((1-w)\,\Upsilon),Z_j)-\frac{\rho}{1-w}\,w_i\,w_j-g(\nabla_{Z_i}(a^k\,Z_k),Z_j).\end{array}$$

Now, recall that

$$a^k\, g(Z_k, Z_j) = -\rho\, w_j,$$

so that

$$g(\nabla_{Z_i}(a^k\, Z_k), Z_j) = -\rho\, w_{ij} - a^k\, g(Z_k, \nabla_{Z_i} Z_j).$$

We conclude so far that

$$\mathring{\tilde{h}}_{ij} = \tfrac{1}{1-w} g(\nabla_{Z_i}((1-w)\,\Upsilon), Z_j) - \tfrac{\rho}{1-w}\, w_i\, w_j + \rho\, w_{ij} + a^k\, g(Z_k, \nabla_{Z_i} Z_j). \quad (2.5)$$

We already know that

$$a^k\, g(Z_k, \nabla_{Z_i} Z_j) = a^k\, \mathring{g}_{kl}\, \mathring{\Gamma}^l_{ij} = -\rho\, \mathring{\Gamma}^k_{ij}\, w_k.$$

where the $\mathring{\Gamma}^k_{ij}$ are the Christoffel symbols associated to $\mathring{g}_{ij}$. To analyze the first term in this formula, let us consider ρ as a variable instead of just a parameter. Thus we consider

$$\tilde{F}(\rho, z) = F\big(\rho(1 - w(z))\Theta(z)\big).$$

so that the vector fields Z_j are equal to $\tilde{F}_*(\partial_{z^j})$ and hence are coordinate vector fields, but now we also have

$$Z_0 := \tilde{F}_*(\partial_\rho) = (1-w)\,\Upsilon,$$

which is a coordinate vector field. Observe that

$$g(\nabla_{Z_i}((1-w)\,\Upsilon), Z_j) = g(\nabla_{Z_j}((1-w)\,\Upsilon), Z_i).$$

This can be seen either by writing

$$\begin{aligned} g(\nabla_{Z_i}((1-w)\,\Upsilon), Z_j) &= \partial_{z^i}\, g((1-w)\Upsilon, Z_j) - g((1-w)\Upsilon, \nabla_{Z_i} Z_j) \\ &= -(1-w)\, w_{ij} + w_i\, w_j - (1-w)\, g(\Upsilon, \nabla_{Z_i} Z_j), \end{aligned}$$

which is clearly symmetric in i, j or simply by noticing that $\mathring{\tilde{h}}_{ij} = \mathring{\tilde{h}}_{ji}$ and inspection of (2.5).

Therefore, we can write

$$2\, g(\nabla_{Z_i}((1-w)\,\Upsilon), Z_j) = g(\nabla_{Z_i} Z_0, Z_j) + g(\nabla_{Z_j} Z_0, Z_i) = \partial_\rho\, \mathring{g}_{ij}.$$

Inserting the above into (2.5), we have obtained the formula

$$\mathring{\tilde{h}} = \tfrac{1}{2(1-w)}\, \partial_\rho\, \mathring{g} - \tfrac{1}{1-w}\, \rho\, dw \otimes dw + \rho\, \mathrm{Hess}_{\mathring{g}} w.$$

We can now expand the first and last term in this expression.

Using the result of Lemma 2.1, we find

$$\begin{aligned}\frac{1}{2(1-w)}\,\partial_\rho\,\mathring{g}_{ij} &= \rho\,(1-w)\,g(\Theta_i,\Theta_j)+\tfrac{2}{3}\,g(R_p(\Theta,\Theta_i)\,\Theta,\Theta_j)\,\rho^3\,(1-w)^3\\ &+ \mathcal{O}_p(\rho^4)+\rho^4\,L_p(w)+\rho\,Q_p^{(2)}(w).\end{aligned}$$

Using the same Lemma, we also have

$$\mathring{\Gamma}_{ij}^{k}=\Gamma_{ij}^{k}+\mathcal{O}_p(\rho^4)+\rho^2\,L_p(w)+\rho^2\,Q_p^{(2)}(w),$$

where Γ_{ij}^k are the Christoffel symbols of S^m in the parmeterization $z \longmapsto \Theta(z)$. Collecting the above estimates, we conclude that

$$\begin{aligned}\mathring{\tilde{h}}_{ij} &= \rho\,(1-w)\,g(\Theta_i,\Theta_j)+\rho\,(w_{ij}-\Gamma_{ij}^k\,w_k)\\ &+ \tfrac{2}{3}\,g(R_p(\Theta,\Theta_i)\,\Theta,\Theta_j)\,\rho^3\,(1-w)^3\\ &+ \mathcal{O}_p(\rho^4)+\rho^3\,L_p(w)+\rho\,Q_p^{(2)}(w).\end{aligned}$$

It remains to observe that

$$g(\mathring{\tilde{N}},\mathring{\tilde{N}})^{-1/2}=1+Q_p^{(2)}(w),$$

to complete the proof of the estimate. □

Collecting the estimates of the subsection we obtain the expansion of the mean curvature of the hyper-surface $S_{p,\rho}(w)$ in powers of ρ and w by taking the trace of $\mathring{h}$ with respect to $\mathring{g}$. We obtain the :

Lemma 2.4. *The mean curvature of the of the hyper-surface $S_{p,\rho}(w)$ can be expanded as*

$$\begin{aligned}\rho\,H(S_{p,\rho}(w)) &= m+(\Delta_{S^m}+m)\,w-\tfrac{1}{3}\,Ric_p(\Theta,\Theta)\,\rho^2\\ &+ \mathcal{O}_p(\rho^3)+\rho^2\,L_p(w)+Q_p^{(2)}(w),\end{aligned}$$

where Ric_p denotes the Ricci tensor computed at p.

3 Existence of constant mean curvature spheres

We now explain how to perturb small geodesic spheres to obtain constant mean curvature hyper-surfaces with large mean curvature. The proof is divided in two steps. In the first step, given $p \in M$ and ρ small enough, we find a small function $w \in \mathcal{C}^{2,\alpha}(S^m)$ and a vector $\Xi \in T_pM$ such that

$$\rho\, H(S_\rho(p,w)) = m - g(\Xi,\Theta).$$

This is achieved by applying a fixed point theorem for contraction mappings. At this point we will have perturbed any small geodesic sphere into a hyper-surface, whose mean curvature is not necessarily constant but, in some sense, which is as close as possible to a constant. In the second step, we use the variational characterization of constant mean curvature as critical points of the functional Ψ defined in section 2. We compute the value of Ψ for the perturbed geodesic spheres obtained in the first step and this provides a function defined on M and depending on ρ whose critical points are associated to constant mean curvature hyper-surfaces.

3.1 A fixed point argument

We use a fixed point theorem to find a function $w \in \mathcal{C}^{2,\alpha}(S^m)$ and a vector $\Xi \in T_pM$ such that

$$\rho\, H(S_\rho(p,w)) = m - g(\Xi,\Theta).$$

This amount to solve the nonlinear elliptic problem

$$(\Delta_{S^m} + m)\, w + g(\Xi,\Theta) = \tfrac{1}{3}\,\mathrm{Ric}_p(\Theta,\Theta)\,\rho^2 + \mathcal{O}_p(\rho^3) + \rho^2\, L_p(w) + Q_p^{(2)}(w). \tag{3.6}$$

We denote by Π (resp. $\Pi^\perp$) the L^2-orthogonal projections onto $\mathrm{Ker}(\Delta_{S^m} + m)$ (resp. $\mathrm{Ker}(\Delta_{S^m}+m)^\perp$). Recall that the kernel of $\Delta_{S^m}+m$ is spanned by the restriction to the unit sphere of the coordinates functions in $\mathbb{R}^{m+1}$. Therefore elements of $\mathrm{Ker}\,(\Delta_{S^m}+m)$ are precisely of the form $g(\Theta,\Xi)$ for $\Xi \in T_pM$. From now on, we assume that the function $w \in \mathcal{C}^{2,\alpha}(S^m)$ is L^2-orthogonal to $\mathrm{Ker}(\Delta_{S^m}+m)$.

It is easy to rephrase the solvability of the nonlinear equation (3.6) as a fixed point problem since the operator

$$\Delta_{S^m} + m : \mathcal{C}^{2,\alpha}(S^m)^\perp \longrightarrow \mathcal{C}^{0,\alpha}(S^m)^\perp$$

is invertible. Here $\mathcal{C}^{k,\alpha}(S^m)^\perp$ denotes the space of functions in $\mathcal{C}^{k,\alpha}(S^m)$ that are L^2-orthogonal to $\mathrm{Ker}(\Delta_{S^m}+m)$. Applying a standard fixed point theorem for contraction mappings, one finds that there exist constants $\kappa > 0$ and $\rho > 0$, which are independent of the choice of the point $p \in M$, such that, for all $\rho \in (0,\rho_0)$ and $p \in M$, there exists a unique $(w_{p,\rho},\Xi_{p,\rho}) \in$

$\mathcal{C}^{2,\alpha}(S^m)^\perp \times T_pM$, solution of (3.6) which belongs to the closed ball of radius $\kappa\,\rho^2$ in $\mathcal{C}^{2,\alpha}(S^m)^\perp \times T_pM$.

We now derive some expansion of $w_{p,\rho}$ in powers of ρ as well as some estimate for $\Xi_{p,\rho}$. To start with, observe that we have

$$\Pi^\perp\left(g(\Xi,\Theta)\right) = 0.$$

Also observe that

$$\Pi^\perp\left(\mathrm{Ric}_p(\Theta,\Theta)\right) = \mathrm{Ric}_p(\Theta,\Theta),$$

since the function $\Theta \longmapsto \mathrm{Ric}_p(\Theta(z),\Theta)$ is invariant when Θ is changed into $-\Theta$ and hence its L^2-projection over elements of the form $g(\Xi,\Theta)$ is 0. Using this, we conclude that, for all $k \geq 0$

$$\|\nabla_p^k \Xi_\rho\|_{\mathcal{C}^{2,\alpha}(TM)} \leq c_k\,\rho^3,$$

for some constant $c_k > 0$ which does not depend on $\rho \in (0,\rho_0)$ nor on $p \in M$.

If $w_p \in \mathrm{Ker}(\Delta_{S^m} + m)^\perp$ is the unique solution of

$$(\Delta_{S^m} + m)\,w_p = \tfrac{1}{3}\,\mathrm{Ric}_p(\Theta,\Theta),$$

which is L^2-orthogonal to the kernel of $\Delta_{S^m} + m$, we also find that the function $w_{p,\rho}$ can be decomposed into

$$w_{p,\rho} = \rho^2\,w_p + v_{p,\rho}, \tag{3.7}$$

where, for all $k \geq 0$

$$\|v_{p,\rho}\|_{\mathcal{C}^{k,\alpha}(S^m)} \leq c_k\,\rho^3,$$

for some constant $c_k > 0$ which does not depend on $\rho \in (0,\rho_0)$ nor on $p \in M$.

We decompose

$$\mathrm{Ric}_p(\Theta,\Theta) = \mathring{\mathrm{Ric}}_p\,(\Theta,\Theta) + \tfrac{1}{m+1}\,\mathbf{s}(p),$$

so that the function $\Theta \longmapsto \mathring{\mathrm{Ric}}_p\,(\Theta,\Theta)$ is the restriction of a homogeneous polynomial of degree 2 which has mean 0 over S^m. Observe that $\mathring{\mathrm{Ric}}_p\,(\Theta,\Theta)$ belongs to the eigenspaces of $-\Delta_{S^m}$ associated to the eigenvalues $2(m+1)$ and hence we have the explicit formula

$$3\,w_p = \tfrac{1}{m(m+1)}\,\mathbf{s}(p) - \tfrac{1}{m+2}\,\mathring{\mathrm{Ric}}_p\,(\Theta,\Theta). \tag{3.8}$$

We denote by $S^\flat_{p,\rho}$ the hyper-surface $S_{p,\rho}(w_{p,\rho})$. By construction the mean curvature of this hyper-surface is given by

$$H(S^\flat_{p,\rho}) = \frac{m}{\rho} - g(\Xi_{p,\rho}, \Theta),$$

where we recall that $\Xi_{p,\rho} \in T_pM$.

Let us rephrase what we have obtained so far slightly differently. Even though our construction depends on the point p we choose, it is easy to check, reducing the value of ρ_0 if this is necessary, that both $w_{p,\rho}$ and $\Xi_{p,\rho}$ depend smoothly on $p \in M$ and $\rho \in (0, \rho_0)$. Therefore as p varies over M, $\Xi_{p,\rho}$ defines a smooth vector field in TM and $w_{p,\rho}$ defines a function on the spherical tangent bundle STM. Moreover, it follows from the construction that, for all $k \geq 0$

$$\|\nabla^k_p w_{p,\rho}\|_{\mathcal{C}^{2,\alpha}(STM)} + \rho^{-1}\, \|\nabla^k_p \Xi_{p,\rho}\|_{\mathcal{C}^{2,\alpha}(TM)} \leq c_k\, \rho^2,$$

for some constant $c_k > 0$ which does not depend on $\rho \in (0, \rho_0)$ nor on $p \in M$.

3.2 A variational argument

Before completing the second part of the proof of Theorem 1.1, we digress slightly. Given any hyper-surface S that is embedded in Euclidean space $\mathbb{R}^{m+1}$ and bounds a compact domain B_S, we set

$$\mathcal{E}_{eucl}(S) := \mathrm{Vol}_M(S) - H_0\, \mathrm{Vol}_{m+1}(B_S),$$

where $H_0 \in \mathbb{R}$ is fixed.

Given any vector field Ξ we can flow the embedded hyper-surface S along Ξ and define S_t to be the image of S by the flow generated by Ξ, at time t. The first variation of

$$t \longmapsto \mathcal{E}_{eucl}(S_t)$$

is given by

$$\frac{d}{dt}\mathcal{E}_{eucl}(S_t)_{|t=0} = \int_S (H(S) - H_0)\, \Xi \cdot N_S\, dvol_S,$$

where $H(S)$ is the mean curvature and N_S is the unit normal vector field associated to S (the normal vector field is assumed to point towards B_S).

In the case where Ξ is a Killing vector field, the flow generated by Ξ acts by isometries and we get

$$\mathcal{E}(S_t) = \mathcal{E}(S),$$

for all t. Hence, we find that the following identity which holds for any compact embedded hyper-surface S, any Killing vector field Ξ and any constant $H_0 \in \mathbb{R}$

$$\int_S (H(S) - H_0)\, \Xi \cdot N_S \, dvol_S = 0. \tag{3.9}$$

This identity was already used by N. Kapouleas in [5] to prove the following observation : Assume that S is a compact embedded hyper-surface in $\mathbb{R}^{m+1}$ whose mean curvature is given by

$$H(S) = H_0 + \Xi_0 \cdot N_S,$$

for some Killing field Ξ_0. Then $\Xi_0 \equiv 0$ and hence S is a constant mean curvature hyper-surface. This property follows at once from (3.9) with $\Xi = \Xi_0$.

The proof of Theorem 1.1 make use a modified version of (3.9) in a Riemannian setting. We define the function

$$\Psi_\rho(p) := \mathrm{Vol}_m(S^\flat_{p,\rho}) - \frac{m}{\rho}\, \mathrm{Vol}_{m+1}(B^\flat_{p,\rho}),$$

where $B^\flat_{p,\rho}$ is the domain enclosed by $S^\flat_{p,\rho}$ which contains the point p. The result of Theorem 1.1 will follow from the :

Proposition 3.1. *There exists $\rho_0 > 0$ such that, if $\rho \in (0, \rho_0)$ and if p is a critical point of Ψ_ρ then $S^\flat_{p,\rho}$ is a constant mean curvature hyper-surface with mean curvature equal to $\frac{m}{\rho}$.*

Proof : Granted the fact that, by construction, $S^\flat_{p,\rho}$ has mean curvature equal to $\frac{m}{\rho} - g(\Xi_{p,\rho}, \Theta)$, it is enough to show that $\Xi_{p,\rho} = 0$ when p is a critical point of Ψ_ρ (and ρ is chosen small enough).

Assume that p is a critical point of Ψ_ρ. Given $\Xi \in T_pM$ we compute using two different methods the differential of the function Ψ_ρ at p, applied to Ξ. By assumption,

$$D\Psi_\rho|_p(\Xi) = 0, \tag{3.10}$$

since p is a critical point of Ψ_ρ.

Provided t is small enough, the surface $S^\flat_{q,\rho}$ where $q := \mathrm{Exp}_p(t\,\Xi)$, can be written as a normal graph over $S^\flat_{p,\rho}$ for some function $f_{p,\rho,\Xi,t}$ that depends smoothly on t. This defines a vector field on $S^\flat_{p,\rho}$ by

$$Z_{p,\rho,\Xi} := \partial_t f_{p,\rho,\Xi,t}\,|_{t=0}\, N_{S^\flat_{p,\rho}},$$

where $N_{S^\flat_{p,\rho}}$ is the normal vector field about $S^\flat_{p,\rho}$. It is easy to check that the vector field $Z_{p,\rho,\Xi}$ can be estimated by

$$\|Z_{p,\rho,\Xi} - X\|_g \leq c\,\rho^2\,\|\Xi\|_g,$$

where the constant $c > 0$ does not depend on ρ (small enough) nor on Ξ and where X is the parallel transport of Ξ along geodesics issued from p.

The first variation of the m-dimensional volume and $(m+1)$-dimensional volume forms, together with the definition of the mean curvature yields

$$D\Psi_\rho|_p(\Xi) = \int_{\tilde{S}_{p,\rho}} \left(H(S^\flat_{p,\rho}) - \frac{m}{\rho} \right) g(Z_{p,\rho,\Xi}, N_{S^\flat_{p,\rho}})\, dvol_{\tilde{S}_{p,\rho}}. \tag{3.11}$$

By construction

$$H(S^\flat_{p,\rho}) = \frac{m}{\rho} - g(\Xi_{\rho,p}, \Theta),$$

where $\Xi_{p,\rho} \in T_pM$ has been defined in the previous section. Therefore, using (3.11) and (3.10), we conclude that

$$\int_{S^\flat_{p,\rho}} g(\Xi_{p,\rho}, \Theta)\; g(Z_{p,\rho,\Xi}, N_{S^\flat_{p,\rho}})\; dvol_{S^\flat_{p,\rho}} = 0.$$

We can write

$$g(Z_{p,\rho,\Xi}, N_{S^\flat_{p,\rho}}) = g(X + (Z_{p,\rho,\Xi} - X), -\Upsilon + (N_{S^\flat_{p,\rho}} + \Upsilon))$$

and, making use of the expansions of the normal vector field together with the expansion of the metric given in Section 2.2, we conclude that

$$|g(Z_{p,\rho,\Xi}, N_{S^\flat_{p,\rho}}) + g(\Xi, \Theta)| \leq c\,\rho^2\,\|\Xi\|_g,$$

for some constant $c > 0$ which does not depend on ρ, provided ρ is chosen small enough. Therefore, we can write

$$\int_{S^\flat_{p,\rho}} g(\Xi_{p,\rho}, \Theta)\, g(\Xi, \Theta)\, dvol_{S^\flat_{p,\rho}} \leq c\,\rho^2\,\|\Xi\|_g \int_{S^\flat_{p,\rho}} |g(\Xi_{p,\rho}, \Theta)|\, dvol_{S^\flat_{p,\rho}}. \tag{3.12}$$

Observe that, in Euclidean space, we have the equality

$$\mathrm{Vol}_m(S^m)\,\|\Xi\|^2 = (m+1) \int_{S^m} g(\Xi \cdot \Theta)^2\, dvol_{S^m},$$

for any vector $\Xi \in \mathbb{R}^{m+1}$. Using the expansion of the metric on $S^\flat_\rho(p)$, we find

$$\tfrac{1}{2}\,\mathrm{Vol}_m(S^m)\,\rho^m\,\|\Xi\|_g^2 \leq (m+1) \int_{S^\flat_\rho(p)} g(\Xi, \Theta)^2\, dvol_{S^\flat_{p,\rho}}, \tag{3.13}$$

for all ρ small enough. Therefore, (3.12) yields

$$\int_{S^\flat_{p,\rho}} g(\Xi_{p,\rho},\Theta)\, g(\Xi,\Theta)\, dvol_{S^\flat_{p,\rho}} \quad \leq \quad c\,\rho^{2-m/2} \left(\int_{S^\flat_{p,\rho}} |g(\Xi_{p,\rho},\Theta)|\, dvol_{S^\flat_{p,\rho}}\right) \times \left(\int_{S^\flat_{p,\rho}} |g(\Xi,\Theta)|^2\, dvol_{S^\flat_{p,\rho}}\right)^{1/2}.$$

Taking $\Xi = \Xi_{p,\rho}$ and using Cauchy-Schwarz inequality, we get

$$\int_{S^\flat_{p,\rho}} |g(\Xi_{\rho,p},\Theta)|^2\, dvol_{S^\flat_{p,\rho}} \leq c\,\rho^2 \int_{S^\flat_{p,\rho}} |g(\Xi_{\rho,p},\Theta)|^2\, dvol_{S^\flat_{p,\rho}}.$$

Clearly, this together with (3.13) implies that $\Xi_{\rho,p} = 0$ for all ρ small enough. This completes the proof of the result. □

In order to complete the proof of Theorem 1.1 it is enough to use the result of Lemma 2.2 with $w = w_{p,\rho,}$. Indeed, using the fact that $w_{p,\rho} = \rho^2\, w_p + v_{p,\rho}$ as in (3.7), we get

$$\begin{aligned} &\tfrac{1}{3}\,\rho^2 \int_{S^m} \mathrm{Ric}_p(\Theta,\Theta)\, w_{\rho,p}\, d\sigma - \tfrac{m}{2} \int_{S^m} w^2_{\rho,p}\, d\sigma + \tfrac{1}{2} \int_{S^m} |\nabla w_{\rho,p}|^2\, d\sigma \\ &\quad = \rho^4 \left(\tfrac{1}{3} \int_{S^m} \mathrm{Ric}_p(\Theta,\Theta)\, w_p\, d\sigma - \tfrac{m}{2} \int_{S^m} w_p^2\, d\sigma + \tfrac{1}{2} \int_{S^m} |\nabla w_p|^2\, d\sigma\right) \\ &\qquad + \mathcal{O}_p(\rho^5). \end{aligned}$$

Since w_p satisfies, $(\Delta_{S^m} + m)\, w_p = \frac{1}{3}\,\mathrm{Ric}_p(\Theta,\Theta)$, we can multiply this equation by w_p and get after integration

$$\int_{S^m} |\nabla w_p|^2\, d\sigma = m \int_{S^m} w_p^2\, d\sigma - \tfrac{1}{3} \int_{S^m} \mathrm{Ric}(\Theta,\Theta)\, w_p\, d\sigma$$

and hence we conclude that

$$\begin{aligned} &\tfrac{1}{3}\,\rho^2 \int_{S^m} \mathrm{Ric}_p(\Theta,\Theta)\, w_{\rho,p}\, d\sigma - \tfrac{m}{2} \int_{S^m} w^2_{\rho,p}\, d\sigma + \tfrac{1}{2} \int_{S^m} |\nabla w_{\rho,p}|^2\, d\sigma \\ &\quad = \rho^4 \left(\tfrac{1}{6} \int_{S^m} \mathrm{Ric}_p(\Theta,\Theta)\, w_p\, d\sigma\right) + \mathcal{O}_p(\rho^5). \end{aligned}$$

Now, from (3.8) we have

$$3\, w_p = \tfrac{1}{m(m+1)}\,\mathbf{s} - \tfrac{1}{m+2}\ \overset{\circ}{\mathrm{Ric}}_p\,(\Theta,\Theta).$$

So, finally

$$\frac{1}{3}\,\rho^2 \int_{S^m} \mathrm{Ric}_p(\Theta,\Theta)\, w_{\rho,p}\, d\sigma - \frac{m}{2} \int_{S^m} w_{\rho,p}^2\, d\sigma + \frac{1}{2} \int_{S^m} |\nabla w_{\rho,p}|^2\, d\sigma$$
$$= \rho^4\, \frac{1}{18(m+1)(m+2)(m+3)}\, \mathrm{Vol}(S^m)\, \left(\tfrac{m+6}{m} s^2(p) - 2\,||\mathrm{Ric}_p||^2\right) + \mathcal{O}_p(\rho^5).$$

We have used the formula of the Appendix to obtain the last equality. Now, it is enough to insert this expression in the expansion of Lemma 2.2 and us the identities given in the Appendix to obtain the expansion

$$\rho^{-m}\Psi(S^\flat_{p,\rho}) = \frac{1}{m+1}\,\mathrm{Vol}(S^m)\,\Big(1 - \frac{1}{2(m+3)}\,\rho^2\,\mathbf{s}(p)$$
$$+\frac{1}{72(m+3)(m+5)}\,\rho^4\,\left(5\,\mathbf{s}^2(p) + 8\,||\mathrm{Ric}_p||^2 - 3\,||R_p||^2 - 18\,\Delta_g \mathbf{s}(p)\right)$$
$$+\frac{1}{18(m+3)(m+2)}\,\rho^4\,\left(\tfrac{m+6}{m} s^2(p) - 2\,||\mathrm{Ric}_p||^2\right) + \mathcal{O}_p(\rho^5)\Big)$$

and by definition

$$\phi(p,\rho) := \frac{2(m+1)(m+3)}{\mathrm{Vol}(S^m)}\,\rho^{-2}\,\left(\frac{1}{m+1}\,\mathrm{Vol}(S^m) - \rho^{-m}\Psi(S^\flat_\rho(p))\right).$$

4 Appendix

4.1 Proof of Lemma 2.2:

To simplify the computation, let us assume that $g(\Theta_i,\Theta_j) = \delta_{ij}$ at the point where the computation is done. Using the classical expansion

$$\sqrt{\det(I+A)} = 1 + \frac{1}{2}\,\mathrm{tr}A + \frac{1}{8}\,(\mathrm{tr}A)^2 - \frac{1}{4}\,\mathrm{tr}(A^2) + \mathcal{O}(|A|^3)$$

we obtain

$$\rho^{-m}\,(1-w)^{-m}\,\sqrt{\det \mathring{g}}$$
$$= 1 + \frac{1}{2}\,|\nabla_{S^m} w|^2 - \frac{1}{6}\,\mathrm{Ric}_p(\Theta,\Theta)\,\rho^2\,(1-w)^2 - \frac{1}{12}\,\nabla_\Theta\,\mathrm{Ric}_p(\Theta,\Theta)\,\rho^3\,(1-w)^3$$
$$-\frac{1}{40}\,\nabla^2_\Theta\,\mathrm{Ric}_p(\Theta,\Theta)\,\rho^4\,(1-w)^4 + \frac{1}{72}\,(\mathrm{Ric}_p(\Theta,\Theta))^2\,\rho^4(1-w)^4$$
$$+\frac{1}{45}\,\sum_{\mu,i} g(R_p(\Theta,\Theta_i)\,\Theta, E_\mu)^2\,\rho^4(1-w)^4$$
$$-\frac{1}{36}\,\sum_{i,j} g(R_p(\Theta,\Theta_i)\,\Theta,\Theta_j)^2\,\rho^4(1-w)^4$$
$$+\mathcal{O}_p(\rho^5) + \rho^3\,L_p(w) + \rho^2\,Q_p^{(2)}(w) + Q_p^{(3)}(w).$$

It is easy to check that

$$\sum_{\mu,i} g(R_p(\Theta,\Theta_i)\Theta, E_\mu)^2 = \sum_{i,j} g(R_p(\Theta,\Theta_i)\Theta,\Theta_k)^2. \tag{4.14}$$

Hence we obtain the expansion of the m-dimensional volume

$$\begin{aligned}
&\rho^{-m}\,\mathrm{Vol}(S_{p,\rho}(w)) = \\
&\quad \mathrm{Vol}(S^m) - \tfrac{1}{6}\,\rho^2 \int_{S^m} \mathrm{Ric}_p(\Theta,\Theta)\, d\sigma - m \int_{S^m} w\, d\sigma \\
&\quad -\rho^4 \int_{S^m} \big(\tfrac{1}{40}\,\nabla^2_\Theta \,\mathrm{Ric}_p(\Theta,\Theta) + \tfrac{1}{180} \sum_{i,j} g(R_p(\Theta,\Theta_i)\,\Theta,\Theta_j)^2 \\
&\qquad\qquad -\tfrac{1}{72}\,(\mathrm{Ric}_p(\Theta,\Theta))^2\big)\, d\sigma \\
&\quad +\tfrac{m(m-1)}{2} \int_{S^m} w^2\, d\sigma + \tfrac{1}{2} \int_{S^m} |\nabla w|^2\, d\sigma + \tfrac{m+2}{6}\,\rho^2 \int_{S^m} \mathrm{Ric}_p(\Theta,\Theta)\, w\, d\sigma \\
&\quad + \int_{S^m} \Big(\mathcal{O}_p(\rho^5) + \rho^3\, L_p(w) + \rho^2\, Q_p^{(2)}(w) + Q_p^{(3)}(w)\Big)\, d\sigma,
\end{aligned}$$

where we have used the fact that

$$\int_{S^m} \nabla_\Theta \mathrm{Ric}(\Theta,\Theta)\, d\sigma = 0, \tag{4.15}$$

since the function being integrated changes sign under the action of the symmetry $\Theta \in S^m \longmapsto -\Theta \in S^m$.

Using the formula in [4] or [14] for the expansion of the volume of geodesic spheres, we find that

$$\begin{aligned}
&\rho^{-m}\,\mathrm{Vol}(S_{p,\rho}(w)) = \\
&\quad \mathrm{Vol}(S^m)\, \Big(1 - \tfrac{1}{6(m+1)}\,\rho^2\, \mathbf{s}(p) \\
&\qquad\qquad + \tfrac{\rho^4}{360(m+1)(m+3)}\, \big(5\,\mathbf{s}^2(p) + 8\,||\mathrm{Ric}_p||^2 - 3\,||R_p||^2 - 18\,\Delta_g \mathbf{s}(p)\big)\Big) \\
&\quad +\tfrac{m(m-1)}{2} \int_{S^m} w^2\, d\sigma + \tfrac{1}{2} \int_{S^m} |\nabla w|^2\, d\sigma + \tfrac{m+2}{6}\,\rho^2 \int_{S^m} \mathrm{Ric}_p(\Theta,\Theta)\, w\, d\sigma \\
&\quad -m \int_{S^m} w\, d\sigma + \int_{S^m} \Big(\mathcal{O}_p(\rho^5) + \rho^3\, L_p(w) + \rho^2\, Q_p^{(2)}(w) + Q_p^{(3)}(w)\Big)\, d\sigma,
\end{aligned}$$

since, when $w = 0$ this formula should agree with the formula in [4] and [14].

Next, to compute the volume of the domain $B_{p,\rho}(w)$ enclosed by $S_{p,\rho}(w)$, we consider polar geodesic normal coordinates (r,Θ) centered at p. Using the expansion (2.3), the volume form can be expanded as

$$\begin{aligned} r^{-m}\sqrt{|g|} &= 1 - \tfrac{1}{6}\,\mathrm{Ric}_p(\Theta,\Theta)\,r^2 - \tfrac{1}{12}\,\nabla_\Theta\,\mathrm{Ric}_p(\Theta,\Theta)\,r^3 \\ &- \tfrac{1}{40}\,\nabla^2_\Theta\,\mathrm{Ric}_p(\Theta,\Theta)\,r^4 - \tfrac{1}{180}\sum_{\mu,\nu} g(R_p(\Theta,E_\mu)\,\Theta,E_\nu)^2\,r^4 \\ &+ \tfrac{1}{72}\,(\mathrm{Ric}_p(\Theta,\Theta))^2\,r^4 + \mathcal{O}_p(r^5). \end{aligned}$$

Integration over the set $r \leq \rho\,(1-w)$ give the expansion of the volume enclosed by $S_\rho(p,w)$ as

$$\begin{aligned} &\rho^{-m-1}\,\mathrm{Vol}(B_{p,\rho}(w)) = \\ &\tfrac{1}{m+1}\,\mathrm{Vol}(S^m) - \tfrac{1}{6(m+3)}\,\rho^2\int_{S^m}\mathrm{Ric}(\Theta,\Theta)\,d\sigma - \int_{S^m} w\,d\sigma \\ &-\tfrac{1}{m+5}\,\rho^4\int_{S^m}\Big(\tfrac{1}{40}\,\nabla^2_\Theta\,\mathrm{Ric}_p(\Theta,\Theta) + \tfrac{1}{180}\sum_{\mu,\nu} g(R_p(\Theta,E_\mu)\,\Theta,E_\nu)^2 \\ &\qquad -\tfrac{1}{72}\,(\mathrm{Ric}_p(\Theta,\Theta))^2\Big)\,d\sigma \\ &+\tfrac{m}{2}\int_{S^m} w^2\,d\sigma + \tfrac{1}{6}\,\rho^2\int_{S^m}\mathrm{Ric}_p(\Theta,\Theta)\,w\,d\sigma \\ &+\int_{S^m}\Big(\mathcal{O}_p(\rho^5) + \rho^3\,L_p(w) + \rho^2\,Q_p^{(2)}(w) + Q_p^{(3)}(w)\Big)\,d\sigma, \end{aligned}$$

where again we have used (4.15). Using the formula in [4] or [14] for the expansion of the volume of geodesic balls, we find that

$$\begin{aligned} &\rho^{-m-1}\,\mathrm{Vol}(B_{p,\rho}(w)) = \\ &\tfrac{1}{m+1}\,\mathrm{Vol}(S^m)\,\Big(1 - \tfrac{1}{6(m+3)}\,\rho^2\,\mathbf{s}(p) \\ &\qquad +\tfrac{\rho^4}{360(m+3)(m+5)}\,\big(5\,\mathbf{s}^2(p) + 8\,||\mathrm{Ric}_p||^2 - 3\,||R_p||^2 - 18\,\Delta_g\mathbf{s}(p)\big)\Big) \\ &-\int_{S^m} w\,d\sigma + \tfrac{m}{2}\int_{S^m} w^2\,d\sigma + \tfrac{1}{6}\,\rho^2\int_{S^m}\mathrm{Ric}_p(\Theta,\Theta)\,w\,d\sigma \\ &+\int_{S^m}\Big(\mathcal{O}_p(\rho^5) + \rho^3\,L_p(w) + \rho^2\,Q_p^{(2)}(w) + Q_p^{(3)}(w)\Big)\,d\sigma, \end{aligned}$$

since when $w = 0$, this expansion should agree with the corresponding expansion given in [4], [14]. The estimate for $\Psi(S_{p,\rho}(w))$ given in Lemma 2.2 then follows at once.

4.2 Some formula

Recall that

$$\int_{S^m} (x^\mu)^2\, d\sigma = \tfrac{1}{m+1}\, \mathrm{Vol}(S^m)$$

and also that

$$\int_{S^m} (x^\mu)^4\, d\sigma = 3 \int_{S^m} (x^\mu\, x^\nu)^2\, d\sigma = \tfrac{3}{(m+1)(m+3)}\, \mathrm{Vol}(S^m),$$

if $\mu \neq \nu$.

Using these, we get

$$\begin{aligned} \int_{S^m} \mathrm{Ric}_p(\Theta,\Theta)\, d\sigma &= \sum_{\mu,\nu} \int_{S^m} \mathrm{Ric}_p(E_\mu, E_\nu)\, x^\mu\, x^\nu\, d\sigma \\ &= \sum_{\mu} \int_{S^m} \mathrm{Ric}_p(E_\mu, E_\mu)\, (x^\mu)^2\, d\sigma \\ &= \tfrac{1}{m+1}\, \mathrm{Vol}(S^m)\, \mathbf{s}(p). \end{aligned} \tag{4.16}$$

Similarly, we have

$$\begin{aligned} &\int_{S^m} (\mathrm{Ric}_p(\Theta,\Theta))^2\, d\sigma = \\ &\sum_{\mu,\nu,\xi,\eta} \int_{S^m} \mathrm{Ric}_p(E_\mu, E_\nu)\, \mathrm{Ric}_p(E_\xi, E_\eta)\, x^\mu\, x^\nu\, x^\xi\, x^\eta\, d\sigma \\ &= \sum_{\mu} \int_{S^m} (\mathrm{Ric}_p(E_\mu, E_\mu))^2\, (x^\mu)^4\, d\sigma \\ &\quad + \sum_{\mu\neq\nu} \int_{S^m} \mathrm{Ric}_p(E_\mu, E_\mu)\, \mathrm{Ric}_p(E_\nu, E_\nu)\, (x^\mu)^2\, (x^\nu)^2\, d\sigma \\ &\quad +2 \sum_{\mu\neq\nu} \int_{S^m} (\mathrm{Ric}_p(E_\mu, E_\nu))^2\, (x^\mu)^2\, (x^\nu)^2\, d\sigma \\ &= \mathrm{Vol}(S^m)\, \tfrac{3}{(m+1)(m+3)}\, \sum_{\mu} (\mathrm{Ric}_p(E_\mu, E_\mu))^2 \\ &\quad + \mathrm{Vol}(S^m)\, \tfrac{1}{(m+1)(m+3)}\, \sum_{\mu\neq\nu} \mathrm{Ric}_p(E_\mu, E_\mu)\, \mathrm{Ric}_p(E_\nu, E_\nu) \\ &\quad + \mathrm{Vol}(S^m)\, \tfrac{2}{(m+1)(m+3)}\, \sum_{\mu\neq\nu} (\mathrm{Ric}_p(E_\mu, E_\nu))^2\, d\sigma \\ &= \tfrac{1}{(m+1)(m+3)}\, \mathrm{Vol}(S^m)\, \Bigg(2 \sum_{\mu,\nu} (\mathrm{Ric}_p(E_\mu, E_\nu))^2 \\ &\qquad + \textstyle\sum_{\mu,\nu} \mathrm{Ric}_p(E_\mu, E_\mu)\, \mathrm{Ric}_p(E_\nu, E_\nu) \Bigg) \\ &\quad = \tfrac{1}{(m+1)(m+3)}\, \mathrm{Vol}(S^m)\, \left(2\, ||\mathrm{Ric}_p||^2 + \mathbf{s}^2(p) \right). \end{aligned} \tag{4.17}$$

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