

## CONSTANT MEAN CURVATURE SURFACES IN $M^2 \times \mathbf{R}$

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ABSTRACT. The subject of this paper is properly embedded  $H$ -surfaces in Riemannian three manifolds of the form  $M^2 \times \mathbf{R}$ , where  $M^2$  is a complete Riemannian surface. When  $M^2 = \mathbf{R}^2$ , we are in the classical domain of  $H$ -surfaces in  $\mathbf{R}^3$ . In general, we will make some assumptions about  $M^2$  in order to prove stronger results, or to show the effects of curvature bounds in  $M^2$  on the behavior of  $H$ -surfaces in  $M^2 \times \mathbf{R}$ .

### 1. INTRODUCTION

There is an enormous difference between the theory of minimal surfaces ( $H = 0$ ) and nonzero constant-mean curvatures surfaces ( $H \neq 0$ ), and this is most evident in  $\mathbf{R}^3$ . A properly embedded minimal surface in  $\mathbf{R}^3$  cannot be compact; if it is simply connected, it must be the plane or the helicoid; and if it is an annulus, it must be the catenoid [15, 3]. For  $\Sigma$ , a properly embedded surface of nonzero constant mean curvature in  $\mathbf{R}^3$ : if  $\Sigma$  is compact, it must be a round sphere; if  $\Sigma$  is noncompact, it cannot be simply connected; and, if  $\Sigma \sim S^1 \times \mathbf{R}$ , it must be rotationally symmetric (a Delaunay surface). For both  $H = 0$  and  $H \neq 0$ , there are many known examples of finite topology with genus greater than zero and more than two ends, all discovered in the last twenty years [6, 7, 27, 26, 28].

The theory of minimal surfaces in  $M^2 \times \mathbf{R}$  is now well developed from the point of view of examples and theory [17, 21, 16]. However, the theory of  $H$ -surfaces in  $M^2 \times \mathbf{R}$  is just beginning. (We will assume throughout, unless we say so explicitly, that  $H \neq 0$ .) With one exception, there are no general theorems in the literature, but there is useful information in some special cases. The exception is when  $M^2 = T^2$  with the flat metric. Here we are considering doubly periodic  $H$ -surfaces in  $\mathbf{R}^3$ , and there are existence results that come from the theory of triply periodic  $H$ -surfaces in  $\mathbf{R}^3$  [9, 10, 5]. There are many known examples. Additional examples come from the existence of doubly periodic  $H$ -surfaces in  $\mathbf{R}^3$  [12].

When  $M^2 = S^2$ , there is a class of examples due to Hsiang and Hsiang, [8], and Pedrosa and Ritoré, [20], who studied the isoperimetric problem in  $S^n \times \mathbf{R}$ . Among

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other things, they analyzed the rotationally-invariant  $H$ -surfaces in  $S^n \times \mathbf{R}$ . Let

$$B = \frac{S^n \times \mathbf{R}}{O(n)} = \{(x, y) \mid x \in [0, \pi], y \in \mathbf{R}\},$$

and consider the generating curves  $\gamma(s) = (x(s), y(s))$  satisfying

$$\gamma'(s) = (\cos \sigma, \sin \sigma),$$

where  $\sigma = \sigma(s)$  satisfies

$$\frac{d\sigma}{ds} = H + (n-1) \cot(x) \sin(\sigma).$$

Their  $O(n)$ -orbits are  $H$ -surfaces. Pedrosa and Ritoré found first integrals and solutions of Delaunay-type, i.e. properly embedded and periodic  $H$ -hypersurfaces that were annular (topologically  $S^{n-1} \times \mathbf{R}$ ). They also found closed profile curves that give rise to compact tori (topologically  $S^1 \times S^{n-1}$ ) that are unstable, and bounded profile curves that define examples that are hyperspheres.

A motivating question for this paper is the following: *Are there properly embedded annular  $H$ -surfaces in  $S^2 \times \mathbf{R}$  that are not of Delaunay type (i.e., rotational)?* Korevaar, Kusner and Solomon proved that, in  $\mathbf{R}^3$ , all such surfaces are Delaunay [11].

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## 2. HEIGHT ESTIMATES IN $M^2 \times \mathbf{R}$

For an interval,  $I \subset \mathbf{R}$ , we define  $M_I := M^2 \times I \subset M^2 \times \mathbf{R}$ . When  $I = \{a\}$ , we will write  $M_a^2 = M^2 \times \{a\}$ .

We prove height estimates for compact embedded  $H$ -surfaces  $\Sigma$  with boundary in some  $M_a^2$ . The estimates depend on curvature bounds for  $M^2$  and on the value of  $H$ . We also prove that when  $M^2$  is compact and  $\Sigma$  is a noncompact and embedded  $H$ -surface, that  $\Sigma$  has both a top and a bottom end.

**Proposition 1.** *Suppose  $\Sigma$  is a compact and embedded  $H$ -surface in  $N = M_{[a, \infty)}$  with boundary in  $M_a^2$ . If the Gauss curvature of  $M^2$  satisfies  $K_M \geq 2\tau$  ( $\tau \leq 0$ ) and  $H^2 > |\tau|$ , then*

$$(1) \quad \Sigma \subset M_{[a, a+2c]},$$

where  $c = \frac{H}{H^2 - |\tau|}$ . In particular, if the sectional curvature of  $M^2$  is nonnegative and  $H \neq 0$ , then

$$(2) \quad \Sigma \subset M_{[a, a + \frac{2}{H}]}$$

Our proof of Proposition 1 (and of Proposition 2 below) uses a height estimate for  $H$ -surfaces that are graphs in  $M^2 \times \mathbf{R}$  with zero boundary values over compact domains in  $M_0^2$ . Let  $h : \Sigma \rightarrow \mathbf{R}$  be the restriction of the projection  $t : M^2 \times \mathbf{R} \rightarrow \mathbf{R}$ , to  $\Sigma$ . We refer to  $h$  as the height function.

**Lemma 1.** *Suppose that the Gauss curvature of  $M^2$  satisfies  $K_M \geq 2\tau$  ( $\tau \leq 0$ ). Let  $W$  be an  $H$ -surface with  $H^2 \geq |\tau|$ , which is a graph over a compact region in  $M_0^2$ , with zero boundary values. Then*

$$(3) \quad h \leq \frac{H}{H^2 - |\tau|}$$

on  $W$ . In particular, if the Gauss curvature of  $M^2$  is nonnegative and  $H \neq 0$ , then

$$(4) \quad h \leq \frac{1}{H}.$$

We will provide the proof of Lemma 1 at the end of the section.

*Proof of Proposition 1.* Observe that translation by  $t_0$  (i.e.  $M_t^2 \rightarrow M_{t+t_0}^2$ ) is an isometry of  $N = M^2 \times \mathbf{R}$ . Therefore, without loss of generality, we may assume that there exists  $b > 0$  for which  $\Sigma \subset M_{[0,b]}$ , with  $\partial\Sigma \subset M_0^2$  and  $\Sigma \cap M_b^2 \neq \emptyset$  and  $\Sigma \cap M_s^2 = \emptyset$  for  $s > b$ . To prove (1), we must show that  $\frac{b}{2} < c$ .

For  $|b-t|$  small,  $\Sigma \cap M_{[t,b]}$  is a vertical graph over a compact domain in  $M_0^2$ . By the Alexandrov reflection technique, the part of  $\Sigma$  in  $M_{[\frac{b}{2},b]}$  is also a graph over a compact domain in  $M_0^2$ . To see this, first observe that since reflection in  $M_t$  cannot be a symmetry of  $\Sigma$  for  $t > \frac{b}{2}$ , one can do Alexandrov reflection through all the  $M_\tau^2$ , as  $\tau$  decreases from  $b$  to  $\frac{b}{2}$ , and no accident occurs. That is, the part of  $\Sigma$  above each  $M_\tau^2$  is a vertical graph for  $\frac{b}{2} \leq \tau \leq b$ . By Lemma 1,  $\frac{b}{2} < c$ , which gives (1). The second statement, (2), follows immediately from the first.  $\square$

We will say that a surface in  $M^2 \times \mathbf{R}$  lies in a halfspace if it is contained in a region of  $M^2 \times \mathbf{R}$  of the form

$$M_{[a,\infty)} = \{(p,t) \in M^2 \times \mathbf{R} \mid t \geq a\} \quad \text{or} \quad M_{(-\infty,a]} = \{(p,t) \in M^2 \times \mathbf{R} \mid t \leq a\}.$$

**Proposition 2.** Suppose  $M^2$  is a compact surface without boundary whose Gauss curvature is bounded below by  $2\tau$ , for some real number  $\tau$ , and suppose  $\Sigma$  is a noncompact properly embedded  $H$  surface in  $N = M^2 \times \mathbf{R}$ . If  $\tau < 0$ , assume that  $H^2 \geq |\tau|$ . Then  $\Sigma$  cannot lie in a halfspace. In particular,  $\Sigma$  must have at least one “top” and one “bottom” end.

*Proof.* We first prove that  $H \neq 0$  if  $\Sigma$  lies in a halfspace. This is because we can find an  $M_t^2$  tangent to  $\Sigma$  at a point, with  $M^2$  lying on one side of  $\Sigma$ . By the maximum principle we would have  $M_t^2 = \Sigma$ , a contradiction since  $M^2$  is compact and  $\Sigma$  is not. This means that we satisfy the conditions of Lemma 1 even when  $M^2$  has nonnegative curvature.

Observe that translation by  $t_0$  (i.e.,  $M_t^2 \rightarrow M_{t+t_0}^2$ ) and reflection  $R_{t_0}$  in  $M_{t_0}^2$  (i.e.,  $M_t^2 \rightarrow M_{2t_0-t}^2$ ) are isometries of  $N = M^2 \times \mathbf{R}$ . Therefore, without loss of generality, we may assume that  $\Sigma$  lies in the halfspace  $M_{[0,\infty)}$  and has a nonempty intersection with  $M_0$ .

Since  $\Sigma_a := \{(x,t) \in \Sigma \mid t \leq a\}$  is compact, we may do Alexandrov-reflection coming up from  $M_0^2$ . For any value of  $a$ ,  $R_a(\Sigma_a)$  is compact. Since  $\Sigma$  is not compact,  $R_a$  is not a symmetry of  $\Sigma$ . This implies that for every  $a > 0$ , the compact surface  $\Sigma_a$  must be a vertical graph over a domain in  $M_a$ . However, the hypotheses of this proposition on the curvature of  $M^2$  and the lower bounds for  $H$  allow us to use Lemma 1 for such a vertical graph, providing a height estimate that is violated for  $a > 0$  sufficiently large. This contradiction establishes the first statement of the proposition. The second statement is an immediate consequence of the first.  $\square$

*Proof of Lemma 1.* We begin with some necessary general observations. For any immersion  $X : \Sigma \rightarrow N$  of one Riemannian manifold into another, the Laplacian of

$X$  is the divergence of the vector-valued one form  $DX$

$$\Delta X := \sum_{i=1}^m \left[ \nabla_{DX(e_i)} DX(e_i) - DX(\hat{\nabla}_{e_i} e_i) \right],$$

where  $\nabla$  and  $\hat{\nabla}$  are the Riemannian connections on  $N$  and  $\Sigma$ , respectively, and the  $\{e_i\}$  are an orthonormal framing of  $T\Sigma$ . If  $X$  is an isometric immersion, then  $DX(\hat{\nabla}_{e_i} e_i) = [\nabla_{DX(e_i)} DX(e_i)]^t$ , where  $[\ ]^t$  is the projection of  $TN|_{X(\Sigma)}$  onto  $X_*(T\Sigma)$ . Then

$$\Delta X = \sum_{i=1}^m [\nabla_{DX(e_i)} DX(e_i)]^n = \sum_{i=1}^m B(e_i, e_i),$$

where  $[\ ]^n$  is projection onto the normal bundle of  $X(\Sigma) \subset N$  and  $B$  is the second fundamental form of  $X$ . [13] Therefore, the Laplacian of  $X$  is the trace of the second fundamental form of  $\Sigma$ ,

$$\Delta X = m\vec{H},$$

where  $m$  is the dimension of  $\Sigma$  and  $\vec{H}$  is the mean curvature vector field of the isometric immersion  $X$ .

In our context,  $N = M^2 \times \mathbf{R}$ . Let  $\vec{n}$  denote the unit normal to  $\Sigma$  and define

$$H := \langle \vec{H}, \vec{n} \rangle \text{ and } n := \langle \vec{n}, \frac{\partial}{\partial t} \rangle.$$

We have

$$\Delta X = 2H\vec{n}.$$

A simple calculation using either the fact that the height function,  $h$ , is the projection onto a one-dimensional subspace in a Riemannian splitting, or that  $\frac{\partial}{\partial t}$  is a Killing field, gives

$$(5) \quad \Delta h = 2Hn.$$

We will also have need of an equation for the Laplacian of  $n = \langle \vec{n}, \frac{\partial}{\partial t} \rangle$ :

$$(6) \quad \Delta n = -(|A|^2 + Ric(\vec{n}))n,$$

where  $A$  is the shape operator of  $\Sigma$ . (Recall that  $A : T\Sigma \rightarrow T\Sigma$  with  $A(U) = \nabla_U \vec{n}$ .) Formula (6) can be derived as in [22] or [18] by looking at the second variation of area of an  $H$ -surface in the direction of a normal component of a Killing field (in this case  $\frac{\partial}{\partial t}$ ). Alternatively, one can derive this equation, directly using the fact that  $\frac{\partial}{\partial t}$  is a Killing field. From general considerations,

$$\Delta n = \Delta \langle \vec{n}, \frac{\partial}{\partial t} \rangle = \langle \Delta \vec{n}, \frac{\partial}{\partial t} \rangle + 2 \langle \nabla \vec{n}, \nabla \frac{\partial}{\partial t} \rangle + \langle \vec{n}, \Delta \frac{\partial}{\partial t} \rangle.$$

Because  $\frac{\partial}{\partial t}$  is Killing,  $\langle \nabla_U \frac{\partial}{\partial t}, V \rangle$  is skew symmetric, which is enough to show that the middle term on the right-hand side of the above expression vanishes. It also allows expression of the last summand in terms of  $Ric(\vec{n}, \frac{\partial}{\partial t})$ . Simplification leads to (6).

Now we restrict our attention to an  $H$ -surface  $W$  that satisfies the assumptions of the lemma. Define

$$(7) \quad \phi = ch + n,$$

where  $c$  is a real-valued positive constant. Since  $W$  is a graph we may choose a “downward-pointing” unit normal vector field  $\vec{n}$  so that

$$n := \langle \vec{n}, \frac{\partial}{\partial t} \rangle \leq 0$$

on  $W$ . Since  $W$  is assumed to be a graph with zero boundary values, it follows that

$$\phi = n \leq 0$$

on  $\partial W$ . If we can find a value of  $c$  for which  $\Delta\phi \geq 0$  on  $W$ , it will follow from the maximum principle that  $\phi = ch + n \leq 0$  on  $W$ ; hence

$$(8) \quad h \leq -n/c \leq 1/c.$$

We will now find such a value of  $c$ . From (5) and (6), we get

$$\Delta\phi = (2cH - |A|^2 - Ric(\vec{n}))n.$$

Since  $n \leq 0$ , we can assert that  $\Delta\phi \geq 0$  if and only if

$$(9) \quad |A|^2 + Ric(\vec{n}) \geq 2cH.$$

Using the simple estimate

$$\begin{aligned} |A|^2 &= tr^2(A) - 2det(A) \\ &= 2\left(\frac{tr(A)}{2}\right)^2 + \left(\frac{1}{2}tr^2(A) - 2det(A)\right) \\ &= 2H^2 + \frac{1}{2}(a_{11} - a_{22})^2 + 2a_{12}^2 \\ &\geq 2H^2, \end{aligned}$$

it suffices to choose  $c$  so that

$$H^2 + \frac{1}{2}Ric(\vec{n}) \geq cH.$$

Since  $Ric(\vec{n}) = |Pr(\vec{n})|^2 K_M$ , where  $Pr(\cdot)$  is the projection from  $TN$  onto  $TM^2$ , and  $|n| = 1$ , it follows from our assumption that  $K_M \geq 2\tau$ ,  $\tau \leq 0$ , and that  $Ric(\vec{n}) \geq 2\tau$ . Therefore we can satisfy the condition on  $c$  above with

$$(10) \quad \frac{H^2 - |\tau|}{H} \geq c.$$

The left-hand side of (10) is positive by assumption and so choosing  $c$  to equal the left-hand side is sufficient to force  $\Delta\phi \geq 0$ . The height estimate (3) follows from (8).

If  $M^2$  has nonnegative sectional curvature, then  $Ric(N) \geq 0$ , the requirement (10) holds for  $c = H$ , and (8) gives the height estimate (4).  $\square$

*Remark 1.* The height estimate in Lemma 1 in the case that  $M^2 \times \mathbf{R} = H^2 \times \mathbf{R}$  works for  $H > \frac{1}{\sqrt{2}}$ . However the rotational  $H$ -surfaces in  $H^2 \times \mathbf{R}$  are compact for  $H > 1/2$ , and noncompact for  $H \leq \frac{1}{2}$ , [17]. So one suspects our height estimates are not sharp.

## 3. FIRST VARIATIONS OF AREA AND VOLUME

Let  $U$  be bounded domain in a Riemannian three manifold  $N$ , whose boundary,  $\partial U$ , consists of a smooth connected surface,  $\Sigma$ , and the union  $Q$  of finitely many smooth, compact and connected surfaces. The closed surface  $\partial U$  is piecewise-smooth and smooth except perhaps on  $\partial\Sigma = \partial Q$ . Let

$$\begin{aligned}\vec{n} &= \text{the outward-pointing unit normal vector field on } \partial U = \Sigma \cup Q; \\ \vec{n}_\Sigma &= \text{the restriction of } \vec{n} \text{ to } \Sigma; \\ \vec{n}_Q &= \text{the restriction of } \vec{n} \text{ to } Q.\end{aligned}$$

Suppose  $Y$  is a vector field defined on a region of  $N$  that contains  $U$ . The first-variation of the volume,  $|U|$ , of  $U$  is given by

$$\begin{aligned}(11) \quad \delta_Y|U| &= \int_U \text{Div} Y = \int_{\partial U} Y \cdot \vec{n} \\ &= \int_\Sigma Y \cdot \vec{n}_\Sigma + \int_Q Y \cdot \vec{n}_Q,\end{aligned}$$

where  $\text{Div} = \text{Div}_N$  is the divergence operator on  $N$ . To write down  $\delta_Y(|\Sigma|)$ , the first variation of area of  $\Sigma$  under  $Y$ , we introduce the following notation:

$$\begin{aligned}\nu &= \text{the outward-pointing unit conormal to } \Sigma \text{ along } \partial\Sigma; \\ Y^t &= \text{the tangential projection of } Y \text{ onto } T\Sigma; \\ Y^n &= Y - Y^t, \text{ the projection of } Y \text{ onto the normal bundle of } \Sigma \text{ in } N.\end{aligned}$$

We may then write

$$\begin{aligned}\delta_Y(|\Sigma|) &= \int_\Sigma \text{div} Y \\ &= \int_\Sigma \text{div} Y^t + \int_\Sigma \text{div} Y^n \\ &= \int_{\partial\Sigma} Y \cdot \nu - \int_\Sigma Y \cdot \vec{H}_\Sigma,\end{aligned}$$

where  $\text{div} = \text{div}_\Sigma$ , is the divergence operator on  $\Sigma$ .

If  $\Sigma$  is an  $H$ -surface,  $H := \langle \vec{H}, \vec{n} \rangle$  is a constant and therefore

$$(12) \quad \delta_Y(|\Sigma|) = \int_{\partial\Sigma} Y \cdot \nu - H \int_\Sigma Y \cdot \vec{n}.$$

Putting (11) and (12) together, we have for  $H$ -surfaces,

$$(13) \quad \delta_Y(|\Sigma| + H|U|) = \int_{\partial\Sigma} Y \cdot \nu + H \int_Q Y \cdot \vec{n}_Q.$$

The following proposition is well known and is immediate from (13).

**Proposition 3.** *If  $Y$  is a Killing vector field on  $N$ , and  $\Sigma$  is an  $H$ -surface, then*

$$(14) \quad \int_{\partial\Sigma} Y \cdot \nu + H \int_Q Y \cdot \vec{n}_Q = 0,$$

where  $Q$ ,  $\nu$ ,  $H$  and  $\vec{n}_Q$  are as defined above.

Suppose  $\Sigma$  is a properly embedded  $H$ -surface in  $N = M^2 \times \mathbf{R}/$ , and assume that  $\Sigma$  bounds a region  $U \subset N$ . Define:

$$\begin{aligned}
 (15) \quad M_t &= M^2 \times \{t\}; \\
 \Sigma_t &= \Sigma \cap M_t; \\
 \Sigma_{[a,b]} &= \Sigma \cap (M^2 \times [a,b]) = \bigcup_{a \leq s \leq b} \Sigma_s; \\
 (16) \quad U_a^b &= U \cap (M^2 \times [a,b]); \\
 Q_t &= U \cap M_t
 \end{aligned}$$

Clearly,

$$\partial U_a^b = \Sigma_{[a,b]} \cup Q_a \cup Q_b, \quad \partial \Sigma = \Sigma_a \cup \Sigma_b, \quad \vec{n}_{Q_b} = +\frac{\partial}{\partial t}, \quad \vec{n}_{Q_a} = -\frac{\partial}{\partial t}.$$

We now apply Proposition 3 to the vector field  $\frac{\partial}{\partial t}$ . Since vertical translation  $M_t \rightarrow M_{t+t_0}$  is an isometry,  $\frac{\partial}{\partial t}$  is a Killing field and we have from (14)

$$\begin{aligned}
 (17) \quad 0 &= \int_{\Sigma_0 \cup \Sigma_t} \frac{\partial}{\partial t} \cdot \nu + H \int_Q \frac{\partial}{\partial t} \cdot \vec{n}_Q \\
 &= \int_{\Sigma_0} \frac{\partial}{\partial t} \cdot \nu + \int_{\Sigma_t} \frac{\partial}{\partial t} \cdot \nu + H(|Q_t| - |Q_0|).
 \end{aligned}$$

We will state an important consequence of this computation as a proposition for later use.

**Proposition 4.** *Suppose  $\Sigma$  is a properly embedded  $H$ -surface in  $N = M^2 \times \mathbf{R}$ ,  $M^2$  compact. Then the vertical flux across  $\Sigma_t$ ,*

$$\int_{\Sigma_t} \frac{\partial}{\partial t} \cdot \nu,$$

*varies within a bounded range*

The proposition follows immediately from (17) and the assumption that  $M^2$  is compact: since  $Q_t \subset M_t$ , the volume  $|Q_t|$  is bounded independent of  $t$ .

*Remark 2.* If  $\Sigma$  is a minimal surface ( $H = 0$ ), then it follows from (17) that the  $\int_{\Sigma_t} \frac{\partial}{\partial t} \cdot \nu$ , the vertical flux across  $\Sigma_t$ , is a constant, independent of  $t$ .

#### 4. LINEAR AREA GROWTH

In this section, we prove that a properly embedded  $H$ -surface in  $N = S^2 \times \mathbf{R}$  will have linear area growth provided it has one additional property. For any choice of antipodal points on  $S^2$ , consider the rotations of  $S^2$  that fix these two points. These extend to rotations of  $N$  whose common fixed point set consists of the two vertical geodesics through these points. Choose a geodesic arc  $\gamma$  joining these antipodal points and let  $\mathcal{P}$  be the union of  $\gamma \times \mathbf{R}$  and any tubular neighborhoods of the two vertical geodesics.

**Theorem 1.** *Let  $\Sigma$  be a properly embedded  $H$ -surface in  $N = S^2 \times \mathbf{R}$ . If  $\Sigma$  is disjoint from  $\mathcal{P}$ , then  $\Sigma$  has linear area growth.*

A generalization of this theorem is given in Section 4.1.

*Remark 3.* In the complement of  $\mathcal{P}$  we may define  $\theta$ , the angle measured from  $\gamma$ . The gradient of  $\theta$  is a Killing field whose length is bounded above by 1, and below by a positive constant that depends on the radius of the tubular neighborhoods chosen in the definition of  $\mathcal{P}$ .

*Proof.* Given  $a > 0$ , we will show that

$$(18) \quad |\Sigma_{[-a,a]}| \leq ca,$$

for some constant  $c$  that does not depend on  $a$ .

Note that  $(x, t) \rightarrow (x, -t)$  is an isometry of  $N$ , and therefore

$$|\Sigma_{[-a,a]}| = |-\Sigma_{[-a,a]}|,$$

where  $-\Sigma = \{(x, -t) \mid (x, t) \in \Sigma\}$ . Hence, by working with  $-\Sigma$  if necessary, we may assume, without loss of generality, that

$$|\Sigma_{[-a,0]}| \leq |\Sigma_{[0,a]}|.$$

Since  $(x, t) \rightarrow (x, t+a)$  is an isometry of  $N$ , the inequality (18) will follow if we establish

$$(19) \quad |\Sigma_{[0,2a]}| \leq ca, \text{ assuming } |\Sigma_{[0,a]}| \leq |\Sigma_{[a,2a]}|.$$

We will now prove (19).

On  $S^2 - \gamma$ , we may define polar coordinates,  $(r, \theta)$ , with  $\pi < \theta < 3\pi$ . The function  $\theta$  is well defined on  $N - \mathcal{P}$ , a region that contains  $\Sigma$  by assumption.

Let  $\frac{\partial}{\partial \theta}$  be the rotational Killing field for the axis defined by the end-points of  $\gamma$ , and let  $\frac{\partial}{\partial t}$  be the Killing field defined by vertical translation. Clearly,  $|\frac{\partial}{\partial t}| = 1$  and, as noted in Remark 3,  $|\frac{\partial}{\partial \theta}|$  is bounded above and also bounded away from zero. Define the vector fields

$$X = t \frac{\partial}{\partial t}, \quad Y = \theta \frac{\partial}{\partial \theta}, \quad Z = tY.$$

A direct calculation using

$$\operatorname{div} W = \operatorname{Div} W - \langle \nabla_n W, n \rangle, \quad \operatorname{Div} f W = \langle \nabla f, W \rangle + f \operatorname{Div} W,$$

and the fact that for a Killing field  $V$ ,  $\langle \nabla_e V, e \rangle = 0$  and therefore

$$\operatorname{Div} V = \operatorname{div} V = 0, \text{ and } \langle \nabla_n V, n \rangle = 0,$$

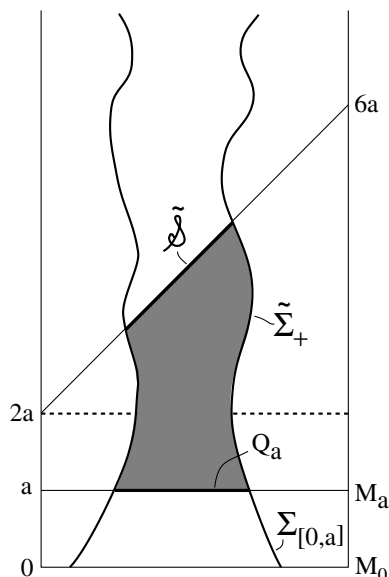
gives

$$(20) \quad \begin{aligned} \operatorname{Div} X &= 1, & \operatorname{div} X &= 1 - n_t^2, \\ \operatorname{Div} Y &= |\frac{\partial}{\partial \theta}|^2, & \operatorname{div} Y &= |\frac{\partial}{\partial \theta}|^2 - n_\theta^2, \\ \operatorname{Div} Z &= t |\frac{\partial}{\partial \theta}|^2, & \operatorname{div} Z &= t(|\frac{\partial}{\partial \theta}|^2 - n_\theta^2) - \theta n_t n_\theta, \end{aligned}$$

where  $n_t = \vec{n}_\Sigma \cdot \frac{\partial}{\partial t}$  and  $n_\theta = \vec{n}_\Sigma \cdot \frac{\partial}{\partial \theta}$ . Using  $X$  as the vector field in the first variation formula (13) on  $\Sigma_{[0,2a]}$ , we have

$$\begin{aligned} \int_{\Sigma_{[0,2a]}} (1 - n_t^2) + H|U_0^{2a}| &= \int_{\Sigma_0 \cup \Sigma_{2a}} t \frac{\partial}{\partial t} \cdot \nu + H \int_{Q_0 \cup Q_{2a}} t \frac{\partial}{\partial t} \cdot \vec{n}_Q \\ &= 2a \left[ \int_{\Sigma_{2a}} \frac{\partial}{\partial t} \cdot \nu + H \int_{Q_{2a}} \frac{\partial}{\partial t} \cdot \vec{n}_Q \right] \\ &= 2a \left[ \int_{\Sigma_{2a}} \frac{\partial}{\partial t} \cdot \nu + H|Q_{2a}| \right]. \end{aligned}$$




 FIGURE 1.  $\tilde{\Sigma}$ .

By Proposition 3, equation (14), the bracketed term on the right-hand side is a constant. Moreover, since  $S^2$  has finite area,  $|U_0^{2a}|$  grows at most linearly in  $a$ . Hence

$$(21) \quad \int_{\Sigma_{[0,2a]}} 1 - n_t^2 \leq c_1 a,$$

where  $c_1$  does not depend on  $a$ .

We now use (13) with the variation vector field  $Z$ . This time we will cut off  $\Sigma$  below by  $S_0^2 = S^2 \times \{0\}$ , and above by a surface,  $\mathcal{S}$ , defined by

$$t\theta = 6\pi a.$$

Note that when  $t = 2a$ , we have  $\theta = 3\pi$ , and when  $t = 6a$ , we have  $\theta = \pi$ . (See Figure 1.) We will denote the parts of  $\Sigma$  and  $U$  between  $S_0^2$  and  $\mathcal{S}$  by  $\tilde{\Sigma}$  and  $\tilde{U}$ , respectively, and define  $\tilde{\mathcal{S}} = \mathcal{S} \cap U$  and  $\Sigma_{\mathcal{S}} = \Sigma \cap \mathcal{S}$ . (See Figure 1, again.) We get from (13)

$$\begin{aligned} \int_{\tilde{\Sigma}} \left( t \left( \left| \frac{\partial}{\partial \theta} \right|^2 - n_\theta^2 \right) - \theta n_t n_\theta \right) - H \int_{\tilde{U}} \left| \frac{\partial}{\partial \theta} \right|^2 t \\ = \int_{\Sigma_0 \cup \Sigma_{\mathcal{S}}} Z \cdot \nu + H \int_{Q_0 \cup \tilde{\mathcal{S}}} Z \cdot \vec{n}_Q = 6\pi a \left[ \int_{\Sigma_{\mathcal{S}}} \frac{\partial}{\partial \theta} \cdot \nu + H \int_{\tilde{\mathcal{S}}} \frac{\partial}{\partial \theta} \cdot \vec{n}_Q \right]. \end{aligned}$$

Since  $\frac{\partial}{\partial \theta}$  is a Killing field, it follows from Proposition 3, equation (14), that the bracketed term on the right is a constant that does not depend on  $a$ . Hence

$$\int_{\tilde{\Sigma}} t \left( \left| \frac{\partial}{\partial \theta} \right|^2 - n_\theta^2 \right) = c_2 a + H \int_{\tilde{U}} t \left| \frac{\partial}{\partial \theta} \right|^2 + \int_{\tilde{\Sigma}} \theta n_t n_\theta.$$

Now observe that  $\pi < \theta < 3\pi$  on  $\tilde{\Sigma}$ ; the vector fields  $\vec{n}$  and  $\frac{\partial}{\partial t}$  both have length 1. Moreover, the Killing field  $\frac{\partial}{\partial \theta}$  has length bounded above and is also bounded away

from zero on  $\tilde{\Sigma}$ :

$$(22) \quad 0 < c_* \leq \left| \frac{\partial}{\partial \theta} \right| \leq c^*.$$

Therefore

$$|\theta n_t n_\theta| \leq 3\pi n_t = 3\pi \left\langle n, \frac{\partial}{\partial \theta} \right\rangle \leq 3\pi c^* := c_4.$$

Also note that  $0 \leq t \leq 6a$  on  $\tilde{U}$ . Using these estimates we have

$$(23) \quad \begin{aligned} \int_{\tilde{\Sigma}} t \left( \left| \frac{\partial}{\partial \theta} \right|^2 - n_\theta^2 \right) &\leq c_2 a + c_3 a^2 + \int_{\tilde{\Sigma}} \theta n_t n_\theta \\ &\leq c_2 a + c_3 a^2 + c_4 |\tilde{\Sigma}|. \end{aligned}$$

Here we have used the fact that  $S^2$  has finite area and therefore the volume of  $|\tilde{U}|$  is linear in  $a$ .

Let  $\tilde{\Sigma}_+ = \tilde{\Sigma} \cap M_{[a, 6a]}$ . We are assuming that  $|\Sigma_{[0, a]}| \leq |\Sigma_{[a, 2a]}|$ , and since  $\Sigma_{[a, 2a]} \subset \tilde{\Sigma}_+$ ,

$$(24) \quad |\tilde{\Sigma}| = |\Sigma_{[0, a]}| + |\tilde{\Sigma}_+| \leq 2|\tilde{\Sigma}_+|.$$

From (24), (23) and (22), we get

$$\begin{aligned} ac_*^2 \int_{\tilde{\Sigma}_+} \left( 1 - \left( \frac{n_\theta}{\left| \frac{\partial}{\partial \theta} \right|} \right)^2 \right) &\leq \int_{\tilde{\Sigma}_+} t \left| \frac{\partial}{\partial \theta} \right|^2 (1 - n_\theta^2) \\ &\leq \int_{\tilde{\Sigma}} t \left| \frac{\partial}{\partial \theta} \right|^2 (1 - n_\theta^2) \leq c_2 a + c_3 a^2 + 2c_4 |\tilde{\Sigma}_+|. \end{aligned}$$

Dividing by  $ac_*^2$  gives

$$\int_{\tilde{\Sigma}_+} \left( 1 - \left( \frac{n_\theta}{\left| \frac{\partial}{\partial \theta} \right|} \right)^2 \right) \leq \frac{1}{c_*^2} (c_2 + c_3 a + \frac{2c_4}{a} |\tilde{\Sigma}_+|).$$

In estimate (21), we saw that  $\int_{\Sigma_{[0, 2\hat{a}]}} 1 - n_t^2 \leq c_1 \hat{a}$ , where  $c_1$  does not depend on  $\hat{a}$ . Using  $\hat{a} = 3a$  and observing that  $\tilde{\Sigma}_+ \subset \Sigma_{[0, 6a]}$ , we get (using the fact that  $n_t^2 + \left( \frac{n_\theta}{\left| \frac{\partial}{\partial \theta} \right|} \right)^2 \leq 1$ )

$$|\tilde{\Sigma}_+| \leq \int_{\tilde{\Sigma}_+} \left( 2 - (n_t^2 + \left( \frac{n_\theta}{\left| \frac{\partial}{\partial \theta} \right|} \right)^2) \right) \leq \left( \frac{1}{c_*^2} (c_2 + c_3 a) + 3c_1 a \right) + \frac{2c_4}{ac_*^2} |\tilde{\Sigma}_+| = c_5 + c_6 a + \frac{2c_4}{ac_*^2} |\tilde{\Sigma}_+|.$$

If  $a > \frac{4c_4}{c_*^2}$ , then

$$|\tilde{\Sigma}_+| \leq 2(c_5 + c_6 a).$$

This estimate together with (24) produces

$$|\tilde{\Sigma}| \leq 4(c_5 + c_6 a),$$

from which (19) follows because  $\Sigma_{[0, 2a]} \subset \tilde{\Sigma}$ . This completes the proof of the theorem.  $\square$

**4.1. Linear area growth in  $M^2 \times \mathbf{R}$  when  $M^2$  has a pole.** The proof of Theorem 1 will give linear area growth in more general situations. Suppose  $M^2$  has a pole,  $\mathbf{p}$ , a fixed point for a one-parameter group of rotational isometries. Assume there is a region  $\mathcal{R} \subset N$ , inside of which the angle of rotation is well defined and the gradient of that angle (a Killing field) has length bounded above and also away from zero. Let  $\Sigma \subset \mathcal{R}$  be a properly embedded  $H$ -surface, and assume that  $\Sigma$  bounds a region  $U \subset \mathcal{R}$  with the property that the area of  $U_t = U \cap M_t^2$  is uniformly bounded. Then the proof of Theorem 1 is directly generalizable to a proof of the linear area growth of  $\Sigma$ . In particular

**Theorem 2.** *Suppose  $M^2$  is a complete surface with a pole  $\mathbf{p}$ . The Killing field  $\frac{\partial}{\partial \theta}$  is defined in a punctured neighborhood, say  $D$ , of  $\mathbf{p}$ , and the angle  $\theta$  is defined in the complement in  $D$  of a geodesic, say  $\gamma$ , starting at  $\mathbf{p}$  and running to the cut locus  $C_{\mathbf{p}}$  of  $\mathbf{p}$ . Let  $\mathcal{R} = (D \setminus \gamma) \times \mathbf{R}$ , and suppose that  $\Sigma = \partial U$ ,  $U \subset \mathcal{R}$ , is a properly embedded  $H$ -surface. Suppose further that  $|U_t| = |U \cap M_t^2|$  is bounded independent of  $t$  and that the Killing field  $\frac{\partial}{\partial \theta}$  associated with the pole  $\mathbf{p}$  satisfies the boundedness condition (22). Then  $\Sigma$  has linear area growth.*

This theorem has an immediate corollary when  $M^2 = H^2$ .

**Corollary 1.** i) *Suppose  $\gamma \subset H^2$  is a geodesic ray beginning at  $\mathbf{p}$ . Let  $\Sigma = \partial U \subset \mathcal{R} = (H^2 \setminus \gamma) \times \mathbf{R}$  be a properly embedded  $H$ -surface with the property that  $|U_t|$  is bounded independent of  $t$ . Suppose further that the Killing field  $\frac{\partial}{\partial \theta}$  associated with the pole  $\mathbf{p}$  satisfies the boundedness condition of (22). Then  $\Sigma$  has linear area growth.*  
 ii) *Suppose  $\alpha : \mathbf{R} \rightarrow H^2$  is either a constant map or a constant-speed ( $= b$ ) parametrization of a complete geodesic. Let  $\beta(t) = (\alpha(t), ct)$ ,  $c \neq 0$ ,  $b^2 + c^2 = 1$ , be a parametrization of a complete unit-speed geodesic  $\beta \subset H^2 \times \mathbf{R}$ . Suppose  $\Sigma \subset H^2 \times \mathbf{R}$  is a properly embedded  $H$ -surface which is cylindrically bounded in the sense that it stays a bounded distance from  $\beta$ . Then  $\Sigma$  has linear area growth.*

*Proof.* The first statement follows directly from Theorem 2 and the fact that any point  $\mathbf{p} \in H^2$  is a pole whose cut locus is empty.

To prove the second statement, we work in the disk model of  $H^2$ . We first observe that without loss of generality, we may assume that either  $\alpha(t)$  is identically equal to zero, or is a constant-speed parametrization (in  $H^2$ ) of the real axis. Furthermore, we may assume that  $\Sigma$  is within distance one of  $\beta$ .

If  $\beta(t) = (0, ct)$ , choose  $\mathbf{p}$  to be a point on the imaginary axis at distance  $d > 3$  from the origin, and let  $\gamma$  be the geodesic ray beginning at  $\mathbf{p}$  and diverging along the imaginary axis. Let  $\mathcal{R} = B_2(0) \times \mathbf{R} \subset H^2 \times \mathbf{R}$ . All the conditions of statement (i) are now satisfied, so  $\Sigma$  has linear area growth.

If  $\alpha$  is the real axis and the cylindrical radius is one, we again choose  $\gamma$  as before. Let  $\mathcal{B} = B \times \mathbf{R}$ , where  $B$  is the geodesic disc in  $\mathbb{H}^2$  centered at  $\beta(0) = O$ , and of radius 3. Clearly  $\Sigma_{[0,c]} \subset \mathcal{B}$ . Our previous calculations in Theorem 1 show that the area of  $\Sigma_{[0,c]}$  is bounded by  $kc$ , where the constant  $k$  depends on the bounds for  $|\partial_\theta|$  in  $\mathcal{B}$ . Observe that the same estimate holds for the area of  $\Sigma_{[c,2c]}$ , since there is an isometry  $\phi$  of  $\mathbb{H}^2 \times \mathbf{R}$  to itself, leaving  $\beta$  invariant, taking  $\beta(1)$  to  $\beta(0)$ ;  $\phi^{-1}$  is the composition of the (hyperbolic) translation of  $\mathbb{H}^2$ , taking  $O$  to  $\alpha(1)$ , with the vertical translation taking  $\alpha(1)$  to  $\beta(1)$ . This map takes  $\beta'(0)$  to  $\beta'(1)$ , and hence

leaves  $\beta$  invariant. Then  $\phi(\Sigma_{[c, 2c]}) \subset \mathcal{B}$ , so its area is also at most  $k c$ . Continuing this way, we conclude  $\Sigma$  has linear (vertical) area growth.  $\square$

## 5. CURVATURE BOUNDS

We will be concerned in this section with properly embedded  $H$ -surfaces,  $\Sigma \subset N = M^2 \times R$ , of finite topology with the additional property that the restriction of the height function on  $M^2 \times \mathbf{R}$  to  $\Sigma$  is proper. If  $M^2$  is compact, every such  $\Sigma$  has this property. Note that an annular end of such an  $H$ -surface must be either a top end or a bottom end.

We will assume that  $M^2$  is complete, that  $M^2$  has bounded Gauss curvature, and that the injectivity radius of  $M^2$  is bounded away from zero. Again, all these properties hold if  $M^2$  is compact.

**Theorem 3.** *Let  $\Sigma$  and  $M^2$  satisfy the above conditions. If  $\Sigma$  has linear area growth, then  $|A|$  is bounded on  $\Sigma$ .*

**Corollary 2.** *Suppose  $\Sigma$  is a properly embedded  $H$ -surface with finite topology in  $S^2 \times \mathbf{R}$ , with the additional property that  $\Sigma \subset D \times R \subset S^2 \times \mathbf{R}$ , where  $D$  is a geodesic disk of radius strictly less than  $\pi$  (or, more generally satisfies the conditions of Theorem 1). Then  $|A|$  is bounded on  $\Sigma$ .*

The corollary follows immediately from Theorem 3, since  $S^2$  is compact and Theorem 1 gives linear area growth. Similar results for  $H$ -surfaces  $\Sigma \subset H^2 \times R$  on which the height function is proper follow from Theorem 3 and Corollary 1.

Theorem 3 will be proved after we present two preliminary results.

We begin by observing as we did in (10), that  $|A|^2 = \text{tr}^2(A) - 2\det(A) = 4H^2 - 2\det(A)$ —a local formula not requiring any of our hypotheses—and that the Gauss curvature  $K(p)$  of  $\Sigma$  at a point  $p \in \Sigma$  satisfies  $\det(A_p) = K(p) - K_N(p)$ , where  $K_N(p)$  is the sectional curvature of  $N$  on the plane  $T_p\Sigma$ . Therefore

$$(25) \quad |A|^2 = 4H^2 + 2K_N - 2K.$$

The geometric quantity  $4H^2 + 2K_N$  is bounded on  $\Sigma$  if  $M^2$  has bounded curvature. Therefore

**Lemma 2.** *Let  $\Sigma = \partial U$ ,  $U \subset N = M^2 \times \mathbf{R}$ , be a properly embedded  $H$ -surface. Assume  $\Sigma$  has linear area growth and that  $M^2$  has bounded curvature. Then*

$$\int_{\Sigma} 4H^2 + 2K_N$$

*has linear growth.*

**Proposition 5.** *Let  $\Sigma$  be an  $H$ -surface in  $M^2 \times \mathbf{R}$  satisfying the assumptions of Theorem 3. Then  $\int_{\Sigma} |A|^2$  has linear growth.*

*Remark 4.* For the proof of this proposition, we do not require that the injectivity radius of  $M^2$  be bounded below.

*Proof.* Since we are assuming that  $\Sigma$  has finite topology, we may decompose it into a compact piece and a finite number of annular ends. Let  $E$  be an annular end of  $\Sigma$ . We may assume, without loss of generality, that  $E$  is a top end, i.e.,  $h(x_n) \rightarrow \infty$ , as  $x_n \in E$  diverges in  $E$ , where  $h$  is the height function. Then if  $h|_{\partial E} < 0$  and  $t \geq 0$ ,  $E$  transverse to  $M_t^2$ , it is clear that  $E \cap M_t^2$  contains at least one Jordan curve  $\alpha$  that is a generator of  $\pi_1(E)$ . We call such an  $\alpha$  an essential curve. We can

assume  $E \pitchfork M_0^2$  and (after replacing  $E$  by a subend) that  $\partial E \subset M_0^2$ . Note that  $E$  may intersect  $M_0^2$  in other Jordan curves, but the part of  $E$  below height zero is compact and at bounded distance from  $M_0^2$ , by our assumption that  $h$ , restricted to  $\Sigma$ , is proper.

Let  $\phi : \mathbf{S}^1 \times \mathbf{R}^+ \rightarrow E$  be a parametrization of  $E$  with the property that

$$\phi(\mathbf{S}^1 \times \{0\}) = \partial E \subset M_0.$$

For  $a, b \in \mathbf{R}^+$  with  $b > a$ , we will assume that  $E$  is transverse to  $M_a^2$  and  $M_b^2$ , an assumption that is true generically. The set  $E_a = E \cap M_a^2$  consists of a finite number of loops, and the same is true of  $E_b$ .

The parametrization  $\phi$  orders the essential loops by proximity to  $\mathbf{S}^1 \times \{0\}$  and, in particular, it orders the essential loops in  $E_a$  and  $E_b$ . Let  $\alpha_1$  be the first essential loop in  $E_a$  and let  $\beta$  be the first essential loop in  $E_b$ . The annulus  $\hat{E}$  bounded by  $\alpha_1$  and  $\beta$  is not necessarily contained in  $E_{[a,b]}$ . However, by Proposition 1 in Section 2, there exists a constant  $C$  (that depends on  $H$  and the bounds on the curvature of  $M^2$ , but not on  $t$ ) such that regions of  $E$  bounded by inessential loops in  $M_t^2$  must lie at a distance of at most  $C$  from  $M_t^2$ . Hence we may assert that

$$\hat{E} \subset E_{[a-C, b+C]}.$$

Using Proposition 1 again, we can assert that if  $b - a > 2C$ , any essential loop that is past  $\beta \subset E_b$  in the ordering cannot lie in  $E_a$ . Let  $\beta_1$  be the last essential loop in  $E_b$ . The loops  $\alpha_1$  and  $\beta_1$  bound an annulus  $\tilde{E} \subset E$ , and we can use Proposition 1 again to assert that

$$(26) \quad E_{[a+C, b-C]} \subset \tilde{E} \subset E_{[a-C, b+C]}.$$

Since we are assuming that  $\Sigma$  has linear area growth, we know that  $E \subset \Sigma$  has linear area growth. This implies that there exists a constant  $c$  such that for all  $t \geq 0$ ,

$$|E_{[t, t+1]}| < c.$$

By the co-area formula we know that there exists, for every integer  $k \geq 0$ , an  $a_k$ ,  $k \leq a_k \leq k + 1$ , with the property that

$$|E_{a_k}| \leq c,$$

where  $|E_{a_k}|$  is the length of  $E_{a_k}$ . Choose  $k_0 > c + 2C$ , let  $k > k_0 + 1 + 2c + 2C$ , and consider  $\tilde{E}_{[a_0, a_k]}$ , the annulus bounded by the essential curves  $\alpha_1 \subset E_{a_0}$ , and  $\beta_1 \subset E_{a_k}$ . Choose points  $x \in \alpha_1$  and  $y \in \beta_1$ , and let  $\overline{\alpha}_1$  (respectively  $\overline{\beta}_1$ ) be the minimizing geodesics on  $E$  homotopic to  $\alpha_1$  relative to  $x$  (respectively homotopic to  $\beta_1$  relative to  $y$ ). Both  $\overline{\alpha}_1$  and  $\overline{\beta}_1$  cannot have length greater than  $c$ . Let  $\overline{E}$  be the annulus in  $E$  bounded by  $\overline{\alpha}_1$  and  $\overline{\beta}_1$ . The exterior angle of  $\overline{\alpha}_1$  (with respect to  $\overline{E}$ ) at  $x$ , and the exterior angle of  $\overline{\beta}_1$  at  $y$  cannot be bigger than  $\pi$  in absolute value. It is an annulus because we have chosen  $k$  large enough to keep  $\overline{\alpha}_1$  and  $\overline{\beta}_1$  disjoint. By Gauss-Bonnet, the total curvature of  $\overline{E}$  is at most  $2\pi$  in absolute value. By the definition of  $\overline{E}$ ,

$$(27) \quad E_* := E_{[a_0+c+C, a_k-c-C]} \subset \overline{E} \subset E_{[a_{k_0}-c-C, a_k+c+C]} =: E^*.$$

Let  $c_0 = |4H^2 + \sup_{M^2} K_N|$ . Using (25), (27) and Lemma 2, we have

$$\begin{aligned} \int_{E_*} |A|^2 &\leq \int_{\overline{E}} |A|^2 = \int_{\overline{E}} (4H^2 + 2K_N) - 2 \int_{\overline{E}} K \\ &\leq c_0 |\overline{E}| + 4\pi \\ &\leq c_0 |E^*| + 4\pi \\ &\leq c_0 c(a_k - a_0 + 2(C + c)) + 4\pi \\ &\leq c_0 c(k + 1 + 2(C + c)) + 4\pi \\ &\leq \text{constant} \times k, \end{aligned}$$

for  $k$  large. Hence  $\int_{E_*} |A|^2$  has linear growth, which implies that  $\int_E |A|^2$  also has this property.  $\square$

Now we present the

*Proof of Theorem 3.* Because  $\Sigma$  has finite topology, it suffices to show that  $|A|$  is bounded on each annular end  $E$  of  $M$ . Assume that  $E$  is a top annular end of  $M$ ,  $\partial E \subset M_0 = M^2 \times \{0\}$ . Recall that  $E_{[a,b]}$  denotes  $E \cap (M^2 \times [a, b])$  and  $E_t = E \cap M_t$ . We will assume  $|A|$  is unbounded on  $E$  and obtain a contradiction.

Let  $p_n \in E$  satisfy  $|A(p_n)| \geq n$ ,  $n = 1, 2, \dots$ , and let  $p_n$  be the lowest point of  $E$  with this property. For each integer  $n$ , let  $m = m(n)$  be the smallest integer larger than  $h(p_n) + 1$ . For  $q \in E_{[0,m]}$  let  $d(q) = m - h(q)$ .

Consider the continuous function  $F = F_n$  defined on  $E_{[0,m]}$ :

$$(28) \quad F(q) = \begin{cases} |A(q)|, & \text{if } h(q) \leq m - 1, \\ |A(q)| d(q)^2, & \text{if } h(q) \geq m - 1. \end{cases}$$

Let  $\tilde{q} = \tilde{q}_n \in E$  be a point where  $F$  achieves its maximum value. Clearly  $|A(\tilde{q})| \geq n$ .

Let  $a_n = h(\tilde{q}) - \frac{d(\tilde{q})}{2}$ ,  $b(n) = h(\tilde{q}) + \frac{d(\tilde{q})}{2}$  and consider  $\mathcal{E} = \mathcal{E}_n = E_{[a_n, b_n]}$ , the part of  $E$  between heights  $a_n$  and  $b_n$ . If  $b_n \leq m - 1$ , then for each  $p \in \mathcal{E}$ ,

$$|A(p)| \leq |A(\tilde{q})|.$$

If  $b_n > m - 1$  and  $h(\tilde{q}) < m$ , then for each  $p \in E_{[a_n, b_n]}$  we have  $d(p) \geq \frac{1}{2}$ . So, if  $h(p) \geq m$ , then  $|A(p)| d(p)^2 \leq |A(\tilde{q})|$ , hence

$$|A(p)| \leq 4|A(\tilde{q})|.$$

If  $h(p) \leq m$ , then  $|A(p)| \leq |A(\tilde{q})|$ . Similarly, the reader can easily check that this inequality also holds if  $h(\tilde{q}) \geq m$ ; so in all cases, when  $p \in \mathcal{E}$ , one has

$$(29) \quad |A(p)| \leq 4|A(\tilde{q})|.$$

At this point, we have produced a sequence  $\{\tilde{q}_n\}$  whose heights  $h(\tilde{q}_n)$  diverge with  $n$ , and which have the property that  $|A(\tilde{q}_n)| \geq n$ . Moreover, on the subset  $\mathcal{E}_n = E_{[a_n, b_n]} \subset E$ , we have the estimate

$$|A(p)| \leq 4|A(\tilde{q}_n)|.$$

Since  $p_n \in E$  was chosen to be a lowest point where  $|A(p_n)| \geq n$ , it follows that  $d(\tilde{q}_n) \leq 2$ , which in turn implies that  $b_n - a_n = d(\tilde{q}) \leq 2$ . Therefore, it follows from

Proposition 5 that

$$(30) \quad \int_{\mathcal{E}_n} |A|^2 \leq c,$$

where  $c$  does not depend on  $n$ .

For each  $n \geq 0$ , translate  $E$  down by  $h(\tilde{q}_n)$ . We may consider  $\{\tilde{q}_n\}$  to be a sequence of points in  $M_0^2$ , and by abuse of notation, we will refer to these points by the same notation. We also consider the subsets  $\mathcal{E}_n$  to be similarly translated and we similarly abuse notation. For each  $n$ , define  $\lambda_n = |A(\tilde{q}_n)|$ . Rescale the metric on  $M^2 \times \mathbf{R}$  by  $\lambda_n$  and denote by  $\mathcal{E}_n^*$  the surface  $\mathcal{E}_n$  in the rescaled metric. The distance between the top and the bottom of  $\mathcal{E}_n^*$  is

$$\lambda_n d(\tilde{q}_n) = |A(\tilde{q}_n)| d(\tilde{q}_n) \geq n.$$

We also have from (29) the estimate

$$|A^*(p)| \leq 4$$

on  $\mathcal{E}_n^*$  and the normalization  $|A(\tilde{q}_n)| = 1$ , where  $|A^*|$  denotes the second fundamental form in the rescaled metric. The integral in (30) is scale invariant:

$$(31) \quad \int_{\mathcal{E}_n^*} |A^*|^2 \leq c,$$

independent of  $n$ . Since we are assuming that  $\Sigma$  has linear area growth, it follows that  $E$  also has linear area growth, and this remains true in the rescaled metrics.

For each  $n$ , denote by  $U_n$  the simply connected neighborhood of the origin in  $T_{\tilde{q}_n}(M^2 \times \mathbf{R})$ , which corresponds via the exponential map to  $M^2 \times \mathbf{R}$  minus the cut locus of  $\tilde{q}_n$ . We consider  $U_n$  with the metric of  $M^2 \times \mathbf{R}$  rescaled by  $\lambda_n$ , and we denote this metric space by  $U_n^*$ . Since the injectivity radius of  $M^2$  is bounded below, and the Gauss curvature of  $M^2$  is bounded in absolute value, the  $U_n^*$  converge to  $\mathbf{R}^2 \times \mathbf{R} = \mathbf{R}^3$  with the flat metric. (The distance from the origin to  $\partial U_n^*$  goes to infinity with  $n$  because of the global lower bound on the injectivity radius and the fact that  $\lambda_n \rightarrow \infty$ . The curvature of  $U_n^*$  behaves like  $\frac{\text{const.}}{\lambda_n^2}$  since there is a global bound on the curvature of  $M^2$ , and hence of  $M^2 \times \mathbf{R}$ .)

For each  $n$ , consider the part of  $\mathcal{E}_n$  inside the cut locus of  $\tilde{q}_n^*$ , and let  $\mathcal{E}_n^*$  be the pullback by the exponential map to  $U_n^*$  of the component of this surface containing the point  $\tilde{q}_n^*$ . The mean curvature of  $\mathcal{E}_n^*$  is  $\frac{H}{\lambda_n}$ , which goes to zero as  $n \rightarrow \infty$ . We have established in (31) bounds for the integral of  $|A^*|^2$ . We are assuming linear growth for  $\Sigma$ , and this implies that  $\mathcal{E}_n^*$  has linear area growth. Therefore, we may choose a subsequence of the  $\mathcal{E}_n^*$  which converges—with finite multiplicity—to an embedded minimal surface,  $\mathcal{S}$ , in  $\mathbf{R}^3$  with the flat metric. Moreover,  $\mathcal{S}$  has  $|A^*(0)| = 1$  and  $\int_{\mathcal{S}} |A^*|^2$  is finite by (31). Since  $|A|^2 = -2K$ ,  $\mathcal{S}$  has finite total curvature, but is not the plane because  $|A^*(0)| \neq 0$ . It is complete because  $\Sigma$  was complete, and the distance from the origin to the boundary of  $\Sigma_n^*$  is not less than the distance from the origin to the boundary of  $U_n^*$ , which diverges with  $n$ .

In the previous paragraph we have shown that  $\mathcal{S} \subset \mathbf{R}^3$  is a complete and non-planar embedded minimal surface with finite total curvature. Such a surface must have a top and a bottom catenoidal end. Consider a top catenoidal end  $\tilde{\Sigma}$  of  $\mathcal{S}$ . Let  $\gamma$  be a Jordan curve on  $\tilde{\Sigma}$ , which is the transverse intersection of  $\Sigma$  with some plane orthogonal to the axis of the top catenoid end. The flux of  $\tilde{\Sigma}$  along  $\gamma$  is a vector

whose component,  $a$ , in the direction of the axis of the catenoidal end, is equal to the logarithmic growth of the end. Since the end is catenoidal,  $a \neq 0$ .

Now  $k\gamma = \lim \gamma_n$ , where  $\gamma_n$  are Jordan curves on  $\Sigma_n^*$  and  $k$  is the (fixed) multiplicity with which the  $\mathcal{E}_n^*$  (a subsequence) are converging to  $\mathcal{S}$ . The  $\gamma_n$  are coming from loops  $\beta_n$  on  $E \subset \Sigma$  by scaling by  $\lambda_n$  with  $\lambda_n \rightarrow \infty$ . We know that this flux is zero if  $\beta_n$  is homologous to zero on  $E$  and equals some fixed vector when  $\beta_n$  is a generator of  $\pi_1(E)$ . Let  $b$  be the component of this vector in the direction determined by the top catenoid end. Then one has  $\lambda_n b \rightarrow ka$ , as  $\lambda_n \rightarrow \infty$ . Since  $b$  is finite, this is a contradiction.  $\square$

*Remark 5.* We conjecture that a properly embedded  $H$ -surface,  $\Sigma$ , of finite topology in  $M^2 \times \mathbf{R}$  must have linear area growth if  $M^2$  is compact. (This is true when  $H = 0$ , [16].) Using Theorem 3 and arguing as in Corollary 2, this conjecture, if true, implies that such a  $\Sigma$  would have bounded second fundamental form.

## REFERENCES

- [1] J. L. Barbosa and R.S. Earp. Prescribed mean curvature in  $H^{n+1}(-1)$  with convex planar boundary, I. *Geom. Dedicata*, 71(1):61–74, 1998. MR1624726 (99d:53064)
- [2] M. Crandall and P. Rabinowitz. Bifurcation, perturbation of simple eigenvalues and linearized stability. *Arch. Rational Mech. Anal.*, 52:161—180, 1973. MR0341212 (49:5962)
- [3] P. Collin. Topologie et courbure des surfaces minimales proprement plongées de  $\mathbf{R}^3$ . *Ann. of Mathematics*, 145:1–31, 1997. MR1432035 (98d:53010)
- [4] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Springer-Verlag, New York, 2nd edition, 1983. MR0737190 (86c:35035)
- [5] Karsten Grosse-Braukmann and Meinhard Wohlgemuth. The gyroid is embedded and has constant mean curvature companions. *Calc. Var.*, 4:499–523, 1996. MR1415998 (97k:53011)
- [6] D. Hoffman and H. Karcher. Complete embedded minimal surfaces of finite total curvature. In *Encyclopedia of Mathematics*, pages 5–93, 1997. R. Osserman, editor, Springer Verlag. MR1490038 (98m:53012)
- [7] D. Hoffman and W. H. Meeks III. Embedded minimal surfaces of finite topology. *Annals of Mathematics*, 131:1–34, 1990. MR1038356 (91i:53010)
- [8] W.-T. Hsiang and W.-Y. Hsiang. On the uniqueness of isoperimetric solutions and imbedded soap bubbles in non-compact symmetric spaces. *Inventiones Math.* 98:39-58, 1989. MR1010154 (90h:53078)
- [9] H. Karcher. Embedded minimal surfaces derived from Scherk's examples. *Manuscripta Math.*, 62:83–114, 1988. MR0958255 (89i:53009)
- [10] H. Karcher. The triply periodic minimal surfaces of Alan Schoen and their constant mean curvature companions. *Manuscripta Math.*, 64:291–357, 1989. MR1003093 (90g:53010)
- [11] N. Korevaar, R. Kusner, and B. Solomon. The structure of complete embedded surfaces with constant mean curvature. *Journal of Differential Geometry*, 30:465–503, 1989. MR1010168 (90g:53011)
- [12] H. B. Lawson. Complete minimal surfaces in  $S^3$ . *Annals of Mathematics*, 92:335–374, 1970. MR0270280 (42:5170)
- [13] H. B. Lawson, Jr. *Lectures on Minimal Submanifolds*. Publish or Perish Press, Berkeley, 1971. MR0576752 (82d:53035b)
- [14] R. Mazzeo and F. Pacard. Bifurcating nodoids. *Contemp. Math.*, 314:169-186, 2002. MR1941630 (2004b:53015)
- [15] W. H. Meeks III and H. Rosenberg. The uniqueness of the helicoid. *Annals of Mathematics* 161(2):723-754, 2005. MR2153399
- [16] W. H. Meeks III and H. Rosenberg. The theory of minimal surfaces in  $M \times \mathbf{R}$ . To appear in *Comment. Math. Helv.*
- [17] B. Nelli and H. Rosenberg. Minimal surfaces in  $\mathbf{H}^2 \times \mathbf{R}$ . *Bull. Bras. Math. Soc., New Series*, 33(2):263-292, 2002. MR1940353 (2004d:53014)
- [18] B. Nelli and H. Rosenberg. Global Properties of Constant Mean Curvature Surfaces in  $\mathbf{H}^2 \times \mathbf{R}$ . To appear in *Pacific J. of Mathematics*.



- [19] P. A. Nitsche. Existence of prescribed mean curvature graphs in hyperbolic space. *Manuscripta Math.*, 108(3):349–367, 2002. MR1918082 (2003f:53015)
- [20] R. Pedrosa and M. Ritoré. Isoperimetric domains in the Riemannian product of a circle with simply connected space forms and applications to free boundary problems. *Indiana Univ. J. of Mathematics*, 48:1357–1394, 1999. MR1757077 (2001k:53120)
- [21] H. Rosenberg. Minimal surfaces in  $M^2 \times \mathbf{R}$ . *Illinois J. Math*, 46(4):1177–1195, 2002. MR1988257 (2004d:53015)
- [22] H. Rosenberg. Hypersurfaces of constant curvature in space forms. *Bull. Sc. Math*, 117:211–239, 1993. MR1216008 (94b:53097)
- [23] R. Schoen. Analytic aspects of the harmonic map problem. In *Seminar on Nonlinear Partial Differential Equations*, 1983. MR0765241 (86b:58032)
- [24] J. Serrin. The problem of Dirichlet for quasilinear elliptic differential equations with many independent variables. *Philos. Trans. Roy. Soc. London Ser. A*, 264:413–496, 1969. MR0282058 (43:7772)
- [25] J. Smoller. *Shock waves and reaction-diffusion equations*. Springer-Verlag, New York, 1983. MR0688146 (84d:35002)
- [26] M. Weber and M. Wolf. Teichmüller theory and handle addition for minimal surfaces. *Annals of Math.* 156:713–795, 2002. MR1954234 (2005j:53012)
- [27] M. Weber and M. Wolf. Minimal surfaces of least total curvature and moduli spaces of plane polygonal arcs. *GAF*, 8:1129–1170, 1998. MR1664793 (99m:53020)
- [28] M. Wohlgemuth. Higher genus minimal surfaces by growing handles out of a catenoid. *Manuscripta Math.*, 70:397–428, 1991. MR1092145 (91k:53021)

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