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CONSTANT POSITIVE 2-MEAN CURVATURE HYPERSURFACES

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Abstract. Hypersurfaces of constant 2-mean curvature in spaces of constant sectional curvature are known to be solutions to a variational problem. We extend this characterization to ambient spaces which are Einstein. We then estimate the 2-mean curvature of certain hypersurfaces in Einstein manifolds. A consequence of our estimates is a generalization of a result, first proved by Chern, showing that there are no complete graphs in the Euclidean space with positive constant 2-mean curvature.

1. Introduction

Let M^n be an oriented Riemannian n-manifold and let $x\colon M^n \longrightarrow \overline{M}^{n+1}$ be an isometric immersion of M^n into an orientable Riemannian $(n+1)$-manifold $\overline{M}^{n+1}$. Let $p \in M$ and let $B_r(p)$ be a geodesic ball of M with center p and radius r. We say that the volume of M has *polynomial growth* if there are positive numbers α and c such that $\operatorname{vol}(B_r(p)) \leq cr^\alpha$, for large r. We have the following result, first proved in a special case by Alencar and do Carmo ([AdC], and later generalized by do Carmo and Zhou [dCZ]).

Theorem A. *Let $x\colon M^n \longrightarrow \overline{M}^{n+1}$ be as above. Assume that x has constant mean curvature H. Assume further that* $\operatorname{Ind}(M) < \infty$ *and that the volume of M is infinite and has polynomial growth. Then*

$$H^2 \leq -\frac{1}{n} \inf_M \overline{\operatorname{Ricc}}(N).$$

Here N is a smooth unit normal field along M, $\overline{\operatorname{Ricc}}(N)$ is the value of the (non-normalized) Ricci curvature of $\overline{M}$ in the vector N, and the index of M, $\operatorname{Ind}(M)$, is defined as follows. Let

$$T = \Delta + \|A\|^2 + \overline{\operatorname{Ricc}}(N),$$

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where Δ is the Laplacian and A is the linear operator associated with the second fundamental form of M. For each compact domain $K \subset M$, define $\mathrm{Ind}_K(L)$ to be the index of the quadratic form

$$I(f) = -\int_M fTf\, dM, \tag{1}$$

for smooth functions f on M that have support in K. Then $\mathrm{Ind}(M)$ is defined as

$$\mathrm{Ind}(M) = \sup_{K\subset M} \mathrm{Ind}_K(L),$$

where K runs over all compact domains in M.

Theorem A has a number of interesting consequences. For instance, if $x\colon M \longrightarrow \overline{M}^{n+1}$ is as in Theorem A and, in addition, it is assumed that the Ricci curvature of $\overline{M}^{n+1}$ satisfies $\overline{\mathrm{Ricc}} > 0$, then the immersion is minimal (cf. [AdC, Corollary 1.3]). In case $\overline{M}^{n+1}$is the Euclidean space, this fact was first observed by Chern [C].

In view of its applications, we want to extend Theorem A to hypersurfaces with constant 2-mean curvature. We first observe that the quadratic form (1) is (modulo a constant) the second variation of the variational problem that characterizes the hypersurfaces with $H =$ *constant*. The hypersurfaces with $H_2 =$ *constant* are also characterized by a variational problem. To show this, it is convenient to consider the following more general situation.

Let S_r be the rth symmetric function of the eigenvalues $k_1, \dots, k_n$ of A, defined as

$$\begin{aligned} &S_0 = 1, \\ &S_r = \sum_{i_1<\cdots<i_r} k_{i_1}\dots k_{i_r}, \quad 1 \le r \le n, \\ &S_r = 0, \quad r > n, \end{aligned}$$

and define the r-mean curvature H_r of x by

$$S_r = \binom{n}{r} H_r.$$

Thus $H_1 = H$ is the mean curvature, H_n is the Gauss-Kronecker curvature, and when the ambient space is Einstein, H_2 is, modulo a constant, the scalar curvature (see Remark 3.9).

It is known (see Section 3) that if $\overline{M}^{n+1}$ has constant sectional curvature, immersions with constant $(r+1)$-mean curvature are critical points of the functional

$$\mathcal{A}_r = \int_M F_r\left(S_1, \dots, S_r\right)\, dM \tag{2}$$

for compactly supported volume-preserving variations. Here the functions F_r are well defined functions that are described in Section 3. For instance, for the mean curvature we have $F_0 = 1$ and for the 2-mean curvature we have $F_1 = S_1$.

Our first goal is to extend the above variational problem, for the case of 2-mean curvature, to ambient spaces more general than spaces of constant sectional curvature. In Section 3, we will show that it is possible to extend the variational problem that characterizes hypersurfaces with constant 2-mean curvature to ambient spaces that are Einstein manifolds. It will be clear in this section that this is probably as far as one can go with the functional (2).

In the above situation, the quadratic form that corresponds to (1) is given as follows. Define the linear map P_1 by $P_1 = S_1 I - A$ and define a differential operator L_1, that corresponds to the Laplacian Δ, by

$$L_1 = \operatorname{trace}\left(P_1 \operatorname{Hess} f\right).$$

Then the differential operator corresponding to T is shown to be (see Section 3)

$$T_1 = L_1 + (S_1 S_2 - 3S_3) + \operatorname{trace}\left(P_1 \overline{R}_N\right),$$

where $\overline{R}_N(Y) = \overline{R}(N, Y)N$ and $\overline{R}$ is the curvature of $\overline{M}$. Finally, our quadratic form is given by

$$I_1(f) = -\int_M f T_1 f \, dM,$$

for smooth functions on M that are compactly supported. The definition of $\operatorname{Ind}_1(M)$ is exactly the same as before.

By definition, $\mathcal{A}_0$ is the volume of M and $\mathcal{A}_1 = \int_M S_1 \, dM$ is what we call the *1-volume of* M. We observe that under the hypothesis $H_2 > 0$, H_1, and therefore S_1, can be made positive (see Proposition 2.3(a)). We say that the 1-volume of M *has polynomial growth* if there are positive numbers α and c such that $\int_{B_r(p)} S_1 \, dM \leq c r^\alpha$, for large r. We can now state our main theorem.

Theorem 1.1. *Let* $x \colon M^n \longrightarrow \overline{M}^{n+1}$ *be an isometric immersion of* M *into an oriented complete Einstein manifold with* $H_2 = constant > 0$. *Assume that* $\operatorname{Ind}_1 M < \infty$ *and that the 1-volume of* M *is infinite and has polynomial growth. Then*

$$H_2^{3/2} \leq -\frac{1}{n(n-1)} \left(\inf_M \left\{ \operatorname{trace}\left(P_1 \overline{R}_N\right) \right\} \right).$$

When $\overline{M}$ has constant sectional curvature c, we write $\overline{M}^{n+1}(c)$. As a corollary of the proof of Theorem 1.1 we obtain:

Theorem 1.2. *Let $x\colon M^n \longrightarrow \overline{M}^{n+1}(c)$ be an isometric immersion with $H_2 = constant > 0$. Assume that $\operatorname{Ind}_1 M < \infty$ and that the 1-volume of M is infinite and has polynomial growth. Then c is negative and*

$$H_2 \leq -c.$$

Theorem 1.2 generalizes the fact, first proved by S. S. Chern ([C, commentary after Theorem 2]), that there are no complete graphs in Euclidean spaces with positive constant 2-mean curvature. This is so because complete graphs in Euclidean spaces with $H_2 = constant > 0$ have index zero (since they are stable) and infinite 1-volume of polynomial growth.

As a byproduct of our proof, we obtain estimates for the first eigenvalue of the elliptic differential operator L_1 defined above. We should observe that one can define L_1 on a Riemmanian manifold M equipped with a symmetric Codazzi tensor A as follows: define $P_1 = (\operatorname{trace} A)\, I - A$ and set $L_1 = \operatorname{trace}(P_1 \operatorname{Hess} f)$. To guarantee that L_1 is elliptic, P_1 must be definite. Our estimates of the first eigenvalue of L_1 hold equally well for this situation.

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2. Preliminaries

A *domain* $D \subset M$ is an open connected subset with compact closure $\overline{D}$ and smooth boundary ∂D. Let us denote by $C_0^\infty(D)$ (respectively $C_c^\infty(D)$) the set of smooth real functions which are zero on ∂D (respectively with compact support in D).

Now we will state some definitions and results concerning the first eigenvalue of an elliptic self-adjoint linear differential operator

$$T\colon C_0^\infty(D) \longrightarrow C^\infty(D)$$

of second order. We recall that the first eigenvalue $\lambda_1^T(D)$ of T is defined as the smallest λ that satisfies

$$T(g) + \lambda g = 0, \tag{3}$$

for some nonzero function $g \in C_0^\infty(D)$. A nonzero function g in $C_0^\infty(D)$ that satisfies (3) for $\lambda = \lambda_1^T$ is called a first eigenfunction of T in D.

Lemma 2.1. *If D and D' are domains in M with $D \subset D'$ then $\lambda_1^T(D) \geq \lambda_1^T(D')$ and equality holds iff $D = D'$.*

For a proof see [Sm, Lemma 2]. We just notice that T satisfies the unique continuation principle (see [A]).

Set

$$\|u\|_{H^1} = \left(\int_D \left(|u|^2 + |\nabla u|^2 \right) dM \right)^{1/2}$$

and let $H^1(D)$ denote the completion of $C_c^\infty(D)$ with respect to the norm $\| \ \|_{H^1}$. $H^1(D)$ is the Sobolev Space over D.

LEMMA 2.2.

$$\lambda_1^T(D) = \inf\left\{ \frac{-\int_D fT(f)\,dM}{\int_D f^2\,dM} : f \in H^1(D), f \not\equiv 0 \right\}.$$

For a proof see [Sm, Lemma 4(a)].

Suppose that M is complete and noncompact. Let $\Omega \subset M$ be a compact subset. The first eigenvalue of M (resp. $M \setminus \Omega$) is defined by

$$\lambda_1(M) = \inf\left\{\lambda_1(D) : D \subset M \text{ is a domain}\right\},$$

and

$$\lambda_1(M \setminus \Omega) = \inf\left\{\lambda_1(D) : D \subset M \setminus \Omega \text{ is a domain}\right\},$$

respectively. We will need the following proposition.

PROPOSITION 2.3. *For an immersion that satisfies $H_2 > 0$ we have*

(a) $H_1^2 \geq H_2$,
(b) $H_1 H_2 \geq H_3$,

and equality holds only at the umbilic points.

Proof. First we recall that

$$H_{k-1}H_{k+1} \leq H_k^2, \quad k = 1, \dots, n, \tag{4}$$

where equality occurs only at umbilic points (cf. [BMV, p. 285, Remark 3]). Taking $k = 1$ in (4) we obtain (a). To prove (b) we proceed as follows. First, we notice that by (a) and by the hypothesis, $H_1 \neq 0$. Multiplying both sides of the inequality in (a) by H_2/H_1 and using (4) again with $k = 2$ gives (b). $\square$

For future reference, we state in the following lemma (see [BC, Lemma 2.1]) some properties of the Newton transformations P_r, defined inductively by

$$P_0 = I,$$
$$P_1 = S_r I - AP_{r-1}.$$

LEMMA 2.4. *For each $1 \leq r \leq n-1$ we have:*

(i) $P_r(e_i) = S_r(A_i)e_i$, *for each* $1 \leq i \leq n$;
(ii) $\operatorname{trace}(P_r) = \sum_{i=1}^n S_r(A_i) = (n-r)S_r$;
(iii) $\operatorname{trace}(AP_r) = \sum_{i=1}^n k_i S_r(A_i) = (r+1)S_{r+1}$;
(iv) $\operatorname{trace}(A^2P_r) = \sum_{i=1}^n k_i^2 S_r(A_i) = S_1 S_{r+1} - (r+2)S_{r+2}$.

3. The variational problem

Let $x\colon M^n \longrightarrow \overline{M}^{n+1}$ be as in the Introduction. Let $D \subset M$ be a domain. By a *variation of* D we mean a differentiable map $\phi\colon (-\varepsilon, \varepsilon) \times \overline{D} \longrightarrow \overline{M}^{n+1}$, $\varepsilon > 0$, such that for each $t \in (-\varepsilon, \varepsilon)$ the map $\phi_t\colon \{t\} \times \overline{D}^n \longrightarrow \overline{M}^{n+1}$ defined by $\phi_t(p) = \phi(t,p)$ is an immersion and $\phi_0 = x|_{\overline{D}}$. Set

$$E_t(p) = \frac{\partial \phi}{\partial t}(t,p) \quad \text{and} \quad f_t = \langle E_t, N_t \rangle,$$

where N_t is the unit normal vector field in $\phi_t(D)$. E is called the *variational vector field* of ϕ.

We say that a variation ϕ of D is *compactly supported* if $\operatorname{supp}\phi_t \subset K$, for all $t \in (-\varepsilon, \varepsilon)$, where $K \subset D$ is a compact domain. The volume associated with ϕ is the function $V\colon (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}$ defined by

$$V(t) = \int_{[0,t]\times D} \phi^* \left(d\overline{M} \right),$$

where $d\overline{M}$ is the volume element of $\overline{M}$. We say that the variation is *volume-preserving* if $V(t) \equiv 0$.

When $\overline{M}$ has constant sectional curvature c, we recall that immersions with constant $(r+1)$-mean curvature are critical points (cf. [BC]) of the variational problem of minimizing the integral

$$\mathcal{A}_r = \int_M F_r \left(S_1, \ldots, S_r \right) dM,$$

for compactly supported volume-preserving variations. The functions F_r are defined inductively by

$$\begin{aligned}
&F_0 = 1, \\
&F_1 S_1, \\
&F_r S_r + \frac{c(n-r+1)}{r-1} F_{r-2}, \quad 2 \le r \le n-1.
\end{aligned}$$

Our aim is to extend the variational problem of hypersurfaces with $H_2 =$ *constant* to a more general ambient space. To this end we first suppose that $\overline{M}$ is an orientable Riemannian $(n+1)$-manifold and compute the first and second variation for the functional

$$\mathcal{A}_1 = \int_M S_1 \, dM.$$

From the computation of the first variation we will see that if we want the functional $\mathcal{A}_1 = \int H_1 \, dM$ to characterize hypersurfaces of constant H_2, we must restrict ourselves to ambient spaces with constant Ricci curvature, that is to Einstein spaces.

We remark that for the r-mean curvatures with $r > 1$ the definition of the functional $\mathcal{A}_r = \int F_r \, dM$ requires that the ambient space has constant sectional curvature c. So an attempt to extend the variational problem for H_r, $r > 1$, to more general ambient spaces seems hopeless unless one changes the functional $\mathcal{A}_r$.

We use $(\cdot)^T$ and $(\cdot)^N$, to denote, respectively, the tangent and normal components, and ∇ and $\overline{\nabla}$, to denote, respectively, the connection of M in the metric induced by ϕ_t and the connection of $\overline{M}$. Let $A(t)$ be the second fundamental form of ϕ_t.

Lemma 3.1.

$$A'(t) = \operatorname{Hess} f + f\overline{R}_N + fA^2 + \nabla_{E^T}(A).$$

Here $\overline{R}_N(Y) = \overline{R}(N,Y)N$, *where* $\overline{R}$ *is the curvature of* $\overline{M}$.

Proof. Let $p \in M$ and let u, v be tangent vector fields defined in a neighborhood of p. Set $u_t = d\phi_t(u), v_t = d\phi_t(v)$ and

$$I(t)\,(u_t, v_t) = -\langle \overline{\nabla}_{u_t} N_t, v_t\rangle = \langle A(t)u_t, v_t\rangle.$$

We now drop the subscript t and differentiate the expression $I(t)\,(u_t, v_t) = -\langle \overline{\nabla}_{u_t} N_t, v_t\rangle$ to obtain

$$\begin{aligned}
(5)\qquad -\left(I(u,v)\right)' &= \langle \overline{\nabla}_E \overline{\nabla}_u N, v\rangle + \langle \overline{\nabla}_u N, \overline{\nabla}_E v\rangle \\
&= \langle \overline{\nabla}_{E^T} \overline{\nabla}_u N, v\rangle + \langle \overline{\nabla}_{E^N} \overline{\nabla}_u N, v\rangle - \langle A(u), \overline{\nabla}_E v\rangle \\
&= -\langle \overline{\nabla}_{E^T}(Au), v\rangle + \langle \overline{\nabla}_u \overline{\nabla}_{E^N} N, v\rangle - \langle \overline{R}(E^N, u)N, v\rangle \\
&\quad + \langle \overline{\nabla}_{[E^N,u]} N, v\rangle - \langle A(u), \overline{\nabla}_E v\rangle.
\end{aligned}$$

Since $[E, u] = 0$, we have

$$(6)\qquad \left[E^N, u\right] = -\left[E^T, u\right]$$

and therefore

$$(7)\qquad \left\langle \overline{\nabla}_{[E^N,u]} N, v\right\rangle = \left\langle A\left(\left[E^T, u\right]\right), v\right\rangle.$$

Also, since $\langle \overline{\nabla}_Z N, N\rangle = 0$ for every vector field Z, we have

$$\begin{aligned}
\left\langle \overline{\nabla}_{E^N} N, u\right\rangle = -\left\langle N, \overline{\nabla}_{E^N} u\right\rangle = \left\langle \overline{\nabla}_{E^N} N, u\right\rangle &= -\left\langle N, \overline{\nabla}_u E^N - \left[E^T, u\right]\right\rangle \\
= -\left\langle N, \overline{\nabla}_u E^N\right\rangle = -df(u)&
\end{aligned}$$

and thus

$$(8)\qquad \overline{\nabla}_{E^N} N = -\nabla f.$$

Substituting (7) and (8) into (5) and using (6) again, we obtain

$$\begin{aligned}
(9)\qquad -\left(I(u,v)\right)' = &-\left\langle \overline{\nabla}_{E^T}(A)u, v\right\rangle - \left\langle A\overline{\nabla}_u E^T, v\right\rangle - \langle \operatorname{Hess} f(u), v\rangle \\
&- f\left\langle \overline{R}(N,u)N, v\right\rangle - \left\langle Au, \overline{\nabla}_E v\right\rangle.
\end{aligned}$$

On the other hand, if we use $I(t) = \langle A(t)u, v\rangle$ we obtain

$$(I(u,v))' = \left\langle \overline{\nabla}_E(Au), v\right\rangle + \left\langle Au, \overline{\nabla}_E v\right\rangle \tag{10}$$
$$= \langle A'(u), v\rangle + \left\langle A\overline{\nabla}_E u, v\right\rangle + \left\langle Au, \overline{\nabla}_E v\right\rangle$$
$$= \langle A'(u), v\rangle + \left\langle A\overline{\nabla}_u E, v\right\rangle + \left\langle Au, \overline{\nabla}_E v\right\rangle$$
$$= \langle A'(u), v\rangle - f\left\langle A^2u, v\right\rangle + \left\langle A\overline{\nabla}_u E^T, v\right\rangle + \left\langle Au, \overline{\nabla}_E v\right\rangle.$$

Notice that we are identifying A with an extended linear map in $\overline{M}$. Comparing (9) and (10) completes the proof. □

Set

$$L_r f = \operatorname{trace}(P_r \operatorname{Hess} f). \tag{11}$$

PROPOSITION 3.2. *We have*

$$\frac{\partial}{\partial t}(S_{r+1}) = L_r(f) + f\left(S_1 S_{r+1} - (r+2)S_{r+2}\right)$$
$$+ f \operatorname{trace}\left(P_r \overline{R}_N\right) + E^T\left(S_{r+1}\right).$$

Proof. Combining Lemma 3.1 and the equation

$$\frac{\partial}{\partial t}(S_{r+1}) = \operatorname{trace}\left(A'(t)P_r\right)$$

(cf. [Re, Equation (2)]) we obtain

$$\frac{\partial}{\partial t}(S_{r+1}) = \operatorname{trace}(P_r \operatorname{Hess} f) + f \operatorname{trace}\left(P_r \overline{R}_N\right)$$
$$+ f \operatorname{trace}\left(P_r A^2\right) + \operatorname{trace}\left(P_r \nabla_{E^T}(A)\right).$$

Now we use Lemma 2.4(iv) and the fact that

$$\operatorname{trace}(P_r \nabla_{E^T} A) = E^T\left(S_{r+1}\right)$$

(cf. [Ro, Equation (4.4)]) to obtain the result. □

The following lemma is well known and can be found in [Re].

LEMMA 3.3. *We have* $\frac{\partial}{\partial t}(dM_t) = \left(-S_1 f + \operatorname{div}\left(E^T\right)\right) dM_t$, *where* dM_t *is the volume element of* $\phi_t(M)$.

Now we have all the ingredients to compute the formulas for the first and second variations for

$$\mathcal{A}_1(t) = \int_D S_1\, dM_t.$$

PROPOSITION 3.4 (First Variation Formula). *For any compactly supported variation of* D *we have*

$$\mathcal{A}_1'(t) = \int_D \left\{-2S_2(t) + \overline{\operatorname{Ric}}\left(N_t\right)\right\} f\, dM_t,$$

where $\overline{\mathrm{Ric}}\,(N_t)$ *is the (non-normalized) Ricci curvature of* $\overline{M}$ *in the direction of* N_t.

Proof. Differentiating the expression

$$\mathcal{A}_1(t) = \int_D S_1 \, dM_t$$

we obtain, using Proposition 3.2 and Lemma 3.3,

$$\begin{aligned}\mathcal{A}'{}_1(t) &= \int_D \left\{\Delta f + f\left(S_1^2 - 2S_2\right) + f \operatorname{trace}\left(\overline{R}_{N_t}\right) + E^T\left(S_1\right)\right\} dM_t \\ &\quad + \int_D \left\{S_1\left(-S_1 f + \operatorname{div}\left(E^T\right)\right)\right\} dM_t \\ &= \int_D \left\{\Delta f - 2S_2 f + f\overline{\mathrm{Ric}}\,(N_t) + \operatorname{div}\left(S_1 E^T\right)\right\} dM_t.\end{aligned}$$

Now, Stokes' Theorem implies that

$$\mathcal{A}'{}_1(t) = \int_D \left\{-2S_2 + \overline{\mathrm{Ric}}\,(N_t)\right\} f \, dM_t + \int_{\partial D} \langle \nabla f + S_1 E^T, \nu\rangle \, ds_t,$$

where ν is the unit exterior normal to ∂D and ds_t is the volume element of ∂D. Since we are working with compactly supported variations, the result follows. □

From Proposition 3.4 we see that if we are looking for a variational problem in $\overline{M}$ for which the critical points are the hypersurfaces of constant 2-mean curvature, the functional $\mathcal{A}_1 = \int_D S_1 \, dM$ is not suitable, unless we require the ambient space to be Einstein, so that the Ricci curvature of $\overline{M}$ is constant. Thus we restrict ourselves to Einstein spaces and compute the second derivative of $\mathcal{A}_1$ at a critical point x for volume-preserving variations. It is known that for volume-preserving variations we have (cf. [BdCE])

$$\int_D f_t \, dM_t = 0, \tag{12}$$

where dM_t is the volume element of M in the metric induced by ϕ_t.

PROPOSITION 3.5 (The Second Variation Formula). *Let* $x\colon M^n \longrightarrow \overline{M}^{n+1}$ *be an isometric immersion with* $S_2 =$ *constant. Suppose that* $\overline{M}$ *is Einstein. Then for every volume-preserving variation we have*

$$A_1''(0) = -2\int_M \left\{f L_1(f) + (S_1 S_2 - 3S_3)\, f^2 + \operatorname{trace}\left(P_1 \overline{R}_N\right) f^2\right\} dM.$$

Proof. We differentiate the expression

$$\mathcal{A}_1'(t) = \int_D \left\{-2S_2(t) + \overline{\mathrm{Ric}}\,(N_t)\right\} f \, dM_t.$$

To obtain the result, we use Proposition 3.2, (12), and the fact that S_2 is constant. □

In the present situation, the differential operator associated with the second variation formula, the Jacobi operator, is given by

$$T_1 = L_1 + (S_1 S_2 - 3S_3) + \operatorname{trace}\left(P_1 \overline{R}_N\right),$$

which reduces to the operator $T_1 = L_1 + (S_1S_2 - 3S_3) + c(n-1)S_1$ in the case when $\overline{M}$ has constant sectional curvature c. In this case, L_1, and therefore T_1, turns out to be self-adjoint. We will prove that this is also true when $\overline{M}$ is Einstein (see Corollary 3.7).

Let ∇ denote the connection of M in the metric induced by the immersion $x\colon M^n \longrightarrow \overline{M}^{n+1}$. By $\langle , \rangle$ we denote both the metric of $\overline{M}^{n+1}$ and the induced metric in M.

Proposition 3.6. *If $\overline{M}$ is Einstein then*

$$\operatorname{trace}(u \to P_1 \nabla_u v) = \operatorname{trace}(u \to \nabla_u P_1 v)$$

for all $v \in T(M)$.

Proof. Let us fix $p \in M$ and let $\{e_i\}_{i=1}^n$ be an orthonormal frame in a neighborhood of p such that $\{e_i\}_{i=1}^n$ is geodesic at p, that is, $\nabla_{e_i} e_j(p) = 0$ for $i, j \in \{1, \dots, n\}$. Without loss of generality, it suffices to prove the proposition for $v = e_j$, $1 \le j \le n$. Since $\operatorname{trace}(u \to P_1 \nabla_u e_j)(p) = \sum_i \langle e_i, P_1 \nabla_{e_i} e_j \rangle (p) = 0$, we have to show that

$$\operatorname{trace}(u \to \nabla_u P_1 e_j)(p) = 0. \tag{13}$$

But

$$\begin{aligned}
\operatorname{trace}(u \to \nabla_u P_1 e_j)(p) &= \sum_{i=1}^n \langle e_i, \nabla_{e_i}(S_1 e_j - A e_j)\rangle \\
&= \sum_{i=1}^n \langle e_i, e_i(S_1) e_j \rangle - \sum_{i=1}^n \langle e_i, \nabla_{e_i} A e_j \rangle \\
&= e_j(S_1) - \sum_{i=1}^n \langle e_i, \nabla_{e_j} A e_i \rangle + \sum_{i=1}^n \langle \overline{R}(e_j, e_i) N, e_i \rangle \\
&= e_j(S_1) - \sum_{i=1}^n e_j \langle e_i, A e_i \rangle + \operatorname{Ric}_{\overline{M}}(e_j, N) \\
&= \operatorname{Ric}_{\overline{M}}(e_j, N),
\end{aligned}$$

where in the third equality we used the Codazzi equation. Since $\overline{M}$ is Einstein, the result follows. □

Corollary 3.7. *If $\overline{M}$ is Einstein then $L_1(f) = \operatorname{div}(P_1 \nabla f)$; in particular, L_1 is self-adjoint.*

Proof. We just take $v = \nabla f$ in Proposition 3.6. $\square$

Remark 3.8. Actually, some restriction on the curvature of the ambient space is necessary if L_1 is to be self-adjoint. Indeed, let (W, g) be a Riemannian manifold with metric g. Let $\phi_p \colon T_pW \longrightarrow T_pW$ be a linear operator and let us write $\phi_p(X, Y) = g(\phi_p X, Y)$, $X, Y \in T_pW$. Let us consider, in W, the operator $\square = \operatorname{trace}(\phi \operatorname{Hess} f)$. S. Y. Cheng and S. T. Yau ([CY, Proposition 1]) proved that $\square$ is self-adjoint in $C_0^\infty(D)$ for a domain $D \subset M$ if and only if, for each $i = 1, \ldots, n$ and for each point $p \in D$,

$$\sum_i \nabla\phi\,(e_j, e_i, e_i)\,(p) = 0,$$

where $\{e_i\}_{i=1}^n$ is a local frame defined in a neighborhood of p. Here $\nabla\phi$ is the 3-tensor that is the covariant derivative of the tensor ϕ. If $\{e_i\}_{i=1}^n$ is geodesic at p, then

$$\sum_i \nabla P_1\,(e_j, e_i, e_i)\,(p) = \operatorname{trace}\,(u \rightarrow \nabla_u P_1 e_j)\,(p).$$

Following the proof of (13) above, we see that

$$\operatorname{trace}\,(u \rightarrow \nabla_u P_1 e_j) = 0 \text{ if and only if } \operatorname{Ric}_{\overline{M}}(e_j, N) = 0$$

for all j, as we claimed.

Remark 3.9. When the ambient space is Einstein, H_2 is up to a constant equal to the scalar curvature of M. In fact, if S and $\overline{S}$ denote the (non-normalized) scalar curvatures of M and $\overline{M}$, the Gauss equation gives

$$S = \overline{S} - \overline{\operatorname{Ric}}(N) + S_2.$$

This, together with the easily verified relation

$$\overline{S} - \overline{\operatorname{Ric}}(N) = nk_0$$

gives

$$S_2 = S - nk_0.$$

By the definition of L_r (see (11)), we see that L_1 is elliptic if and only if P_1 is definite. We prove:

Lemma 3.10. *If $H_2 > 0$ then L_1 is elliptic. Furthermore, $-L_1$ is nonnegative; that is, $-\int_D f L_1 f \, dM > 0$ for all nonzero functions $f \in C_c^\infty(D)$.*

Proof. It is well known that

$$S_1^2 - |A|^2 = 2S_2.$$

Thus, since $S_1 = nH_1$ and $S_2 > 0$, we have

$$nH_1 > |A|$$

(note that we can orient M so that $H_1 > 0$), which we can rewrite as

$$k_1 + k_2 + \cdots + k_n > \sqrt{k_1^2 + k_2^2 + \cdots + k_n^2}.$$

Thus, $k_1 + k_2 + \cdots + k_n > |k_i| \geq k_i$ for each i, which implies that

$$S_1(A_i) = k_1 + \cdots + k_{i-1} + k_{i+1} + \cdots + k_n > 0.$$

But $S_1(A_i)$, $i = 1, \ldots, n$, are the eigenvalues of P_1 (see Lemma 2.4(i)). So L_1 is elliptic and Corollary 3.7 together with Stokes' Theorem gives the rest of the lemma. □

Most results in Sections 3 and 4 depend essentially on the ellipticity of L_1. Therefore, in view of Lemma 3.10, unless otherwise stated, we will be assuming that the immersion $x\colon M^n \longrightarrow \overline{M}^{n+1}$ satisfies $H_2 > 0$ and that M is oriented so that P_1 is positive definite (see the proof of Lemma 3.10). In view of Lemma 2.4(ii), this choice of orientation is the one that makes H_1, and so $\mathcal{A}_1$, positive.

Propositions 3.11, 3.13 and 3.16 and Lemma 3.12 below are already known for Δ. Their proofs are essentially the same for L_1 and we will include them here for completeness.

For Δ, Proposition 3.11 is proved in [CY, Theorem 4].

Proposition 3.11. *Let f and g be two smooth functions defined on a domain D of M. Suppose that $g \in C_0^\infty(D)$ with $g > 0$ on D, and that $f > 0$ on $\overline{D}$. Then*

$$\inf_{x \in D} \left\{ \frac{L_1(g)}{g}(x) - \frac{L_1(f)}{f}(x) \right\} < 0.$$

Proof. Consider the function $h = g/f$ defined on M. Applying Corollary 3.7, we get

$$\begin{aligned} L_1(h) &= \operatorname{div}\left(P_1\left(\nabla\left(\frac{g}{f}\right)\right)\right) \\ &= \operatorname{div}\left(P_1\left(\frac{\nabla g}{f} - \frac{g\nabla f}{f^2}\right)\right) \\ &= \operatorname{div}\left(\frac{1}{f}P_1(\nabla g) - \frac{g}{f^2}P_1(\nabla f)\right) \\ &= -\left\langle \frac{\nabla f}{f^2}, P_1(\nabla g)\right\rangle + \frac{1}{f}L_1(g) - \left\langle \frac{\nabla g}{f^2} - \frac{2g\nabla f}{f^3}, P_1(\nabla f)\right\rangle - \frac{g}{f^2}L_1(f) \end{aligned}$$

$$= -\frac{1}{f^2}\langle \nabla f, P_1(\nabla g)\rangle - \frac{1}{f^2}\langle \nabla g, P_1(\nabla f)\rangle + \frac{2g}{f^3}\langle \nabla f, P_1(\nabla f)\rangle + \frac{1}{f}L_1(g) - \frac{g}{f^2}L_1(f).$$

Since P_1 is self-adjoint we obtain

$$\begin{aligned} L_1(h) &= -\frac{2}{f^2}\langle P_1(\nabla f), \nabla g\rangle + \frac{2g}{f^3}\langle \nabla f, P_1(\nabla f)\rangle + h\left[\frac{L_1(g)}{g} - \frac{L_1(f)}{f}\right] \\ &= -\frac{2}{f}\langle P_1(\nabla f), \nabla h\rangle + h\left[\frac{L_1(g)}{g} - \frac{L_1(f)}{f}\right]. \end{aligned}$$

We now consider the operator G, defined by

$$G(h) = L_1(h) + \frac{2}{f}\langle P_1(\nabla f), \nabla h\rangle - h\left[\frac{L_1(g)}{g} - \frac{L_1(f)}{f}\right].$$

Since L_1 is elliptic, if $\left[\frac{L_1(g)}{g} - \frac{L_1(f)}{f}\right] \geq 0$ on D, we can use the Hopf maximum principle to conclude that the solution h of $G(h) = 0$ cannot attain its maximum in the interior of D unless h is constant. Since $h \geq 0$ and $h(\partial D) = 0$, we conclude that $h \equiv 0$. This implies $g \equiv 0$, which is a contradiction. □

For Δ, the following lemma was proved in [AdC].

Lemma 3.12. *Suppose that M is complete and noncompact. Let f be a positive smooth function defined on M and let $\Omega \subset M$ be a compact subset. Then*

$$\lambda_1^{L_1}(M \setminus \Omega) \geq \inf_{M\setminus\Omega}\left(\frac{-L_1(f)}{f}\right).$$

Proof. Let $D \subset M\setminus\Omega$ be a domain. Let $g \in C_0^\infty(D)$ be a first eigenfunction of L_1 in D. It is known that $g \neq 0$ in D. From Proposition 3.11 we have

$$\inf_{x\in D}\left\{\frac{L_1(g)}{g} - \frac{L_1(f)}{f}\right\} < 0,$$

and therefore

$$\inf_{x\in D}\left\{-\lambda_1^{L_1}(D) - \frac{L_1(f)}{f}\right\} < 0.$$

Thus

$$\lambda_1^{L_1}(D) > \inf_{x\in D}\left(-\frac{L_1(f)}{f}\right),$$

and by taking the infimum over all domains $D \subset M\setminus\Omega$ the lemma follows. □

For Δ, the following proposition was established in [FC-S].

Proposition 3.13. *Suppose that M is complete and noncompact. The following statements are equivalent:*

(i) $\lambda_1^{T_1}(D) \geq 0$ *for every domain $D \subset M$.*

(ii) $\lambda_1^{T_1}(D) > 0$ *for every domain* $D \subset M$.
(iii) *There exists a positive smooth function* f *on* M *satisfying the equation* $T_1 f = 0$.

Proof. (i) $\Longrightarrow$ (ii): Let $D \subset M$ be a domain. Fix $x_0 \in M$ and choose $R > 0$ large enough so that $D \subsetneq B_{x_0}(R)$. Then, by Lemma 2.1, $\lambda_1^{T_1}(D) > \lambda_1^{T_1}(B_{x_0}(R))$. But $\lambda_1^{T_1}(B_{x_0}(R)) \geq 0$ by hypothesis, so the conclusion follows.

(ii) $\Longrightarrow$ (iii): We want to prove the existence of a function f as described in the statement. Let $x_0 \in M$ be a fixed point. We start by proving the following lemma.

LEMMA 3.14. *For each* $R > 0$, *there exists a unique positive solution of the problem*

$$\begin{cases} T_1 u = 0 & \text{on } B_{x_0}(R), \\ u = 1 & \text{on } \partial B_{x_0}(R). \end{cases} \tag{14}$$

Proof. Let us fix $R > 0$. Since $\lambda_1^{T_1}(B_{x_0}(R)) > 0$ by hypothesis, there is no nonzero solution of

$$\begin{cases} T_1 u = 0 & \text{on } B_{x_0}(R), \\ u = 0 & \text{on } \partial B_{x_0}(R). \end{cases}$$

Set $q = -(S_1 S_{r+1} - (r+2)S_{r+2} + c(n-r)S_r)$. The Fredholm Alternative ([GT, Theorem 6.15, p. 102]) implies the existence of a unique solution v on $B_{x_0}(R)$ of

$$\begin{cases} T_1 v = q & \text{on } B_{x_0}(R), \\ v = 0 & \text{on } \partial B_{x_0}(R). \end{cases}$$

It follows that $u = v + 1$ is a unique solution of (14). We still need to prove that $u > 0$ on $B_{x_0}(R)$. We will first show that $u \geq 0$ on $B_{x_0}(R)$. To this end, set $\Omega = \{x \in B_{x_0}(R) : u(x) < 0\}$ and suppose $\Omega \neq \emptyset$. Ω is open. Without loss of generality, we can suppose Ω is connected. By the definition of Ω, u satisfies

$$\begin{cases} T_1 u = 0 & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{15}$$

Since $\lambda_1^{T_1}(\Omega) > 0$ by hypothesis and since u satisfies (15), we have $u \equiv 0$ in Ω. Hence, by the unique continuation principle (cf. [A]), $u = 0$ on $B_{x_0}(R)$, contradicting the fact that $u = 1$ on $\partial B_{x_0}(R)$. We have thus shown that $u \geq 0$ on $B_{x_0}(R)$, and since u is not identically zero, the maximum principle ([Sp, vol. V, Corollary 19, p. 187]) implies that $u > 0$ on $B_{x_0}(R)$, which proves the lemma. $\square$

For each $R > 0$ let us denote by u_R the function given by Lemma 3.14. Set $f_R(x) = u_R(x_0)^{-1} u_R(x)$ for $x \in B_{x_0}(R)$. Thus, f_R satisfies

$$\begin{cases} T_1 f_R = 0 & \text{on } B_{x_0}(R), \\ f_R(x_0) = 1, \quad f_R > 0 & \text{on } B_{x_0}(R). \end{cases}$$

Fix a ball $B_{x_0}(\sigma) \subset M$ and let $\Omega \subset M$ be a domain such that $B_{x_0}(4\sigma) \subset \Omega$. Since T_1 is elliptic with smooth coefficients in M, T_1 is strictly elliptic with bounded coefficients in Ω. From the Harnack inequality ([GT, Theorem 8.20, p. 189]), we conclude that there exists a positive constant b independent of R such that, for $R > 4\sigma$,

$$\sup_{B_{x_0}(\sigma)} f_R \leq b \inf_{B_{x_0}(\sigma)} f_R \leq b, \tag{16}$$

where in the last inequality we used that $f_R(x_0) = 1$. By [LM, Theorem 5.4, p. 194] (see also [GT, Problem 6.1, p. 134]) we have

$$\sup_{B_{x_0}(\sigma)} \left| D^\delta f_R \right| \leq d_{\sigma,|\delta|} \int_\Omega |f_R|^2 \, dM,$$

where $\delta = (\delta_1, \ldots, \delta_n)$, with nonnegative integers δ_i, $|\delta| = \sum \delta_i$, and

$$D^\delta u = \frac{\partial^{|\delta|} u}{\partial x_1^{\delta_1} \ldots \partial x_n^{\delta_n}}$$

for local coordinates $(x_1, \ldots, x_n)$. Here $d_{\sigma,|\delta|}$ is a positive constant depending on σ and $|\delta|$ (but independent of R). Then, in view of (16), we see that all derivatives of f_R are bounded uniformly (independent of R) on $\overline{B_{x_0}(\sigma)}$. Since σ is arbitrary, we conclude that all derivatives of f_R are bounded uniformly (independent of R) on compact subsets of M. Using the Theorem of Arzelá-Ascoli and the Cantor diagonal method we conclude that for each compact subset K of M, there exists a sequence $R_i \to \infty$ so that f_{R_i} converges uniformly, along with its derivatives, on K. Using the diagonal method again, we can arrange that $\{f_{R_i}\}$, along with its derivatives, converges uniformly on compact subsets of M to a function f satisfying $T_1 f = 0$ on M and $f(x_0) = 1$. Since f is not identically zero and $f \geq 0$, the maximum principle ([Sp, vol. V, Corollary (19), p. 187]) implies that $f > 0$ on M.

(iii) $\Longrightarrow$ (i): Suppose that $\lambda_1^{T_1}(D) < 0$ for some $D \subset M$. Then, since $C_c^\infty(D)$ is dense in $H^1(D)$, Lemma 2.2 implies that there exists $g \in C_c^\infty(D)$ with $I_r(g,g) < 0$. We conclude, using Smale's version of the Morse Index Theorem [Sm], that there exist a domain $D' \subsetneq D$ (in fact, $D' \subset \operatorname{supp} g$) and a function $v \in C_0^\infty(D')$ with $v > 0$ in D' such that $T_1 v = 0$. We will prove in a moment that we can choose positive constants k_1 and k_2 such that $w = k_1 f - k_2 v \geq 0$ and $w(p) = k_1 f(p) - k_2 v(p) = 0$ for some point p in D'. Since $T_1 w = 0$, by the maximum principle ([Sp, vol. V, Corollary 19, p. 187]), it follows that $w \equiv 0$. This is a contradiction since $v(\partial D') = 0$

and $f > 0$ on M. In order to complete the argument, we now describe explicitly the constants k_1 and k_2. Set $k_1 = \max_{\overline{D'}} v$ and $k = \min_{\overline{D'}} f$. Define an auxiliary function $g = kv/(k_1 f) > 0$. Let p be such that $g(p) = \max_{\overline{D'}} g$. Define $k_2 = k/g(p)$. Then we have

$$w(t) = k_1 f(t) - k_2 v(t) = k_1 f(t) - \frac{k_1 f(p) v(t)}{v(p)} = \frac{k_1 \left(f(t)v(p) - f(p)v(t)\right)}{v(p)}$$

for all $t \in D'$. By the choice of p, it is clear that $w(t) \geq 0$ for all $t \in D'$ and that $w(p) = 0$. □

In order to state the next proposition we recall the definition of stability.

Definition 3.15. Let $x\colon M \longrightarrow \overline{M}^{n+1}$ satisfy $H_2 =$ *constant* > 0 and let $D \subset M$ be a domain. We say that D is *1-stable* if $I_1(f) > 0$ for all $f \in C_c^\infty(D)$. Otherwise, we say that D is *1-unstable*.

For Δ, the following proposition was proved in [F-C].

Proposition 3.16. *Suppose that the immersion $x\colon M \longrightarrow \overline{M}^{n+1}$ satisfies $H_2 = constant > 0$ and that M is complete and noncompact. If $\operatorname{Ind}_1 M < \infty$ then there exist a compact set $K \subset M$ and a positive function f on M so that $M \setminus K$ is 1-stable and $T_1 f = 0$ on $M \setminus K$.*

Proof. The proof of the existence of a compact set K_1 so that $M \setminus K_1$ is 1-stable is standard and we will omit it (cf. [G] or [F-C]). The proof of the existence of the function f is similar to that of the implication (ii) ⇒ (iii) of Proposition 3.13. For completeness, we sketch the argument.

Let $R_0 > 0$ be sufficiently large so that $K_1 \subset B_{x_0}(R_0)$ for some $x_0 \in M$. Let Ω be a connected component of $M \setminus B_{x_0}(R_0)$ and set

$$D_R(\Omega) = \Omega \cap A(R_0, R), \quad R > R_0,$$

where $A(R_0, R) = B_{x_0}(R) \setminus B_{x_0}(R_0)$. Since $M \setminus K_1$ is 1-stable, by Lemma 2.2, $\lambda_1^{T_1}(D_R(\Omega)) \geq 0$ for each $R > R_0$. Here we used the fact that $C_c^\infty(D_R(\Omega))$ is dense in $H^1(D_R(\Omega))$. By Lemma 2.1, $\lambda_1^{T_1}(D_{R'}(\Omega)) > \lambda_1^{T_1}(D_R(\Omega)) \geq 0$ for $R > R'$ and, in particular, $\lambda_1^{T_1}(D_R(\Omega)) > 0$ for any $R > R_0$. For each $R > R_0$ there exists a positive solution u_R of the problem

$$\begin{cases} T_1 u = 0 & \text{on } D_R(\Omega), \\ u = 1 & \text{on } \partial D_R(\Omega). \end{cases}$$

(This can be proved in the same way as Lemma 3.14.) Fix $x_1 \in \Omega$ and set

$$f_R(x) = \left(u_R(x_1)\right)^{-1} u_R(x), \text{ for } x \in D_R(\Omega), \text{ R large enough.}$$

Proceeding as in the proof the implication (ii) ⇒ (iii) of Theorem 3.13, we construct a positive function f in Ω such that $T_1 f = 0$. Doing this for every connected component of $M \setminus B_{x_0}(R_0)$, we obtain a positive function f defined on $M \setminus B_{x_0}(R_0)$ that satisfies $T_1 f = 0$. Now we set $K = \overline{B_{x_0}(R_0)}$ and extend the function to a positive function f on M. □

4. Proof of Theorem 1.1

We need some more preparations before we can begin with the proof of Theorem 1.1.

Consider the second order ordinary differential equation

$$(v(t)y'(t))' + \lambda v(t)y(t) = 0, \quad t \geq R_0 > 0, \tag{17}$$

where $v(t)$ is a positive continuous function on $[R_0, +\infty)$ and λ is a positive constant.

Definition 4.1. We say that (17) is *oscillatory* if its solutions $y(t)$ have zeros for t arbitrarily large.

The following lemma was proved in [dCZ, Theorem 2.1].

Lemma 4.2. *Assume that $v(t)$ is a positive continuous function on $[R_0, +\infty)$ and that $\int_{T_0}^{+\infty} v(\tau)d\tau = +\infty$. Then (17) is oscillatory provided that one of the following two conditions holds:*

(i) *$\lambda > 0$ and $V(t) = \int_{R_0}^{t} v(\tau)d\tau \leq at^{\alpha}$ for some positive constants a and α.*
(ii) *$\lambda > a^2/4$ and $V(t) = \int_{R_0}^{t} v(\tau)d\tau \leq ae^{t\alpha}$ for some positive constants a and α.*

Theorem 4.3 below generalizes Theorem 3.1 of [dCZ]. It yields estimates on the first eigenvalue of L_1 for M minus a compact set under certain conditions on the growth of the 1-volume of M.

We say that the *1-volume of M has exponential growth* if there exist positive numbers α, R_0 and a such that

$$\int_{B_p(R)} S_1 \, dM \leq ae^{\alpha R} \text{ for any } R \geq R_0.$$

Theorem 4.3. *Assume that M is complete noncompact with infinite 1-volume. Let $\Omega \subset M$ be a compact subset. Then*

(i) *If the 1-volume of M has polynomial growth then $\lambda_1^{L_1}(M \setminus \Omega) = 0$.*
(ii) *If the 1-volume of M has exponential growth then*

$$\lambda_1^{L_1}(M \setminus \Omega) \leq \frac{\alpha^2}{4}(n-1).$$

Proof. Let $T_1 < T_2$ be positive numbers, $p \in M$, and set $A(T_1, T_2) = B_p(T_2) \setminus B_p(T_1)$. Using Stokes' Theorem, Corollary 3.7 and Lemma 2.2 we see that for any $f \in C_0^\infty(A(T_1, T_2))$,

$$\lambda_1^{L_1}(A(T_1, T_2)) \leq \frac{\int_{A(T_1,T_2)} \langle P_1 \nabla f, \nabla f \rangle \, dM}{\int_{A(T_1,T_2)} f^2 \, dM}. \tag{18}$$

The ellipticity of L_1 (equivalently, the positiveness of the eigenvalues of P_1) yields

$$\int_{A(T_1,T_2)} \langle P_1 \nabla f, \nabla f \rangle \, dM \leq \int_{A(T_1,T_2)} \operatorname{trace}(P_1) |\nabla f|^2 \, dM. \tag{19}$$

Using the estimate (19) in (18) and Lemma 2.4(ii) we obtain

$$\lambda_1^{L_1}(A(T_1, T_2)) \leq \frac{\int_{A(T_1,T_2)} (n-1) S_1 |\nabla f|^2 \, dM}{\int_{A(T_1,T_2)} f^2 \, dM}. \tag{20}$$

Let $v(R) = \int_{\partial B_p(R)} S_1 \, ds$, where ds is the volume element of $\partial B_p(R)$. Then,

$$\int_{B_p(R)} S_1 \, dM = \int_0^R v(t) \, dt.$$

Since the 1-volume is infinite, we have $\int_T^{+\infty} v(t) \, dt = +\infty$ for any constant $T > 0$. Since Ω is compact we can find a constant T_0 such that $\Omega \subset B_p(T_0)$.

If (i) holds, Lemma 4.2(i) says that for any $\lambda > 0$ there exists a nontrivial oscillatory solution $y_\lambda(t)$ of (17) on $[R_0, +\infty)$. Thus there exist two numbers $R_1^\lambda < R_2^\lambda$ in $[R_0, +\infty)$ such that $y_\lambda(R_1^\lambda) = y_\lambda(R_2^\lambda) = 0$, and $y_\lambda(t) \neq 0$ for any $t \in (R_1^\lambda, R_2^\lambda)$. Set $R(s) = \operatorname{dist}(s, p)$ and write $\varphi_\lambda(s) = y_\lambda(R(s))$. Using Lemma 2.1 and (20) we obtain

$$\begin{aligned}
\lambda_1^{L_1}(M \setminus \Omega) &\leq \lambda_1^{L_1}\left(A\left(T_1^\lambda, T_2^\lambda\right)\right) \\
&\leq \frac{(n-1) \int_{\left(A\left(T_1^\lambda, T_2^\lambda\right)\right)} S_1 |\nabla \varphi_\lambda|^2 \, dM}{\int_{\left(A\left(T_1^\lambda, T_2^\lambda\right)\right)} |\varphi_\lambda|^2 \, dM} \\
&= \frac{(n-1) \int_{R_1^\lambda}^{R_2^\lambda} \left(y_\lambda'(R)\right)^2 v(R) \, dR}{\int_{R_1^\lambda}^{R_2^\lambda} \left(y_\lambda(R)\right)^2 v(R) \, dR} \\
&= \frac{-(n-1) \int_{R_1^\lambda}^{R_2^\lambda} \left(v(R) y_\lambda'(R)\right)' y_\lambda(R) \, dR}{\int_{R_1^\lambda}^{R_2^\lambda} \left(y_\lambda(R)\right)^2 v(R) \, dR} \\
&= \lambda(n-1).
\end{aligned}$$

By Lemma 2.2 and Stokes' Theorem we have $\lambda_1^{L_1}(M \setminus \Omega) \geq 0$ and therefore

$$0 \leq \lambda_1^{L_1}(M \setminus \Omega) \leq \lambda(n-1).$$

Since λ is an arbitrary positive constant, it follows that $\lambda_1^{L_1}(M \setminus \Omega) = 0$.

If (ii) holds, Lemma 4.2(ii) says that for any $\lambda > \alpha^2/4$ there exists a nontrivial oscillatory solution $y_\lambda(t)$ of (17) on $[R_0, +\infty)$. As in the case (i) we obtain

$$\lambda_1^{L_1}(M \setminus \Omega) \leq \lambda(n-1).$$

Since λ is an arbitrary positive constant larger than $\alpha^2/4$, it follows that

$$\lambda_1^{L_1}(M \setminus \Omega) \leq \frac{\alpha^2}{4}(n-1). \qquad \square$$

We are now ready to prove Theorem 1.1. In fact, Theorem 1.1 follows from the following theorem.

THEOREM 4.4. *Let $x\colon M^n \longrightarrow \overline{M}^{n+1}$ be an isometric immersion of M into an oriented complete Einstein manifold with $H_2 = constant > 0$. Assume that the 1-volume of M is infinite and that $\operatorname{Ind}_1 M < \infty$. Then:*

(i) *If the 1-volume of M has polynomial growth then*

$$H_2^{3/2} \leq -\frac{1}{n(n-1)}\left(\inf_M \left\{\operatorname{trace}\left(P_1 \overline{R}_N\right)\right\}\right).$$

(ii) *If the 1-volume of M has exponential growth then*

$$H_2^{3/2} \leq \frac{\alpha^2}{4n} - \frac{1}{n(n-1)}\left(\inf_M \left\{\operatorname{trace}\left(P_1 \overline{R}_N\right)\right\}\right).$$

Proof. By Proposition 3.16 there exist a compact set K and a positive function f on M such that on $M \setminus K$, f satisfies

$$0 = T_1 f = L_1 f + (S_1 S_2 - 3S_3)\, f + \left\{\operatorname{trace}\left(P_1 \overline{R}_N\right)\right\} f.$$

By Lemma 3.12 we have

$$\begin{aligned} \lambda_1^{L_1}(M \setminus K) &\geq \inf_{M\setminus K}\left(-\frac{L_1(f)}{f}\right) \\ &= \inf_{M\setminus K}\left\{(S_1S_2 - 3S_3) + \left\{\operatorname{trace}\left(P_1\overline{R}_N\right)\right\}\right\} \\ &= \inf_{M\setminus K}\left\{n\tbinom{n}{2} H_1H_2 - 3\tbinom{n}{3} H_3 + \left\{\operatorname{trace}\left(P_1\overline{R}_N\right)\right\}\right\} \\ &\geq \inf_{M}\left\{n\tbinom{n}{2} H_1H_2 - 3\tbinom{n}{3} H_3 + \left\{\operatorname{trace}\left(P_1\overline{R}_N\right)\right\}\right\}. \end{aligned}$$

Proposition 2.3(b) yields

$$\lambda_1^{L_1}(M) \geq \inf_M\left\{n(n-1)H_1H_2 + \left\{\operatorname{trace}\left(P_1\overline{R}_N\right)\right\}\right\}. \tag{21}$$

Using Proposition 2.3(a) we obtain

$$\lambda_1^{L_1}(M) \geq \inf_M\left\{n(n-1)H_2^{3/2} + \left\{\operatorname{trace}\left(P_1\overline{R}_N\right)\right\}\right\}.$$

If (i) is satisfied, then by Theorem 4.3(i) we have

$$0 \geq \inf_M \left\{ n(n-1)H_2^{3/2} + \left\{ \operatorname{trace}\left(P_1 \overline{R}_N\right) \right\} \right\},$$

and since $H_2 = cte$ we obtain

$$H_2^{3/2} \leq -\frac{1}{n(n-1)} \inf_M \left\{ \operatorname{trace}\left(P_1 \overline{R}_N\right) \right\}.$$

If (ii) is satisfied, Theorem 4.3(ii) implies that

$$\frac{\alpha^2(n-1)}{4} \geq \inf_M \left\{ n(n-1)H_2^{3/2} + \left\{ \operatorname{trace}\left(P_1 \overline{R}_N\right) \right\} \right\}.$$

Hence, since $H_2 = cte$, we obtain

$$H_2^{3/2} \leq \frac{\alpha^2}{4n} - \frac{1}{n(n-1)} \inf_M \left\{ \operatorname{trace}\left(P_1 \overline{R}_N\right) \right\}. \qquad \square$$

Corollary 4.5. *Let $x\colon M^n \longrightarrow \overline{M}^{n+1}(c)$ be an isometric immersion with $H_2 = constant > 0$. Assume that* $\operatorname{Ind}_1 M < \infty$ *and that the 1-volume of M is infinite and has polynomial growth. Then c is negative and*

$$H_2^{3/2} \leq -c \inf_M \{H_1\}.$$

Remark 4.6. It follows that there is no hypersurface in Euclidean spaces or in the unit sphere satisfying the hypotheses of Corollary 4.5.

Remark 4.7. If we are willing to restrict ourselves to ambient spaces of constant sectional curvature c, Theorem 1.1, and in fact Corollary 4.5, can be extended to $(r+1)$-mean curvatures with $r > 1$. We point out that in order to guarantee the ellipticity of L_1, $r > 1$, we have to require that M contains a point at which all principal curvatures have the same sign. We also note that the r-volume of M is $\int_M S_r \, dM$ and that ellipticity of L_1 implies $S_r > 0$. The proof is analogous to the case $r = 1$; most details can be found in [E].

To conclude this paper, we give a proof of Theorem 1.2 of the Introduction, which we now recall.

Let $x\colon M^n \longrightarrow \overline{M}^{n+1}(c)$ be an isometric immersion with $H_2 = constant > 0$. Assume that $\operatorname{Ind}_1 M < \infty$ *and that the 1-volume of M is infinite and has polynomial growth. Then c is negative and $H_2 \leq -c$.*

Proof. The result follows from the proof of Theorem 1.1. In fact, by (21) we see that

$$0 \geq \lambda_1^{L_1}(M \setminus K) \geq n(n-1) \inf_M \{H_1 H_2 + c H_1\}.$$

Thus

$$\inf_M \{H_1(H_2 + c)\} \leq 0,$$

and since $H_1 \geq H_2^{1/2}$ we obtain the result. □

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