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Constant-to-one and onto global maps of homomorphisms between strongly connected graphs

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Abstract. The global maps of homomorphisms of directed graphs are very closely related to homomorphisms of a class of symbolic dynamical systems called subshifts of finite type. In this paper, we introduce the concepts of 'induced regular homomorphism' and 'induced backward regular homomorphism' which are associated with every homomorphism between strongly connected graphs whose global map is finite-to-one and onto, and using them we study the structure of constant-to-one and onto global maps of homorphisms between strongly connected graphs and that of constant-to-one and onto homomorphisms of irreducible subshifts of finite type. We determine constructively, up to topological conjugacy, the subshifts of finite type which are constant-to-one extensions of a given irreducible subshift of finite type. We give an invariant for constant-to-one and onto homomorphisms of irreducible subshifts of finite type.

0. *Introduction*

A homomorphism between graphs (the word 'graph' means 'directed graph' throughout this paper), naturally induces a mapping between the bisequence spaces over the graphs, which is called the global map of the homomorphism. The bisequence spaces $\Omega(G)$ over graphs G with the shift homeomorphisms σ on them constitute a class of symbolic dynamical systems $(\Omega(G), \sigma)$ called subshifts of finite type (or topological Markov chains) and hence the global map of a homomorphism of graphs is a homomorphism of subshifts of finite type. The converse of this is almost valid by the theorem of Curtis, Hedlund and Lyndon [**8**]. Therefore, many properties of the global maps of homomorphisms of graphs can straightforwardly be interpreted as those of homomorphisms of subshifts of finite type, so that the study of the global maps of homomorphisms of graphs provides useful combinatorial approaches to that of homomorphisms of subshifts of finite type (cf. [**17**]). In fact, in [**1**] and others, a notion similar to that of a homomorphism of graphs was used as a one-block map together with other graph theoretical notions for the study of homomorphisms of subshifts of finite type.

On the other hand, the global maps of homomorphisms of graphs and homomorphisms of subshifts of finite type can be considered as a new area of graph theory which investigates relations between graphs, especially in connection with the spectral properties of the adjacency matrices of graphs. In fact, [**17**] treated this, and moreover the results in the classification theories for subshifts of finite type of [**21**], [**18**], and [**1**] and the results on homomorphisms of subshifts of finite type of [**13**] and [**10**] can be interpreted as results in graph theory concerning the above area.

In this paper, we introduce the concepts of 'induced regular homomorphism' and 'induced backward regular homomorphism' which are associated with every homomorphism between strongly connected graphs whose global map is finite-to-one and onto. Using them we study constant-to-one and onto global maps of homomorphisms between strongly connected graphs and constant-to-one and onto homomorphisms of irreducible subshifts of finite type. (The term 'constant-to-one' means 'k-to-one for some k'.) We give some necessary and sufficient conditions for the global map of a homomorphism between strongly connected graphs to be constant-to-one and onto, one of which immediately gives a structure result for constant-to-one and onto homomorphisms of irreducible subshifts of finite type (corollary 6.4). Using this we obtain our main theorem (theorem 7.3) which determines constructively, up to topological conjugacy, the subshifts of finite type which are constant-to-one extensions of a given irreducible subshift of finite type (that is, the subshifts of finite type such that there are constant-to-one and onto homomorphisms from them to a given irreducible subshift of finite type). It is also shown that if there exists a constant-to-one and onto homomorphism from a subshift of finite type $(\Omega(G_1), \sigma_1)$ to an irreducible subshift of finite type $(\Omega(G_2), \sigma_2)$, then the elementary divisors not divisible by λ (the indeterminate) of the adjacency matrix of G_2 are contained in the elementary divisors of the adjacency matrix of G_1.

Many extended notions, techniques and results of those in [**8**] appear in this paper; the reader is assumed to be familiar with [**8**].

Many statements of the theorems, propositions, and lemmas after § 2 contain second versions. But proofs will be given only for the first versions because the proofs of the second versions are similar.

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1. *Background*

A *graph* (directed graph with labelled points and labelled arcs) G is defined to be a triple (P, A, ζ) where P is a finite set of elements called *points*, A is a finite set of elements called *arcs* and ζ is a mapping of A into $P \times P$. If $\zeta(a) = (u, v)$ for $a \in A$ and $u, v \in P$, then u and v are the *initial endpoint* of a and the *terminal endpoint* of a, respectively, which are denoted by $\mathrm{i}(a)$ and $\mathrm{t}(a)$, respectively.

A sequence $x = a_1 \cdots a_p$ $(p \geq 1)$ with $a_i \in A$, $i = 1, \ldots, p$, is a *path of length p* in G if

$$\mathrm{t}(a_i) = \mathrm{i}(a_{i+1}) \qquad \text{for } i = 1, \ldots, p-1.$$

We call $\mathrm{i}(a_1)$ and $\mathrm{t}(a_p)$ the *initial endpoint* of x and the *terminal endpoint* of x, respectively. Every point u of G is *a path of length* 0 in G whose initial and terminal endpoint is u. For any path x in G, we denote by $\mathrm{i}(x)$ and $\mathrm{t}(x)$ the initial endpoint of x and the terminal endpoint of x, respectively, and if $\mathrm{i}(x) = u$ and $\mathrm{t}(x) = v$, then we often say that x *goes from* u *to* v. The set of all paths in G is denoted by $\Pi(G)$. The set of all paths of length $p \geq 0$ in G is denoted by $\Pi^{(p)}(G)$.

Standing hypothesis. Throughout the remainder of this paper, we assume that a graph has at least one point and for each point u, there exists at least one arc going to u and at least one arc going from u.

Let $\mathbb{Z}$ be the set of integers. Let $G=(P, A, \zeta)$ be a graph. A mapping $\alpha: \mathbb{Z} \to A$ is a *bisequence* over G if

$$t(\alpha(i)) = i(\alpha(i+1)) \qquad \text{for all } i \in \mathbb{Z}.$$

Let $\Omega(G)$ denote the set of all bisequences over G. If $\alpha \in \Omega(G)$ and $i \in \mathbb{Z}$, then $\alpha(i)$ will often be denoted by α_i. For $\alpha \in \Omega(G)$ and $i, j \in \mathbb{Z}$ with $i \leq j$, let

$$\alpha[i, j] = \alpha_i \alpha_{i+1} \cdots \alpha_j.$$

Clearly $\alpha[i, j] \in \Pi^{(j-i+1)}(G)$. We define a metric d on $\Omega(G)$ as follows: let $\alpha, \beta \in \Omega(G)$,

$$d(\alpha, \beta) = 0 \qquad \text{if } \alpha = \beta,$$
$$d(\alpha, \beta) = (1+k)^{-1} \qquad \text{if } \alpha \neq \beta.$$

where

$$k = \min\{i \geq 0 \mid \alpha[-i, i] \neq \beta[-i, i]\}.$$

With this metric, $\Omega(G)$ is compact.

Let $G_1=(P, A, \zeta_1)$ and $G_2=(Q, B, \zeta_2)$ be graphs. A *homomorphism* of G_1 into G_2 is a pair (h, ϕ) of mappings $h: A \to B$ and $\phi: P \to Q$ such that for any $a \in A$, if $\zeta_1(a) = (u, v)$ with $u, v \in P$, then

$$\zeta_2(h(a)) = (\phi(u), \phi(v)).$$

By our standing hypothesis for graphs, the homomorphisms (h, ϕ) of G_1 into G_2 is uniquely determined by h. Therefore, we say that h is a homomorphism of G_1 into G_2 and we denote by ϕ_h the unique mapping ϕ such that (h, ϕ) is a homomorphism of G_1 into G_2.

A homomorphism $h: A \to B$ of a graph $G_1=(P, A, \zeta_1)$ into a graph $G_2=(Q, B, \zeta_2)$ is naturally extended to a mapping

$$h^*: \Pi(G_1) \to \Pi(G_2).$$

That is, we define $h^*: \Pi(G_1) \to \Pi(G_2)$ as follows: for each $x \in \Pi(G_1)$, if the length of x is 0, i.e. x is a point of G_1, then $h^*(x) = \phi_h(x)$, and if $x = a_1 \cdots a_p$ ($p \geq 1$) with $a_i \in A$, $i = 1, \ldots, p$, then $h^*(x) = h(a_1) \cdots h(a_p)$. The mapping h^* is called the *extension* of h. Another mapping is naturally induced by h. We define

$$h_\infty: \Omega(G_1) \to \Omega(G_2)$$

as follows: for $\alpha \in \Omega(G_1)$, $h_\infty(\alpha) = \beta$, where $\beta_i = h(\alpha_i)$ for all $i \in \mathbb{Z}$. We call h_∞ the *global map* of h. A graph $G=(P, A, \zeta)$ is *strongly connected* if for any $u, v \in P$, there exists a path going from u to v.

For a positive integer k, a mapping $f: X \to Y$ is k*-to-one* if $|f^{-1}(y)| = k$ for all $y \in f(X)$. A mapping $f: X \to Y$ is *constant-to-one* if there exists a positive integer k such that f is k-to-one; *uniformly finite-to-one* if there exists a positive integer k such that $|f^{-1}(y)| \leq k$ for all $y \in Y$; and *finite-to-one* if $|f^{-1}(y)| < \infty$ for all $y \in Y$.

Let h be a homomorphism of a graph G_1 into a graph G_2. Two paths x and y in G_1 are *indistinguishable by h* if

$$\mathrm{i}(x)=\mathrm{i}(y), \quad \mathrm{t}(x)=\mathrm{t}(y), \quad \text{and} \quad h^*(x)=h^*(y).$$

The following results are found in [**17**]. Similar results also appear in [**1**].

PROPOSITION 1.1. *Let h be a homomorphism of a strongly connected graph G_1 into a graph G_2. Then the following statements are equivalent.*

(1) *No two distinct paths in G_1 are indistinguishable by h.*

(2) *h^* is uniformly finite-to-one.*

(3) *h_∞ is uniformly finite-to-one.*

(4) *h_∞ is finite-to-one.*

PROPOSITION 1.2. *For any homomorphism h of a graph G_1 into a graph G_2, h^* is onto iff h_∞ is onto.*

For a graph G, we denote by $M(G)$ the *adjacency matrix* of G (i.e. if G has n points $u_1, \ldots, u_n$, then $M(G)$ is the square matrix (m_{ij}) of order n such that m_{ij} is the number of arcs going from u_i to u_j). Since $M(G)$ is a non-negative matrix, by the Perron–Frobenius Theorem, $M(G)$ has the non-negative characteristic value that the moduli of all the other characteristic values do not exceed (cf. [**6**, Vol. II]). We denote by $r(G)$ that 'maximal' characteristic value of $M(G)$.

The following result is found in [**17**]. In view of the above propositions and the facts stated later in this section, it is essentially the same as the well-known result on symbolic flows (see, e.g., [**1**]) that a finite-to-one and onto homomorphism (of symbolic flows) between subshifts of finite type preserves topological entropy.

PROPOSITION 1.3. *If there is a homomorphism h of a graph G_1 into a graph G_2 with h^* uniformly finite-to-one and onto, then $r(G_1)=r(G_2)$.*

In [**17**], a stronger result has been given. That is, it has been proved there that with the same condition as in proposition 1.3, not only $r(G_1)=r(G_2)$ but also the characteristic polynomial of $M(G_2)$ divides that of $M(G_1)$. Furthermore, Kitchens ([**10**]) has given a still stronger result. He has proved that if G_1 and G_2 are strongly connected graphs and $M(G_1)$ and $M(G_2)$ are 0–1 matrices and if there is a homomorphism h of G_1 into G_2 with h_∞ finite-to-one and onto, then the block of the Jordan form of $M(G_2)$ is a principal submatrix of the Jordan form of $M(G_1)$. The condition that $M(G_1)$ and $M(G_2)$ are 0–1 matrices can be eliminated.

The following result has been proved in [**17**] using a graph-theoretical method. In view of propositions 1.1 and 1.2, it can also follow from a result in [**4**].

THEOREM 1.4. *Let G_1 and G_2 be two strongly connected graphs with $r(G_1)=r(G_2)$. Then for any homomorphism h of G_1 into G_2, h^* is uniformly finite-to-one iff h^* is onto.*

Let G_1 and G_2 be strongly connected graphs and let h be a homomorphism of G_1 into G_2. Then, by the above propositions and theorem 1.4, we have many equivalent statements; the following are several of them.

(1) $r(G_1)=r(G_2)$ and h^* is onto.

(2) h^* is uniformly finite-to-one and onto.

(3) h_∞ is finite-to-one and onto.

(4) h^* is onto and there exist no two distinct paths which are indistinguishable by h.

(5) $r(G_1)=r(G_2)$ and there exist no two distinct paths which are indistinguishable by h.

We will use (1) as a representative of these and the other equivalent statements in most statements of conditions in our results. Moreover the equivalence of them will be used frequently without reference.

Let $G=(P, A, \zeta)$ be a graph. For any non-negative integer p, we define a graph $L^{(p)}(G)$ as follows: $L^{(0)}(G)=G$. For $p\geq 1$, $L^{(p)}(G)=(\Pi^{(p)}(G), \Pi^{(p+1)}(G), \zeta^{(p)})$, where $\zeta^{(p)}(a_1\cdots a_{p+1})=(a_1\cdots a_p, a_2\cdots a_{p+1})$ for $a_1\cdots a_{p+1}\in\Pi^{(p+1)}(G)$ with $a_i\in A$, $i=1,\ldots,p+1$. We call $L^{(p)}(G)$ the *path graph of length p* of G. ($L^{(1)}(G)$ is usually known as the *line digraph* of G (cf. [**7**]) or the *adjoint* of G (cf. [**2**]).) Essentially the same notion as $L^{(p)}(G)$ was also used for 'higher block system' of [**1**].) Clearly, if G is strongly connected, then $L^{(p)}(G)$ is strongly connected for all $p\geq 0$. For any integers p and q with $p\geq q\geq 1$, we define a mapping

$$h_{G,p,q}:\Pi^{(p)}(G)\to A$$

as follows. For any $a_1\cdots a_p\in\Pi^{(p)}(G)$ with $a_i\in A$,

$$h_{G,p,q}(a_1\cdots a_p)=a_q.$$

Then clearly $h_{G,p,q}$ is a homomorphism of $L^{(p-1)}(G)$ into G and $(h_{G,p,q})^*$ is uniformly finite-to-one and onto. Hence by proposition 1.3,

$$r(L^{(p-1)}(G))=r(G).$$

Furthermore $(h_{G,p,q})_\infty$ is a homeomorphism of $\Omega(L^{(p-1)}(G))$ onto $\Omega(G)$.

Let A be a finite non-empty set (of symbols). Let $G_0(A)$ be the graph defined by

$$G_0(A)=(\{\Lambda\}, A, \zeta_A)$$

where $\zeta_A(a)=(\Lambda,\Lambda)$ for all $a\in A$. Then $G_0(A)$ is a strongly connected graph having only one point Λ and each element of A is an arc (loop) going from Λ to itself. Clearly $\Pi(G_0(A))$ is the set of all finite sequences of elements of A and

$$\Omega(G_0(A))=A^{\mathbf{Z}}.$$

Let $\Omega_A=A^{\mathbf{Z}}$. Each element of Ω_A is a *bisequence* over A. Of course, Ω_A is a compact metric space with the metric defined before. The homeomorphism $\sigma:\Omega_A\to\Omega_A$ defined by

$$(\sigma(\alpha))_i=\alpha_{i+1},\qquad \alpha\in\Omega_A,\quad i\in\mathbb{Z},$$

is called the *shift*. The dynamical system (Ω_A,σ) is called the *full shift system* over A. Let X be a closed non-empty subset of Ω_A such that $\sigma^{-1}(X)=X$. The dynamical system (X,σ) is called a *subdynamical system* of (Ω_A,σ) or a *symbolic flow* over A. (For simplicity, we denote $\sigma|X$ by σ.)

Let $G=(P, A, \zeta)$ be a graph. Then $(\Omega(G), \sigma)$ is a subdynamical system of (Ω_A, σ). A symbolic flow of this type is called a *subshift of finite type*, (cf. [**21**] and [**19**]). If G is strongly connected, then $(\Omega(G), \sigma)$ is an *irreducible subshift of finite type.* (Cf. [**3**]. Note that G is strongly connected iff the adjacency matrix $M(G)$ of G is irreducible.)

Let (X, σ_1) and (Y, σ_2) be two symbolic flows. A *homomorphism* $\pi:(X, \sigma_1)\to(Y, \sigma_2)$ is a continuous mapping of X into Y such that $\pi\sigma_1=\sigma_2\pi$. We say that (X, σ_1) and (Y, σ_2) are *topologically conjugate* if there exists an isomorphism of (X, σ_1) onto (Y, σ_2).

Clearly global maps of homomorphisms of graphs are homomorphisms of symbolic flows. The converse is almost valid.

Let $G_1=(P, A, \zeta_1)$ and $G_2=(Q, B, \zeta_2)$ be two graphs and let p be a positive integer. A mapping $f:\Pi^{(p)}(G_1)\to B$ is an *admissible p-block map* if for any $a_1\cdots a_{p+1}\in\Pi^{(p+1)}(G_1)$ with $a_1,\ldots, a_{p+1}\in A$,

$$\mathrm{t}(f(a_1\cdots a_p))=\mathrm{i}(f(a_2\cdots a_{p+1})).$$

Corresponding to any admissible p-block map $f:\Pi^{(p)}(G_1)\to B$, we define a mapping $f_\infty:\Omega(G_1)\to\Omega(G_2)$ by

$$f_\infty(\alpha)=\beta \qquad \text{where } \beta_i=f(\alpha_i\cdots\alpha_{i+p-1})$$

for all $i\in\mathbb{Z}$. Clearly an admissible 1-block map f is a homomorphism of G_1 into G_2 and f_∞ is its global map. The well-known theorem of Curtis, Hedlund & Lyndon [**8**] for homomorphisms of symbolic flows (as pointed out by Klein [**12**]) implies the following.

THEOREM 1.5. (Curtis, Hedlund & Lyndon) *Let G_1 and G_2 be graphs. Then a mapping $\pi:\Omega(G_1)\to\Omega(G_2)$ is a homomorphism of $(\Omega(G_1), \sigma_1)$ into $(\Omega(G_2), \rho_2)$ iff there exist integers $p\geq 1$ and k and an admissible p-block map*

$$f:\Pi^{(p)}(G_1)\to\Pi^{(1)}(G_2)$$

such that

$$\pi=\sigma^{-k}f_\infty.$$

The following, which appears in [**17**], is a graph-theoretical interpretation of the above theorem.

COROLLARY 1.6. *Let G_1 and G_2 be graphs. Then a mapping $\pi:\Omega(G_1)\to\Omega(G_2)$ is a homomorphism of $(\Omega(G_1), \sigma_1)$ into $(\Omega(G_2), \sigma_2)$ iff there exist integers p and q with $p\geq q\geq 1$ and a homomorphism h of $L^{(p-1)}(G_1)$ into G_2 such that*

$$\pi=h_\infty(h_{G_1,p,q})_\infty^{-1}.$$

We remark that for a graph G and integers p and q with $p\geq q\geq 1$, $(h_{G,p,q})_\infty$ is an isomorphism of $(\Omega(L^{(p-1)}(G)), \sigma')$ onto $(\Omega(G), \sigma)$.

2. *Regular homomorphisms and biregular homomorphisms*

A homomorphism h of a graph G_1 into a graph G_2 is *regular* [*backward-regular* (abbreviated to *b-regular*)] if for each point u of G_1 and for each arc b going from

[to] $\phi_h(u)$, there exists exactly one arc a going from [to] u with $h(a)=b$. The same notions appear in [**1**] as 'right-resolving' ['left-resolving'] together with the following.

PROPOSITION 2.1. *If h is a regular [b-regular] homomorphism of a graph G_1 into a strongly connected graph G_2, then h^* and h_∞ are uniformly finite-to-one and onto and $r(G_1)=r(G_2)$.*

A homomorphism h is *biregular* if h is both regular and b-regular.

PROPOSITION 2.2. *Let h be a biregular homomorphism of a graph $G_1=(P, A, \zeta_1)$ into a strongly connected graph $G_2=(Q, B, \zeta_2)$. Then h^* and h_∞ are $|P|/|Q|$-to-one and onto.*

Proof. By proposition 2.1, h^* and h_∞ are onto. Since h is regular, it follows that

$$|(h^*)^{-1}(y)|=|\phi_h^{-1}(\mathrm{i}(y))| \qquad \text{for all } y\in\Pi(G_2).$$

Since h is b-regular, it also follows that

$$|(h^*)^{-1}(y)|=|\phi_h^{-1}(\mathrm{t}(y))| \qquad \text{for all } y\in\Pi(G_2).$$

Let $v_1, v_2\in Q$. Since G_2 is strongly connected, there exists a path y in G_2 going from v_1 to v_2. Hence, by the above,

$$|\phi_h^{-1}(v_1)|=|(h^*)^{-1}(y)|=|\phi_h^{-1}(v_2)|.$$

Thus, for all $v\in Q$, $|\phi_h^{-1}(v)|=|P|/|Q|$, and hence for all $y\in\Pi(G_2)$,

$$|(h^*)^{-1}(y)|=|P|/|Q|.$$

Thus h^* is $|P|/|Q|$-to-one.

For each $\beta\in\Omega(G_2)$,

$$|h_\infty^{-1}(\beta)|\leq(h^*)^{-1}(\beta[-i, i])$$

for some sufficiently large integer $i\geq 0$. Hence $|h_\infty^{-1}(\beta)|\leq|P|/|Q|$. Thus it suffices to show that for each $\beta\in\Omega(G_2)$, there exist at least $|P|/|Q|$ bisequences α such that $h_\infty(\alpha)=\beta$. This is proved in a similar way to that used in the proof of theorem 6.7 of Hedlund [**8**]. □

For a graph G, we call a graph G_1 such that there exists a biregular homomorphism h of G_1 into G a *biregular extension* of G. Given a strongly connected graph G, it is easy to determine all biregular extensions G_1 and biregular homomorphisms h of G_1 into G.

Let h be a biregular homomorphism of a graph $G_1=(P, A, \zeta_1)$ into a strongly connected graph $G_2=(Q, B, \zeta_2)$. Since h is regular, for each $b\in B$, we can define a mapping

$$\mu_b: \phi_h^{-1}(\mathrm{i}(b))\to\phi_h^{-1}(\mathrm{t}(b))$$

as follows. For each $u\in\phi_h^{-1}(\mathrm{i}(b))$, define $\mu_b(u)=\mathrm{t}(a)$, where a is the unique arc of G_1 such that $\mathrm{i}(a)=u$ and $h(a)=b$. Since h is b-regular, it follows that μ_b is a bijection. Since G_2 is strongly connected, by proposition 2.2 there exists a positive integer k such that $|\phi_h^{-1}(v)|=k$ for all $v\in Q$. Thus for each $b\in B$, μ_b is a bijection of a k-point set onto a k-point set.

Let the adjacency matrix $M(G_2)$ of G_2 be (m_{ij}). Then it is easy to see that the adjacency matrix of G_1 can be written in the form

$$M(G_1)=\begin{pmatrix} M_{11} & \cdots & M_{1q} \\ & \cdots & \\ M_{q1} & \cdots & M_{qq} \end{pmatrix}$$

where $q=|Q|$ and M_{ij} is the square matrix of order k obtained by summing the m_{ij} permutation matrices corresponding to the μ_b's such that b's are the arcs of G_2 going from point $i\in Q$ to point $j\in Q$. (If $m_{ij}=0$, then M_{ij} is the zero-matrix of order k.)

Conversely, if we are given an assignment of some permutation μ_b on $\{1,\ldots,k\}$ to each arc $b\in B$ of a strongly connected graph $G_2=(Q,B,\zeta_2)$, then we can straightforwardly obtain a biregular extension of G_2 and a biregular homomorphism h of G_1 into G_2. (Let the set of points of G_1 be $\{(i,j)\,|\,i\in Q, 1\le j\le k\}$ and for each $b\in B$, make k arcs a_{bj}, $j=1,\ldots,k$, of G_1 such that

$$\mathrm{i}(a_{bj})=(\mathrm{i}(b),j),\quad \mathrm{t}(a_{bj})=(\mathrm{t}(b),\mu_b(j)),\quad \text{and}\quad h(a_{bj})=b.)$$

Thus we have the following proposition.

PROPOSITION 2.3. *Let $G=(Q,B,\zeta)$ be a strongly connected graph with $|Q|=q$ and $M(G)=(m_{ij})$. Then G_1 is a biregular extension of G iff for some positive integer k, $M(G_1)$ is written as a square matrix of order qk of the form*

$$M(G_1)=\begin{pmatrix} M_{11} & \cdots & M_{1q} \\ & \cdots & \\ M_{q1} & \cdots & M_{qq} \end{pmatrix}$$

where M_{ij} is the sum of some m_{ij} permutation matrices of order k.

A rectangular 0–1 matrix with non-zero columns and with exactly one 1 in each row, is called an *amalgamation matrix*.

Let h be a homomorphism of a graph G_1 with m points, $u_1,\ldots,u_m$, into a graph G_2 with n points $v_1,\ldots,v_n$, and let ϕ_h be onto. Let R be the $m\times n$ matrix with $R=(r_{ij})$ where $r_{ij}=1$ if $\phi_h(u_i)=v_j$ and otherwise $r_{ij}=0$ for $i=1,\ldots,m$ and $j=1,\ldots,n$. Clearly R is an amalgamation matrix. We call R the *amalgamation matrix associated with ϕ_h*.

A similar result to the following appears in [**18**].

PROPOSITION 2.4. *Let G_1 and G_2 be graphs. If h is a regular [b-regular] homomorphism of G_1 into G_2, then $M(G_1)R=RM(G_2)$ [$R^tM(G_1)=M(G_2)R^t$], where R is the amalgamation matrix associated with ϕ_h. [R^t denotes the transpose of R.] Conversely if R is an amalgamation matrix satisfying $M(G_1)R=RM(G_2)$ [$R^tM(G_1)=M(G_2)R^t$], then there exists a regular [b-regular] homomorphism h of G_1 into G_2 such that R is the amalgamation matrix associated with ϕ_h.*

Proof. Let $G_1=(P,A,\zeta_1)$ and $G_2=(Q,B,\zeta_2)$ with $P=\{u_1,\ldots,u_m\}$ and $Q=\{v_1,\ldots,v_n\}$. For any $1\le i,j\le m$, let A_{ij} be the set of arcs of G_1 going from u_i to u_j. For any $1\le k,l\le n$, let B_{kl} be the set of arcs of G_2 going from v_k to v_l.

Let h be a regular homomorphism of G_1 into G_2 and let R be the amalgamation matrix associated with ϕ_h. Since h is a regular homomorphism, for any i, j with $1 \le i \le m$ and $1 \le j \le n$, we have

$$\sum_{u_l \in \phi_h^{-1}(v_j)} |A_{il}| = |B_{k_i j}|$$

where k_i is the index such that $\phi_h(u_i) = v_{k_i}$. Since the left-hand-side of the above equation equals the (i, j) entry of $M(G_1)R$ and the right-hand-side equals the (i, j) entry of $RM(G_2)$, we have $M(G_1)R = RM(G_2)$.

Conversely, assume that $R = (r_{ij})$ is an amalgamation matrix such that $M(G_1)R = RM(G_2)$. Let $\phi: P \to Q$ be the mapping such that $r_{ij} = 1$ iff $\phi(u_i) = v_j$, $(i = 1, \ldots, m, j = 1, \ldots, n)$. Then since $M(G_1)R = RM(G_2)$, for any i, j with $1 \le i \le m$ and $1 \le j \le n$, we have

$$\sum_{u_l \in \phi^{-1}(v_j)} |A_{il}| = |B_{k_i j}|$$

where k_i is the index such that $\phi(u_i) = v_{k_i}$. For any i, j with $1 \le i \le m$ and $1 \le j \le n$, let

$$h_{ij}: \bigcup_{u_l \in \phi^{-1}(v_j)} A_{il} \to B_{k_i j}$$

be any bijection. Let $h: A \to B$ be defined as follows: $h(a) = h_{ij}(a)$ if $a \in \bigcup_{u_l \in \phi^{-1}(v_j)} A_{il}$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. It is easy to see that h is a regular homomorphism of G_1 into G_2 with $\phi_h = \phi$. □

PROPOSITION 2.5. *Let G_1 and G_2 be graphs such that G_1 is a biregular extension of G_2. Then the elementary divisors of $M(G_2)$ is contained in the elementary divisors of $M(G_1)$.*

Proof. By proposition 2.4, there is an amalgamation matrix R such that

$$M(G_1)R = RM(G_2) \quad \text{and} \quad R^t M(G_1) = M(G_2)R^t.$$

Since R is an amalgamation matrix, the columns of R are non-zero and any distinct two of them are orthogonal. Hence if R is a $p \times q$ matrix, we can choose a $p \times (p-q)$ matrix S with non-zero columns such that any two distinct columns of the $p \times p$ matrix T of the form $(R \quad S)$ are orthogonal. That is, $R^t S = 0$, and $T^t T$ and $S^t S$ together with $R^t R$ are diagonal matrices. Therefore, since $M(G_1)R = RM(G_2)$ and $R^t M(G_1) = M(G_2)R^t$, it follows that

$$\begin{aligned}
T^{-1}M(G_1)T &= (T^tT)^{-1}T^tM(G_1)T \\
&= (T^tT)^{-1}\begin{pmatrix} R^t \\ S^t \end{pmatrix} M(G_1)(R \quad S) \\
&= (T^tT)^{-1}\begin{pmatrix} R^tM(G_1)R & R^tM(G_1)S \\ S^tM(G_1)R & S^tM(G_1)S \end{pmatrix} \\
&= \begin{pmatrix} (R^tR)^{-1} & 0 \\ 0 & (S^tS)^{-1} \end{pmatrix}\begin{pmatrix} R^tRM(G_2) & M(G_2)R^tS \\ S^tRM(G_2) & S^tM(G_1)S \end{pmatrix} \\
&= \begin{pmatrix} M(G_2) & 0 \\ 0 & M_1 \end{pmatrix}
\end{aligned}$$

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where $M_1 = (S^tS)^{-1}S^tM(G_1)S$. Thus the result follows. (See [**6**, Vol. I, Chap. VI, theorem 5.) □

3. *A maximal compatible set is a minimal complete set*

Let $G_1 = (P, A, \zeta_1)$ and $G_2 = (Q, B, \zeta_2)$ be two graphs and let h be a homomorphism of G_1 into G_2. Let $U \subset P$ and let $y \in \Pi(G_2)$. Define

$$\mathrm{C}_h(U, y) = \{\mathrm{t}(x) \mid x \in \Pi(G_1), \quad \mathrm{i}(x) \in U, \quad h^*(x) = y\}$$

and

$$\bar{\mathrm{C}}_h(y, U) = \{\mathrm{i}(x) \mid x \in \Pi(G_1), \quad \mathrm{t}(x) \in U, \quad h^*(x) = y\}.$$

For $u \in P$ and $y \in \Pi(G_2)$, we denote $\mathrm{C}_h(\{u\}, y)$ $[\bar{\mathrm{C}}_h(y, \{u\})]$ by $\mathrm{C}_h(u, y)$ $[\bar{\mathrm{C}}_h(y, u)]$. A subset U of P is called a *compatible set* [*a backward-compatible* (abbreviated to *b-compatible*) *set*] *for h* if $U = \mathrm{C}_h(u, y)$ $[U = \bar{\mathrm{C}}_h(y, u)]$ for some $u \in P$ and $y \in \Pi(G_2)$. A subset U of P is called a *complete set* [*a backward-complete* (abbreviated to *b-complete*) *set*] *for h*, if there exists $v \in Q$ such that $U \subset \phi_h^{-1}(v)$ and $\mathrm{C}_h(U, y) \neq \varnothing$ $[\bar{\mathrm{C}}_h(y, U) \neq \varnothing]$ for all $y \in \Pi(G_2)$ with $\mathrm{i}(y) = v$ $[\mathrm{t}(y) = v]$.

LEMMA 3.1. *Let G_1 and G_2 be graphs and let h be a homomorphism of G_1 into G_2. If h^* is onto, then there exists a compatible [b-compatible] set for h which is a complete [b-complete] set for h.*

Proof. Assume that h^* is onto but that any compatible set for h is not a complete set for h. Let v be a point of G_2. Since h^* is onto, $\phi_h^{-1}(v)$ is a complete set for h. Let $\phi_h^{-1}(v) = \{u_1, \ldots, u_p\}$. Since a compatible set $\{u_1\}$ is not a complete set, there exists $y_1 \in \Pi(G_2)$ such that $\mathrm{i}(y_1) = v$ and $\mathrm{C}(u_1, y_1) = \varnothing$. (If h is understood, we shall often omit 'for h' and the suffix h of $\mathrm{C}_h(u, y)$.) Since a compatible set $\mathrm{C}(u_2, y_1)$ is not a complete set, there exists $y_2 \in \Pi(G_2)$ such that $\mathrm{i}(y_2) = \mathrm{t}(y_1)$ and

$$\mathrm{C}(u_2, y_1y_2) = \mathrm{C}(\mathrm{C}(u_2, y_1), y_2) = \varnothing.$$

Proceeding in this way, we have $y_1, \ldots, y_p$ in $\Pi(G_2)$ such that $y_1 \cdots y_p \in \Pi(G_2)$ and

$$\mathrm{C}(u_i, y_1 \cdots y_i) = \varnothing \quad \text{for } i = 1, \ldots, p.$$

Hence we have

$$\mathrm{C}(\phi_h^{-1}(v), y_1 \cdots y_p) = \bigcup_{i=1}^{p} \mathrm{C}(u_i, y_1 \cdots y_p) = \varnothing,$$

which is a contradiction. □

THEOREM 3.2. *Let $G_1 = (P, A, \zeta_1)$ and $G_2 = (Q, B, \zeta_2)$ be two strongly connected graphs with $r(G_1) = r(G_2)$. Let h be a homomorphism of G_1 into G_2 with h^* onto. Then every maximal compatible [b-compatible] set for h is a minimal complete [b-complete] set for h.*

Proof. Let U be a maximal compatible set for h. Then there exists $u \in P$ and $y \in \Pi(G_2)$ such that $U = \mathrm{C}(u, y)$. By lemma 3.1, there exists a complete set written as $\mathrm{C}(v, z)$ with $v \in P$ and $z \in \Pi(G_2)$. Let $v' \in \mathrm{C}(v, z)$. Since G_1 is strongly connected, there exists a path x going from v' to u. Clearly

$$\mathrm{C}(v, zh^*(x)y) \supset U.$$

Since U is a maximal compatible set, $U = \mathrm{C}(v, zh^*(x)y)$. Since $\mathrm{C}(v, z)$ is a complete set,

$$U = \mathrm{C}(\mathrm{C}(v, z), h^*(x)y).$$

is a complete set.

Assume that there exists $s \in U$ such that $U' = U - \{s\}$ is a complete set for h. Let $\mathrm{C}(t, w)$ be a maximal compatible set with $t \in P$ and $w \in \Pi(G_2)$. Since G_1 is strongly connected, there exists a path x' in G_1 going from s to t. Clearly we have

$$\mathrm{C}(t, w) \subset \mathrm{C}(s, h^*(x')w) \subset \mathrm{C}(U, h^*(x')w).$$

Since U is a compatible set, $\mathrm{C}(U, h^*(x')w)$ is a compatible set. Since $\mathrm{C}(t, w)$ is a maximal compatible set, it follows that $\mathrm{C}(t, w) = \mathrm{C}(U, h^*(x')w)$, so that

$$\mathrm{C}(s, h^*(x')w) = \mathrm{C}(U, h^*(x')w).$$

Since U' is a complete set, $\mathrm{C}(U', h^*(x')w) \neq \varnothing$. Hence there exist $s' \in U'$ and $p \in \mathrm{C}(U', h^*(x')w)$ and a path x_1 going from s' to p with

$$h^*(x_1) = h^*(x')w.$$

Since $\mathrm{C}(U', h^*(x')w) \subset \mathrm{C}(U, h^*(x')w) = \mathrm{C}(s, h^*(x')w)$, $p \in \mathrm{C}(s, h^*(x')w)$. Hence there exists a path x_2 going from s to p with

$$h^*(x_2) = h^*(x')w.$$

Since U is a compatible set and $s, s' \in U$, there exist two paths x_3 and x_4 in G_1 such that

$$\mathrm{i}(x_3) = \mathrm{i}(x_4), \quad \mathrm{t}(x_3) = s', \quad \mathrm{t}(x_4) = s \quad \text{and} \quad h^*(x_3) = h^*(x_4).$$

Hence x_3x_1 and x_4x_2 are two distinct paths in G_1 which are indistinguishable by h. This is a contradiction (see § 1). Thus we conclude that U is a minimal complete set. □

As a corollary of the above theorem, we have the following basic result, which can be viewed as a generalization of a result of L. R. Welch, [**8**, theorem 14.4]. (Cf. [**14**, lemma 2].)

COROLLARY 3.3. *Let G_1 and G_2 be two strongly connected graphs with $r(G_1) = r(G_2)$. Let h be a homomorphism of G_1 into G_2 with h^* onto. If U is a maximal compatible [b-compatible] set for h, then for any path y in G_2 with $\mathrm{i}(y) \in \phi_h(U)[\mathrm{t}(y) \in \phi_h(U)]$, $\mathrm{C}_h(U, y)$ $[\bar{\mathrm{C}}_h(y, U)]$ is a maximal compatible [b-compatible] set for h.*

Proof. Let U be a maximal compatible set for h. Let y be a path in G_2 with $\mathrm{i}(y) \in \phi_h(U)$. From theorem 3.2, U is a complete set. Hence $\mathrm{C}(U, y)$ is a complete set. Since U is a compatible set, $\mathrm{C}(U, y)$ is a compatible set. Let V be a maximal compatible set such that $V \supset \mathrm{C}(U, y)$. Then from theorem 3.2, V is a minimal complete set. Therefore since $\mathrm{C}(U, y)$ is a complete set, we have $V = \mathrm{C}(U, y)$. Thus $\mathrm{C}(U, y)$ is a maximal compatible set. □

4. *Induced regular and b-regular homomorphisms*

By virtue of corollary 3.3, we can introduce the notions of 'induced regular homomorphism' and 'induced b-regular homomorphism' which are associated with every homomorphism h between two strongly connected graphs such that h^* is uniformly finite-to-one and onto. (These are a generalization of 'right λ-bundle-graph' and 'left λ-bundle-graph' of [**14**].)

Throughout this section, we assume that $G_1 = (P, A, \zeta_1)$ and $G_2 = (Q, B, \zeta_2)$ are two strongly connected graphs with $r(G_1) = r(G_2)$ and h is a homomorphism of G_1 into G_2 such that h^* is onto.

Denote by $\mathscr{C}_h[\bar{\mathscr{C}}_h]$ the set of all maximal compatible [b-compatible] sets for h. For any $U \subset P$ and $y \in \Pi(G_2)$, we define

$$\mathrm{B}_h(U, y) = \{x \in \Pi(G_1) \mid \mathrm{i}(x) \in U, \quad h^*(x) = y\}$$

and

$$\bar{\mathrm{B}}_h(y, U) = \{x \in \Pi(G_1) \mid \mathrm{t}(x) \in U, \quad h^*(x) = y\}.$$

We define the *bundle-graph induced by h* as the graph

$$\mathscr{G}_h = (\mathscr{C}_h, \mathscr{E}_h, \zeta_h)$$

where $\mathscr{E}_h$ is the set of all pairs of the form $(U, \mathrm{B}_h(U, b))$ where $U \in \mathscr{C}_h$ and $b \in B$ with $\mathrm{i}(b) \in \phi_h(U)$, and $\zeta_h: \mathscr{E}_h \to \mathscr{C}_h \times \mathscr{C}_h$ is defined as follows:

$$\zeta_h((U, \mathrm{B}_h(U, b))) = (U, \mathrm{C}_h(U, b))$$

for all $U \in \mathscr{C}_h$ and $b \in B$ with $\mathrm{i}(b) \in \phi_h(U)$. By corollary 3.3, $\mathrm{C}_h(U, b) \in \mathscr{C}_h$ for any $U \in \mathscr{C}_h$ and $b \in B$ with $\mathrm{i}(b) \in \phi_h(U)$. Hence ζ_h is well-defined. Furthermore, we define a mapping $\tilde{h}: \mathscr{E}_h \to B$ as follows:

$$\tilde{h}((U, \mathrm{B}_h(U, b))) = b$$

for all $U \in \mathscr{C}_h$ and $b \in B$ with $\mathrm{i}(b) \in \phi_h(U)$.

Similarly, the *backward bundle-graph* (abbreviated to *b-bundle-graph*) *induced by h* is defined to be the graph

$$\bar{\mathscr{G}}_h = (\bar{\mathscr{C}}_h, \bar{\mathscr{E}}_h, \bar{\zeta}_h)$$

where $\bar{\mathscr{E}}_h$ is the set of all pairs of the form $(\bar{\mathrm{B}}_h(b, U), U)$ where $U \in \bar{\mathscr{C}}_h$ and $b \in B$ with $\mathrm{t}(b) \in \phi_h(U)$ and $\bar{\zeta}_h: \bar{\mathscr{E}}_h \to \bar{\mathscr{C}}_h \times \bar{\mathscr{C}}_h$ is defined as follows:

$$\bar{\zeta}_h((\bar{\mathrm{B}}_h(b, U), U)) = (\bar{\mathrm{C}}_h(b, U), U)$$

for all $U \in \bar{\mathscr{C}}_h$ and $b \in B$ with $\mathrm{t}(b) \in \phi_h(U)$. Also, by virtue of corollary 3.3, $\bar{\zeta}_h$ is well-defined. We define a mapping $\bar{\tilde{h}}: \bar{\mathscr{E}}_h \to B$ as follows:

$$\bar{\tilde{h}}((\bar{\mathrm{B}}_h(b, U), U)) = b$$

for all $U \in \bar{\mathscr{C}}_h$ and $b \in B$ with $\mathrm{t}(b) \in \phi_h(U)$.

PROPOSITION 4.1. *The bundle-graph $\mathscr{G}_h$ [b-bundle-graph $\bar{\mathscr{G}}_h$] is strongly connected and the mapping $\tilde{h}[\bar{\tilde{h}}]$ is a regular [b-regular] homomorphism of $\mathscr{G}_h[\bar{\mathscr{G}}_h]$ into G_2.*

Proof. Let $U, V \in \mathscr{C}_h$. If $\mathrm{C}(U, z) = V$ for $z = b_1 \cdots b_p \in \Pi(G_2)$ with $b_1, \ldots, b_p \in B$, then there exists a path $E_1 \cdots E_p$ in $\mathscr{G}_h$ such that

$$E_i = (U_i, \mathrm{B}_h(U_i, b_i)) \qquad \text{for } i = 1, \ldots, p$$

where $U_1 = U$, $U_{i+1} = \mathrm{C}(U_i, b_i)$ for $i = 1, \ldots, p$ and $U_{p+1} = V$. Hence to prove that $\mathscr{G}_h$ is strongly connected, it suffices to show that there exists a path z in G_2 such that $\mathrm{C}(U, z) = V$. Since V is a compatible set, there exists $v \in P$ and $y \in \Pi(G_2)$ such that $V = C(v, y)$. Since G_1 is strongly connected, there exists a path x such that $i(x) \in U$ and $t(x) = v$. Clearly,

$$C(U, h^*(x)y) \supset V.$$

Since $C(U, h^*(x)y)$ is a compatible set and V is a maximal compatible set, we have

$$C(U, h^*(x)y) = V.$$

The remainder is clear from the construction. □

Remark 4.2. Each of $(\tilde{h})^*$ and $(\bar{\tilde{h}})^*$ is uniformly finite-to-one and onto, and $r(\mathcal{G}_h) = r(\bar{\mathcal{G}}_h) = r(G_2)$. This follows from propositions 4.1 and 2.1.

We call $\tilde{h}[\bar{\tilde{h}}]$ the *induced regular* [*b-regular*] *homomorphism* of h.

It follows from corollary 3.3 that each subset of paths in G_1 of the form $B_h(U, y)$ $[\bar{B}_h(y, U)]$ where $U \in \mathcal{C}_h$ $[U \in \bar{\mathcal{C}}_h]$ and $y \in \Pi(G_2)$ with $i(y) \in \phi_h(U)$ $[t(y) \in \phi_h(U)]$, is non-empty. To each path Z of length $p \geq 0$ in $\mathcal{G}_h[\bar{\mathcal{G}}_h]$ corresponds the non-empty subset of paths $B_h(\underline{U}, y)$ $[\bar{B}_h(y, U)]$ of length p in G_1 where $i(Z) = U$ $[t(Z) = U]$ and $y = \tilde{h}(Z)$ $[y = \bar{\tilde{h}}(Z)]$. It is called the *bundle* of Z and is denoted by $B(Z)$. Clearly each subset of paths in G_1 of the form $B_h(U, y)[\bar{B}_h(y, U)]$ where $U \in \mathcal{C}_h[U \in \bar{\mathcal{C}}_h]$ and $y \in \Pi^{(p)}(G_2)$ with $i(y) \in \phi_h(U)$ $[t(y) \in \phi_h(U)]$, is the bundle of some path of length p in $\mathcal{G}_h[\bar{\mathcal{G}}_h]$, and is also called a *bundle* [*backward bundle*, abbreviated to *b-bundle*] *of length p for h*.

For $\Gamma \in \Omega(\mathcal{G}_h)[\Gamma \in \Omega(\bar{\mathcal{G}}_h)]$ and $\alpha \in \Omega(G_1)$, we say that Γ *contains* α if $B(\Gamma_i) \ni \alpha_i$ for all $i \in \mathbb{Z}$.

LEMMA 4.3. *For each* $\Gamma \in \Omega(\mathcal{G}_h)$ $[\Gamma \in \Omega(\bar{\mathcal{G}}_h)]$, *there exists* $\alpha \in \Omega(G_1)$ *such that* Γ *contains* α, *and for each* $\alpha \in \Omega(G_1)$, *there exists* $\Gamma \in \Omega(\mathcal{G}_h)$ $[\Gamma \in \Omega(\bar{\mathcal{G}}_h)]$ *such that* Γ *contains* α.

Proof. Let $\Gamma \in \Omega(\mathcal{G}_h)$. For each non-negative integer k, there exists an element x_k of $B(\Gamma[-k, k])$, and there exists $\alpha^{(k)} \in \Omega(G_1)$ such that

$$\alpha^{(k)}[-k, k] = x_k$$

(because any point of G_1 has an arc going from it and an arc going to it by our standing hypothesis for graphs). Since $\Omega(G_1)$ is a compact metric space, there exists a sequence $0 \leq k_0 \leq k_1 < \cdots$ of integers and $\alpha \in \Omega(G_1)$ such that

$$\lim_{j\to\infty} \alpha^{(k_j)} = \alpha.$$

It is easy to see that Γ contains α.

Conversely, let $\alpha \in \Omega(G_1)$. Let k be any non-negative integer. Let U_k be a maximal compatible set for h such that $U_k \ni i(\alpha[-k, k])$. Since $\tilde{h}$ is regular, there exists $Z_k \in \Pi^{(2k+1)}(\mathcal{G}_h)$ going from U_k with $\tilde{h}^*(Z_k) = h^*(\alpha[-k, k])$. Clearly $B(Z_k) \ni \alpha[-k, k]$. There exists $\Gamma^{(k)} \in \Omega(\mathcal{G}_h)$ such that $\Gamma^{(k)}[-k, k] = Z_k$ (because, by proposition 4.1, any point of $\mathcal{G}_h$ has an arc going from it and an arc going to it). Since $\Omega(\mathcal{G}_h)$ is a compact metric space, there exists a sequence $0 \leq k_0 < k_1 < \cdots$ of integers and $\Gamma \in \Omega(\mathcal{G}_h)$ such that

$$\lim_{j\to\infty} \Gamma^{(k_j)} = \Gamma.$$

It is also easy to see that Γ contains α. □

5. *Mergible homomorphisms*

For paths x and y in a graph G, y is an *initial subpath* [*a terminal subpath*] of x, if there exists a path w in G such that $x = yw$ [$x = wy$]. (Here we assume that $\mathrm{i}(x)x = x\mathrm{t}(x) = x$ for each path x in G.)

Let h be a homomorphism of a graph G_1 into a graph G_2. Let p be a non-negative integer. We say that h is *p bundle-mergible* [*p backward-bundle-mergible* (abbreviated to *p b-bundle-mergible*)] if for any two paths x_1 and x_2 of length $l \geq p$ in G_1, if $\mathrm{i}(x_1) = \mathrm{i}(x_2)$ [$\mathrm{t}(x_1) = \mathrm{t}(x_2)$] and $h^*(x_1) = h^*(x_2)$, then x_1 and x_2 have the same initial [terminal] subpath of length $l - p$. We say that h is *mergible*, if for some non-negative integers p and q, h is both p bundle-mergible and q b-bundle-mergible. (The notion of 'p bundle-[b-bundle]-mergible' corresponds to 'nonexistence of a right [left] f-branch of length p' in [**8**].)

Remark 5.1. Let G_1 and G_2 be strongly connected graphs with $r(G_1) = r(G_2)$, and let h be a homomorphism of G_1 into G_2. Then h is p bundle-mergible [p b-bundle-mergible] iff h^* is onto and each bundle [b-bundle] X of length $l \geq p$ for h, all paths in X have the same initial [terminal] subpath of length $l - p$.

Proof. If h is p bundle-mergible, then h^* is onto because no two distinct paths in G_1 are indistinguishable by h. Hence h^* is onto (see § 1). The proof of the remainder is straightforward. □

Remark 5.2. Let G_1 and G_2 be two strongly connected graphs with $r(G_1) = r(G_2)$ and let h be a homomorphism of G_1 into G_2. Then h is 0 bundle-mergible [0 b-bundle-mergible] iff h is regular [b-regular].

Proof. Assume that h is 0 bundle-mergible. Then h^* is onto by remark 5.1. Since h is 0 bundle-mergible, it follows that for each point u of G_2, $\{u\}$ is a maximal compatible set for h, and for each arc b with $\mathrm{i}(b) = \phi_h(u)$, the arc a such that $\mathrm{i}(a) = u$ and $h(a) = b$, is unique; such an arc a always exists because $\{u\}$ is a complete set for h by theorem 3.2. Thus h is regular. The converse is clear. □

The terminology of p bundle-mergible [p b-bundle-mergible] is based on remark 5.1. Another restatement of the property of being p bundle-mergible [p b-bundle-mergible], is given as the following lemma. (This can be considered as a generalization of [**8**, theorem 16.9].)

LEMMA 5.3. *Let G_1 and G_2 be two strongly connected graphs with $r(G_1) = r(G_2)$ and let h be a homomorphism of G_1 into G_2. Let p be a non-negative integer. Then h is p bundle-mergible [p b-bundle-mergible] iff h^* is onto and for each point u of G_1 and each path y of length at least p in G_2 with* $\mathrm{i}(y) = \phi_h(u)$ [$\mathrm{t}(y) = \phi_h(u)$], $\mathrm{C}_h(u, y)$ [$\bar{\mathrm{C}}_h(y, u)$] *is either empty or a maximal compatible* [*b-compatible*] *set.*

Proof. Assume that h is p bundle-mergible. By remark 5.1, h^* is onto. Let u be a point of G_1 and let y be a path of length $l \geq p$ in G_2 with $\mathrm{i}(y) = \phi_h(u)$. Suppose that $\mathrm{C}(u, y) \neq \varnothing$. Let U be a maximal compatible set which contains u. Then by corollary 3.3, $\mathrm{C}(U, y)$ is a maximal compatible set. Therefore it suffices to show that $\mathrm{C}(u, y) = \mathrm{C}(U, y)$.

Clearly $C(u, y) \subset C(U, y)$. Let v be an arbitrary element of $C(U, y)$. Then there exists $x_1 \in \Pi(G_1)$ such that

$$i(x_1) \in U, \quad t(x_1) = v, \quad \text{and} \quad h^*(x_1) = y.$$

Since $C(u, y) \neq \varnothing$, there exists $x_2 \in \Pi(G_1)$ such that $i(x_2) = u$ and $h^*(x_2) = y$. Clearly $x_1, x_2 \in B_h(U, y)$. Since h is p bundle-mergible, all paths in $B_h(U, y)$ have the same initial subpath of length $l - p$ (remark 5.1). Hence

$$i(x_1) = i(x_2) = u$$

so that $v \in C(u, y)$. Hence we have $C(u, y) \supset C(U, y)$. Thus $C(u, y) = C(U, y)$. The proof of the converse is omitted (because this will not be used in this paper). □

LEMMA 5.4. *Let G_1 and G_2 be two strongly connected graphs with $r(G_1) = r(G_2)$, and let h be a homomorphism of G_1 into G_2. If h is p bundle-mergible and q b-bundle-mergible, then the induced regular homomorphism $\tilde{h}$ of h is 0 bundle-mergible and $p+q$ b-bundle-mergible.*

Proof. Since $\tilde{h}$ is regular, $\tilde{h}$ is 0 bundle-mergible (remark 5.2). Let Z_1 and Z_2 be paths in $\mathscr{G}_h$ such that $t(Z_1) = t(Z_2)$, $\tilde{h}^*(Z_1) = \tilde{h}^*(Z_2)$, and Z_1 and Z_2 are of length l with $l \geq p+q$. To show that $\tilde{h}$ is $p+q$ b-bundle mergible, we shall show that Z_1 and Z_2 have the same terminal subpath of length $l-(p+q)$. To show this, it suffices to prove that the initial subpaths of length $p+q$ of Z_1 and Z_2, say $Z_1^{(p+q)}$ and $Z_2^{(p+q)}$ respectively, have the same terminal endpoint, because $\tilde{h}$ is regular.

Let $U_1 = i(Z_1)$ and let $U_2 = i(Z_2)$. Let $y = \tilde{h}^*(Z_1) = \tilde{h}^*(Z_2)$ and write $y = b_1 \cdots b_l$ where $b_1, \ldots, b_l$ are arcs of G_2. Then

$$C_h(U_1, b_1 \cdots b_{p+q}) = t(Z_1^{(p+q)}),$$
$$C_h(U_2, b_1 \cdots b_{p+q}) = t(Z_2^{(p+q)}),$$

and

$$C_h(U_1, y) = t(Z_1) = t(Z_2) = C_h(U_2, y).$$

Let $v \in C_h(U_1, y) = C_h(U_2, y)$. Then there are paths x_1 and x_2 in G_1 such that

$$i(x_1) \in U_1, \quad i(x_2) \in U_2, \quad t(x_1) = t(x_2) = v, \quad \text{and} \quad h^*(x_1) = h^*(x_2) = y.$$

Let $x_1^{(q)}$ and $x_2^{(q)}$ be the initial subpaths of length q of x_1 and x_2, respectively. Since h is q b-bundle-mergible, x_1 and x_2 have the same terminal subpath of length $l-q$ so that

$$t(x_1^{(q)}) = t(x_2^{(q)}).$$

Let $s = t(x_1^{(q)}) = t(x_2^{(q)})$. Then $C_h(s, b_{q+1} \cdots b_{p+q})$ is not empty because it contains the terminal endpoint of the initial subpath of length $p+q$ of x_1. Since h is p bundle-mergible, it follows from lemma 5.3 that $C_h(s, b_{q+1} \cdots b_{p+q})$ is a maximal compatible set. Since U_1 and U_2 are compatible sets for h, so are $C_h(U_1, b_1 \cdots b_{p+q})$ and $C_h(U_2, b_1 \cdots b_{p+q})$. Moreover $C_h(U_1, b_1 \cdots b_{p+q})$ and $C_h(U_2, b_1 \cdots b_{p+q})$ contain the maximal compatible set $C_h(s, b_{q+1} \cdots b_{p+q})$. Therefore

$$C_h(U_1, b_1 \cdots b_{p+q}) = C_h(s, b_{q+1} \cdots b_{p+q}) = C_h(U_2, b_1 \cdots b_{p+q}).$$

Thus we have $t(Z_1^{(p+q)}) = t(Z_2^{(p+q)})$. □

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LEMMA 5.5. *Let G_1 and G_2 be two strongly connected graphs with $r(G_1)=r(G_2)$ and let h be a homomorphism of G_1 into G_2. If h is p bundle-mergible [p b-bundle-mergible] for a non-negative integer p, then there exists an admissible $(p+1)$-block map*

$$f:\Pi^{(p+1)}(\mathcal{G}_h)\to\Pi^{(1)}(G_1)$$

$[f:\Pi^{(p+1)}(\bar{\mathcal{G}}_h)\to\Pi^{(1)}(G_1)]$ *such that*

$$\tilde{h}_\infty = h_\infty f_\infty$$

$[\bar{\tilde{h}}_\infty = h_\infty f_\infty \sigma^{-p}]$ *and f_∞ is one-to-one and onto [where $\sigma:\Omega(\bar{\mathcal{G}}_h)\to\Omega(\bar{\mathcal{G}}_h)$ is the shift].*

Proof. Since h is p bundle-mergible, it follows from remark 5.1 that for each $Z\in\Pi^{(p+1)}(\mathcal{G}_h)$, all paths in $\mathrm{B}(Z)$ have the same initial subpath of length 1. (Recall that $\mathrm{B}(Z)$ is the bundle of Z (cf. § 4).) Hence we can define a block map

$$f:\Pi^{(p+1)}(\mathcal{G}_h)\to\Pi^{(1)}(G_1)$$

as follows. For each $Z\in\Pi^{(p+1)}(\mathcal{G}_h)$, $f(Z)$ is the initial subpath of length 1 of the paths in $\mathrm{B}(Z)$. It is straightforward to see that f is an admissible $(p+1)$-block map and for each $\Gamma\in\Omega(\mathcal{G}_h)$, $f_\infty(\Gamma)$ is a unique bisequence which is contained in Γ. Hence we have $\tilde{h}_\infty(\Gamma)=h_\infty(f_\infty(\Gamma))$ for each $\Gamma\in\Omega(\mathcal{G}_h)$ and it suffices to show that for any $\alpha\in\Omega(G_1)$, there exists a unique element Γ of $\Omega(\mathcal{G}_h)$ which contains α.

Let $\alpha\in\Omega(G_1)$. By lemma 4.3, there exists $\Gamma\in\Omega(\mathcal{G}_h)$ such that Γ contains α. Suppose that Γ' in $\Omega(\mathcal{G}_h)$ contains α. Let $i\in\mathbb{Z}$. If $p=0$, then h is regular. Hence each maximal compatible set for h consists of a single point of G_1. Hence

$$\mathrm{i}(\Gamma_i)=\mathrm{i}(\alpha_i)=\mathrm{i}(\Gamma'_i).$$

Assume that $p\geq 1$. Since $\mathrm{i}(\Gamma_{i-p})\ni\mathrm{i}(\alpha_{i-p})$,

$$\mathrm{t}(\Gamma_{i-1})=\mathrm{C}_h(\mathrm{i}(\Gamma_{i-p}),\, h^*(\alpha[i-p,i-1]))\supset \mathrm{C}_h(\mathrm{i}(\alpha_{i-p}),\, h^*(\alpha[i-p,i-1])).$$

Since h is p bundle-mergible, it follows from lemma 5.3 that

$$\mathrm{C}_h(\mathrm{i}(\alpha_{i-p}),\, h^*(\alpha[i-p,i-1]))$$

is a maximal compatible set. Since $\mathrm{t}(\Gamma_{i-1})$ is a compatible set for h, we have

$$\mathrm{t}(\Gamma_{i-1})=\mathrm{C}_h(i(\alpha_{i-p}),\, h^*(\alpha[i-p,i-1])).$$

For the same reason,

$$\mathrm{t}(\Gamma'_{i-1})=\mathrm{C}_h(\mathrm{i}(\alpha_{i-p}),\, h^*(\alpha[i-p,i-1])).$$

Hence we have

$$\mathrm{i}(\Gamma_i)=\mathrm{t}(\Gamma_{i-1})=\mathrm{t}(\Gamma'_{i-1})=\mathrm{i}(\Gamma'_i).$$

Since $\tilde{h}(\Gamma_i)=h(\alpha_i)=\tilde{h}(\Gamma'_i)$ and $\tilde{h}$ is a regular homomorphism, we have $\Gamma_i=\Gamma'_i$. Since i was arbitrary, we have $\Gamma=\Gamma'$. □

Recently, the author learned that in **[11]**, Kitchens has a similar result to lemma 5.5.

Let G be a graph and let n be a non-negative integer. We consider the path graph

$$L^{(n)}(G)=(\Pi^{(n)}(G),\Pi^{(n+1)}(G),\zeta^{(n)})$$

of length n of G (cf. § 1). For each path x of length at least n in G, we define $(x)_n$ as follows. If x is of length n, then $(x)_n=x$, and if $x=a_1\cdots a_l$ where $l\geq n+1$ and $a_1,\ldots,a_l$ are arcs of G, then

$$(x)_n=(a_1\cdots a_{n+1})(a_2\cdots a_{n+2})\cdots(a_{l-n}\cdots a_l).$$

Then if x is a path of length $l \geq n$ in G, then $(x)_n$ is a path of length $l-n$ in $L^{(n)}(G)$. Obviously, each path in $L^{(n)}(G)$ is written as $(x)_n$ for some path x of length at least n in G.

Let h be a homomorphism of a graph G_1 into a graph G_2. Let n be a non-negative integer. We define a mapping

$$h^{(n)}: \Pi^{(n+1)}(G_1) \to \Pi^{(n+1)}(G_2)$$

by

$$h^{(n)}(x) = h^*(x) \qquad x \in \Pi^{(n+1)}(G_1).$$

Clearly $h^{(n)}$ is a homomorphism of $L^{(n)}(G_1)$ into $L^{(n)}(G_2)$ and for each path x of length at least n in G,

$$(h^{(n)})^*((x)_n) = (h^*(x))_n.$$

One readily gets the following.

LEMMA 5.6. *Let h be a homomorphism of a graph G_1 into a graph G_2, and let n and p be non-negative integers. If h is p bundle-mergible [p b-bundle-mergible], then $h^{(n)}$ is a p bundle-mergible [p b-bundle-mergible] homomorphism of $L^{(n)}(G_1)$ into $L^{(n)}(G_2)$.*

LEMMA 5.7. *Let G_1 and G_2 be two strongly connected graphs with $r(G_1) = r(G_2)$ and let h be a homomorphism of G_1 into G_2. Assume that h* is *p bundle-mergible [p b-bundle-mergible] for a non-negative integer p. Then any two distinct maximal compatible [b-compatible] sets for $h^{(p)}$ are disjoint.*

Proof. First we note that $L^{(p)}(G_1)$ and $L^{(p)}(G_2)$ are strongly connected and

$$r(L^{(p)}(G_1)) = r(G_1) = r(G_2) = r(L^{(p)}(G_2)).$$

From lemma 5.6, $h^{(p)}$ is p bundle-mergible. Let W be a maximal compatible set for $h^{(p)}$. Then since $h^{(p)}$ is p bundle-mergible, it follows from lemma 5.3 that there exists a point $w \in \Pi^{(p)}(G_1)$ of $L^{(p)}(G_1)$ and a path s of length p in $L^{(p)}(G_2)$ such that

$$W = C_{h^{(p)}}(w, s).$$

(There exists $z \in \Pi(L^{(p)}(G_1))$ such that

$$W = C_{h^{(p)}}(\mathrm{i}(z), (h^{(p)})^*(z)).$$

We may assume that the length of z is not less than p. Let $\hat{z}$ be the terminal subpath of length p of z. Then since $h^{(p)}$ is p bundle-mergible, it follows from lemma 5.3 that

$$W = C_{h^{(p)}}(\mathrm{i}(\hat{z}), (h^{(p)})^*(\hat{z})).$$

Put $w = \mathrm{i}(\hat{z})$ and put $s = (h^{(p)})^*(\hat{z})$.)

There exists $y \in \Pi^{(2p)}(G_2)$ such that $(y)_p = s$. It is straightforward to see that

$$C_{h^{(p)}}(w, s) = \{x \in \Pi^{(p)}(G_1) \mid wx \in \Pi^{(2p)}(G_1), \quad h^*(wx) = y\}.$$

This implies that if $x \in W$, then we can write $W = B_h(\{\mathrm{i}(x)\}, h^*(x))$. Thus we conclude that if W_1 and W_2 are maximal compatible sets for $h^{(p)}$ and $W_1 \cap W_2 \neq \varnothing$, then $W_1 = W_2$. Hence any two distinct maximal compatible sets for $h^{(p)}$ are disjoint. □

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LEMMA 5.8. *Let G_1 and G_2 be two strongly connected graphs and let h be a regular homomorphism of G_1 into G_2. If every two distinct maximal b-compatible sets for h are disjoint, then the induced b-regular homomorphism $\bar{\bar{h}}$ is biregular.*

Proof. By proposition 4.1, it suffices to show that $\bar{\bar{h}}$ is regular. Let U be any point of $\bar{\mathscr{G}}_h$ (i.e. any maximal b-compatible set for h). Let $v = \phi_{\bar{\bar{h}}}(U)$ and let b be any arc of G_2 going from v. Let $u \in U$. Then, $\phi_h(u) = v$. Since h is regular, there exists an arc a of G_1 such that $\mathrm{i}(a) = u$ and $h(a) = b$. Let V be a maximal b-compatible set for h which contains $\mathrm{t}(a)$. Let $U' = \bar{\mathrm{C}}_h(b, V)$ and let $E = (\bar{\mathrm{B}}_h(b, V), V)$. Then by definition, E is an arc of $\bar{\mathscr{G}}_h$ going from U' to V with $\bar{\bar{h}}(E) = b$. Since $V \ni \mathrm{t}(a)$ and $h(a) = b$,

$$\bar{\mathrm{C}}_h(b, V) \ni \mathrm{i}(a) = u.$$

Hence $U \cap U' \ni u$. Since U and U' are maximal b-compatible sets and $U \cap U' \neq \emptyset$, it follows from the assumption of the lemma that $U = U'$. Thus E is an arc of $\bar{\mathscr{G}}_h$ going from U and $\bar{\bar{h}}(E) = b$.

Assume that there exists an arc E' of $\bar{\mathscr{G}}_h$ with $E' \neq E$ such that $\mathrm{i}(E') = U$, and $\bar{\bar{h}}(E') = b$. Then there exists a maximal b-compatible set V' with $V' \neq V$ such that $E' = (\bar{\mathrm{B}}_h(b, V'), V')$. Since $\bar{\mathrm{C}}_h(b, V') = \mathrm{i}(E') = U$, there exists an arc a' of G_1 such that

$$\mathrm{i}(a') = u, \quad h(a') = b, \quad \text{and} \quad \mathrm{t}(a') \in V'.$$

Since V' and V are distinct maximal b-compatible sets for h, $V' \cap V = \emptyset$ so that $a' \neq a$. But this is impossible because h is regular. Thus E is a unique arc of $\bar{\mathscr{G}}_h$ with $\mathrm{i}(E) = U$ and $\bar{\bar{h}}(E) = b$. We have proved that $\bar{\bar{h}}$ is regular. □

THEOREM 5.9. *Let G_1 and G_2 be strongly connected graphs with $r(G_1) = r(G_2)$, and let h be a mergible homomorphism of G_1 into G_2. Then there exist a strongly connected graph H, an integer $p \geq 0$, a biregular homomorphism g of H into $L^{(p)}(G_2)$ and an isomorphism $\rho: (\Omega(G_1), \sigma_1) \to (\Omega(H), \sigma)$ such that*

$$h_\infty = (h_{G_2, p+1, 1})_\infty g_\infty \rho.$$

Proof. Let $G_3 = \bar{\mathscr{G}}_h$ and let $g_1 = \bar{\bar{h}}$. Then from proposition 4.1 and remark 4.2, G_3 is a strongly connected graph with $r(G_3) = r(G_2)$, and h_1 is a regular homomorphism of G_3 into G_2. Since h is mergible, it follows from lemma 5.4 that h_1 is 0 bundle-mergible and there exists a non-negative integer p such that h_1 is p b-bundle-mergible. Moreover, it follows from lemma 5.5 and theorem 1.5 that there exists an isomorphism ρ' of $(\Omega(G_3), \sigma_3)$ onto $(\Omega(G_1), \sigma_1)$ such that

$$(h_1)_\infty = h_\infty \rho'.$$

Let $G_4 = L^{(p)}(G_3)$, let $G_5 = L^{(p)}(G_2)$, and let $h_2 = h_1^{(p)}$. Then G_4 and G_5 are strongly connected graphs with

$$r(G_4) = r(G_3) = r(G_2) = r(G_5)$$

and h_2 is a homomorphism of G_4 into G_5. Since h_1 is 0 bundle-mergible and p b-bundle-mergible, it follows from lemma 5.6 that h_2 is 0 bundle-mergible and p b-bundle-mergible. Moreover, from lemma 5.7, any two distinct maximal b-compatible sets for h_2 are disjoint. Let $\rho_1 = (h_{G_3, p+1, 1})_\infty$ and let $\rho_2 = (h_{G_2, p+1, 1})_\infty$ (cf. § 1).

Then ρ_1 is an isomorphism of $(\Omega(G_4), \sigma_4)$ onto $(\Omega(G_3), \sigma_3)$ (and ρ_2 is an isomorphism of $(\Omega(G_5), \sigma_5)$ onto $(\Omega(G_2), \sigma_2)$) and we have

$$\rho_2(h_2)_\infty = (h_1)_\infty \rho_1.$$

Let $H = \bar{\mathscr{G}}_{h_2}$ and let $g = \bar{\tilde{h}}_2$. Then, by proposition 4.1, H is a strongly connected graph and g is a homomorphism of H into G_5. Since h_2 is regular (because h_2 is 0 bundle-mergible (remark 5.2)) and any two distinct maximal b-compatible sets for h_2 are disjoint, it follows from lemma 5.8 that g is biregular. Since h_2 is p b-bundle-mergible, it follows from lemma 5.5 and theorem 1.5 that there exists an isomorphism ρ'' of $(\Omega(H), \sigma)$ onto $(\Omega(G_4), \sigma_4)$ such that

$$g_\infty = (h_2)_\infty \rho''.$$

Thus we have

$$h_\infty \rho' \rho_1 \rho'' = \rho_2 g_\infty.$$

Put $\rho = (\rho' \rho_1 \rho'')^{-1}$. Then ρ is an isomorphism of $(\Omega(G_1), \sigma_1)$ onto $(\Omega(H), \sigma)$ and we have

$$h_\infty = (h_{G_2, p+1, 1})_\infty g_\infty \rho. \qquad \square$$

6. *Characterizations of constant-to-one and onto global maps*

In [**8**, § 9–§ 12] Hedlund describes the properties of inverses of onto endomorphisms of full shift dynamical systems. With minor modifications in the statements and the proofs, many of them are extended to onto global maps of homomorphisms between strongly connected graphs whose adjacency matrices have the same characteristic value. (Extensions of them to onto endomorphisms of irreducible subshifts of finite type were pointed out by Coven and Paul [**3**], and extensions of them to finite-to-one and onto homomorphisms between TPPD sofic systems were mentioned in [**4**].) Many of Hedlund's discussions on a block map $f: A^n \to A$, where A is a non-empty finite set of symbols and n is a positive integer, and the mapping $f_\infty: \Omega_A \to \Omega_A$ defined by

$$(f_\infty(\alpha))_i = f(\alpha_i \alpha_{i+1} \cdots \alpha_{i+n-1}) \qquad \alpha \in \Omega(A),\ i \in \mathbb{Z},$$

can be interpreted naturally as discussions on the homomorphism h_f of $L^{(n-1)}(G_0(A))$ into $G_0(A)$ defined by $h_f(x) = f(x)$, $x \in A^n$, and its global map $(h_f)_\infty$. (Cf. § 1. Note that $L^{(n-1)}(G_0(A))$ has point set A^{n-1} and arc set A^n. Hence, for example, 'totally $(n-1)$-separated' in [**8**] for bisequences in Ω_A corresponds to 'point-separated' defined below for bisequences in $\Omega(L^{(n-1)}(G_0(A)))$.) These discussions on h_f and $(h_f)_\infty$ can straightforwardly be extended to any homomorphism h of a strongly connected graph G_1 into a strongly connected graph G_2 with $r(G_1) = r(G_2)$, and h_∞.

Let G be a graph. Two bisequences $\alpha, \beta \in \Omega(G)$ are *point-separated* if $\mathrm{i}(\alpha_i) \neq \mathrm{i}(\beta_i)$ for all $i \in \mathbb{Z}$.

The following lemma is proved in the same way as [**8**, lemma 16.7].

LEMMA 6.1. *Let G_1 and G_2 be graphs and let h be a homomorphism of G_1 into G_2. If for each $\beta \in h_\infty(\Omega(G_1))$, any two distinct members of $h_\infty^{-1}(\beta)$ are point-separated, then h is mergible.*

Let G be a strongly connected graph. A bisequence $\alpha \in \Omega(G)$ is *positively transitive* [*negatively transitive*] if for each positive integer l and each $x \in \Pi^{(l)}(G)$, there exists $i \in \mathbb{Z}$ with $i \geq 0$ [$i \in \mathbb{Z}$ with $i \leq -l+1$] such that

$$\alpha[i, i+l-1] = x.$$

A bisequence $\alpha \in \Omega(G)$ is *bilaterally transitive if* α is both positively transitive and negatively transitive.

The result of L. R. Welch and A. M. Gleason given as theorems 11.1 and 11.2 of [**8**], can straightforwardly be extended to the following theorem; a similar extension in a more general setting was stated in [**5**].

THEOREM 6.2. *Let G_1 and G_2 be strongly connected graphs with $r(G_1) = r(G_2)$, and let h be a homomorphism of G_1 into G_2 with h^* onto. Then there exists a positive integer $m(h)$ such that if $\beta \in \Omega(G_2)$ is bilaterally transitive, then*

$$|h_\infty^{-1}(\beta)| = m(h).$$

Furthermore, for each $\beta \in \Omega(G_2)$,

$$|h_\infty^{-1}(\beta)| \geq m(h)$$

and the set $h_\infty^{-1}(\beta)$ contains $m(h)$ members which are mutually point-separated.

Now we reach our first goal.

THEOREM 6.3. *Let G_1 and G_2 be strongly connected graphs and let h be a homomorphism of G_1 into G_2. Then the following statements are equivalent.*

(1) *h_∞ is constant-to-one and onto.*

(2) *$r(G_1) = r(G_2)$ and for each $\beta \in \Omega(G_2)$, any two distinct members in $h_\infty^{-1}(\beta)$ are point-separated.*

(3) *$r(G_1) = r(G_2)$ and h is mergible.*

(4) *h^* is onto and h is mergible.*

(5) *There is a strongly connected graph H, an integer $p \geq 0$, a biregular homomorphism g of H into $L^{(p)}(G_2)$, and an isomorphism $\rho: (\Omega(G_1), \sigma_1) \to (\Omega(H), \sigma)$ such that $h_\infty = (h_{G_2,p+1,1})_\infty g_\infty \rho$.*

Proof. By proposition 1.3 and theorem 6.2, (1) implies (2). By lemma 6.1, (2) implies (3). By theorem 5.9, (3) implies (5). By proposition 2.2, (5) implies (1). If h is mergible, then no two distinct paths in G_1 are indistinguishable by h. Hence (3) and (4) are equivalent. □

Thus we have a structure result for constant-to-one and onto homomorphisms of irreducible subshifts of finite type.

COROLLARY 6.4. *Let G_1 and G_2 be strongly connected graphs and let $\pi: (\Omega(G_1), \sigma_1) \to (\Omega(G_2, \sigma_2)$ be a homomorphism. Then π is constant-to-one and onto iff there exists a strongly connected graph H, an integer $p \geq 0$, a biregular homomorphism g of H into $L^{(p)}(G_2)$, and an isomorphism $\rho: (\Omega(G_1), \sigma_1) \to (\Omega(H), \sigma)$ such that*

$$\pi = (h_{G_2,p+1,1})_\infty g_\infty \rho.$$

Proof. This follows from theorem 6.3 and corollary 1.6 □

We remark that there exists a finite procedure to determine whether (4) of theorem 6.3 holds or not for a given homomorphism between strongly connected graphs. We also remark that we can obtain completely analogous results to theorems 16.1 (a theorem of O. S. Rothaus) and 16.11 in [**8**] for homomorphisms of irreducible subshifts of finite type, a part of which was stated in [**5**] without proof.

Let G be a graph and let $\alpha, \beta \in \Omega(G)$. We say that α and β are *totally* 0*-separated* if α and β are point-separated. For a positive integer p, α and β are *totally* p*-separated* if

$$\alpha[i, i+p-1] \neq \beta[i, i+p-1] \qquad \text{for all } i \in \mathbb{Z}.$$

For other terminology see [**8**].

THEOREM 6.5. *Let G_1 and G_2 be strongly connected graphs with $r(G_1) = r(G_2)$. Let p be a positive integer. Let $f: \Pi^{(p)}(G_1) \to \Pi^{(1)}(G_2)$ be an admissible p-block map. Then the following statements are equivalent.*

(1) *f_∞ is constant-to-one.*

(2) *f_∞ is open and onto.*

(3) *f_∞ has a cross-section.*

(4) *For each $\beta \in \Omega(G_2)$, any two distinct members of $f_\infty^{-1}(\beta)$ are totally $(p-1)$-separated.*

Proof. Using the equivalence of (1) and (2) of theorem 6.3 and straightforwardly modifying a part of the discussions in § 16 of [**8**] (see lemmas and theorems from 16.2 to 16.6 and their proofs together with a theorem of E. A. Michael), we first have the theorem for $p = 1$. The general case can obviously be reduced to this. □

COROLLARY 6.6. *Let G_1 and G_2 be two strongly connected graphs with $r(G_1) = r(G_2)$ and let $\pi: (\Omega(G_1), \sigma_1) \to (\Omega(G_2), \sigma_2)$ be a homomorphism. The following statements are equivalent.*

(1) *π is constant-to-one.*

(2) *π is open and onto.*

(3) *π has a cross-section.*

(4) *For each $\beta \in \Omega(G_2)$, any two distinct members of $\pi^{-1}(\beta)$ are separated.*

Proof. This is proved using theorems 6.5 and 1.5 in the same way as [**8**, theorem 16.11]. □

Furthermore, we remark that the following generalization of [**15**, theorem 2] is obtained in the same way as in [**15**].

THEOREM 6.7. *Let G_1, G_2, and G_3 be strongly connected graphs with $r(G_1) = r(G_2) = r(G_3)$. Let $\pi_1: (\Omega(G_1), \sigma_1) \to (\Omega(G_2), \sigma_2)$ and $\pi_2: (\Omega(G_2), \sigma_2) \to (\Omega(G_3), \sigma_3)$ be homomorphisms. Then if $\pi_2\pi_1$ is constant-to-one, each of π_1 and π_2 is constant-to-one.*

7. *Constant-to-one extensions of irreducible subshifts of finite type*

In this section, we determine, up to topological conjugacy, the subshifts of finite type which are constant-to-one extensions of a given irreducible subshift of finite type. We say that $(\Omega(G_1), \sigma_1)$ is a *constant-to-one extension* of $(\Omega(G_2), \sigma_2)$ if there exists a constant-to-one homomorphism of $(\Omega(G_1), \sigma_1)$ onto $(\Omega(G_2), \sigma_2)$. If $(\Omega(G_1), \sigma_1)$ is a constant-to-one extension of $(\Omega(G_2), \sigma_2)$ and $(\Omega(G_2), \sigma_2)$ is irreducible (i.e. G_2 is strongly connected), $(\Omega(G_1), \sigma_1)$ is not necessarily irreducible (i.e. G_1 is not necessarily strongly connected). But we do have proposition 7.1 below.

A graph $G' = (P', A', \zeta')$ is a *subgraph* of a graph $G = (P, A, \zeta)$ if $P' \subset P$, $A' \subset A$, and $\zeta'(a) = \zeta(a)$ for all $a \in A'$. A maximal strongly connected subgraph of a graph G is called a *component* of G.

PROPOSITION 7.1. *Let G_1 be a graph whose components are $G_{11}, \cdots, G_{1m}$, and let G_2 be a strongly connected graph. Let there exist a constant-to-one homomorphism π of $(\Omega(G_1), \sigma_1)$ onto $(\Omega(G_2), \sigma_2)$. Then G_1 is the union of $G_{11}, \ldots, G_{1m}$, that is, there exists no path in G_1 going from a point of G_{1i} to a point of G_{1j} for any distinct i, j with $1 \le i, j \le m$. Moreover, $\pi_i = \pi | \Omega(G_{1i})$ is constant-to-one and onto for $i = 1, \ldots, m$.*

To prove proposition 7.1, we shall use the following lemma.

LEMMA 7.2. *Let G_1 be a graph, let G_2 be a strongly connected graph and let h be a homomorphism of G_1 into G_2 with h_∞ finite-to-one and onto. Let $\beta \in \Omega(G_2)$ be bilaterally transitive and let $\alpha \in h_\infty^{-1}(\beta)$. Then $\alpha \in \Omega(G_{1i})$ for some component G_{1i} of G_1.*

Proof. Assume that α is not contained in $\Omega(G_{1i})$ for any component G_{1i} of G_1. Then there exist components G_{1k} and G_{1l} of G_1 and $s, t \in \mathbb{Z}$ such that

$$\Pi^{(1)}(G_{1k}) \ni \alpha_j \qquad \text{for all } j < s,$$
$$\Pi^{(1)}(G_{1l}) \ni \alpha_j \qquad \text{for all } j > t,$$

but neither $\Pi(G_{1k})$ nor $\Pi(G_{1l})$ contains $\alpha[s, t]$. Let

$$h_k = h|\Pi^{(1)}(G_{1k} \quad \text{and} \quad h_l = h|\Pi^{(1)}(G_{1l}).$$

Then h_k and h_l are homomorphisms of G_{1k} into G_2 and of G_{1l} into G_2, respectively. Since β is negatively transitive, $(h_k)^*$ is onto, and also since β is positively transitive, $(h_l)^*$ is onto. Since h_∞ is finite-to-one, so are both of $(h_k)_\infty$ and $(h_l)_\infty$. Hence both of $(h_k)_\infty$ and $(h_l)_\infty$ are uniformly finite-to-one and onto.

It is known [**4**, p. 175]) that the inverses of a negatively [positively] transitive point (bisequence) through a finite-to-one and onto homomorphism between irreducible subshifts of finite type are also negatively [positively] transitive. Therefore since β is negatively transitive, α is negatively transitive in G_{1k}. (Consider a bisequence $\alpha' \in \Omega(G_{1k})$ such that $\alpha'_j = \alpha_j$ for all $j < s$ and apply the above fact to $(h_k)_\infty(\alpha')$.) Similarly, since β is positively transitive, α is positively transitive in G_{1l}.

There exists $x_1 \in \Pi(G_{1k})$ such that $\bar{C}_{h_k}(h_k^*(x_1), t(x_1))$ is a maximal b-compatible set for h_k. Since α is negatively transitive in G_{1k}, there exist $s_1, s_2 \in \mathbb{Z}$ with $s_1 \le s_2 \le s$ such that $\alpha[s_1, s_2] = x_1$. (By corollary 3.3, we may assume that the length of x_1 is greater than 0.) Also, there exists $x_2 \in \Pi(G_{1l})$ such that $C_{h_l}(\mathrm{i}(x_2), h^*(x_2))$ is a maximal compatible set for h_l. Since α is positively transitive in G_{1l}, there exist

$t_1, t_2 \in \mathbb{Z}$ with $t \le t_1 \le t_2$ such that $\alpha[t_1, t_2] = x_2$. Let

$$U = \bar{C}_{h_k}(\beta[s_1, s_2], t(\alpha_{s_2})) \quad \text{and} \quad V = C_{h_l}(i(\alpha_{t_1}), \beta[t_1, t_2]).$$

By the above, U is a maximal b-compatible set for h_k and V is a maximal compatible set for h_l. Since β is positively transitive, there exists $t_3 \in \mathbb{Z}$ with $t_3 > t_2$ such that

$$t(\beta_{t_3}) = i(\beta_{s_1}).$$

Let $y = \beta[s_1, t_3]$. Then $i(y) = t(y)$. For each $j \ge 0$, let

$$U_j = \bar{C}_{h_k}(y^j, U) \quad \text{and} \quad V_j = C_{h_l}(V, \beta[t_2+1, t_3]y^j).$$

Then by corollary 3.3, U_j is a maximal b-compatible set for h_k and V_j is a maximal compatible set for h_l. Hence, $U_j \ne \emptyset$ and $V_j \ne \emptyset$ for all $j \ge 0$. Therefore, there exists $\tilde{\alpha} \in \Omega(G_1)$ such that

$$\tilde{\alpha}[s_2+1, t_1-1] = \alpha[s_2+1, t_1-1],$$

$$\tilde{\alpha}[s_1 - j(t_3 - s_1 + 1), s_2] \in \bar{B}_{h_k}(y^j\beta[s_1, s_2], \{t(\alpha_{s_2})\}) \qquad \text{for all } j \ge 0,$$

and

$$\tilde{\alpha}[t_1, t_3 + j(t_3 - s_1 + 1)] \in B_{h_l}(\{i(\alpha_{t_1})\}, \beta[t_1, t_3]y^j) \qquad \text{for all } j \ge 0.,$$

Let $\tilde{\beta} = h_\infty(\tilde{\alpha})$. Then clearly $\tilde{\beta}$ is a periodic bisequence of period $t_3 - s_1 + 1$ with $\tilde{\beta}[s_1, t_3] = y$. Clearly $\tilde{\alpha}$ is not periodic. But

$$h_\infty(\sigma^{j(t_3 - s_1 + 1)}(\tilde{\alpha})) = \tilde{\beta} \qquad \text{for all } j \in \mathbb{Z}$$

where σ is the shift on $\Omega(G_1)$. Hence $h_\infty^{-1}(\tilde{\beta})$ is infinite, which is a contradiction. Thus $\alpha \in \Omega(G_{1i})$ for some component G_{1i} of G_1. □

Proof of proposition 7.1. By corollary 1.6, there exist positive integers $p, q, p \ge q$, and a homomorphism h of $L^{(p-1)}(G_1)$ into G_2 such that

$$\pi = h_\infty(h_{G_1,p,q})_\infty^{-1}.$$

Clearly h_∞ is constant-to-one and onto. Put $H = L^{(p-1)}(G_1)$ and put $H_i = L^{(p-1)}(G_{1i})$ for $i = 1, \ldots, m$. Then it is easy to see that $H_1, \ldots, H_m$ are all the components of H. Let

$$h_i = h|\Pi^{(1)}(H_i) \qquad \text{for } i = 1, \ldots, m.$$

Then clearly, h_i is a homomorphism of H_i into G_2 with $(h_i)_\infty$ finite-to-one.

Let β be a bilaterally transitive bisequence in $\Omega(G_2)$. Then, by lemma 7.2, each $\alpha \in \Omega(H)$ such that $h_\infty(\alpha) = \beta$, is contained in some $\Omega(H_i)$, $1 \le i \le m$. Let

$$\{i_1, \ldots, i_l\} = \{i \mid 1 \le i \le m, \quad \Omega(H_i) \cap h_\infty^{-1}(\beta) \ne \emptyset\}.$$

Assume that there exists $\alpha' \in \Omega(H)$ such that $\alpha' \notin \Omega(H_{i_1}) \cup \cdots \cup \Omega(H_{i_l})$. Let $\gamma = h_\infty(\alpha')$. Since β is bilaterally transitive, $h_{i_j}^*$ is onto and so is $(h_{i_j})_\infty$ for $j = 1, \cdots, l$. Hence $(h_{i_j})_\infty$ is finite-to-one and onto for $j = 1, \ldots, l$. By theorem 6.2,

$$|(h_{i_j})_\infty^{-1}(\beta)| \le |(h_{i_j})_\infty^{-1}(\gamma)|$$

for $j = 1, \ldots, l$. Hence

$$|h_\infty^{-1}(\beta)| = \sum_{j=1}^{l} |(h_{i_j})_\infty^{-1}(\beta)| < 1 + \sum_{j=1}^{l} |(h_{i_j})_\infty^{-1}(\gamma)| \le |h_\infty^{-1}(\gamma)|,$$

which is a contradiction because h_∞ is constant-to-one and onto. Therefore $\Omega(H_{i_1}) \cup \cdots \cup \Omega(H_{i_l}) = \Omega(H)$. This implies that $\{i_1, \ldots, i_l\} = \{1, \ldots, m\}$,

$\Omega(H)=\Omega(H_1)\cup\cdots\cup\Omega(H_m)$ and $(h_i)_\infty$ is finite-to-one and onto for $i=1,\ldots,m$. Hence for each $\beta'\in\Omega(G_2)$,

$$|h_\infty^{-1}(\beta')|=\sum_{i=1}^{m}|(h_i)_\infty^{-1}(\beta')|,$$

and by theorem 6.2,

$$|(h_i)_\infty^{-1}(\beta)|\le|(h_i)_\infty^{-1}(\beta')|,$$

$i=1,\ldots,m$. Therefore, since h_∞ is constant-to-one, each $(h_i)_\infty$ must be constant-to-one.

Since $\Omega(H)=\Omega(H_1)\cup\cdots\cup\Omega(H_m)$, $\Omega(G_1)=\Omega(G_{11})\cup\cdots\cup\Omega(G_{1m})$. Hence there is no path in G_1 going from a point of G_{1i} to a point of G_{1j} for any distinct i,j with $1\le i,j\le m$. Since

$$\pi_i=(h_i)_\infty(h_{G_{1i},p,q})_\infty^{-1},$$

π_i is constant-to-one and onto. □

The following is the main theorem of this paper. In view of proposition 2.3, it determines constructively, up to topolological conjugacy, the subshifts of finite type which are constant-to-one extensions of a given irreducible subshift of finite type.

THEOREM 7.3. *Let G_1 be a graph and G_2 a strongly connected graph. Then $(\Omega(G_1),\sigma_1)$ is a constant-to-one extension of $(\Omega(G_2),\sigma_2)$ iff there is a biregular extension H of $L^{(p)}(G_2)$ for some integer $p\ge 0$ such that $(\Omega(G_1),\sigma_1)$ is topologically conjugate to $(\Omega(H),\sigma)$.*

Proof. Assume that there is a constant-to-one homomorphism π of $(\Omega(G_1),\sigma_1)$ onto $(\Omega(G_2),\sigma_2)$. Then from proposition 7.1, G_1 is the union of its components $G_{11},\ldots,G_{1m}$ and $\pi|\Omega(G_{1i})$ is constant-to-one and onto for $i=1,\ldots,m$. Put

$$\pi_i=\pi|\Omega(G_{1i})\qquad\text{for } i=1,\ldots,m.$$

It follows from corollary 6.4 that for each $i=1,\ldots,m$, there exists a strongly connected graph H_i, an integer $p_i\ge 0$, and a biregular homomorphism g_i of H_i into $L^{(p_i)}(G_2)$ such that $(\Omega(H_i),\sigma_i')$ and $(\Omega(G_{1i}),\sigma_{1i})$ are topologically conjugate. Let $p=\max\{p_1,\ldots,p_m\}$. Consider the homomorphism $g_i^{(p-p_i)}$ of $L^{(p-p_i)}(H_i)$ into $L^{(p-p_i)}(L^{(p_i)}(G_2))$, (see the paragraph before lemma 5.6). It is easy to see that $g_i^{(p-p_i)}$ is biregular. Therefore, since $L^{(p)}(G_2)$ is isomorphic to $L^{(p-p_i)}(L^{(p_i)}(G_2))$, $L^{(p-p_i)}(H_i)$ is a biregular extension of $L^{(p)}(G_2)$. It follows that $(\Omega(L^{(p-p_i)}(H_i)),\sigma_i'')$ is topologically conjugate to $(\Omega(G_{1i}),\sigma_{1i})$ for $i=1,\ldots,m$. Let H be the union of the graphs $L^{(p-p_i)}(H_i)$, $i=1,\ldots,m$. Then, by the above, $(\Omega(H),\sigma)$ is topologically conjugate to $(\Omega(G_1),\sigma_1)$. Moreover, since H is the union of its components each of which is a biregular extension of $L^{(p)}(G_2)$, H is a biregular extension of $L^{(p)}(G_2)$.

The converse is clear from proposition 2.2. □

There is a remarkable spectral property of matrices concerning topological conjugacy of subshifts of finite type. It follows directly from the well-known theorem of Williams, [**21**], (characterizing topological conjugacy of subshifts of finite type by 'strong shift equivalence' of matrices defining the subshifts), and a result of Flanders (see [**9**, p. 106]) that if two subshifts of finite type $(\Omega(G_1), \sigma_1)$ and $(\Omega(G_2, \sigma_2)$ are topologically conjugate, then the elementary divisors not divisible by λ of $M(G_1)$ and those of $M(G_2)$ are the same, where λ is the indeterminate. (See also [**19**].) But the converse of this does not hold by example 3 of [**21**].

As stated in § 1, there are also remarkable spectral properties of matrices concerning finite-to-one extensions of subshifts of finite type. In [**17**], the author showed that for two subshifts of finite type $(\Omega(G_1), \sigma_1)$ and $(\Omega(G_2), \sigma_2)$, if there is a finite-to-one homomorphism of $(\Omega(G_1), \sigma_1)$ onto $(\Omega(G_2), \sigma_2)$, then the characteristic polynomial of $M(G_2)$ divides the characteristic polynomial of $M(G_1)$, mod powers of λ. Furthermore, Kitchens, [**10**], showed that if $(\Omega(G_1), \sigma_1)$ and $(\Omega(G_2), \sigma_2)$ are irredicible subshifts of finite type and there is a finite-to-one homomorphism of $(\Omega(G_1), \sigma_1)$ onto $(\Omega(G_2), \sigma_2)$, then the block of the Jordan form of $M(G_2)$ with non-zero eigenvalues is a principal submatrix of the Jordan form of $M(G_1)$. In [**10**], Kitchens also showed that the converse does not hold. The following theorem is a result along the above lines.

THEOREM 7.4. *Let G_1 and G_2 be strongly connected graphs. If there exists a constant-to-one homomorphism of $(\Omega(G_1), \sigma_1)$ onto $(\Omega(G_2), \sigma_2)$, then the elementary divisors not divisible by λ of $M(G_2)$ is contained in the elementary divisors of $M(G_1)$.*

Proof. The result follows from theorem 7.3, proposition 2.5, the spectral property of matrices concerning topological conjugacy stated above, and the fact that $(\Omega(G_2), \sigma_2)$ and $(\Omega(L^{(p)}(G_2)), \sigma_2')$ are topologically conjugate. □

Marcus [**13**] proved that for any strongly connected graph G with $r(G) = n$, where n is a positive integer, there is a strongly connected graph G' such that each row sum of $M(G')$ is n, each column sum of $M(G')$ is n, and $(\Omega(G'), \sigma')$ is topologically conjugate to $(\Omega(G), \sigma)$. It is easy to see that G' is a biregular extension of $G_0(A)$ where A is the set of n symbols. (See [**16**, lemma 1]. As for $G_0(A)$, see § 1.) Therefore, by proposition 2.2, every irreducible subshift of finite type $(\Omega(G), \sigma)$ with $r(G) = n$ is a constant-to-one extension of the full shift system on n symbols. Since $r(G) = n$ for every subshift of finite type $(\Omega(G), \sigma)$ which is a finite-to-one extension of the full shift system on n symbols, we conclude that every irreducible subshift of finite type which is a finite-to-one extension of a full shift system, is also a constant-to-one extension of the full shift system. The question arises of whether every irreducible subshift of finite type which is a finite-to-one extension of an irreducible subshift of finite type, is also a constant-to-one extension of the irreducible subshift of finite type. The following example shows that the answer is negative.

Let G_1 and G_2 be graphs with

$$M(G_1) = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad M(G_2) = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}.$$

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Clearly G_1 and G_2 are strongly connected. Since $M(G_1)R = RM(G_2)$ where

$$R = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

it follows from proposition 2.4 that there is a regular homomorphism of G_1 into G_2. Hence $(\Omega(G_1), \sigma_1)$ is a finite-to-one extension of $(\Omega(G_2), \sigma_2)$. But the elementary divisors of $M(G_1)$ are $\lambda - 2$, $\lambda - 1$, and $(\lambda+1)^2$, whereas the elementary divisors of $M(G_2)$ are $\lambda - 2$ and $\lambda + 1$. Therefore, by theorem 7.4, $(\Omega(G_1), \sigma_1)$ is not a constant-to-one extension of $(\Omega(G_2), \sigma_2)$. (Note also that both $(\Omega(G_1), \sigma_1)$ and $(\Omega(G_2), \sigma_2)$ are constant-to-one extensions of the full shift system on 2 symbols.)

The converse of theorem 7.4 does not hold. An example of Kitchens [**10**] shows that if G_1 and G_2 are strongly connected graphs such that the elementary divisors of $M(G_1)$ contains the elementary divisors of $M(G_2)$, $(\Omega(G_1), \sigma_1)$ is not necessarily even an extension of $(\Omega(G_2), \sigma_2)$.

Question. Does the converse of theorem 7.4 hold under the condition that $(\Omega(G_1), \sigma_1)$ is a finite-to-one extension of $(\Omega(G_2), \sigma_2)$?

8. *Concluding remarks*

Finally, to show that there are other applications of induced regular homomorphisms and induced b-regular homomorphisms, we state some results omitting proofs.

Let G_1 and G_2 be strongly connected graphs with $r(G_1) = r(G_2)$ and let h be a homomorphism of G_1 into G_2 with h^* onto. Let $m(h)$ be as in theorem 6.2. One can prove that

$$m(\tilde{h}) = m(\tilde{\tilde{h}}) = m(h)$$

and if v is any point of G_2, $m(h)$ equals the maximum number of mutually disjoint maximal b-compatible [compatible] sets for $\tilde{h}$ [for $\tilde{\tilde{h}}$] contained in $\phi_{\tilde{h}}^{-1}(v)$ [in $\phi_{\tilde{\tilde{h}}}^{-1}(v)$]. One can also prove that $m(h) = 1$ iff $U \cap V \neq \varnothing$ ($|U \cap V| = 1$) for any maximal compatible set U for h and any maximal b-compatible set V for h with $\phi_h(U) = \phi_h(V)$.

Let h be a regular [b-regular] homomorphism of a graph G_1 into a graph G_2. Let p be a non-negative integer. Then h is said to be *p definite* if for any x_1, $x_2 \in \Pi^{(p)}(G_1)$, $h^*(x_1) = h^*(x_2)$ implies $\mathrm{t}(x_1) = \mathrm{t}(x_2)$ [$\mathrm{i}(x_1) = \mathrm{i}(x_2)$], and h is said to be *definite* if h is p definite for some non-negative integer p.

A definite regular homomorphism is considered to be a generalization of the state transition diagram of a finite automaton having a definite table, which was introduced in [**20**]. The properties of definite tables and a practical decision procedure for definiteness of tables presented in [**20**], can straightforwardly be extended to definite regular [b-regular] homomorphisms of graphs.

Let G_1 and G_2 be strongly connected graphs with $r(G_1) = r(G_2)$, and let h be a homomorphism of G_1 into G_2 with h^* onto. Then one can show that the induced regular [b-regular] homomorphism $\tilde{h}$ [$\bar{\tilde{h}}$] of h is p definite iff h is p b-bundle-mergible [p bundle-mergible] and $m(h) = 1$. (Cf. [**14**, theorem 5].) Therefore a criterion for bijectivity of h_∞ can be obtained, that is, h_∞ is one-to-one and onto iff both $\tilde{h}$ and $\bar{\tilde{h}}$ are definite.

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