Constellations Maximizing Minimal Distance for Physical-Layer Network Coding Multiway Relaying

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***Abstract*—This paper objects a design of constellations suitable for Physical-Layer Network Coding (PLNC) in a Multi-Way Relay Channel (M-WRC). We formulate the constellation design as a general constrained optimization problem maximizing minimal distance of network coding function decoding. The numerically obtained constellations are recognized to possess some regular structure. The optimal constellations for 2-WRC are power-balanced hexagonal constellations with particular type of indexing. As far as the optimization problem is numerically tractable, we declare optimality of power-scaled pre-rotated amplitude shift keying constellations for M-WRC with number of terminals > 2. All optimums decode non-linear modulo-sum network coding function and offer considerable performance gains in comparison to canonical schemes. This work also demonstrates that PLNC strategy tailored to M-WRC overcome PLNC 2-WRC approach.**

I. Introduction

A. Background, Motivation and Goals

Physical-Layer (Wireless) Network Coding (PLNC) attracts attention of research community for almost a decade. Under ideal conditions, PLNC provides the highest achievable rates in a 2-Way Relay Channel (2-WRC) [1]. PLNC concept based on nested lattice codes with linear network coding framework is denoted as Compute-and-Forward (CF) strategy [2]. CF reaches the highest achievable rates and naturally scales into general multi-node wireless network providing a useful network information theoretical tool. In real scenario, PLNC faces several additional issues (system complexity, signaling overhead, time and frequency offset, etc.), however its viability for 2-WRC seem to be proven as demonstrate first practical implementations [3]. Unlike relatively clear situation in a 2-WRC, a scaling into more complex network topology is challenging and needs individual optimization for each particular topology [4]–[6].

This paper considers PLNC in a Multi-Way Relay Channel (M-WRC). M-WRC consists of multiple terminals exchanging information via a relay node, see Fig. 1. M-WRC may represent a basic scenario e.g. for wireless peer-to-peer network, social network or ad hock satellite network. Theoretically, nested lattice codes in M-WRC provides the highest achievable rates [7] which motivates usage of PLNC. Related work [8] analysis error propagation effect of PLNC in M-WRC and claims for optimized higher cardinality constellations which we target here.

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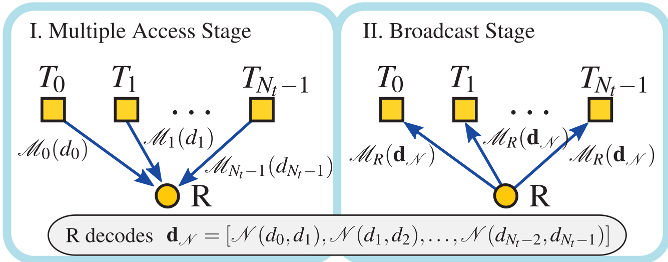

Fig. 1. Physical-layer network coding in a multi-way relay channel.

Design of constellations for PLNC is non-trivial task since the performance even in a simple uncoded AWGN model significantly depends on a constellation indexing and a network coding function [9]. Related work [10] fixes indexing and modulo-sum network coding function and search for optimal lattice constellations in a 2-WRC. The other way around, the authors of [11] fix constellation and indexing and search for optimal network coding function. Clearly, all these parameters need to be designed jointly in order to provide the optimum, which is the goal of our paper.

B. Contributions

We formulate a constellation design for PLNC in a M-WRC as a constrained optimization problem maximizing minimal distance of suitable network coding function decoding. To the best of our knowledge, this constellation design problem has never been presented. Related paper [12] designs precoding of QAM constellations such that each symbol of superposed constellation corresponds to a single data vector. This approach serves as a benchmark to our results and validate the optimality of proposed constellations optimized for modulo-sum function decoding. The numerical results for 2-WRC agree with power-balanced hexagonal constellations as was conjectured in [10]. As far as the problem is numerically tractable, we confirm optimality of power-scaled per-rotated Amplitude Shift Keying (ASK) constellations for M-WRC with number of terminals $>$ 2 as were conjectured in limited case of power balanced 3-WRC [5]. This work also demonstrates that PLNC strategy tailored to M-WRC overcome PLNC 2-WRC approach.

II. System Model

A. Signal Space Model and Used Notation

Let terminals $T_0, T_1, \dots T_{N_t-1}$ use constellations $\mathscr{A}_0, \mathscr{A}_1, \dots \mathscr{A}_{N_t-1}$ where N_t denotes total number of terminals. The notation from perspective of terminal T_0 is following. Alphabet $\mathscr{A}_0$ is formed by baseband signal space points $\mathscr{A}_0 = \{s_0^{(i)}\}_{i=0}^{M_0-1}$, where signal points are complex scalar numbers $s_0^{(i)} \in \mathbb{C}$ and M_0 denotes the constellation cardinality $M_0 = |\mathscr{A}_0|$. We describe all cardinalities by vector $\mathbf{M} = [M_0, M_1, \dots M_{N_t}]$. Upper indexes $\star^{(i)}$ are used when a concrete value of the variable is to be stressed. We define constellation mapper $\mathscr{M}_0$ such that the alphabet indexes directly correspond to data symbols $\mathscr{M}_0(d_0) = s_0^{(d_0)}$, where data symbols are $d_0 \in \mathbb{Z}_{M_0}$ and $\mathbb{Z}_{M_0} = \{0, 1, \dots, (M_0 - 1)\}$.

B. Multi-Way Relay Channel and Model Assumptions

M-WRC consists of terminals $T_0, T_1, \dots T_{N_t-1}$ communicating in a half-duplex manner (each terminal cannot send and receive at the same time) via a relay R. Every terminal wishes to send its message to all other terminals. Unlike in [6], we do not assume a presence of direct links among the terminals. We assume an idealized time and phase synchronized scenario with simple AWGN model and per symbol uncoded relaying.

C. Physical-Layer Network Coding Multi-Way Relaying

PLNC in M-WRC consists of two stages: a Multiple Access (MA) stage and a BroadCast (BC) stage, see Fig. 1. At the first MA stage, both terminals transmit simultaneously to the relay which receives a signal superposition

$$x = \sum_{i=0}^{N_t-1} \mathscr{M}_i(d_i) + w = u(\mathbf{d}) + w, \tag{1}$$

where $u(\mathbf{d})$ denotes a superposed signal $u(\mathbf{d}) = \sum_{i=0}^{N_t-1} \mathscr{M}_i(d_i)$ and w is a complex AWGN noise with variance $2N_0$. The relay decodes a network coded symbol $\mathbf{d}_{\mathscr{N}} = [d_{\mathscr{N}0}, d_{\mathscr{N}1}, \dots, d_{\mathscr{N}(N_t-1)}]$ which is such a symbol that a detection of all symbols $\mathbf{d} = [d_0, d_1, \dots d_{N_t-1}]$ at any terminal is possible providing the knowledge of $\mathbf{d}_{\mathscr{N}}$ and any symbol d_i (known by terminal a priori). We choose a network coded symbol to be $d_{\mathscr{N}} = \mathscr{N}(d_0, d_1)$ for 2-WRC, $\mathbf{d}_{\mathscr{N}} = [\mathscr{N}(d_0, d_1), \mathscr{N}(d_1, d_2)]$ for 3-WRC and

$$\mathbf{d}_{\mathscr{N}} = [\mathscr{N}(d_0, d_1), \mathscr{N}(d_1, d_2), \dots \mathscr{N}(d_{N_t-2}, d_{N_t-1})] \tag{2}$$

for M-WRC where network coding function $\mathscr{N}$ fulfills an exclusive law of network coding [11]

$$\begin{aligned} \mathscr{N}(d, d) &\neq \mathscr{N}(d', d), \quad d \neq d' \\ \mathscr{N}(d, d) &\neq \mathscr{N}(d, d'), \quad d \neq d' \end{aligned} \tag{3}$$

and thus it enables a unique inversion $\mathscr{N}^{-1}$ providing one of the data inputs. For instance, terminal T_0 in 3-WRC obtains the remaining symbols with knowledge of $\mathbf{d}_{\mathscr{N}} = [d_{\mathscr{N}0}, d_{\mathscr{N}1}] = [\mathscr{N}(d_0, d_1), \mathscr{N}(d_1, d_2)]$ and d_0 as $[d_1, d_2] = [\mathscr{N}^{-1}(d_{\mathscr{N}0}, d_0), \mathscr{N}^{-1}(d_{\mathscr{N}1}, \mathscr{N}^{-1}(d_{\mathscr{N}0}, d_0))]$. Typical minimum cardinality exclusive operations are bit-wise XOR and modulo-sum function

$$\mathscr{N}_{\mathrm{XOR}}(d, d') = d \oplus d', \tag{4}$$

$$\mathscr{N}_{\mathrm{MOD}}(d, d') = (d + d')_{\mathrm{mod}\, M}, \tag{5}$$

respectively. The relay decodes network coded symbol as

$$\hat{\mathbf{d}}_{\mathscr{N}} = \arg\max_{\mathbf{d}_{\mathscr{N}}} p(x|\mathbf{d}_{\mathscr{N}}), \tag{6}$$

where likelihood function of $\mathbf{d}_{\mathscr{N}}$ is given by

$$p(x|\mathbf{d}_{\mathscr{N}}) = \frac{1}{M_{\mathscr{N}}} \sum_{\mathbf{d}:\mathbf{d}_{\mathscr{N}}} \frac{1}{2\pi N_0} e^{-\frac{|x-u(\mathbf{d})|^2}{2N_0}}. \tag{7}$$

Summation (7) runs over all $\mathbf{d}$ such that $[\mathscr{N}(d_0, d_1), \dots \mathscr{N}(d_{N_t-2}, d_{N_t-1})]$ equals to $\mathbf{d}_{\mathscr{N}}$ where $M_{\mathscr{N}}$ denotes the cardinality of network coded symbols.

At the second BC stage, the relay broadcasts network coded symbol $\mathbf{d}_{\mathscr{N}}$ and every terminal subsequently performs detection using network coding function invertibility and knowledge of its own data symbol.

III. Constellation Design for PLNC in M-WRC

A. Objective Function

Natural goal is to design constellations minimizing the error performance of (6). We can simplify this complex task by assumption of medium to high Signal-to-Noise Ratio (SNR) where one of the exponential functions becomes considerably higher and we can neglect the impact of others. This leads to the classical approximation by pairwise symbol error

$$P_{se} \simeq N_{\min} Q\left(\sqrt{\frac{\Delta^2_{\min}}{4N_0}\mathscr{E}}\right), \tag{8}$$

where $\mathscr{E} = {}^1\!/_M \sum_{i=1}^{M-1} |s^{(i)}|^2$ denotes a mean symbol energy, $N_{\min}$ is an average number of nearest neighbors[1] and $Q(x) = {}^1\!/\sqrt{2\pi} \int_x^\infty \exp\left(-{}^{u^2}\!/_2\right) \mathrm{d}u$. The minimal distance of network coded symbol decoding is defined as

$$\Delta^2_{\min} = \min_{\mathbf{d}_{\mathscr{N}} \neq \mathbf{d}'_{\mathscr{N}}} |u(\mathbf{d}) - u(\mathbf{d}')|^2, \tag{9}$$

where $\mathbf{d}_{\mathscr{N}}$ is given by (2) and $\mathbf{d}'_{\mathscr{N}} = \left[\mathscr{N}(d'_0, d'_1), \mathscr{N}(d'_1, d'_2), \dots \mathscr{N}(d'_{N_t-2}, d'_{N_t-1})\right]$ and data vectors are $\mathbf{d} = [d_0, d_1, \dots d_{N_t-1}]$, $\mathbf{d}' = [d'_0, d'_1, \dots d'_{N_t-1}]$.

B. Energy Conditions

We distinguish two cases a) all terminals use the same alphabets $\mathscr{A}_i = \mathscr{A}_j$ and b) the alphabets are arbitrary $\mathscr{A}_i \neq \mathscr{A}_j$. Case a) is easier to solve since lower number of variables need to be find and we can numerically solve more complex tasks. A fair comparison requires constellations to possess identical

[1]The number of nearest neighbors of a network coding function decoding is rather a number of nearest neighboring regions corresponding to distinct network coded symbols, for details see Sec. 3.4.2 in [9].

energy properties. In case a) all signal points are normalized to the unit mean symbol energy

$$\mathscr{E}_0 = \mathscr{E}_1 = \cdots = \mathscr{E}_{N_t-1} = 1. \tag{10}$$

In case b), we assume a more general condition where total energy remain constant

$$\sum_{i=0}^{N_t-1} \mathscr{E}_i = N_t. \tag{11}$$

Case b) means that some terminals consume more energy in favor of some other terminals. To avoid this energy disbalance, we propose a periodic switching of the role which terminal has higher energy. The enforced periodization brings average energy consumption in balance.

Let us assume an example with 3 terminals using constellations with different symbol energies. Let the terminals use alphabets $[\mathscr{A}_0, \mathscr{A}_1, \mathscr{A}_2]$ in the first stage , $[\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_0]$ in the second stage and $[\mathscr{A}_2, \mathscr{A}_0, \mathscr{A}_1]$ in the last stage. Thus, every terminal uses each constellation one third of time balancing the mean energy. Theory of Latin squares solves the problem for arbitrary number of terminals.

C. Optimization Problem

Our optimization problem is the following, we search for arbitrary constellations $[\mathscr{A}_0, \mathscr{A}_1, \ldots, \mathscr{A}_{N_t-1}]$ which

$$\text{maximize} \quad \Delta^2_{\min}, \tag{12}$$

subject to (10) in case a) $\mathscr{A}_i = \mathscr{A}_j$ or (11) in case b) $\mathscr{A}_i \neq \mathscr{A}_j$; $\Delta^2_{\min}$ is given by (9). The problem is naturally parametrized by number of terminals N_t and constellation cardinalities $\mathbf{M}$.

Next important parameter is network coding function $\mathscr{N}$. We consider 3 basic types: bit-wise XOR $\mathscr{N}_{\mathrm{XOR}}$ (4), modulo-sum $\mathscr{N}_{\mathrm{MOD}}$ (5) and

$$\mathscr{N}_{\mathrm{JOINT}}(d, d') = d' + M'd, \tag{13}$$

where $d' \in \mathbb{Z}_{M'}$. Function (13) means that the decision is a joint decision about pair $[d, d']$. This type of objective function use strategy [12].

D. Optimization Tools

Optimization problem (12) is a standard constellation design problem which is a non-convex quadratically constrained NP hard minimax problem [13]. First of all we reformulate the minimax problem: instead of maximizing minimum, we maximize auxiliary variable and include extra conditions that the auxiliary variable is lower or equal than all the conditions over which minimum operator was assumed. We solve the reformulated problem by global meta-heuristic differential evolution algorithm and its results are further post-processed by local interior-points method[2]. Global optimization method runs for several thousands of initial starting values of its pseudo-random number generator utilizing the power of parallel programing. The fact that the highest minimal distance possess constellations with zero mean enables us to force zero mean a priori and thus lower the number of variables.

Later, we observed that all results for number of terminals $N_t > 2$ resemble linearly precoded ASK constellations so we consider an optimization where instead of constellations we look for linear precoding coefficients. This problem has lower number of variables providing solutions to more complex tasks. Even though we cannot guarantee that we find the global optimum (although it is very likely), we can expect that the results perform close to the optimum.

E. Permuted Latin Squares and Constellation Indexing

Network coding functions satisfying (3) with minimal cardinality of the output form an algebraic structure of Latin squares [9]. Unfortunately, theory of Latin squares is not so complete as e.g. group theory and many statements are still unknown (e.g. total number of Latin squares). Nonetheless (as long as the size of Latin squares are computationally manageable), we may let computer to list all distinct Latin squares. By the exhaustive search, we have found:

- All Latin squares are row or column permutations of bit-wise XOR or modulo-sum Latin square for cardinality 2 and 4.
- All linear Latin squares are row or column permutations of bit-wise XOR Latin square for cardinality 2, 4 and 8, where every output of linear Latin square can be described as $a \otimes d \oplus b \otimes d'$ where $a, b \in \mathbb{F}_M$ and symbols $\otimes$ and $\oplus$ denote product and addition in finite-field $\mathbb{F}_M$.

Remark 1. The search for arbitrary constellation naturally includes its all possible indexing. Obviously, an exchange of two constellation indexes is equivalent to the exchange of relevant rows (or columns) of the Latin square. It means that if we select some network coding function for the optimization, then the result includes all row and column permutations of that network coding function. As a consequence, we search over all possible Latin squares by choosing bit-wise XOR and modulo-sum for cardinality 2 and 4, and bit-wise XOR constrained search includes all linear network coding functions for cardinalities 2, 4 and 8.

IV. Numerical Results

Table I summarizes minimal distances of numerically optimized constellations for PLNC in a 2-WRC in case a) $\mathscr{A}_i = \mathscr{A}_j$ for cardinalities $\mathbf{M}$. Labels 'mod', 'xor', 'joint', 'lattice + mod' stand for network coding function $\mathscr{N}_{\mathrm{MOD}}$ (4), $\mathscr{N}_{\mathrm{XOR}}$ (5), $\mathscr{N}_{\mathrm{JOINT}}$ (13) and power-balanced hexagonal constellations [10] with $\mathscr{N}_{\mathrm{MOD}}$, respectively. In case a), joint detection of data pairs cannot be made without error implying a zero minimal distance. We confirm that constellations having the highest minimal distance are hexagonal constellations [10]. Namely binary antipodal, 4HEX and 8HEX constellation depicted in Fig. 2. Interestingly, maximal possible minimal distance for cardinality 8 of bit-wise XOR decoding (according to Remark 1 also any linear network coding decoding) is much lower than for $\mathscr{N}_{\mathrm{MOD}}$.

[2]We employ build in procedures in Wolfram's Mathematica 10.

TABLE I

MINIMAL DISTANCES OF NUMERICALLY OBTAINED CONSTELLATIONS FOR CASE A) $\mathscr{A}_i = \mathscr{A}_j$ IN 2-WRC

M	mod	xor	joint	lattice + mod
[2,2]	4.0	4.0	0.0	4.0
[4,4]	2.0	2.0	0.0	2.0
[8,8]	0.93 Fig. 2	0.64 Fig. 2	0.0	0.93

TABLE II

MINIMAL DISTANCES OF NUMERICALLY OBTAINED CONSTELLATIONS FOR CASE B) $\mathscr{A}_i \neq \mathscr{A}_j$ IN 2-WRC

M	mod	xor	joint	lattice + mod
[2,2]	4.0	4.0	4.0	4.0
[2,4]	2.67	2.67	1.68	2.67
[4,4]	2.0	2.0	0.89	2.0
[2,8]	1.51	1.33	0.91	1.51

Results for general case b) $\mathscr{A}_i \neq \mathscr{A}_j$ are listed in Table II. Arbitrary alphabets now enable joint decoding to have non-zero minimal distance. Power balanced hexagonal constellations with modulo-sum decoding [10] leads again to the highest minimal distance. Let us explain power balancing at the following example with binary antipodal and 8HEX constellation. The binary antipodal constellation is more robust to the noise and its minimal distance equals to 4. 8HEX constellation has minimal distance only 0.93. So, we scale binary constellation down and 8HEX up keeping (11) in order to have the same minimal distance 1.51. Both scaled constellations are then taken from identical lattice and using indexes which form an arithmetic progression in each lattice dimension [10] yields the final minimal distance to be 1.51. Assumption of asymmetric cardinalities is benefitial in channels with asymmetric strength of wireless links.

Table III summarizes minimal distances of numerically optimized constellations for M-WRC with number of terminals $N_t > 2$. Label 'ask + mod' stands for optimally, linearly precoded (i.e. power and phase precoded) ASK constellations with $\mathscr{N}_{\text{MOD}}$ decoding. We do not present results for case a) $\mathscr{A}_i = \mathscr{A}_j$ because then the minimal distances are always zero. We confirm that constellations having the highest minimal distance are decoding $\mathscr{N}_{\text{MOD}}$ and can be described as linearly precoded ASK constellations. Since the results of general design and linearly precoded ASK perfectly match, we present optimally linear precoding of ASK in Tab IV. Proposed schemes provide considerable nominal gains over some canonical constellations (PSK, QAM) with optimized linear precoding as present Table V.

All numerical results share some common features: the optimal constellations are points in hexagonal lattice and modulo-sum decoding leads to the highest minimal distances.

TABLE III

MINIMAL DISTANCES OF NUMERICALLY OBTAINED CONSTELLATIONS FOR CASE B) $\mathscr{A}_i \neq \mathscr{A}_j$ IN M-WRC WITH NUMBER OF TERMINALS > 2

M	mod	xor	joint	ask + mod
[2,2,2]	4.0	4.0	2.4	4.0
[2,2,4]	1.71	1.33	1.33	1.71
[2,4,4]	1.09	0.86	0.60	1.09
[2,2,2,2]	2.0 Fig. 4	2.0	1.78	2.0
[2,2,2,4]	1.0	0.94	0.84	1.0
[2,2,2,2,2]	1.05	1.05	1.05	1.05
[2,2,2,2,2,2]	0.65	0.65	0.65	0.65

TABLE IV

OPTIMAL LINEAR PRECODING OF ASK FOR $\mathscr{N}_{\text{MOD}}$ DECODING IN M-WRC.

M	$[\Delta^2_{\min}, N_{\min}]$	$[\rho_0, \rho_1, \ldots, \rho_{N_t-1}]$	$[\varphi_0, \varphi_1, \ldots, \varphi_{N_t-1}]$
[2,2,2]	[4.0,3.75]	[1.0,1.0,1.0]	$[0, \pi/3, 2/3\pi]$
[2,2,4]	[1.71,4.0]	[0.65,0.65,1.5]	$[0, \pi/3, 2/3\pi]$
[2,4,4]	[1.09,4.9]	[0.52,1.2,1.2]	$[0, \pi/3, 2/3\pi]$
[4,4,4] Fig. 3	[0.80,5.3]	[1.0,1.0,1.0]	$[0, \pi/3, 2/3\pi]$
[4,4,8]	[0.39,6.0]	[0.70,0.70,1.4]	$[0, \pi/3, 2/3\pi]$
[4,8,8]	[0.26,5.7]	[0.56,1.2,1.2]	$[0, \pi/3, 2/3\pi]$
[8,8,8]	[0.19,5.8]	[1.0,1.0,1.0]	$[0, \pi/3, 2/3\pi]$
[2,2,2,2] Fig. 4	[2.0,4.2]	[0.71,0.71,1.22,1.22]	[4.71,0.52,5.24,0]
[2,2,2,4]	[1.0,4.7]	[0.87,1.32,0.5,1.12]	[1.57,1.76,1.05,0]
[2,2,4,4]	[0.53,4.7]	[0.63,0.97,0.82,1.41]	[5.24,4.38,4.71,0]
[2,4,4,4]	[0.28,4.7]	[0.45,1.02,0.59,1.55]	[0.86,1.9,0.33,0]
[4,4,4,4] Fig. 4	[0.17,4.7]	[1.21,0.92,0.46,1.21]	[1.05,1.76,5.95,0]
[2,2,2,8]	[0.50,5.1]	[0.35,0.94,0.61,1.62]	[1.05,1.76,1.57,0]
[2,2,4,8]	[0.28,5.3]	[1.15,0.92,0.59,1.21]	[4.83,4.71,1.05,0]
[2,2,2,2,2]	[1.05,4.5]	[0.89,0.51,0.51, 1.36,1.36]	[0.86,2.43,4.52, 1.71,0]
[2,2,2,2,4]	[0.59,5.0]	[0.77,1.38,0.38, 0.38,1.49]	[1.57,1.33,2.62, 4.71,0]
[2,2,2,2,2,2]	[0.65,5.0]	[1.39,0.81,0.7, 1.61,0.4,0.4]	[0.4,4.71,6.28, 1.57,1.57,0]

TABLE V

NOMINAL CODING GAIN $10\log_{10}\left(\Delta^2_{\min,\text{ASK+mod}}/\Delta^2_{\min}\right)$ [DB] OF THE PROPOSED LINEARLY PRECODED ASK + $\mathscr{N}_{\text{MOD}}$ OVER SOME TYPICAL SCHEMES WITH OPTIMIZED LINEAR PRECODING

M	QAM + mod	QAM + xor	PSK + mod	PSK + xor
[4,4,4] Fig. 3	2.5	2.5	4.2	2.5
[8,8,8]	9.7	9.7	12.4	12.4
[4,4,4,4] Fig. 4	2.2	2.2	2.2	2.2

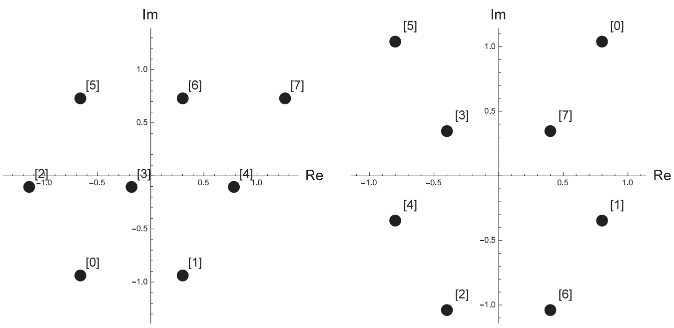

Fig. 2. Primary constellations and its indexing maximizing minimal distance of $\mathscr{N}_{\text{MOD}}$ decoding and $\mathscr{N}_{\text{XOR}}$ decoding, respectively, for PLNC in a 2-WRC with cardinalities $\mathbf{M} = [8,8]$. The minimal distance for $\mathscr{N}_{\text{XOR}}$ decoding is lower because the constellation shape is not so compact.

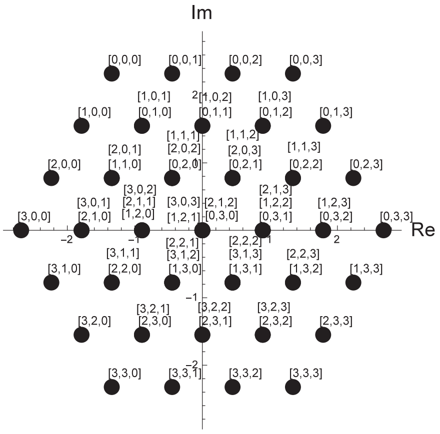

Fig. 3. Superposed constellation and its indexing maximizing minimal distance for PLNC in 3-WRC with $\mathbf{M} = [4,4,4]$ and $\mathscr{N}_{\mathrm{MOD}}$ decoding. Note that some signal space points correspond to multiple data symbols (e.g. point 0 corresponds to data vector $\mathbf{d}$ of $[3,0,3]$, $[2,1,2]$, $[0,3,0]$, $[1,2,1]$ but all of them have the same $\mathbf{d}_{\mathscr{N}} = [\mathscr{N}_{\mathrm{MOD}}(d_0,d_1), \mathscr{N}_{\mathrm{MOD}}(d_1,d_2)] = [3,3]$).

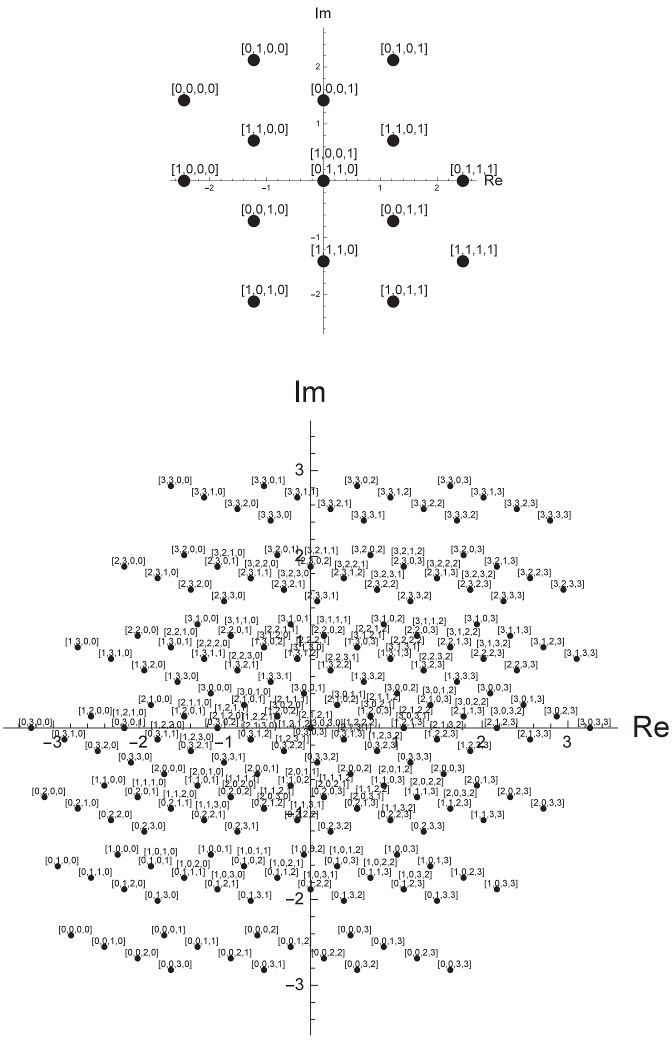

Fig. 4. Superposed constellations and its indexing maximizing minimal distance of $\mathscr{N}_{\mathrm{MOD}}$ decoding for PLNC in 4-WRC with cardinalities $\mathbf{M} = [2,2,2,2]$ and $\mathbf{M} = [4,4,4,4]$, respectively.

A. *Optimal Linear Precoding of ASK for 3-WRC*

For several particular cardinalities, we have observed that numerically optimized linear precoding of ASK for 3-WRC possesses a regular structure. Here, we explain validity for arbitrary cardinalities.

Proposition 2. *All superposed constellation points of the optimally precoded ASK constellations which fall to the same signal space point correspond to identical network coded symbol $[\mathscr{N}_{\mathrm{MOD}}(d_0,d_1), \mathscr{N}_{\mathrm{MOD}}(d_1,d_2)]$. The optimal precoding coefficients are such that each precoded constellation lies in one dimensional subspaces of hexagonal lattice. It means that power scaling ρ_i is such that all minimal distances of primary ASK constellations are equal*

$$\rho_i \delta^2_{\min,i} = \rho_j \delta^2_{\min,j}, \quad \forall i,j \in \mathbb{Z}_3 \tag{14}$$

subject to energy constrain $\sum_{i=0}^{2} \rho_i^2 = 3$ and phase pre-rotation is $\varphi_i = \frac{\pi}{3} i$.

Proof: Let signal space point of ith terminal constellation be described by hexagonal lattice generator matrix $\mathbf{G}$ and vector of lattice coordinates $\mathbf{a}$ as $s_i = \mathbf{G}\mathbf{a}_i$, $\mathbf{a}_i \in \mathbb{Z}^2$ and $\mathbf{G} = [1, 1/2 + \mathrm{j}\sqrt{3}/2]$. The fact that each constellation forms one dimensional subspace means that we can describe any lattice coordinates as

$$\mathbf{a}_i = \boldsymbol{\nu}_i k_i \tag{15}$$

where $\mathbf{v}_i \in \mathbb{Z}^2$ and $k_i \in \mathbb{Z}$ is the subspace base vector and integer coefficient of ith constellation. Base vector $\mathbf{v}_1$ is linearly dependent due to the $\pi/3$ symmetry of hexagonal lattice $\mathbf{v}_1 = \mathbf{v}_0 + \mathbf{v}_2$. When two superposed constellation points equal $u = u'$ where $u = \sum_{i=0}^{2} \mathbf{s}_i$ and $u' = \sum_{i=0}^{2} \mathbf{s}'_i$ then

$$\mathbf{a}_0 + \mathbf{a}_1 + \mathbf{a}_2 = \mathbf{a}'_0 + \mathbf{a}'_1 + \mathbf{a}'_2 \tag{16}$$

which yields $\mathbf{a}_0 + \mathbf{a}_1 + \mathbf{a}_2 = \boldsymbol{\nu}_0 (k_0 + k_1) + \boldsymbol{\nu}_1 (k_1 + k_2)$. If $u = u'$ then $k_0 + k_1 = k'_0 + k'_1$ and $k_1 + k_2 = k'_1 + k'_2$ leading to

$$\begin{aligned} \mathbf{a}_0 + \mathbf{a}_1 &= \mathbf{a}'_0 + \mathbf{a}'_1, \\ \mathbf{a}_1 + \mathbf{a}_2 &= \mathbf{a}'_1 + \mathbf{a}'_2. \end{aligned} \tag{17}$$

Now, we use the same concept of proof as in [10]. Let all constellations be indexed by arithmetical progression with increment 1. The inverse indexing function is $\mathscr{I}^{-1}(\mathbf{a}_0) = d_0 = \left(\mathbf{1}^T \mathbf{a}_0\right)_{\mathrm{mod} M_0}$ where $\mathbf{1} = [1,1,\ldots 1]^T$. The network coded symbol $\mathbf{d}_{\mathscr{N}} = [d_{\mathscr{N}0}, d_{\mathscr{N}1}]$ is

$$\begin{aligned} d_{\mathscr{N}0} &= \mathscr{N}_{\mathrm{MOD}}(d_0,d_1) = (d_0 + d_1)_{\mathrm{mod} M} = \\ &= \left(\mathscr{I}^{-1}(\mathbf{a}_0) + \mathscr{I}^{-1}(\mathbf{a}_1)\right)_{\mathrm{mod} M} = \quad \left(\mathbf{1}^T (\mathbf{a}_0 + \mathbf{a}_1)\right)_{\mathrm{mod} M_0} \end{aligned} \tag{18}$$

and similarly for

$$d_{\mathscr{N}1} = \mathscr{N}_{\mathrm{MOD}}(d_1,d_2) = \left(\mathbf{1}^T (\mathbf{a}_1 + \mathbf{a}_2)\right)_{\mathrm{mod} M_0}. \tag{19}$$

When we apply (17) to (18) and (19) for u and u', we conclude that if $u = u'$ then $[d_{\mathscr{N}0}, d_{\mathscr{N}1}] = [d'_{\mathscr{N}0}, d'_{\mathscr{N}1}]$ which proves the proposition. ■

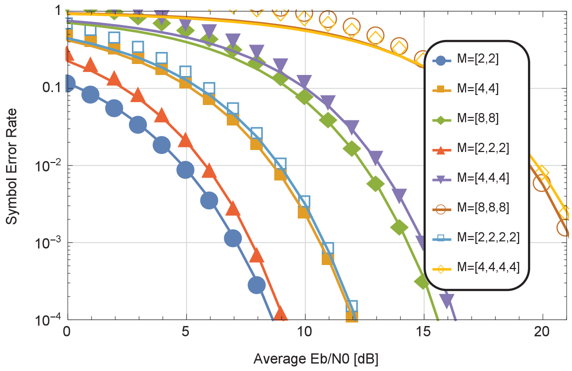

Fig. 5. Symbol error rate at the MA stage of the optimized constellations for several cardinalities **M**. The markers correspond to pairwise error approximation (8) and solid lines are Monte Carlo error evaluations.

V. Performance Evaluation

Figure 5 presents symbol error rate performance at the MA stage of the optimized constellations for several cardinalities **M**. The performance curves fit to pairwise error approximation (8) using parameters from Table IV. Bit throughput from Fig. 6 is simulated on packets of 512 symbols.

In order to avoid impact of bit indexing whose optimal selection is unknown, we show an approximate bit throughput as a symbol throughput multiplied by $\log_2 M$. PLNC using four terminal constellations like case $\mathbf{M} = [2,2,2,2]$ or $[4,4,4,4]$ in a 4-WRC requires a single MA stage for delivering required network coded symbol at the relay

$$\mathbf{d}_{\mathcal{N}} = [\mathcal{N}_{\text{MOD}}(d_0,d_1), \mathcal{N}_{\text{MOD}}(d_1,d_2), \mathcal{N}_{\text{MOD}}(d_2,d_3)]. \quad (20)$$

When only a pair of terminals are transmitting at each MA stage e.g in case of $[2,2]$ or $[4,4]$, 3 orthogonal MA stages are required to deliver (20). Thus its throughput needs to be reduced by $1/3$. Multiplication factor of 3 terminal constellations is not simply $2/3$ because, at the first MA stage we deliver $[\mathcal{N}_{\text{MOD}}(d_0,d_1), \mathcal{N}_{\text{MOD}}(d_1,d_2)]$ but at the second MA stage we cannot deliver $[\mathcal{N}_{\text{MOD}}(d_2,d_3), \mathcal{N}_{\text{MOD}}(d_0',d_1')]$ as we wish. The reason is simply that 3 terminal constellations can deliver $[\mathcal{N}_{\text{MOD}}(d_0,d_1), \mathcal{N}_{\text{MOD}}(d_1,d_2)]$ where for both functions must be symbol d_1 the same. Even though the 3 terminal constellations are slightly more reduced than by $2/3$, they still perform very well. We conclude that PLNC optimized for M-WRC with $N_t > 2$ is attractive and it overcomes PLNC approach optimized for 2-WRC. Especially novel schemes with cardinalities $\mathbf{M} = [2,2,2,2]$ and $[4,4,4,4]$ perform well.

VI. Conclusion

We have numerically designed constellations for PLNC in a M-WRC maximizing minimal distance of network coding function decoding. In certain cases, we simply confirm that constellations previously found ad hoc or under suboptimal approach are in fact optimal (e.g. hexagonal constellations in 2-WRC and linearly precoded ASK in 3-WRC). In other cases, we have found more general solutions like constellations asymmetric cardinalities and we have also found novel schemes e.g. promising linearly precoded ASK in 4-WRC. All optimal constellations are from hexagonal lattice. This is evidently due to restriction to one complex dimension. We conjecture that the optimal multi-dimensional constellations would occupy different type of lattice, e.g. D_4 lattice for two dimensions. Notable consequence is that all optimums require non-linear modulo-sum network coding function and assumption of linearity may considerable decrease performance as observed also by mutual information analysis [14].

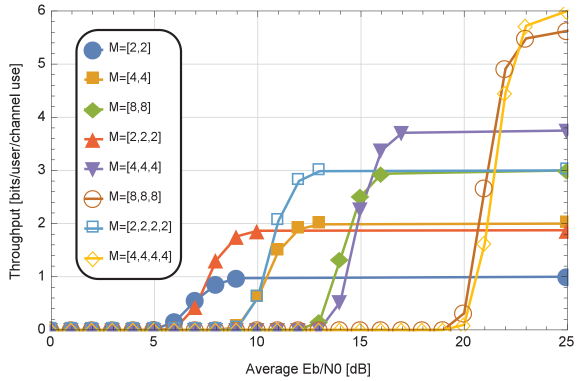

Fig. 6. Bit throughput at the MA stage of the optimized constellations for several cardinalities **M** in 4-WRC and 512 symbols long packets.

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