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Constrained Digital Regulation of Hyperbolic PDE Systems: A Learning Control Approach

Jinhoon Choi†, Beom Joon Seo and Kwang Soon Lee

Department of Chemical Engineering, Sogang University,
1 Sinsoo-Dong, Mapo-Koo, Seoul 121-742, Korea
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Abstract−In this paper, exploiting repetitive properties, a constrained digital regulation technique for first order hyperbolic PDE systems is proposed that guarantees the stability and performance of the closed loop system.

Key words: First Order Hyperbolic PDE's, Q-ILC, Reaction-convection Processes, Constrained Systems

INTRODUCTION

Most of the process control algorithms in practice are based on the finite dimensional control theory. However, many chemical processes are described by partial differential equations (PDE's) and are infinite dimensional in nature due to spatial variation. Especially when the convection is dominant and thus diffusion can be ignored, chemical processes that are described by a system of first order hyperbolic PDE's. Such processes include tubular reactors [Ray, 1981], fixed bed reactors [Stangeland and Foss, 1970] and pressure swinging adsorption [Ruthven and Sircar, 1994]. More examples can be found in [Rhee et al., 1986]. Conventionally such infinite dimensional systems described by PDE's are controlled by finite dimensional controllers that are designed through finite dimensional reduction of the process model via the spatial discretization techniques. However, such spatial discretization leads in general to a high order finite dimensional system. Moreover, the finite dimensional controllers based on the finite dimensional models can lead to an unstable closed loop system. Although stability is achieved, the performance of such controllers can be very poor when they are applied to an infinite dimensional system. For diffusion dominant systems that are described by parabolic PDE's, there are infinitely many discrete modes among which only finite number of modes are slow and all the rests are stable fast [Balas, 1979; Friedman, 1976]. Hence, for such systems, a meaningful low dimensional approximation possible through modal decomposition. However, for first order hyperbolic PDE's, all the modes have the same, or almost the same, energy. Thus, a low dimensional model through modal decomposition is not possible since an infinite number of modes are necessary for accurate approximation of the original system. As a result, even if the finite dimensional high order system model obtained through spatial discretization is reduced by the operation data based model reduction techniques such as Karhunen-Loeve decomposition [Sirovich, 1987], such reduced system may work very poorly under the situation different from operation data. Due to such prob- lems, the optimal control approach was adopted for the control of hyperbolic PDE systems, that leads to infinite dimensional controllers [Wang, 1966; Lo, 1973; Balas, 1986]. Recently the geometric control theory based design of infinite dimensional controller without resorting to optimal control techniques was proposed in Christofides and Daoutidis [1996].

The aforementioned optimal and geometric control strategies for hyperbolic PDE systems are unconstrained, continuous and infinite dimensional ones that are quite complicated. Constraints are always present in any practical control problems. For instance, the physical restriction of the actuator limits the value the input can assume. Moreover due to safety, environmental regulation and so on, the states of the plant are desired to lie within a designated area in the state space. Moreover, since all the chemical processes are controlled by computers nowadays, these control techniques need to be implemented in discrete time. Hence it is clearly desirable to develop some constrained finite dimensional digital control strategies that guarantee the stability and performance of the closed loop system. In this paper, we first show that the hyperbolic PDE solutions are repetitive in nature. Exploiting this, we reformulate the constrained regulation problem of hyperbolic PDE systems as a constrained batch process control problem where the control action is updated in discrete time. For the resulting batch process control problem, we adopt the recently developed Q-ILC techniques [Lee et al., 1996, 2000; Chin et al., 1999; Jung et al. 1999], that is a learning control strategy for batch processes. The resulting regulation strategy is a constrained, digital and finite dimensional one. The convergence of the control law is guaranteed from the convergence of Q-ILC techniques. Moreover, the performance of the controller is also guaranteed. The proposed methodology is illustrated with a tubular reactor example.

PRELIMINARIES ON Q-ILC

In this section, we briefly review the Q-ILC technique [Lee et al., 1996, 2000], which is a model-based iterative learning control technique developed specifically for multivariable batch process control problems. It can be regarded as a generalization of the traditional ILC techniques developed mainly for robot-arm training. The Q-ILC technique can be used to recursively refine the input trajectory based on the tracking error obtained in the previous batches.

Consider an n_u-input, n_y-output discrete-time linear time varying

†To whom correspondence should be addressed.
E-mail: jchoi@ccs.sogang.ac.kr

batch process that is run over a prespecified time interval. Since only the transient behavior exists in the operation of a batch process and most of the chemical processes are nonlinear, the process model for control is typically described by linear time varying difference equations that are obtained for instance through linearization around the desired trajectory. Since a batch operation is defined over a finite time interval, a general and convenient description of process dynamics is a linear algebraic system relating the input sequence to the output sequence over the entire batch horizon. Define

$$y^T=[y^T(1)y^T(2)\cdots y^T(N)]$$
$$u^T=[u^T(0)u^T(1)\cdots u^T(N-1)]. \tag{1}$$

Then, we may describe the effect of u on y as

$$y=Gu+b \tag{2}$$

where G is the dynamic gain matrix which is assumed to be known (through identification or linearization of a nonlinear model with respect to a reference trajectory) and b is the bias vector which may be unknown. The causality of the process restricts G to have a lower triangular structure.

Now define the error trajectory vector:

$$e=y_d-y \tag{3}$$

where y_d is the desired reference output trajectory. The error trajectory may depend on several things including input u and initial condition (i.e., condition at the start of a batch).

Let e_k be the error vector at the *k*th batch. Then, (2) can be rewritten as

$$e_{k+1}=e_k-G\Delta u_{k+1}. \tag{4}$$

where $\Delta u_{k+1}=u_{k+1}-u_k$.

Q-ILC is derived based on model (4) and solves the following minimization at the start of the kth batch:

$$\min_{\Delta u_k}\frac{1}{2}\{e_{k|k-1}^T Q e_{k|k-1}+\Delta u_k^T R\Delta u_k\} \tag{5}$$

where Q and R are PD (positive definite) matrices.

For the unconstrained case, the resulting control law is

$$u_k=u_{k-1}+H^Q e_{k-1|k-1} \tag{6}$$

where

$$H^Q=(G^TQG+R)^{-1}G^TQ. \tag{7}$$

As in model predictive control, we may incorporate constraints imposed on the input and output variables into the above quadratic minimization. In this case, (5) becomes a standard quadratic programming problem.

It has been shown that the Q-ILC algorithm given by (6) has the following properties [Lee et al., 2000]:

1. Convergence

If the desired trajectory is reachable (this is always true when we have sufficient control inputs and thus G has full row rank), the error trajectory e for system (4) converges to zero asymptotically as the number of batch grows for any choice of Q>0 and R>0. Otherwise the Q-ILC minimizes the error. The same is true for the constrained algorithm under some reasonable assumptions on the choice of constraints.

2. Robustness

The convergence property is retained when the model error is within certain limits. The region of attractivity can be increased by increasing the input weight R. However, this slows down the rate of convergence.

3. Disturbance Sensitivity

The severe sensitivity of input signal to high frequency components of the output error in other traditional ILC algorithms can be abated without losing the convergence property. Indeed it can be adjusted at will by the choice of R.

FORMULATION

The behavior of first order hyperbolic PDE systems cannot be reduced to a low dimensional manifold. Hence the available control design techniques for such systems results in a unconstrained infinite dimensional controller that updates the distributed control in continuous time. However as we will show in this section the solution of hyperbolic PDE is repetitive in nature. Hence we show that, updating the control iteratively, we can obtain a simple constrained digital finite dimensional regulation strategy for first order hyperbolic PDE systems that guarantees the stability and performance.

Consider the linear first order hyperbolic partial differential equation:

$$\frac{\partial x}{\partial t}=-A\frac{\partial x}{\partial z}+B(z)x+C(z)u$$

with the boundary condition

$$x(t, 0)=x_i$$

and the initial condition:

$$x(0, z)=x_0(z),\ \forall z\in[0, L].$$

Such system may be obtained from the linearization around the steady state of the quasi-linear first order PDE systems such as reaction convection processes:

$$\frac{\partial x}{\partial t}=-A\frac{\partial x}{\partial z}+F(x)x+G(x)u$$

or the nonlinear first order PDE systems:

$$\frac{\partial x}{\partial t}=-A\frac{\partial x}{\partial z}+H(x, u).$$

Here we assume the matrix A is simple and is in the form

$$A=\begin{bmatrix} a_1 & 0 & \cdots & 0\\ 0 & a_2 & \cdots & 0\\ \vdots & \ddots & \ddots & \vdots\\ 0 & \cdots & 0 & a_n\end{bmatrix}$$

where $a_1\ge a_2\ge\cdots\ge a_n>0$. Hence we have

$$\frac{\partial x_i}{\partial t}=-a_i\frac{\partial x_i}{\partial z}+B_i(z)x+C_i(z)u$$

where $B_i(z)$ and $C_i(z)$ are the ith row of B(z) and C(z), respectively.

In this paper we will consider a finite number of control actuators

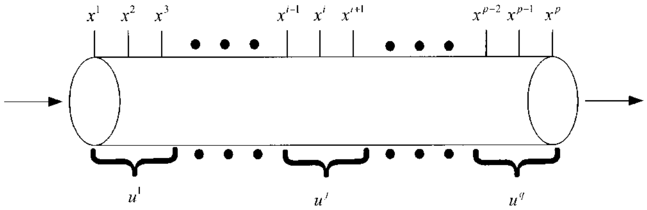

Fig. 1. Sensors and actuators.

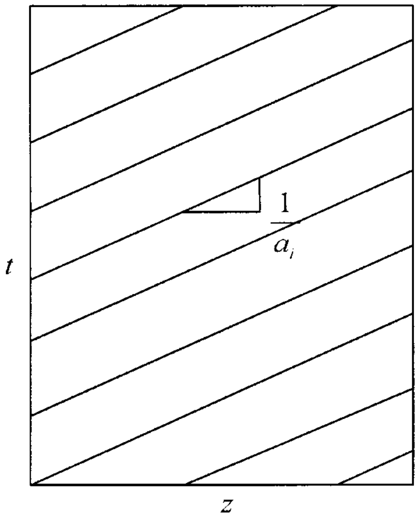

Fig. 2. Characteristic lines for x_i.

and finite number of point sensors. Namely different control inputs are applied in each prespecified intervals and the states are measured many but finite number of locations by point sensors as depicted in Fig. 1.

In this paper, we consider the regulation problem of the hyperbolic PDE systems. The characteristic lines for x_i are shown in Fig. 2.

Employing the method of characteristics [Ray, 1981], the system on the characteristic line can be reduced to an ODE:

$$\left.\frac{dx_i}{dt}\right|_{c_i} = B_i(z)x + C_i(z)u.$$

Instead of continuous update of control input, we will adopt digital control where control input is renewed at discrete times. For this define

$$T = \frac{L}{a_n}.$$

Now consider the region over the time interval $[\tau, \tau+T]$ where a characteristic line passing through $(L, \tau+T)$ is shown for each subsystem for x_i as depicted in Fig. 3.

If the control input is fixed over the above region, the solution above the characteristic line for x_n will be the same for each z. Now consider successive such blocks and assume the control action is fixed over each block. In each block, the solution will be constant at each spatial position z above the characteristic line for x_n whereas the transient behavior will take place below the characteristic line for x_n. Exploiting this, we consider such a block as a batch ignoring the transient behavior below the characteristic line for x_n. Then

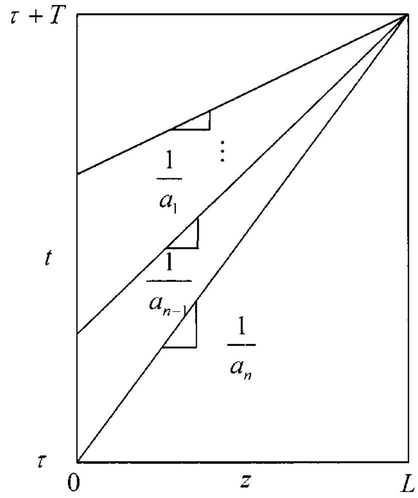

Fig. 3. Representative characteristic lines.

the digital regulation of the hyperbolic PDE systems reduces to a batch process control problem for which numerous techniques are available. In this paper, we will apply the Q-ILC technique that has been developed recently. In each block, the characteristic line for x_i depicted in Fig. 3 will be used as the representative characteristic line for x_i since, above the characteristic line for x_n, x_i will be the same on any characteristic lines for x_i.

Since the Q-ILC technique requires the discrete time model, the ODE's need to be discretized. Exact discretization [Rugh, 1996] is desirable. However, if this is not possible, approximate discretization can also be used. Notice that the time scales are different for dynamics of each x_i on the representative characteristic line. Hence for coordination, sampling times of the ODE's on the representative characteristic lines should be different for each x_i. Namely, if Δt_n is the sampling time for x_n, the sampling time for x_i, Δt_i, must be $(a_n/a_i)\ \Delta t_n$. Then the spatial location associated with the *k*th sampling time will be the same for all x_i's. Now we assume the point sensors are located in each spatial location corresponding to a sampling point so that the location of point sensors are the integer multiple of $a_n\Delta t_n$.

Through discretization, we get the following discrete state space model along characteristic lines:

$$x(i+1) = B(i)x(i) + C(i)u(i).$$

Then G associated with this system is given by

$$G = \begin{bmatrix} C(1) & 0 & 0 & \cdots & 0 \\ C(2) & B(2)C(1) & 0 & \cdots & 0 \\ C(3) & B(3)C(2) & B(3)B(2)C(1) & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ C(N-1) & B(N-1)C(N-2) & B(N-1)B(N-2)C(N-3) & \cdots & B(N-1)\cdots B(2)C(1) \end{bmatrix}$$

With this discretized system, we are now ready to apply the Q-ILC technique.

From the convergence result of Q-ILC, the states at the discretized points on the characteristic lines converge to the desired values. If the discretization is exact and the desired state is reachable, then this in turn implies that the desired trajectory for original hyperbolic systems is achieved along the characteristic lines. Even if the desired state is reachable, the approximate discretization will usually lead to minor off-set despite of the integral action of the Q-ILC. This results from the lack of degree of freedom of control action and, thus, will not happen if we have sufficient degree of freedom. However, when the approximation is good enough, this off-set will be negligible since Q-ILC minimizes the off-set. Clearly if the desired trajectory is achieved above the representative characteristic line for x_i, it is achieved over each entire block. When the desired trajectory is not reachable, Q-ILC will minimizes the error between the desired trajectory and the converged trajectory in the long run.

APPLICATION TO NONISOTHERMAL TUBULAR REACTOR

Consider the nonisothermal tubular reactor that is a reaction convection process. We assume a first order endothermic reaction takes place in the reactor:

$$A \rightarrow B$$

and the associated reaction kinetics follows the Arrhenius Law:

$$-\left(\frac{dC_A}{dt}\right)_{rxn} = k_0 e^{-E/RT} C_A$$

where C_A is the concentration of species A; T is the reactor temperature; k_0 is the pre-exponential constant; E is the activation energy; R is the gas constant. We adopt the following standard assumptions on the ideal tubular reactor:

- Perfect radial mixing takes place
- Diffusion is negligible
- Densities and heat capacities for A and B are the same and constant

Under these assumptions the species balance for A and energy balance become

$$\frac{\partial C_A}{\partial t} = -v\frac{\partial C_A}{\partial z} - k_0 e^{-E/RT} C_A$$

$$\frac{\partial T}{\partial t} = -v\frac{\partial T}{\partial z} - \frac{\Delta H_r}{\rho c_p} k_0 e^{-E/RT} C_A + \frac{U}{\rho c_p V}(T_j - T)$$

with the boundary conditions

$$C_A(0, t) = C_A^0, \; T(0, t) = T^0$$

and initial conditions

$$C_A(z, 0) = C_{A0}(z), \; T(z, 0) = T_0(z)$$

where v is the velocity of the flow; ΔH_r is the heat of reaction; ρ is the density; c_p is the heat capacity; T_j is the jacket temperature, U is the heat transfer coefficient; V is the volume of reactor. The length of the reactor L is assumed 1 m. Notice that these are quasi-linear hyperbolic PDEs. The process parameters are listed in Table 1.

Table 1. Process parameters

Process parameter	Value
v (m/min)	1
E (cal/mol)	2.0×10^4
R (cal/mol·K)	1.987
ρ (kg/*lt*)	0.09
c_p (cal/kg·K)	700.231
k_0 (1/min)	5×10^{12}
U_w (cal/min·K)	2000.0
ΔH (cal/mol)	548.0001
V (*lt*)	10
L (m)	1

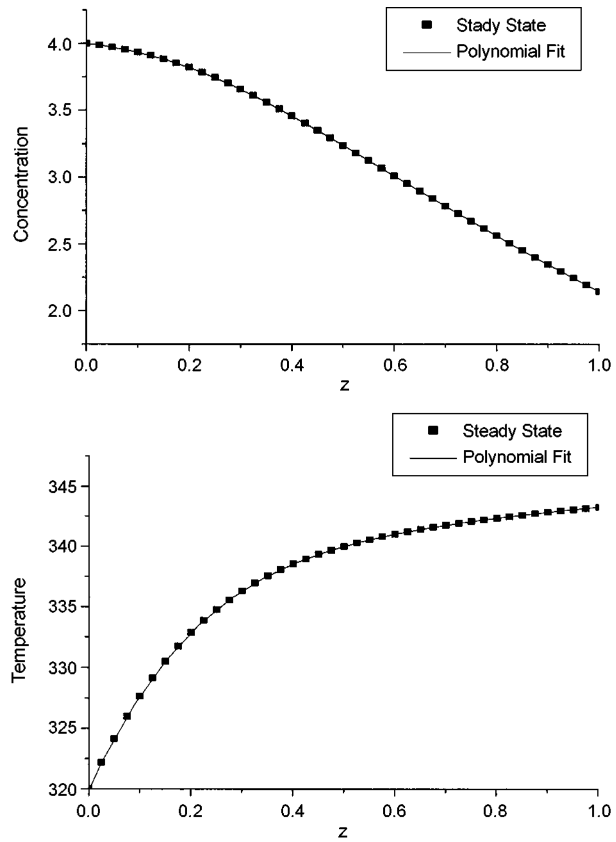

Fig. 4. Steady state profiles.

The desired steady state profile is assumed to be the one when the jacket temperature is constant as 350 °K. It is depicted in Fig. 4.

For the application of the control strategy proposed in this paper, we need linear hyperbolic PDEs. Hence we linearize the quasi-linear hyperbolic PDEs around the desired steady state. Since the exact solution of desired steady state is difficult to find, we obtain the analytic expression of the desired steady state through the regression with the 8th order polynomial (see Fig. 4) and use it for linearization. It is

$$C_{Ass}(z) = 4.00005 - 0.44522z - 1.72573z^2 - 5.06454z^3 + 12.70154z^4 - 8.70048z^5 - 1.92157z^6 + 5.06086z^7 - 1.76172z^8,$$

$$T_{ss}(z) = 320.00048 + 91.32149z - 159.62909z^2 + 122.33974z^3$$

$$+23.97147z^4-115.3329z^5+76.04642z^6$$
$$-13.00251z^7-2.43448z^8.$$

Since the shape of the desired steady state is simple, the fitting relative error with the 8th order polynomial was less than 10^{-4}. Through linearization, we have

$$\frac{\partial x_1}{\partial t}=-v\frac{\partial x_1}{\partial z}-k_0e^{-E/RT_{ss}(z)}x_1-k_0\frac{E}{RT_{ss}^2(z)}e^{-E/RT_{ss}(z)}C_{Ass}(z)x_2$$
$$\frac{\partial x_2}{\partial t}=-v\frac{\partial x_2}{\partial z}-\frac{\Delta H_r}{\rho c_p}k_0e^{-E/RT_{ss}(z)}x_1$$
$$-\left(\frac{(\Delta H_r)}{\rho c_p}k_0\frac{E}{RT_{ss}^2(z)}e^{-E/RT_{ss}(z)}C_{Ass}(z)+\frac{U}{\rho c_p V}\right)x_2+\frac{U}{\rho c_p V}u$$

with the boundary conditions

$$x_1(0, t)=0,\ x_2(0, t)=0$$

and initial conditions

$$x_1(z, 0)=x_{10}(z),\ x_2(z, 0)=x_{20}(z)$$

where

$$x_1(t, z)=C_A(t, z)-C_{Ass}(z),\ x_2(t, z)=T(t, z)-T_{ss}(z),\ u(t, z)=T_j(t, z)-350.$$

Employing the method of characteristics, we have the following ODE's along the characteristic line.

$$\frac{dx_1}{dt}=-k_0e^{-E/RT_{ss}(vt)}x_1-k_0\frac{E}{RT_{ss}^2(vt)}e^{-E/RT_{ss}(vt)}C_{Ass}(vt)x_2$$
$$\frac{dx_2}{dt}=-\frac{\Delta H_r}{\rho c_p}k_0e^{-E/RT_{ss}(vt)}x_1$$
$$-\left(\frac{\Delta H_r}{\rho c_p}k_0\frac{E}{RT_{ss}^2(vt)}e^{-E/RT_{ss}(vt)}C_{Ass}(vt)+\frac{U}{\rho c_p V}\right)x_2+\frac{U}{\rho c_p V}u$$

These ODE's are discretized with the sampling time $\Delta t=0.025$ min. For these ODE's, the exact discretization is not possible and thus the approximate discretization technique is used. To obtain the better approximate discretization, one can employ the more discretization within a sampling time and solve them for x(k+1) with initial condition x(k) and constant control u(k).

Now we are ready to apply the control strategy proposed in this paper. For this, we assume the reactor is divided into five different zones with the same length and each zone is heated by separate heating jacket. Moreover we assume the temperature and the concentration are measured at every discretized point by point sensors. Finally the control inputs are assumed to satisfy the saturation constraints:

$$298\le T_j(z)\le 400$$

or

$$-52\le u\le 50.$$

The weighting matrices associated with Q-ILC are as follows:

$$Q=\begin{bmatrix}7 & 0\\ 0 & 7\end{bmatrix},\ R=1.$$

Since the control input must be the same over the each zone controlled by a heating jacket. We need the following constraints:

$$\Delta u_{5\times i}=\Delta u_{5\times i+1}=\cdots=\Delta u_{5\times i+7},\ \forall i=0, 1,\ldots, 4.$$

The simulation of the closed loop system starting from a nonsteady state trajectory has been carried out. The simulation results with the linearized model are shown in Figs. 5, 6 and 7 whereas those with the nonlinear model in Figs. 8, 9 and 10.

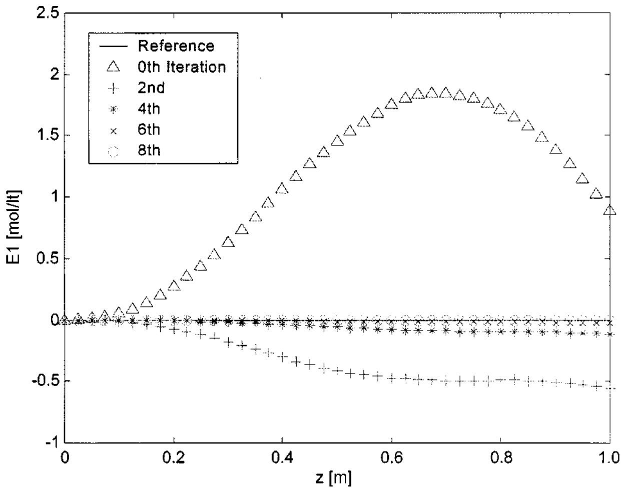

Fig. 5. Convergence of e_1 with linearized model.

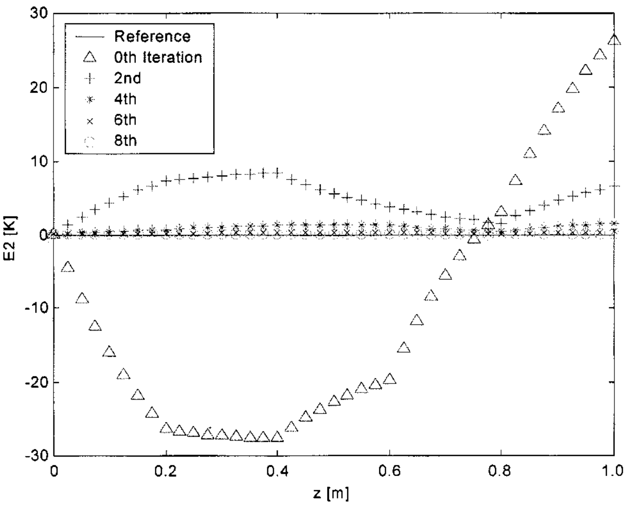

Fig. 6. Convergence of e_2 with linearized model.

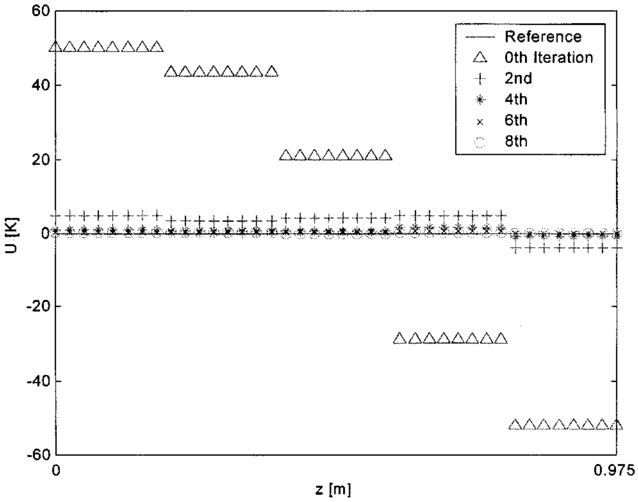

Fig. 7. Convergence of u with linearized model.

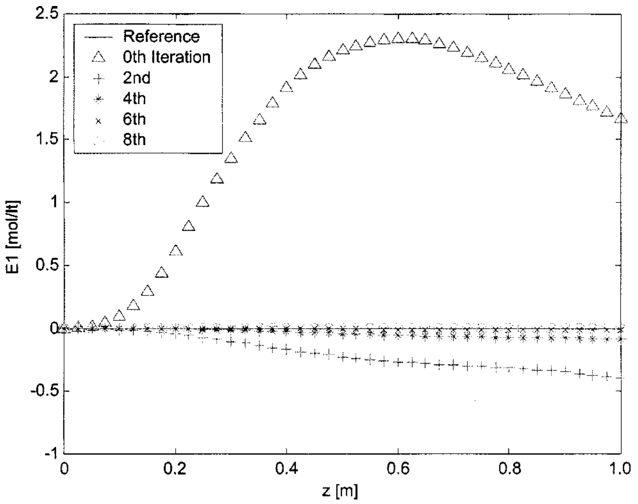

Fig. 8. Convergence of e_1 with nonlinear model.

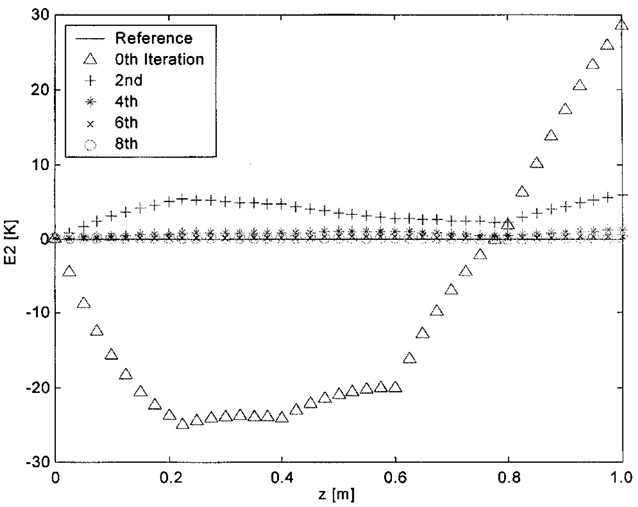

Fig. 9. Convergence of e_2 with nonlinear model.

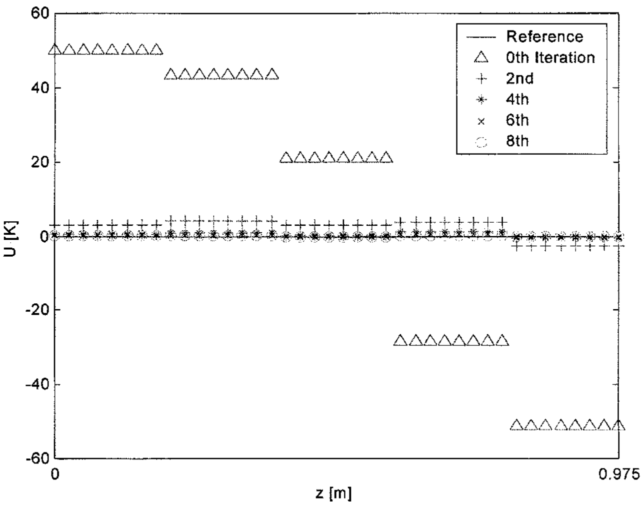

Fig. 10. Convergence of u with nonlinear model.

For both linearized and nonlinear models, the trajectories converge to the desired ones. Due to the step changes of the control input in the spatial direction, the error trajectories of e_2 are not quite smooth. In practice, these trajectories will be smoothen out since the exact step change is not possible physically.

CONCLUSION

The existing control strategies for systems described by first order hyperbolic PDE's are unconstrained, continuous time and infinite dimensional ones that need to be approximated for computer control. In this paper, we have proposed a constrained finite dimensional digital regulation technique that guarantees the stability and performance of the closed loop system. It is illustrated with an example that the proposed technique is promising for computer control of systems described by first order hyperbolic PDE's subject to constraints.

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