CONSTRAINED OPTIMIZATION OF IMAGE RESTORATION FILTERS

T.E. Riemer and C.D. McGillem

Purdue University, W. Lafayette, Indiana

Abstract

A linear shift-invariant preprocessing technique is described which re-
quires no specific knowledge of the image parameters and which is sufficiently
general to allow the effective radius of the composite imaging system to be
minimized while constraining other system parameters to remain within speci-
fied limits.

Introduction. The finite size of the point-spread function of a practical imaging system causes the output to be a two-dimensional spatial smearing or blurring of the original image. Methods of preprocessing the image to reduce this blurring have been widely studied but no fully satisfactory processor has as yet been developed. The preprocessing method proposed here, although new in formulation, can be thought of as a generalization and extension of previous work by Smith [4] and Stuller [5]. This method accomodates the problem of non-circularly symmetric imaging system point-spread functions, provides for controlled extent of the preprocessing filter to minimize distortion due to transients resulting from truncation errors and edge effects, can be used with various kinds of system noise, and can be readily extended to provide constraint of other system parameters. The analysis presented here relates to a line-scanner system although it is applicable in principle to many other system configurations.

Analysis of Preprocessing Technique. Referring to Fig. 1 and noting that a bar over a variable denotes a two-dimensional spatial vector, a basis for choosing the preprocessing filter point-spread function, $h_r(\bar{v})$, in order to increase the resolution of the composite imaging system is to minimize the radius of gyration, R_r, of the composite imaging system point-spread function, $g(\bar{v})$, given by

$$R_r^2 = \frac{\int_{-\infty}^{\infty} |\bar{v}|^2 g(\bar{v}) \, d\bar{v}}{\int_{-\infty}^{\infty} g^2(\bar{v}) \, d\bar{v}} \qquad (1)$$

where the composite system point-spread function is expressed

$$g(\bar{v}) = \int_{-\infty}^{\infty} h_r(\bar{z}) \, h_b(\bar{v}-\bar{z}) \, d\bar{z} . \qquad (2)$$

The magnitude of R_r may be minimized by setting the denominator equal to a constant and minimizing the numerator with respect to $h_r(\cdot)$. The function $|\bar{v}|^2$ in (1) is a penalty weighting function which causes $h_r(\bar{v})$ to decrease as $\bar{v}$ increases.

Additional flexibility in controlling the shape of $g(\bar{v})$ is obtained by using a more general penalty weighting function $w(\bar{v})$ which increases at an arbitrarily controlled rate outside a specified range of $\bar{v}$. With this penalty function the numerator and denominator of (1) can be written as the functionals

$$I_0 = \int_{-\infty}^{\infty} w(\bar{v}) \, g^2(\bar{v}) \, d\bar{v} \tag{3}$$

$$I_1 = \int_{-\infty}^{\infty} g^2(\bar{v}) \, d\bar{v} = K_1 \tag{4}$$

Two additional constraints are considered essential for most imaging systems. The first limits the size, and to some extent the shape, of the preprocessing filter point-spread function, $h_r(\bar{v})$, so that truncation errors are minimized. This constraint is

$$I_2 = \int_{-\infty}^{\infty} s(\bar{v}) \, h_r^2(\bar{v}) \, d\bar{v} = K_2 \tag{5}$$

where $s(\bar{v})$ is a penalty weighting function similar in form to $w(\bar{v})$. The second additional constraint limits the total noise power in the preprocessed image as follows

$$I_3 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{nn}(\bar{z}-\bar{v}) \, h_r(\bar{z}) \, h_r(\bar{v}) \, d\bar{z} \, d\bar{v} = K_3 \tag{6}$$

where it is assumed that the noise source is a stationary ergodic random process with autocorrelation function $R_{nn}(\bar{z}-\bar{v})$.

The required point-spread function is found by minimizing, with respect to $h_r(\cdot)$, the augmented functional formed by adding to (3) the constraints (4), (5) and (6) each of which has been multiplied by an appropriate Lagrange multiplier [1,2]. The required minimization is obtained by converting to the spatial frequency domain, setting the gradient of the augmented functional equal to zero and solving the resulting equation subject to the constraints of (4)-(6). The gradient of the augmented functional is as follows

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{H_b^*(f_x,f_y)\ H_b(\xi_x,\xi_y)\left[W(f_x-\xi_x,f_y-\xi_y)+W^*(\xi_x-f_x,\xi_y-f_y)\right]\right.$$

$$\left.+\ \lambda_2\left[S(f_x-\xi_x,f_y-\xi_y)+S^*(\xi_x-f_x,\xi_y-f_y)\right]\right\}\ H_r(\xi_x,\xi_y)\ d\xi_x\ d\xi_y$$

$$+\left[2\lambda_1|H_b(f_x,f_y)|^2+2\lambda_3\Phi_{nn}(f_x,f_y)\right]H_r(f_x,f_y)=0 \qquad (7)$$

where $H_b(\cdot)$, $W(\cdot)$, $S(\cdot)$, $H_r(\cdot)$ and $\Phi(\cdot)$ are the two-dimensional Fourier transforms of $h_b(\cdot)$, $w(\cdot)$, $s(\cdot)$, $h_r(\cdot)$ and $R_{nn}(\cdot)$ respectively and λ_1, λ_2 and λ_3 are Lagrange multipliers associated with constraint equations (4)-(6) respectively.

The frequency domain representation of the constraint equations is

$$I_1=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|H_b(f_x,f_y)|^2|H_r(f_x,f_y)|^2\ df_x\ df_y=K_1 \qquad (8)$$

$$I_2=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}S(f_x-\xi_x,f_y-\xi_y)H_r(\xi_x,\xi_y)H_r^*(f_x,f_y)\ d\xi_x\ d\xi_y\ df_x\ df_y=K_2 \qquad (9)$$

$$I_3=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi_{nn}(f_x,f_y)|H_r(f_x,f_y)|^2\ df_x\ df_y=K_3\ . \qquad (10)$$

In order to proceed further an appropriate form for the penalty functions $w(\bar{v})$ and $s(\bar{v})$ must be chosen. Several criteria must be satisfied by these functions: they must increase rapidly away from the origin; they must be Fourier transformable; and $w(\bar{v})$ must be convex to assure the existence of a global minimum of (3). A convenient representation centered at the origin is

$$W(\bar{v})=W_x(x)W_y(y)=\left\{(\frac{x}{x_w})^{2k_{wx}}+c_{wx}\right\}\left\{(\frac{y}{y_w})^{2k_{wy}}+c_{wy}\right\}\ . \qquad (11)$$

The constants $0<c_{wx},c_{wy}<1$ are introduced to assure that the overall weighting function will be everywhere greater than zero thus avoiding singular behavior of the solution. The constants k_{wx} and k_{wy} are positive integers. A similar expression may be written for $s(\bar{v})=s_x(x)s_y(y)$. The shape of $w(\bar{v})$ along one dimension is shown in Fig. 2. For the following analysis it is assumed that k_{wx}, k_{wy}, k_{sx}, and k_{sy} are all unity. This assumption does not limit the general applicability of (7)-(10) but is made to illustrate the method of solution of these equations.

With the above assumption (7) can be put into the following form [3]

$$\frac{H_b^*(f_x,f_y)}{8\pi^2 x_w^2 y_w^2}\left\{\frac{\partial^4}{\partial f_x^2 \partial f_y^2} H_b(f_x,f_y)H_r(f_x,f_y) - 4\pi^2 y_w^2 c_w \frac{\partial^2}{\partial f_x^2} H_b(f_x,f_y)H_r(f_x,f_y)\right.$$

$$\left. -4\pi^2 x_w^2 c_{wx}\frac{\partial^2}{\partial f_y^2} H_b(f_x,f_y)H_r(f_x,f_y) + 8\pi^4 x_w^2 y_w^2 c_{wx}c_{wy}H_b(f_x,f_y)H_r(f_x,f_y)\right\}$$

$$+\frac{\lambda_2}{8\pi^4 x_s^2 y_s^2}\left\{\frac{\partial^4}{\partial f_x^2 \partial f_y^2} H_r(f_x,f_y) - 4\pi^2 y_s^2 c_{sy}\frac{\partial^2}{\partial f_x^2} H_r(f_x,f_y)\right.$$

$$\left. - 4\pi^2 x_s^2 c_{sx}\frac{\partial^2}{\partial f_y^2} H_r(f_x,f_y) + 8\pi^4 x_s^2 y_s^2 c_{sx}c_{sy}H_r(f_x,f_y)\right\}$$

$$+\left[2\lambda_1|H_b(f_x,f_y)|^2 + 2\lambda_3 \overline{I}_{nn}(f_x,f_y)\right] H_r(f_x,f_y) = 0 . \qquad (12)$$

Equation (12) is a two-dimensional fourth-order partial differential equation which, when solved simultaneously with (8), (9), and (10), specifies the spatial frequency spectrum of the required preprocessing filter.

Image Collecting Systems with Separable Point-Spread Functions. For many physically realizable image collecting systems, certain assumptions may be made which greatly simplify the form of (12). A large class of physically realizable image collecting system point-spread functions may be modelled as separable point-spread functions, from which it follows that

$$H_b(f_x,f_y) = H_{bx}(f_x) H_{by}(f_y) \qquad (13)$$

$$H_r(f_x,f_y) = H_{rx}(f_x) H_{ry}(f_y) \qquad (14)$$

$$\overline{I}_{nn}(f_x,f_y) = \overline{I}_{nnx}(f_x) \overline{I}_{nny}(f_y) . \qquad (15)$$

With these assumptions, it can be shown that the solution for the two-dimensional point-spread function of $h_r(\cdot)$ defined by (12) and constraint equations (8)-(10) reduces to two sets of equations [3]: one set specifying the x-axis component of $h_r(\cdot)$; the other the y-axis component. Since these equation sets are identical except for appropriate changes in x- and y-axis component variables, only the equation set for the x-axis components will be given.

This equation set is composed of the second-order ordinary differential equation

$$H_{rx}''(f_x) + \frac{2H_{bx}^*(f_x)\,H_{bx}'(f_x)x_s^2}{|H_{bx}(f_x)|^2 x_s^2 + \lambda_{2x}x_w^2}\,H_{rx}'(f_x)$$

$$+ x_s^2 \left\{ \frac{H_{bx}^*(f_x)\,H_b''(f_x) - 4\pi^2 x_w^2\left[(c_{wx}+\lambda_{1x})|H_{bx}(f_x)|^2+\lambda_3 x \Phi_{nn}(f_x) + c_{sx}\right]}{|H_{bx}(f_x)|^2 x_s^2 + \lambda_{2x}x_w^2} \right\} H_{rx}(f_x) = 0$$

$$(16)$$

and the constraint equations

$$\int_{-\infty}^{\infty} |H_{bx}(f_x)|^2 |H_{rx}(f_x)|^2\, df_x = K_{1x} \tag{17}$$

$$-\frac{1}{4\pi^2 x_s^2} \int_{-\infty}^{\infty}\left[H_{rrx}''(f_x)\,H_{rrx}(f_x) + H_{rix}''(f_x)\,H_{rix}(f_x) \right] df_x$$

$$+ c_{sx} \int_{-\infty}^{\infty} |H_{rx}(f_x)|^2\, df_x = K_{2x} \tag{18}$$

$$\int_{-\infty}^{\infty} \Phi_{nn}(f_x)\,|H_r(f_x)|^2\, df_x = K_{3x} \tag{19}$$

where

$$H_{rx}(f_x) = H_{rrx}(f_x) + j\, H_{rix}(f_x)\ . \tag{20}$$

Thus the spatial frequency transform of the x-axis component of $h_r(\cdot)$ is

obtained by the simultaneous solution of (16) and the constraints of (17)-(19).

This system of equations can be solved by standard numerical procedures and

the functions $h_{rx}(\cdot)$ and $h_{ry}(\cdot)$ determined. The preprocessing operation thus

becomes two one-dimensional convolutions carried out sequentially rather than

a single two-dimensional convolution as would generally be the case.

Conclusions. The proposed method for specifying an image preprocessing point-

spread function is sufficiently general to allow a variety of penalty weighting

functions to be used for control of the effective radius of the point-spread

function while constraining other system parameters to remain within specified

limits.

Although (12) in conjunction with (8)-(10) specify the general form of the spatial frequency transform of the optimum preprocessing filter, it was shown that in the case of a separable point-spread function the solution can be considerably simplified. This same simplification also results for the case of a radially symmetric point-spread function, however, the resulting processor must perform a two-dimensional convolution in this case rather than two sequential one-dimensional convolutions.

The simplified solution based upon separable point-spread functions can be extended to a large class of practical systems by approximation techniques. For example, by properly selecting the parameters of a two-dimensional Gaussian function, an elliptical or circular point-spread function can be approximated and preprocessing carried out as sequential one-dimensional convolutions; thus substantially reducing the processing time.

References:
1. Epstein, B., Linear Functional Analysis, W. B. Saunders, Co., 1970.
2. Franks, L. E., Signal Theory, Prentice-Hall, 1969.
3. McGillem, C.D., and T.E. Riemer, "A Linear Shift-Invariant Image Preprocessing Technique for Multispectral Scanner Systems," LARS Information Note: 022673, Laboratory for the Application of Remote Sensing, Purdue University.
4. Smith, H. A., "Improvement of the Resolution of a Linear Scanning Device," SIAM Journal on Applied Mathematics, January 1966, Vol. 14, No. 1.
5. Stuller, J. A., "An Algebraic Approach to Image Restoration Filter Design" Computer Graphics and Image Processing, Dec. 1972, pp. 107-122.
6. Zemanian, A. H., Distribution Theory and Transform Analysis, McGraw-Hill, 1965.
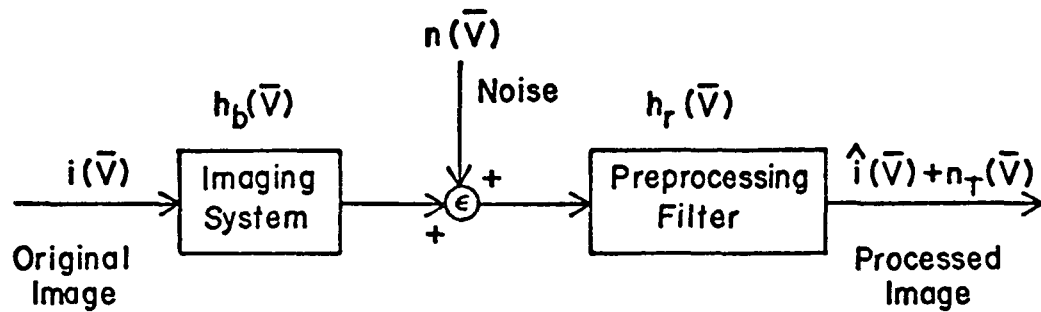
Figure 1.   Block Diagram of Imaging and Preprocessing Systems.
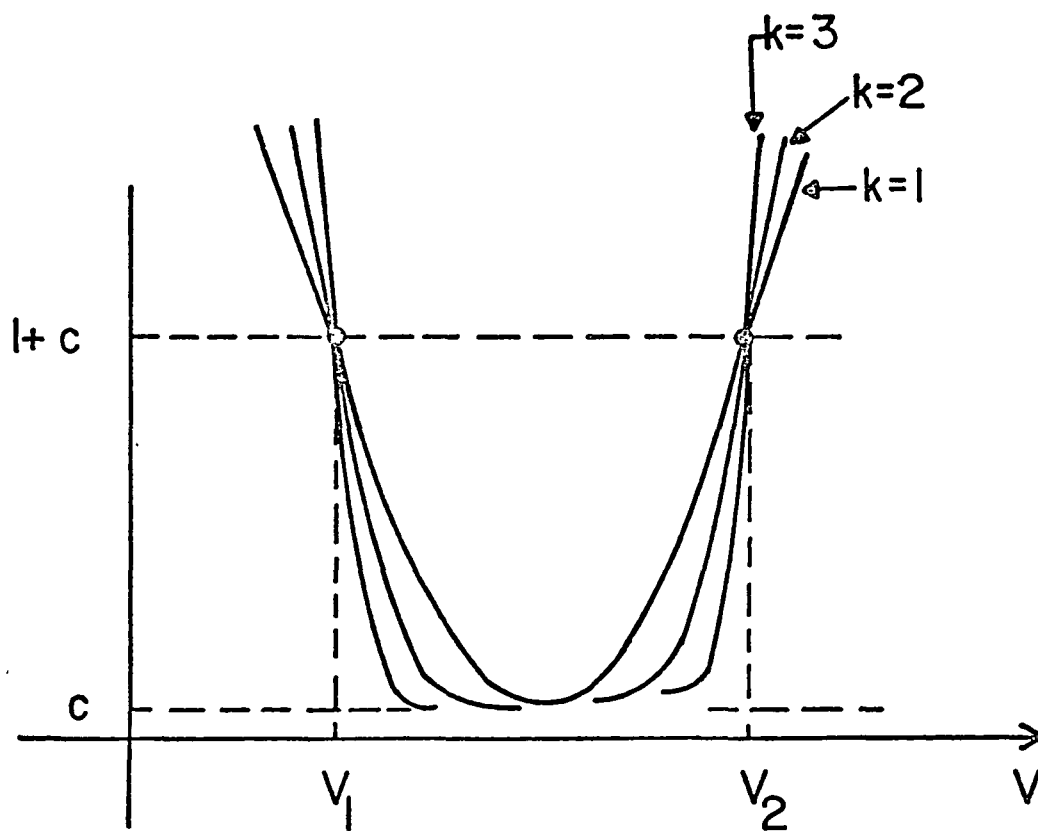

Figure 2.   One-Dimensional Penalty Weighting Function.

022573 Constrained Optimization of Image Restoration Filters.
T. E. Riemer and C. D. McGillem. A linear shift-
invariant preprocessing technique is described which
requires no specific knowledge of the image parameters
and which is sufficiently general to allow the effec-
tive radius of the composite imaging system to be
minimized while constraining other system parameters
to remain within specified limits.