

CONSTRAINT-STYLE PRECONDITIONERS FOR REGULARIZED SADDLE POINT PROBLEMS*

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Abstract. The problem of finding good preconditioners for the numerical solution of an important class of indefinite linear systems is considered. These systems are of a regularized saddle point structure $\begin{bmatrix} A & B^T \\ B & -C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}$, where $A \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{m \times m}$ are symmetric and $B \in \mathbb{R}^{m \times n}$. In [*SIAM J. Matrix Anal. Appl.*, 21 (2000), pp. 1300–1317], Keller, Gould, and Wathen analyze the idea of using constraint preconditioners that have a specific 2 by 2 block structure for the case of C being zero. We shall extend this idea by allowing the (2, 2) block to be symmetric and positive semidefinite. Results concerning the spectrum and form of the eigenvectors are presented, as are numerical results to validate our conclusions.

Key words. preconditioning, indefinite linear systems, Krylov subspace methods

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1. Introduction. Recently, a large amount of work has been devoted to the problem of solving large linear systems in saddle point form. Such systems arise in a wide variety of technical and scientific applications. For example, interior point methods in both linear and nonlinear optimization require the solution of a sequence of systems in saddle point form [27]. Another popular field, which is a major source of saddle point problems, is that of mixed finite element methods in engineering fields; see [9] and [19, Chapters 7 and 9]. An excellent survey of numerical methods for algebraic saddle point problems has been written by Benzi, Golub, and Liesen [4].

We wish to find the solution of block 2×2 linear systems of the form

$$(1.1) \quad \underbrace{\begin{bmatrix} A & B^T \\ B & -C \end{bmatrix}}_{\mathcal{A}} \begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} c \\ d \end{bmatrix}}_b,$$

where $A \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{m \times m}$ are symmetric and $B \in \mathbb{R}^{m \times n}$. We shall assume that $m \leq n$ and $\ker(C) \cap \ker(B^T) = \{0\}$, thus ensuring that \mathcal{A} is nonsingular [4, Theorem 3.1]. If A and C are positive definite, then the matrix \mathcal{A} is a permuted quasi-definite matrix [26]. Vanderbei has shown that quasi-definite matrices are strongly factorizable; i.e., a Cholesky-like factorization LDL^T exists for any symmetric row and column permutation of the quasi-definite matrix [26]. The diagonal matrix has n positive and m negative pivots. However, we shall not confine ourselves to quasi-definite matrices.

It may be attractive to use iterative methods to solve systems such as (1.1), particularly for large m and n . In particular, Krylov subspace methods might be used. It is often advantageous to use a preconditioner, \mathcal{P} , with such iterative methods. The preconditioner should reduce the number of iterations required for convergence but

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not significantly increase the amount of computation required at each iteration [25, Chapter 13].

In section 2, we shall first review the well-known spectral properties of a technique commonly known as constraint preconditioning when $C = 0$ [14, 16]. For the case of $C = 0$, a constraint preconditioner exactly reproduces the (constraint) blocks B , B^T and the $C = 0$ block. It is restrictive to assume that the matrix C in the saddle point systems is always a zero matrix: a number of situations arise in which $C \neq 0$ [1, 15, 23]. In all these cases, C is positive semidefinite, and hence we shall consider the idea of extending constraint preconditioners to the case of C being positive semidefinite. In particular, the preconditioner will exactly reproduce the B , B^T and C blocks, while the A block will be replaced by a symmetric block, which we refer to as G ; this is considered in sections 3 and 4. Such a preconditioner has been considered before; for example, Perugia and Simoncini consider the case of G being diagonal and positive definite [18], while G is assumed to be nonsingular in [22] and positive definite in [3, 8, 24], but we show that these assumptions can be relaxed. In the past couple of years, the use of implicit factorization preconditioners has been proposed [7] with the aim of reducing the cost (both in CPU time and memory usage) of applying a preconditioner of the form suggested in this paper. However, such implicit factorization preconditioners will frequently generate a matrix G which is symmetric and singular or indefinite, and thus the analysis of these preconditioners with such a G is necessary.

2. Constraint preconditioners. Let us initially assume that $C = 0$. Lukšan and Vlček [17] and Keller, Gould, and Wathen [14] investigated the spectral properties of the resulting preconditioned system when we use a preconditioner of the form

$$(2.1) \quad \mathcal{P} = \begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix},$$

where G is symmetric and approximates but (in general) is not the same as A . In [17], G is additionally assumed to be positive definite. They were able to prove various results about the eigenvalues and eigenvectors for the preconditioned systems $\mathcal{P}^{-1}\mathcal{A}$, where \mathcal{A} and \mathcal{P} are defined in (1.1) and (2.1), respectively. \mathcal{P} is called a *constraint preconditioner*. The proof of the following theorem can be found in [14].

THEOREM 2.1. *Let $\mathcal{A} \in \mathbb{R}^{(n+m) \times (n+m)}$ be a symmetric and indefinite matrix of the form*

$$\mathcal{A} = \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix},$$

where $A \in \mathbb{R}^{n \times n}$ is symmetric and $B \in \mathbb{R}^{m \times n}$ is of full rank. Assume Z is an $n \times (n - m)$ basis for the nullspace of B . Preconditioning \mathcal{A} by a matrix of the form

$$\mathcal{P} = \begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix},$$

where $G \in \mathbb{R}^{n \times n}$ is symmetric and $B \in \mathbb{R}^{m \times n}$ is as above, implies that the matrix $\mathcal{P}^{-1}\mathcal{A}$ has

- an eigenvalue at 1 with multiplicity $2m$,
- $n - m$ eigenvalues λ which are defined by the generalized eigenvalue problem $Z^T A Z x_z = \lambda Z^T G Z x_z$.

This accounts for all of the eigenvalues.

If either $Z^T AZ$ or $Z^T GZ$ is positive definite, then the indefinite preconditioner \mathcal{P} applied to the indefinite saddle point matrix \mathcal{A} with $C = 0$ yields a preconditioned matrix $\mathcal{P}^{-1}\mathcal{A}$ which has real eigenvalues [14]. If both $Z^T AZ$ and $Z^T GZ$ are positive definite, then we can use a projected preconditioned conjugate gradient method to find x and y ; see [12]. Results about the associated eigenvectors and the Krylov subspace dimension can also be found in [14].

3. Constraint preconditioners for the case of symmetric and positive definite C . In this section, we shall assume that the matrix C is symmetric and positive definite. The term *constraint preconditioner* was used in [10] and [14] because the (1, 2) and (2, 1) matrix blocks of the preconditioner are exact representations of those in \mathcal{A} , where these blocks represent constraints. However, we also observe that the (2, 2) matrix block is an exact representation when $C = 0$. This motivates the generalization of the constraint preconditioner to take the form

$$(3.1) \quad \mathcal{P} = \begin{bmatrix} G & B^T \\ B & -C \end{bmatrix},$$

where $G \in \mathbb{R}^{n \times n}$ approximates but is, in general, not the same as A .

We shall use the following assumptions in the theorems of this section.

A1 $C \in \mathbb{R}^{m \times m}$ is symmetric and positive definite.

A2 $A \in \mathbb{R}^{n \times n}$ is symmetric.

A3 $B \in \mathbb{R}^{m \times n}$ ($m < n$).

A4 $G \in \mathbb{R}^{n \times n}$ is symmetric.

A5 $\mathcal{A} \in \mathbb{R}^{(n+m) \times (n+m)}$ is as defined in (1.1).

A6 $\mathcal{P} \in \mathbb{R}^{(n+m) \times (n+m)}$ is as defined in (3.1).

In the next section, A1 will be relaxed.

THEOREM 3.1. *Assume that A1–A6 hold; then the matrix $\mathcal{P}^{-1}\mathcal{A}$ has*

- an eigenvalue at 1 with multiplicity m ,
- n eigenvalues which are defined by the generalized eigenvalue problem

$$(A + B^T C^{-1} B)x = \lambda (G + B^T C^{-1} B)x.$$

This accounts for all of the eigenvalues.

Proof. The eigenvalues of the preconditioned coefficient matrix $\mathcal{P}^{-1}\mathcal{A}$ may be derived by considering the generalized eigenvalue problem

$$(3.2) \quad \begin{bmatrix} A & B^T \\ B & -C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} G & B^T \\ B & -C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Expanding this out, we obtain

$$(3.3) \quad Ax + B^T y = \lambda Gx + \lambda B^T y$$

and

$$(3.4) \quad Bx - Cy = \lambda Bx - \lambda Cy.$$

Equation (3.4) implies that either $\lambda = 1$ or $Bx - Cy = 0$. If the former holds, then (3.3) becomes

$$(3.5) \quad Ax = Gx.$$

Equation (3.5) is trivially satisfied by $x = 0$, and hence there are m linearly independent eigenvectors of the form $\begin{bmatrix} 0^T & y^T \end{bmatrix}$ associated with the unit eigenvalue. If there exist any $x \neq 0$ which satisfy (3.5), then there will be i ($0 \leq i \leq n$) linearly independent eigenvectors of the form $\begin{bmatrix} x^T & y^T \end{bmatrix}$, where the components x arise from the generalized eigenvalue problem $Ax = Gx$.

If $\lambda \neq 1$, then (3.4) implies that

$$y = C^{-1}Bx.$$

Substituting this into (3.3) yields the generalized eigenvalue problem

$$(3.6) \quad (A + B^T C^{-1} B) x = \lambda (G + B^T C^{-1} B) x.$$

Thus, the nonunit eigenvalues of $\mathcal{P}^{-1}\mathcal{A}$ are defined as the nonunit eigenvalues of (3.6). Noting that if (3.6) has any unit eigenvalues, then the values of $x (\neq 0)$ which satisfy this are exactly those which arise from the generalized eigenvalue problem $Ax = Gx$, we complete our proof. \square

Theorem 3.1 generalizes the results of [8, Theorem 1] by removing the assumption that G is positive definite. If $A + B^T C^{-1} B$ or $G + B^T C^{-1} B$ is positive definite, then the preconditioned system has real eigenvalues. If both $A + B^T C^{-1} B$ and $G + B^T C^{-1} B$ are positive definite, then we can apply a projected preconditioned conjugate gradient method to find x and y [7, 11]. We also observe that if C has a small 2-norm, $\|A\|_2 = \mathcal{O}(1)$ and $\|G\|_2 = \mathcal{O}(1)$, then the $B^T C^{-1} B$ terms will dominate the generalized eigenvalue problem (3.6) for $Bx \neq 0$, and hence there will be at least m further eigenvalues clustered about 1 for $\|C\|_2 \ll 1$. This additional clustering of part of the spectrum of $\mathcal{P}^{-1}\mathcal{A}$ will often translate into a speeding up of the convergence of a selected Krylov subspace method [2, section 1.3].

THEOREM 3.2. *Assume that A1–A6 hold and $G + B^T C^{-1} B$ is positive definite; then the matrix $\mathcal{P}^{-1}\mathcal{A}$ has $n + m$ eigenvalues as defined in Theorem 3.1 and $m + i + j$ linearly independent eigenvectors. There are*

- m eigenvectors of the form $\begin{bmatrix} 0^T & y^T \end{bmatrix}$ that correspond to the case $\lambda = 1$,
- i ($0 \leq i \leq n$) eigenvectors of the form $\begin{bmatrix} x^T & y^T \end{bmatrix}$ arising from $Ax = Gx$ for which the i vectors x are linearly independent and $\lambda = 1$,
- j ($0 \leq j \leq n$) eigenvectors of the form $\begin{bmatrix} x^T & y^T \end{bmatrix}$ that correspond to the case $\lambda \neq 1$.

Proof. The form of the eigenvectors follows directly from the proof of Theorem 3.1. It remains for us to show that the $m + i + j$ eigenvectors are linearly independent; that is, we need to show that

$$(3.7) \quad \begin{bmatrix} 0 & \cdots & 0 \\ y_1^{(1)} & \cdots & y_m^{(1)} \end{bmatrix} \begin{bmatrix} a_1^{(1)} \\ \vdots \\ a_m^{(1)} \end{bmatrix} + \begin{bmatrix} x_1^{(2)} & \cdots & x_i^{(2)} \\ y_1^{(2)} & \cdots & y_i^{(2)} \end{bmatrix} \begin{bmatrix} a_1^{(2)} \\ \vdots \\ a_i^{(2)} \end{bmatrix} + \begin{bmatrix} x_1^{(3)} & \cdots & x_j^{(3)} \\ y_1^{(3)} & \cdots & y_j^{(3)} \end{bmatrix} \begin{bmatrix} a_1^{(3)} \\ \vdots \\ a_j^{(3)} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

implies that the vectors $a^{(k)}$ ($k = 1, 2, 3$) are zero vectors. Multiplying (3.7) by $\mathcal{P}^{-1}\mathcal{A}$, and recalling that in (3.7) the first matrix arises from the case $\lambda_k = 1$ ($k = 1, \dots, m$),

the second matrix from the case $\lambda_k = 1$ ($k = 1, \dots, i$), and the last matrix from $\lambda_k \neq 1$ ($k = 1, \dots, j$), gives

$$(3.8) \quad \begin{bmatrix} 0 & \cdots & 0 \\ y_1^{(1)} & \cdots & y_m^{(1)} \end{bmatrix} \begin{bmatrix} a_1^{(1)} \\ \vdots \\ a_m^{(1)} \end{bmatrix} + \begin{bmatrix} x_1^{(2)} & \cdots & x_i^{(2)} \\ y_1^{(2)} & \cdots & y_i^{(2)} \end{bmatrix} \begin{bmatrix} a_1^{(2)} \\ \vdots \\ a_i^{(2)} \end{bmatrix} \\ + \begin{bmatrix} x_1^{(3)} & \cdots & x_j^{(3)} \\ y_1^{(3)} & \cdots & y_j^{(3)} \end{bmatrix} \begin{bmatrix} \lambda_1 a_1^{(3)} \\ \vdots \\ \lambda_j a_j^{(3)} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Subtracting (3.7) from (3.8), we obtain

$$\begin{bmatrix} x_1^{(3)} & \cdots & x_j^{(3)} \\ y_1^{(3)} & \cdots & y_j^{(3)} \end{bmatrix} \begin{bmatrix} (\lambda_1 - 1)a_1^{(3)} \\ \vdots \\ (\lambda_j - 1)a_j^{(3)} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The assumption that $G + B^T C^{-1} B$ is positive definite implies that $x_k^{(3)}$ ($k = 1, \dots, j$) are linearly independent and thus that $(\lambda_k - 1)a_k^{(3)} = 0$ ($k = 1, \dots, j$). The eigenvalues λ_k ($k = 1, \dots, j$) are nonunit, which implies that $a_k^{(3)} = 0$ ($k = 1, \dots, j$). We also have linear independence of $x_k^{(2)}$ ($k = 1, \dots, i$), and thus $a_k^{(2)} = 0$ ($k = 1, \dots, i$). Equation (3.7) simplifies to

$$\begin{bmatrix} 0 & \cdots & 0 \\ y_1^{(1)} & \cdots & y_m^{(1)} \end{bmatrix} \begin{bmatrix} a_1^{(1)} \\ \vdots \\ a_m^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

However, $y_k^{(1)}$ ($k = 1, \dots, m$) are linearly independent, and thus $a_k^{(1)} = 0$ ($k = 1, \dots, m$). \square

Krylov subspace theory states that iteration with any method with an optimality property, e.g., GMRES [21], will terminate when the degree of the minimal polynomial is attained. This is also true of some other (nonoptimal) practical iteration methods such as BiCGTAB as long as failure (for example, through irregular convergence [25, Chapter 8]) does not occur. In particular, the degree of the minimal polynomial is equal to the dimension of the corresponding Krylov subspace $\mathcal{K}(\mathcal{P}^{-1}\mathcal{A}, b)$ (for general b) [20, Proposition 6.1], where

$$\mathcal{K}(\mathcal{P}^{-1}\mathcal{A}, b) = \text{span}\{b, \mathcal{P}^{-1}\mathcal{A}b, (\mathcal{P}^{-1}\mathcal{A})^2b, \dots, (\mathcal{P}^{-1}\mathcal{A})^{n+m-1}b\}.$$

THEOREM 3.3. *Assume that A1–A6 hold and $G + B^T C^{-1} B$ is positive definite; then the dimension of the Krylov subspace $\mathcal{K}(\mathcal{P}^{-1}\mathcal{A}, b)$ is at most $\min\{n+2, n+m\}$.*

Proof. As in the proof of Theorem 3.1, the generalized eigenvalue problem is

$$(3.9) \quad \begin{bmatrix} A & B^T \\ B & -C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} G & B^T \\ B & -C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Suppose that the preconditioned matrix $\mathcal{P}^{-1}\mathcal{A}$ takes the form

$$(3.10) \quad \mathcal{P}^{-1}\mathcal{A} = \begin{bmatrix} \Theta_1 & \Theta_3 \\ \Theta_2 & \Theta_4 \end{bmatrix},$$

where $\Theta_1 \in \mathbb{R}^{n \times n}$, $\Theta_2 \in \mathbb{R}^{m \times n}$, $\Theta_3 \in \mathbb{R}^{n \times m}$, and $\Theta_4 \in \mathbb{R}^{m \times m}$. It is straightforward to show that $\Theta_3 = 0$ and $\Theta_4 = I$. The precise forms of Θ_1 and Θ_2 are irrelevant for the argument that follows.

From the earlier eigenvalue derivation, it is evident that the characteristic polynomial of the preconditioned linear system (3.10) is

$$(\mathcal{P}^{-1}\mathcal{A} - I)^m \prod_{i=1}^n (\mathcal{P}^{-1}\mathcal{A} - \lambda_i I).$$

In order to prove the upper bound on the Krylov subspace dimension, we need to show that the order of the minimal polynomial is less than or equal to $\min\{n + 2, n + m\}$. Expanding the polynomial $(\mathcal{P}^{-1}\mathcal{A} - I) \prod_{i=1}^n (\mathcal{P}^{-1}\mathcal{A} - \lambda_i I)$ of degree $n + 1$, we obtain

$$\begin{bmatrix} (\Theta_1 - I) \prod_{i=1}^n (\Theta_1 - \lambda_i I) & 0 \\ \Theta_2 \prod_{i=1}^n (\Theta_1 - \lambda_i I) & 0 \end{bmatrix}.$$

Since Θ_1 has a full set of linearly independent eigenvectors, Θ_1 is diagonalizable. Hence,

$$(\Theta_1 - I) \prod_{i=1}^n (\Theta_1 - \lambda_i I) = 0.$$

We therefore obtain

$$(3.11) \quad (\mathcal{P}^{-1}\mathcal{A} - I) \prod_{i=1}^n (\mathcal{P}^{-1}\mathcal{A} - \lambda_i I) = \begin{bmatrix} 0 & 0 \\ \Theta_2 \prod_{i=1}^n (\Theta_1 - \lambda_i I) & 0 \end{bmatrix}.$$

If $\Theta_2 \prod_{i=1}^n (\Theta_1 - \lambda_i I) = 0$, then the order of the minimal polynomial of $\mathcal{P}^{-1}\mathcal{A}$ is less than or equal to $\min\{n + 1, n + m\}$. If $\Theta_2 \prod_{i=1}^n (\Theta_1 - \lambda_i I) \neq 0$, then the dimension of $\mathcal{K}(\mathcal{P}^{-1}\mathcal{A}, b)$ is at most $\min\{n + 2, n + m\}$ since multiplication of (3.11) by another factor $(\mathcal{P}^{-1}\mathcal{A} - I)$ gives the zero matrix. \square

Theorem 3.3 tells us that with preconditioner

$$\mathcal{P} = \begin{bmatrix} G & B^T \\ B & -C \end{bmatrix}$$

for

$$\mathcal{A} = \begin{bmatrix} A & B^T \\ B & -C \end{bmatrix}$$

the dimension of the Krylov subspace is no greater than $\min\{n + 2, n + m\}$ under appropriate assumptions. Hence, termination (in exact arithmetic) is guaranteed in a number of iterations smaller than this.

4. Constraint preconditioners for the case of symmetric and positive semi-definite C . We shall relax assumption A1 and instead make the following assumptions in the theorems of this section:

B1 $C \in \mathbb{R}^{m \times m}$ is symmetric and positive semidefinite, and has rank p , where $0 < p < m$.

B2 $\ker(C) \cap \ker(B^T) = \{0\}$.

B3 C is factored as $C = EDE^T$, where $E \in \mathbb{R}^{m \times p}$, and $D \in \mathbb{R}^{p \times p}$ is symmetric and positive definite.

B4 The matrix $F \in \mathbb{R}^{m \times (m-p)}$ is such that its columns span the nullspace of C .

B5 $\begin{bmatrix} E & F \end{bmatrix} \in \mathbb{R}^{m \times m}$ is orthogonal.

B6 The columns of $N \in \mathbb{R}^{n \times (n-m+p)}$ span the nullspace of $F^T B$.

Observe that assumption B2 implies that $F^T B$ has full rank $m - p$: if $Cx = 0$, then we can write $x = Fy$ for some vector $y \in \mathbb{R}^{m-p}$. If also $B^T x = 0$, then substituting into $x = Fy$ we obtain $B^T Fy = 0$. Assumption B2 implies that $B^T Fy = 0$ if and only if $y = 0$, and hence $F^T B$ has full rank $m - p$.

The exact form of the factorization of C in B3 is clearly not relevant and, also, clearly not unique—a spectral decomposition is a possibility.

THEOREM 4.1. Assume that A2–A6 and B1–B6 hold; then the matrix $\mathcal{P}^{-1}A$ has

- an eigenvalue at 1 with multiplicity $2m - p$,
- $n - m + p$ eigenvalues which are defined by the generalized eigenvalue problem

$$N^T (A + B^T E D^{-1} E^T B) N z = \lambda N^T (G + B^T E D^{-1} E^T B) N z.$$

This accounts for all of the eigenvalues.

Proof. Any $y \in \mathbb{R}^m$ can be written as $y = Ey_e + Fy_f$. Substituting this into the generalized eigenvalue problem (3.2) and premultiplying by

$$\begin{bmatrix} I & 0 \\ 0 & E^T \\ 0 & F^T \end{bmatrix},$$

we obtain

$$(4.1) \quad \left[\begin{array}{cc|c} A & B^T E & B^T F \\ E^T B & -D & 0 \\ \hline F^T B & 0 & 0 \end{array} \right] \begin{bmatrix} x \\ y_e \\ y_f \end{bmatrix} = \lambda \left[\begin{array}{cc|c} G & B^T E & B^T F \\ E^T B & -D & 0 \\ \hline F^T B & 0 & 0 \end{array} \right] \begin{bmatrix} x \\ y_e \\ y_f \end{bmatrix}.$$

Noting that the (3, 3) block has dimension $(m - p) \times (m - p)$ and is a zero matrix in both coefficient matrices, we can apply Theorem 2.1 from [14] to obtain that $\mathcal{P}^{-1}A$ has

- an eigenvalue at 1 with multiplicity $2(m - p)$,
- $n - m + 2p$ eigenvalues which are defined by the generalized eigenvalue problem

$$(4.2) \quad \bar{N}^T \begin{bmatrix} A & B^T E \\ E^T B & -D \end{bmatrix} \bar{N} w_n = \lambda \bar{N}^T \begin{bmatrix} G & B^T E \\ E^T B & -D \end{bmatrix} \bar{N} w_n,$$

where \bar{N} is an $(n + p) \times (n - m + 2p)$ basis for the nullspace of $\begin{bmatrix} F^T B & 0 \end{bmatrix} \in \mathbb{R}^{(m-p) \times (n+p)}$, and

$$\begin{bmatrix} x \\ y_e \end{bmatrix}^T = \bar{N} w_n + \begin{bmatrix} B^T F \\ 0 \end{bmatrix} w_b.$$

Letting $\bar{N} = \begin{bmatrix} N & 0 \\ 0 & I \end{bmatrix}$, then (4.2) becomes

$$(4.3) \quad \begin{bmatrix} N^T A N & N^T B^T E \\ E^T B N & -D \end{bmatrix} \begin{bmatrix} w_{n1} \\ w_{n2} \end{bmatrix} = \lambda \begin{bmatrix} N^T G N & N^T B^T E \\ E^T B N & -D \end{bmatrix} \begin{bmatrix} w_{n1} \\ w_{n2} \end{bmatrix}.$$

This generalized eigenvalue problem is exactly that of the form considered in Theorem 3.1, and so (4.3) has an eigenvalue at 1 with multiplicity p , and the remaining eigenvalues are defined by the generalized eigenvalue problem

$$(4.4) \quad N^T (A + B^T E D^{-1} E^T B) N w_{n1} = \lambda N^T (G + B^T E D^{-1} E^T B) N w_{n1}.$$

Hence, $\mathcal{P}^{-1}\mathcal{A}$ has an eigenvalue at 1 with multiplicity $2m - p$, and the other eigenvalues are defined by the generalized eigenvalue problem (4.4). \square

Weaker forms of Theorem 4.1 can be found in [3, section 3.7] and [18, Proposition 5] for the case where G is assumed to be symmetric and positive definite (and diagonal in [18]). We have relaxed this assumption to G being symmetric and also increased the lower bound on the number of unit eigenvalues from m to $2m - p$.

As for the cases $C = 0$ and C nonsingular, we are able to obtain conditions which guarantee that the eigenvalues are real and for which a projected preconditioned conjugate gradient method could be applied to find x and y ; respectively, these conditions are

- either $N^T (A + B^T E D^{-1} E^T B) N$ or $N^T (G + B^T E D^{-1} E^T B) N$ is positive definite,
- both $N^T (A + B^T E D^{-1} E^T B) N$ and $N^T (G + B^T E D^{-1} E^T B) N$ are positive definite.

Interestingly, the projected preconditioned conjugate gradient method is also derived by the use of a factorization of C as in assumption B3; transformations are then used to remove the requirement of needing to factorize C [7]. Additionally, in [7] the authors show that it can be easy to establish that $N^T (G + B^T E D^{-1} E^T B) N$ is symmetric and positive definite through the use of implicit factorization constraint preconditioners: we emphasize that G is often singular or indefinite in these cases.

Similarly to the case $p = m$, if C has a small 2-norm, $\|A\| = \mathcal{O}(1)$ and $\|G\| = \mathcal{O}(1)$, then the $N^T B^T E D^{-1} E^T B N$ terms will dominate the generalized eigenvalue problem (4.4) for $E^T B N w_{n1} \neq 0$ and hence there will be at least p further eigenvalues clustered about 1 when $\|C\|_2 \ll 1$.

THEOREM 4.2. *Assume that A2–A6, B1–B6 hold and $G + B^T E D^{-1} E^T B$ is positive definite; then the matrix $\mathcal{P}^{-1}\mathcal{A}$ has $n + m$ eigenvalues as defined in Theorem 3.1 and $m + i + j$ linearly independent eigenvectors. There are*

- m eigenvectors of the form $\begin{bmatrix} 0^T & y^T \end{bmatrix}$ that correspond to the case $\lambda = 1$,
- i ($0 \leq i \leq n$) eigenvectors of the form $\begin{bmatrix} x^T & y^T \end{bmatrix}$ arising from $Ax = Gx$ for which the i vectors x are linearly independent and $\lambda = 1$,
- j ($0 \leq j \leq n$) eigenvectors of the form $\begin{bmatrix} x^T & y^T \end{bmatrix}$ that correspond to the case $\lambda \neq 1$.

Proof. The proof of the form and linear independence of the $m + i + j$ eigenvectors is obtained in a similar manner to the proof of Theorem 3.2. \square

A weaker form of Theorem 4.2 can be found in [3]: this corresponds to the case of G being symmetric and positive definite.

To show that both the lower and upper bounds on the number of linearly independent eigenvectors can be attained, we need only consider variations on Examples 2.5 and 2.6 from [14].

Example 4.1 (minimum bound). Consider the matrices

$$\mathcal{A} = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}, \quad \mathcal{P} = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 3 & 4 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$$

such that $m = 2, n = 2, p = 1$, and G is indefinite. The preconditioned matrix $\mathcal{P}^{-1}\mathcal{A}$ has an eigenvalue at 1 with multiplicity 4 but only two linearly independent eigenvectors which arise from the first case of Theorem 4.2. These eigenvectors may be taken to be $[0 \ 0 \ 1 \ 0]^T$ and $[0 \ 0 \ 0 \ 1]^T$.

Example 4.2 (maximum bound). Let $\mathcal{A} \in \mathbb{R}^{4 \times 4}$ be as defined in Example 4.1, but assume that $G = A$. The preconditioned matrix $\mathcal{P}^{-1}\mathcal{A}$ has an eigenvalue at 1 with multiplicity 4 and clearly a complete set of eigenvectors. These may be taken to be the columns of the identity matrix.

The linear independence of the $m + i + j$ eigenvectors allows us to obtain an upper bound on the dimension of the Krylov subspace $\mathcal{K}(\mathcal{P}^{-1}\mathcal{A}, b)$.

THEOREM 4.3. *Assume that A2–A6, B1–B6 hold and $G + B^T E D^{-1} E^T B$ is positive definite; then the dimension of the Krylov subspace $\mathcal{K}(\mathcal{P}^{-1}\mathcal{A}, b)$ is at most $\min\{n - m + p + 2, n + m\}$.*

Proof. As in the proof of Theorem 3.3, the preconditioned matrix $\mathcal{P}^{-1}\mathcal{A}$ takes the form

$$(4.5) \quad \mathcal{P}^{-1}\mathcal{A} = \begin{bmatrix} \Theta_1 & 0 \\ \Theta_2 & I \end{bmatrix},$$

where $\Theta_1 \in \mathbb{R}^{n \times n}$, and $\Theta_2 \in \mathbb{R}^{m \times n}$. The precise forms of Θ_1 and Θ_2 are irrelevant for the argument that follows.

From the earlier eigenvalue derivation, it is evident that the characteristic polynomial of the preconditioned linear system (4.5) is

$$(\mathcal{P}^{-1}\mathcal{A} - I)^{2m-p} \prod_{i=1}^{n-m+p} (\mathcal{P}^{-1}\mathcal{A} - \lambda_i I).$$

In order to prove the upper bound on the Krylov subspace dimension, we need to show that the order of the minimal polynomial is less than or equal to $\min\{n - m + p + 2, n + m\}$. Expanding the polynomial $(\mathcal{P}^{-1}\mathcal{A} - I) \prod_{i=1}^{n-m+p} (\mathcal{P}^{-1}\mathcal{A} - \lambda_i I)$ of degree $n + 1$, we obtain

$$\begin{bmatrix} (\Theta_1 - I) \prod_{i=1}^{n-m+p} (\Theta_1 - \lambda_i I) & 0 \\ \Theta_2 \prod_{i=1}^{n-m+p} (\Theta_1 - \lambda_i I) & 0 \end{bmatrix}.$$

Since $G + B^T E D^{-1} E^T B$ is positive definite, Θ_1 has a full set of linearly independent eigenvectors and is diagonalizable. Hence, $(\Theta_1 - I) \prod_{i=1}^{n-m+p} (\Theta_1 - \lambda_i I) = 0$. We therefore obtain

$$(4.6) \quad (\mathcal{P}^{-1}\mathcal{A} - I) \prod_{i=1}^{n-m+p} (\mathcal{P}^{-1}\mathcal{A} - \lambda_i I) = \begin{bmatrix} 0 & 0 \\ \Theta_2 \prod_{i=1}^{n-m+p} (\Theta_1 - \lambda_i I) & 0 \end{bmatrix}.$$

If $\Theta_2 \prod_{i=1}^{n-m+p} (\Theta_1 - \lambda_i I) = 0$, then the order of the minimal polynomial of $\mathcal{P}^{-1}\mathcal{A}$ is less than or equal to $\min\{n - m + p + 1, n + m\}$. If $\Theta_2 \prod_{i=1}^{n-m+p} (\Theta_1 - \lambda_i I) \neq 0$, then the dimension of $\mathcal{K}(\mathcal{P}^{-1}\mathcal{A}, b)$ is at most $\min\{n - m + p + 2, n + m\}$ since multiplication of (4.6) by another factor $(\mathcal{P}^{-1}\mathcal{A} - I)$ gives the zero matrix. \square

Thus, in exact arithmetic, iteration with any method with an optimality condition will terminate in at most $\min\{n - m + p + 2, n + m\}$ iterations (in practice, exact arithmetic is not available, and hence this theoretical bound may be exceeded). We observe that if $p = m$, then Theorem 4.3 gives the same bound on the Krylov subspace dimension as that in Theorem 3.3, and if $p = 0$, then we obtain the results of [14].

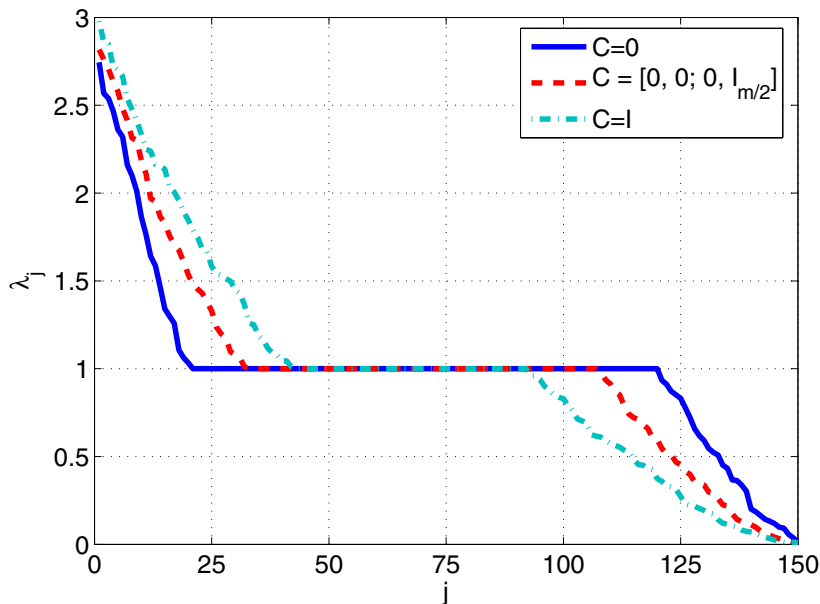


FIG. 5.1. Distribution of the eigenvalues of $\mathcal{P}^{-1}\mathcal{A}$ for the CVXQP1.S problem ($m = 50, n = 100$) with $C = 0$, $C = [0, 0; 0, I_{m/2}]$, and $C = I$. The eigenvalues are sorted such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{n+m}$.

5. Numerical results. The CUTer test set [13] provides a set of quadratic programming problems. We shall use a problem from this set to illustrate how changing the rank of C affects the multiplicity of the unit eigenvalues and the termination of GMRES. All tests were performed in MATLAB 7.01.

The CVXQP1.S problem from the CUTer test set is small with $n = 100$ and $m = 50$. It is a convex quadratic program whose constraints are linear; it is a purely academic problem which has been constructed specifically for test problems. “Barrier” penalty terms (in this case 1.1) are added to the diagonal of A to simulate systems that might arise during an iteration of an interior point method for such problems. We shall set $G = \text{diag}(A)$, $C = \text{diag}(0, \dots, 0, 1, \dots, 1)$ and vary the number of zeros on the diagonal of C so as to change its rank.

In Figure 5.1, we illustrate the change in the eigenvalues of the preconditioned system $\mathcal{P}^{-1}\mathcal{A}$ for three different choices of C . The eigenvalues are sorted so that

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{n+m}.$$

When $C = 0$, we expect there to be at least $2m$ unit eigenvalues [14]. We observe that our example has exactly $2m$ eigenvalues at 1. From Theorem 3.1, when $C = I$, there will be at least m unit eigenvalues. Our example has exactly m unit eigenvalues (Figure 5.1).

When C has rank $\frac{m}{2}$, then the preconditioned system $\mathcal{P}^{-1}\mathcal{A}$ has at least $\frac{3m}{2}$ unit eigenvalues according to Theorem 4.1. Once again, the number of unit eigenvalues for our example is exactly the lower bound given by the theorem.

Now suppose that we use (full) GMRES preconditioned by our extended constraint preconditioner with $G = \text{diag}(A)$ and vary the rank of C by changing the

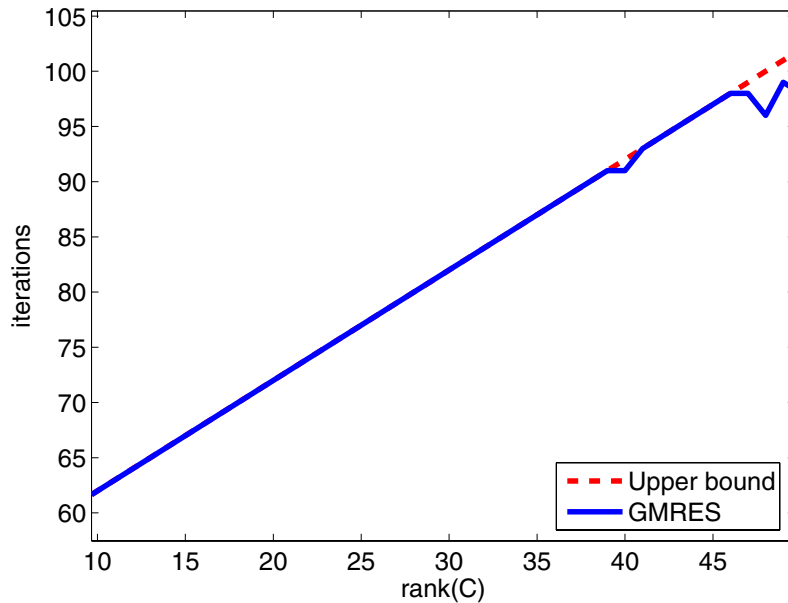


FIG. 5.2. Comparison of upper bound on the Krylov subspace dimension and the number of iterations required to reduce the residual by 10^{-12} .

number of 1's along the diagonal of C (all other entries are zero). Figure 5.2 shows that with this example and choice of G there is a strong correlation between the upper bound on the Krylov subspace dimension and the number of iterations required to reduce the residual by at least a factor of 10^{-12} . This has been chosen as an extreme example, and the number of iterations is often a lot lower than the upper bound on the Krylov subspace dimension. A comprehensive comparison (taking into account both CPU times and the number of iterations) for these preconditioners can be found in [7]: this study reveals the possible advantages of choosing G to be singular or indefinite.

6. Conclusions. In this paper, we investigated a class of preconditioners for regularized saddle point matrix systems that incorporate the $(1, 2)$, $(2, 1)$, and $(2, 2)$ blocks of the original matrix. We showed that the inclusion of these blocks in the preconditioner clusters at least $2m - p$ eigenvalues at 1, regardless of the structure of G . However, the standard convergence theory for Krylov subspace methods is not readily applicable because, in general, $\mathcal{P}^{-1}\mathcal{A}$ does not have a complete set of linearly independent eigenvectors. Using a minimal polynomial argument, we found a general (sharp) upper bound on the number of iterations required to solve linear systems of the form (1.1).

To confirm the analytical results of this paper, we used a subset of problems from the CUTER test set. We used the CVXQP1_S problem and varied the rank of C to confirm the lower bound on the number of unit eigenvalues and the upper bound on the Krylov subspace dimension.

We have assumed that the submatrices B , B^T and $-C$ in (1.1) are exactly reproduced in the preconditioner. For truly large-scale problems, this will be unrealistic

[5, 6, 18], but the theorems in this paper may still be of some interest in the inexact setting as a guide for choosing preconditioners. We wish to investigate this possibility in our future work.

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