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Constructing a Brownian Sheet with Values in a Compact Riemannian Manifold

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In this paper, we propose a new method of constructing a two-parameter random field $\mathbf{W}_M^x(s,t)$, $x \in M$, with values in a compact Riemannian manifold M possessing the property that the random processes $\mathbf{W}_M^x(\,\cdot\,,t)$ and $\mathbf{W}_M^x(s,\,\cdot\,)$ are Brownian motions on the manifold M with parameters t and s, respectively, issuing from the point x. (By a *Brownian motion* on a manifold M with parameter t we mean the diffusion process generated by the operator $-(t/2)\Delta_M$, where Δ_M is the Laplace operator on the manifold M.) For the case in which the manifold is a compact Lie group, the two-parameter random field constructed in the paper coincides with the Brownian sheet defined by Malliavin [1] in 1991. (Malliavin called this random field a Brownian motion with values in $C([0,1],M)$, which is the set of continuous functions defined on the closed interval $[0,1]$ and taking values in M.) Nevertheless, for the case in which the manifold is a compact Lie group, the method proposed in the present paper essentially differs from that used in Malliavin's paper.

1. FIRST STEP IN THE CONSTRUCTION OF THE RANDOM FIELD $\mathbf{W}_M^x$

Suppose that M is a d-dimensional compact Riemannian manifold without boundary isometrically embedded in $\mathbb{R}^m$. By a *Brownian sheet with values in* $\mathbb{R}^m$ we mean the family of m independent standard Brownian sheets. Suppose that $\mathbf{W}_{t,s}$ is an n-dimensional Brownian sheet. Consider $\mathbf{W}_{t,s}$ as a process taking values in the space $C([0,1],\mathbb{R}^m)$. We denote this process by the symbol $\mathbf{W}_t$. We introduce the following notation: if E is a locally convex space, then E^t denotes $C([0,t],E)$; if $y \in C([0,1],\mathbb{R}^m)$ is a continuous function, then $\mathbb{W}^y$ denotes the distribution of the process $\mathbf{W}_t^y = y + \mathbf{W}_t$. If $\psi \in C([0,1],\mathbb{R}^m)$, then we define the process $(\mathbf{W}_\psi^y)_t = \psi(t) + \mathbf{W}_t^y$. Suppose that $\widetilde{\mathbb{W}}_\psi^y$ is the distribution of this process and $\mathsf{E}_{y,\psi}$ is the expectation with respect to the measure $\widetilde{\mathbb{W}}_\psi^y$. Further, $U_\varepsilon(M)$ denotes the ε-neighborhood of the manifold M. We consider $\mathbf{W}_\psi^y$ for functions y and ψ satisfying the conditions: $y(0) \in M$, $\psi(0) = 0$. The goal of this section is to prove the existence of a limit (given below) with respect to the family of bounded continuous cylindrical functions, where by a cylindrical function $C([0,1]\times[0,1],\mathbb{R}^m) \to \mathbb{R}$ we mean a function f for which there exists a finite collection of points $\tau_1,\dots,\tau_n,\xi_1,\dots,\xi_k$ and a function $\widetilde{f}\colon \mathbb{R}^{nk} \to \mathbb{R}$ such that

$$f(\omega) = \widetilde{f}(\omega(\tau_1,\xi_1),\omega(\tau_1,\xi_2),\dots,\omega(\tau_n,\xi_k)).$$

This limit defines the measure $\widetilde{\mathbb{W}}_{M,\psi,s,t}^y$:

$$\int_{C([0,s],\mathbb{R}^m)^t} f(\omega)\widetilde{\mathbb{W}}_{M,\psi,s,t}^y(d\omega) = \lim_{\varepsilon\to 0} \frac{\mathsf{E}_{y,\psi}\{f(\omega)\mathbb{I}_{\{(\mathbf{W}_\psi^y)_t(s)\in U_\varepsilon(M)\}}\}}{\widetilde{\mathbb{W}}_\psi^y\{(\mathbf{W}_\psi^y)_t(s) \in U_\varepsilon(M)\}}. \tag{1}$$

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Before proving the existence of such a limit, we consider the process

$$(\mathbf{W}^z_{\psi,s})_t = \psi(t) + B^s_t,$$

where $\psi\colon [0,1] \to \mathbb{R}^m$ is a continuous function satisfying the condition $\psi(0) = 0$ and B^s_t is a Brownian motion with parameter s issuing from the point z. The results obtained for this process will be used for a subsequent construction.

Some results for the process $(\mathbf{W}^z_{\psi,s})_t$. Suppose that $\mathbb{W}^z_{\psi,s}$ denotes the distribution of the process $(\mathbf{W}^z_{\psi,s})_t$ and $\mathsf{E}_{z,\psi,s}$ is the expectation with respect to this distribution.

Lemma 1. *The limit*

$$\int_{C([0,t],\mathbb{R}^m)} f(\omega)\mathbb{W}^z_{M,\psi,s,t}(d\omega) = \lim_{\varepsilon\to 0} \frac{\mathsf{E}_{z,\psi,s}\{f(\omega)\mathbb{I}_{\{(\mathbf{W}^z_{\psi,s})_t \in U_\varepsilon(M)\}}\}}{\mathbb{W}^z_{\psi,s}\{(\mathbf{W}^z_{\psi,s})_t \in U_\varepsilon(M)\}},$$

with respect to the family of continuous bounded cylindrical functions, exists and defines the measure $\mathbb{W}^z_{M,\psi,s,t}$ *in the integral on the left.*

Sketch of the proof. Let us find a function $\widetilde{f}\colon \mathbb{R}^{k+1} \to \mathbb{R}$ and a finite set of points $\tau_1, \dots, \tau_k$ such that

$$f(\omega) = \widetilde{f}(\omega(\tau_1), \dots, \omega(\tau_k), \omega(t)).$$

We have

$$\begin{aligned}
\int_{C([0,t],\mathbb{R}^m)} f(\omega)\mathbb{W}^z_{M,\psi,s,t}(d\omega) &= \lim_{\varepsilon\to 0} \frac{\int_{C([0,t],\mathbb{R}^m)} f(\omega)\mathbb{I}_{\{\omega\,:\,\omega(t)\in U_\varepsilon(M)\}}\mathbb{W}^z_{\psi,s}(d\omega)}{\mathbb{W}^z_{\psi,s}\{\omega : \omega(t) \in U_\varepsilon(M)\}} \\
&= \lim_{\varepsilon\to 0} \frac{1}{\mathsf{P}^{\mathbb{W}}(t, 0, U_\varepsilon(M - z - \psi(t)))} \int_{\mathbb{R}^m} \mathsf{P}^{\mathbb{W}}(\tau_1, 0, dx_1) \int_{\mathbb{R}^m} \mathsf{P}^{\mathbb{W}}(\tau_2 - \tau_1, x_1, dx_2) \cdots \\
&\quad\times \int_{U_\varepsilon(M-\psi(t)-z)} \mathsf{P}^{\mathbb{W}}(t - \tau_k, x_k, dx_{k+1}) \\
&\qquad\times \widetilde{f}(x_1 + z + \psi(\tau_1), \dots, x_k + z + \psi(\tau_k), x_{k+1} + z + \psi(t)),
\end{aligned}$$

where

$$\mathsf{P}^{\mathbb{W}}(\tau, x, dz) = \frac{1}{(2\pi s\tau)^{m/2}} \exp\biggl\{-\frac{|z-x|^2}{2s\tau}\biggr\}\,dz.$$

Since the function in the integrand is bounded, it suffices, by Lebesgue's theorem, to prove that the following limit exists:

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \frac{\int_{(U_\varepsilon(M-\psi(t)-z))} \widetilde{f}(x_1 + z + \psi(\tau_1), \dots, x_{k+1} + z + \psi(t))\mathsf{P}^{\mathbb{W}}(t - \tau_k, x_k, dx_{k+1})}{\mathsf{P}^{\mathbb{W}}(t, 0, U_\varepsilon(M - z - \psi(t)))} \\
&\quad= \lim_{\varepsilon\to 0} \frac{\int_{U_\varepsilon(M-\psi(t)-z-x_k)} \widetilde{f}(x_1 + z + \psi(\tau_1), \dots, x_{k+1} + x_k + z + \psi(t))\mathsf{P}^{\mathbb{W}}(t - \tau_k, 0, dx_{k+1})}{\mathsf{P}^{\mathbb{W}}(t, 0, U_\varepsilon(M - z - \psi(t))}.
\end{aligned}$$

By M_1 we denote the manifold $M - \psi(t) - z - x_k$ and by M_2 the manifold $M - \psi(t) - z$. Further, suppose that

$$\lambda_\varepsilon = \frac{1}{\mathrm{vol}_{m-d}(\varepsilon)} l\big|_{U_\varepsilon(M_1)}, \qquad \mu_\varepsilon = \frac{1}{\mathrm{vol}_{m-d}(\varepsilon)} l\big|_{U_\varepsilon(M_2)},$$

where l is the Lebesgue measure on $\mathbb{R}^m$. We can easily see that the proof of the existence of this limit can be reduced to that of the existence of the limit

$$\lim_{\varepsilon\to 0}\frac{\int_{\mathbb{R}^m} g(x_{k+1})\exp\Bigl\{-\frac{|x_{k+1}-x_k|^2}{2s(t-\tau_k)}\Bigr\}\lambda_\varepsilon(dx_{k+1})}{\int_{\mathbb{R}^m}\exp\Bigl\{-\frac{|x_{k+1}|^2}{2st}\Bigr\}\mu_\varepsilon(dx_{k+1})},$$

where $g\colon \mathbb{R}\to\mathbb{R}$ is another symbol for the function $\widetilde{f}$ introduced to indicate the dependence on the last variable solely. We can easily show that, as $\varepsilon\to 0$, the measures λ_ε and μ_ε converge weakly to the surface measures on M_1 and M_2, respectively. □

Lemma 2. *The limit* (1) *with respect to the family of continuous bounded cylindrical functions exists.*

Sketch of the proof. Suppose that $\mathsf{P}^{\widetilde{\mathbb{W}}}(t,y,\Gamma)=\widetilde{\mathbb{W}}^y(\omega:\omega(t)\in\Gamma)$ is the transition probability for the measure $\widetilde{\mathbb{W}}^y$, where $y\in C([0,1],\mathbb{R}^m)$. Further, suppose that the function

$$\widetilde{f}\colon C([0,s],\mathbb{R}^m)^{k+1}\to\mathbb{R}$$

and the finite set of points $\tau_1,\tau_2,\dots,\tau_k$ satisfy the relation

$$f(\omega)=\widetilde{f}(\omega(\tau_1),\omega(\tau_2),\dots,\omega(\tau_k),\omega(t)).$$

Let the symbol π_s denote the coordinate mapping. The proof is carried out by using the following formula from [2, p. 204]:

$$\begin{aligned}\int_{C([0,s],\mathbb{R}^m)^t} f(\omega)\widetilde{\mathbb{W}}^0(d\omega)&=\int_{C([0,s],\mathbb{R}^m)}\mathsf{P}^{\widetilde{\mathbb{W}}}(\tau_1,0,dw_1)\int_{C([0,s],\mathbb{R}^m)}\mathsf{P}^{\widetilde{\mathbb{W}}}(\tau_2-\tau_1,w_1,dw_2)\cdots\\ &\times\int_{\pi_s^{-1}(U_\varepsilon(M-\psi(t)-y(s)))}\widetilde{f}(w_1,\dots,w_{k+1})\mathsf{P}^{\widetilde{\mathbb{W}}}(t-\tau_k,w_k,dw_{k+1})\end{aligned}$$

and applying Lemma 1 to the measure in the last integral. □

2. ASYMPTOTICS IN t FOR AN INTEGRAL OF SPECIFIC FORM

Proposition 1. *Let i be an isometric embedding of the manifold M in $\mathbb{R}^m$ and $g\in C^2(M)$. Then*

$$\frac{1}{(2\pi t)^{d/2}}\int_M g(z)\exp\Bigl\{-\frac{|z-y|^2}{2t}\Bigr\}\lambda_M(dz)=g(y)+\frac{t}{8}\,g(y)\bigl(c(y)-\operatorname{scal}(y)\bigr)-\frac{t}{2}\,\Delta_M g(y)+tR(t,y),$$

where $|R(t,y)|<Kt^{1/2}$, K is a constant independent of y, $\operatorname{scal}(y)$ is the scalar curvature at the point y, and the function $c(y)$ is of the form

$$c(y)=\sum_{k,l}\sum_\alpha\biggl(\frac{\partial^2 i^\alpha}{\partial x^k\partial x^l}\biggr)^2(0),$$

where the x^k are the normal coordinates in a neighborhood U_y of the point y which are specified by the homeomorphism of the neighborhood U_y onto a neighborhood of zero U in $\mathbb{R}^d$. Independently of the local coordinates, $c(y)$ can be written as

$$c(y)=-\frac{1}{2}\,\Delta_M\Delta_M|y-\cdot|^2\big|_y-\frac{1}{3}\operatorname{scal}(y)$$

and, therefore, $c(y)$ depends only on the embedding i.

Sketch of the proof. We have obtained a more exact asymptotic expression in comparison with that obtained for an integral of similar form in [3]. The idea of the proof is the same. □

Corollary 1. *Suppose that $g \in C^2(M)$. Then the following asymptotics is valid*:

$$\frac{\int_M g(z)\exp\left\{-\frac{|z-y|^2}{2t}\right\}\lambda_M(dz)}{\int_M \exp\left\{-\frac{|z-y|^2}{2t}\right\}\lambda_M(dz)} = g(y) - \frac{t}{2}\,\Delta_M g(y) + tR_1(t,y),$$

where $|R_1(t,y)| < K_1 t^{1/2}$, and K_1 is a constant independent of y.

Corollary 2. *Suppose that $g \in C^2(M)$, $y \in M$, and ψ is a Hölder function of Hölder order α, $1/3 < \alpha < 1/2$, such that $\psi(0) = 0$. Suppose that Pr_M is the projection mapping onto the manifold M along the subspaces normal to the manifold and defined in a suitable neighborhood of the manifold $\psi_M(t,y) = \mathrm{Pr}_M(y + \psi(t))$. Then the following asymptotics is valid*:

$$\frac{\int_M g(z)\exp\left\{-\frac{|z-y-\psi(t)|^2}{2t}\right\}\lambda_M(dz)}{\int_M \exp\left\{-\frac{|z-y-\psi(t)|^2}{2t}\right\}\lambda_M(dz)} = g(y+\psi_M(t)) - \frac{t}{2}\,\Delta_M g(y) + tR_2(t,y),$$

where $|R_2(t,y)| < K_2 t^{3\alpha-1}$ and K_2 is a constant.

3. SECOND STEP IN THE CONSTRUCTION OF THE RANDOM FIELD $\mathbf{W}_M^x$

Suppose that f is a continuous bounded cylindrical function on $C([0,s],\mathbb{R}^m)^1$ and $\varphi\colon \mathbb{R} \to M$ is a function which is the trajectory of the Brownian motion on M such that $\varphi(0) = x$. Suppose that $\mathcal{P}_1 = \{0 = t_0 \le t_1 \le \dots \le t_n = 1\}$ is a partition of the interval $[0,1]$. If E is a locally convex space, then to each $\omega \in E^1$ we can assign a finite sequence of n elements

$$(\omega_1, \omega_2, \dots, \omega_n) \in E^{t_1} \times E^{t_2 - t_1} \times \dots \times E^{t_n - t_{n-1}},$$

where ω_j is defined on the interval $[0, t_j - t_{j-1}]$ by the formula $\omega_j(t) = \omega(t_{j-1} + t)$. We define the function $\varphi_{t_{i-1}t_i}$ on the interval $[0, t_i - t_{i-1}]$ as follows:

$$\varphi_{t_{i-1}t_i}(t) = \varphi(t_{i-1} + t) - \varphi(t_{i-1}).$$

Let us define the measure $\widetilde{\mathbb{W}}^x_{M,\varphi,s,\mathcal{P}_1}$ by the formula

$$\begin{aligned}\int_{C([0,s],\mathbb{R}^m)^1} f(\omega)\widetilde{\mathbb{W}}^x_{M,\varphi,s,\mathcal{P}_1}(d\omega) &= \int_{C([0,s],\mathbb{R}^m)^{t_1}} \widetilde{\mathbb{W}}^x_{M,\varphi_{0t_1},s,t_1}(d\omega_1) \\ &\times \int_{C([0,s],\mathbb{R}^m)^{t_2-t_1}} \widetilde{\mathbb{W}}^{\omega_1(t_1)}_{M,\varphi_{t_1t_2},s,t_2-t_1}(d\omega_2)\cdots \\ &\times \int_{C([0,s],\mathbb{R}^m)^{t_n-t_{n-1}}} \widetilde{\mathbb{W}}^{\omega_{n-1}(t_{n-1}-t_{n-2})}_{M,\varphi_{t_{n-1}t_n},s,t_n-t_{n-1}}(d\omega_n) f(\omega_1,\omega_2,\dots,\omega_n).\end{aligned}$$

It is readily verified that $\omega_i(t_i - t_{i-1})(0) \in M$, so that the measure $\widetilde{\mathbb{W}}^x_{M,\varphi,s,\mathcal{P}_1}$ is well defined. Further, let $\mathcal{P}_2 = \{0 = s_0 \le s_1 \le \dots \le s_k = 1\}$ be a partition of the interval $[0,1]$. Now, suppose that s is a time parameter. Instead of the symbol $\widetilde{\mathbb{W}}^x_{M,\varphi,s,\mathcal{P}_1}$, we shall write $\widetilde{\mathbb{W}}^\varphi_{M,s,\mathcal{P}_1}$. Let us define the measure $\mathbb{W}^x_{M,\mathcal{P}_1,\mathcal{P}_2}$ by the formula

$$\begin{aligned}\int_{C([0,1],\mathbb{R}^m)^1} f(\omega)\mathbb{W}^x_{M,\mathcal{P}_1,\mathcal{P}_2}(d\omega) &= \int_{C([0,1],\mathbb{R}^m)^{s_1}} \widetilde{\mathbb{W}}^x_{M,s_1,\mathcal{P}_1}(d\omega_1) \\ &\times \int_{C([0,1],\mathbb{R}^m)^{s_2-s_1}} \widetilde{\mathbb{W}}^{\omega_1(s_1)}_{M,s_2-s_1,\mathcal{P}_1}(d\omega_2)\cdots \\ &\times \int_{C([0,1],\mathbb{R}^m)^{s_n-s_{n-1}}} \widetilde{\mathbb{W}}^{\omega_{n-1}(s_{n-1}-s_{n-2})}_{M,s_n-s_{n-1},\mathcal{P}_1}(d\omega_n) f(\omega_1,\dots,\omega_n).\end{aligned}$$

Theorem 1. *For each $x \in M$, if the meshes of the partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ tend to zero, then the sequence of measures $\mathbb{W}^x_{M,\mathcal{P}_1,\mathcal{P}_2}$ is weakly convergent to the measure $\mathbb{W}^x_M$ with respect to the family of continuous bounded cylindrical functions. The measure $\mathbb{W}^x_M$ regarded as the distribution of a process with values in $C([0,1],M)$, possesses a transition probability at time t coinciding with the distribution of the Brownian motion with parameter t on the manifold issuing from the point x.*

Sketch of the proof. Suppose that

$$\begin{aligned} h\cdot \mathbb{W}^x_{M,\varphi,s,\mathcal{P}_1}\Big(= h\cdot \mathbb{W}^\varphi_{M,s,\mathcal{P}_1}\Big) &= \int_{C([0,1],\mathbb{R}^m)} h(\omega)\mathbb{W}^x_{M,\varphi,s,\mathcal{P}_1}(d\omega) \\ &= \int_{C([0,t_1],\mathbb{R}^m)} \mathbb{W}^x_{M,\varphi_{0t_1},s,t_1}(d\omega_1) \int_{C([0,t_2-t_1],\mathbb{R}^m)} \mathbb{W}^{\omega_1(t_1)}_{M,\varphi_{t_1t_2},s,t_2-t_1}(d\omega_2)\cdots \\ &\quad\times \int_{C([0,t_n-t_{n-1}],\mathbb{R}^m)} \mathbb{W}^{\omega_{n-1}(t_{n-1}-t_{n-2})}_{M,\varphi_{t_{n-1}t_n},s,t_n-t_{n-1}}(d\omega_n) h(\omega_1,\omega_2,\dots,\omega_n). \end{aligned}$$

Suppose that there exists a function $\widetilde{f}\colon C([0,1],\mathbb{R}^m)\to\mathbb{R}$ such that $f(\omega)=\widetilde{f}(\omega(t))$. Then

$$\begin{aligned} \int_{C([0,s],\mathbb{R}^m)^1} f(\omega)\widetilde{\mathbb{W}}^\varphi_{M,s,\mathcal{P}_1}(d\omega) &= \int_{C([0,1],\mathbb{R}^m)} \widetilde{f}(w)\widetilde{\mathbb{W}}^\varphi_{M,s,\mathcal{P}_1}\circ\pi_s^{-1}(dw) \\ &= \int_{C([0,1],\mathbb{R}^m)} \widetilde{f}(w)\mathbb{W}^\varphi_{M,s,\mathcal{P}_1}(dw). \end{aligned}$$

This yields

$$\begin{aligned} \int_{C([0,1],\mathbb{R}^m)^1} f(\omega)\mathbb{W}^x_{M,\mathcal{P}_1,\mathcal{P}_2}(d\omega) &= \int_{C([0,1],\mathbb{R}^m)} \mathbb{W}^x_{M,s_1,\mathcal{P}_1}(dw_1) \\ &\quad\times \int_{C([0,1],\mathbb{R}^m)} \mathbb{W}^{w_1}_{M,s_2-s_1,\mathcal{P}_1}(dw_2)\cdots \int_{C([0,1],\mathbb{R}^m)} \mathbb{W}^{w_{n-2}}_{M,s_{n-1}-s_{n-2},\mathcal{P}_1}(dw_{n-1}) \\ &\quad\times \int_{C([0,1],\mathbb{R}^m)} \mathbb{W}^{w_{n-1}}_{M,s_n-s_{n-1},\mathcal{P}_1}(dw_n)\widetilde{f}(w_n). \end{aligned}$$

Consider the integral

$$\int_{C([0,t],\mathbb{R}^m)} g(\omega)\mathbb{W}^z_{M,\psi,s,t}(d\omega),$$

where the function $g \in C([0,t],\mathbb{R}^m)$ is such that there exists a function $\widetilde{g}\in C(\mathbb{R})$ for which $g(\omega)=\widetilde{g}(\omega(t))$. As a result of simple calculations, we obtain

$$\begin{aligned} \int_{C([0,t],\mathbb{R}^m)} g(\omega)\mathbb{W}^z_{M,\psi,s,t}(d\omega) &= \lim_{\varepsilon\to 0}\frac{\int_{C([0,t],\mathbb{R}^m)} g(\omega)\mathbb{I}_{\{\omega\,:\,\omega(t)\in U_\varepsilon(M)\}}(\omega)\,\mathbb{W}^z_{\psi,s}(d\omega)}{\mathbb{W}^z_{\psi,s}\{\omega:\omega(t)\in U_\varepsilon(M)\}} \\ &= \frac{\int_M \exp\Big\{-\frac{|x_1-z-\psi(t)|^2}{2ts}\Big\}\widetilde{g}(x_1)\,\lambda_M(dx_1)}{\int_M \exp\Big\{-\frac{|x_1-z-\psi(t)|^2}{2ts}\Big\}\lambda_M(dx_1)}. \end{aligned}$$

First, suppose that the function f is such that there exists a function $p\colon\mathbb{R}^m\to\mathbb{R}$ and numbers $t,s\in[0,1]$ for which $f(\omega)=p(\omega(t,s))$. The integral

$$\int_{C([0,1],\mathbb{R}^m)^1} f(\omega)\mathbb{W}^x_{M,\mathcal{P}_1,\mathcal{P}_2}(d\omega)$$

is of the form

$$\frac{\int_M \exp\Bigl\{-\frac{|x_1-x|^2}{2\Delta s_1\Delta t_1}\Bigr\}\,dx_1}{\int_M \exp\Bigl\{-\frac{|\bar x_1-x|^2}{2\Delta s_1\Delta t_1}\Bigr\}\,d\bar x_1}\cdots\frac{\int_M \exp\Bigl\{-\frac{|x_{n-1}-x_{n-2}|^2}{2\Delta s_1\Delta t_{n-1}}\Bigr\}\,dx_{n-1}}{\int_M \exp\Bigl\{-\frac{|\bar x_{n-1}-x_{n-2}|^2}{2\Delta s_1\Delta t_{n-1}}\Bigr\}\,d\bar x_{n-1}}\,\frac{\int_M \exp\Bigl\{-\frac{|x_n-x_{n-1}|^2}{2\Delta s_1\Delta t_n}\Bigr\}\,dx_n}{\int_M \exp\Bigl\{-\frac{|\bar x_n-x_{n-1}|^2}{2\Delta s_1\Delta t_n}\Bigr\}\,d\bar x_n}$$

$$\times\frac{\int_M \exp\Bigl\{-\frac{|y_1-x_1|^2}{2\Delta s_2\Delta t_1}\Bigr\}\,dy_1}{\int_M \exp\Bigl\{-\frac{|\bar y_1-x_1|^2}{2\Delta s_2\Delta t_1}\Bigr\}\,d\bar y_1}\cdots\frac{\int_M \exp\Bigl\{-\frac{|y_{n-1}-y_{n-2}-x_{n-1}+x_{n-2}|^2}{2\Delta s_2\Delta t_{n-1}}\Bigr\}\,dy_{n-1}}{\int_M \exp\Bigl\{-\frac{|\bar y_{n-1}-y_{n-2}-x_{n-1}+x_{n-2}|^2}{2\Delta s_2\Delta t_{n-1}}\Bigr\}\,d\bar y_{n-1}}$$

$$\times\frac{\int_M \exp\Bigl\{-\frac{|y_n-y_{n-1}-x_n+x_{n-1}|^2}{2\Delta s_2\Delta t_n}\Bigr\}\,dy_n}{\int_M \exp\Bigl\{-\frac{|\bar y_n-y_{n-1}-x_n+x_{n-1}|^2}{2\Delta s_2\Delta t_n}\Bigr\}\,d\bar y_n}\cdots$$

$$\times\frac{\int_M \exp\Bigl\{-\frac{|u_1-z_1|^2}{2\Delta s_{k-1}\Delta t_1}\Bigr\}\,du_1}{\int_M \exp\Bigl\{-\frac{|\bar u_1-z_1|^2}{2\Delta s_{k-1}\Delta t_1}\Bigr\}\,d\bar u_1}\cdots\frac{\int_M \exp\Bigl\{-\frac{|u_{n-1}-u_{n-2}-z_{n-1}+z_{n-2}|^2}{2\Delta s_{k-1}\Delta t_{n-1}}\Bigr\}\,du_{n-1}}{\int_M \exp\Bigl\{-\frac{|\bar u_{n-1}-u_{n-2}-z_{n-1}+z_{n-2}|^2}{2\Delta s_{k-1}\Delta t_{n-1}}\Bigr\}\,d\bar u_{n-1}}$$

$$\times\frac{\int_M \exp\Bigl\{-\frac{|u_n-u_{n-1}-z_n+z_{n-1}|^2}{2\Delta s_{k-1}\Delta t_n}\Bigr\}\,du_n}{\int_M \exp\Bigl\{-\frac{|\bar u_n-u_{n-1}-z_n+z_{n-1}|^2}{2\Delta s_{k-1}\Delta t_n}\Bigr\}\,d\bar u_n}$$

$$\times\frac{\int_M \exp\Bigl\{-\frac{|v_1-u_1|^2}{2\Delta s_k\Delta t_1}\Bigr\}\,dv_1}{\int_M \exp\Bigl\{-\frac{|\bar v_1-u_1|^2}{2\Delta s_k\Delta t_1}\Bigr\}\,d\bar v_1}\cdots\frac{\int_M \exp\Bigl\{-\frac{|v_{n-1}-v_{n-2}-u_{n-1}+u_{n-2}|^2}{2\Delta s_k\Delta t_{n-1}}\Bigr\}\,dv_{n-1}}{\int_M \exp\Bigl\{-\frac{|\bar v_{n-1}-v_{n-2}-u_{n-1}+u_{n-2}|^2}{2\Delta s_k\Delta t_{n-1}}\Bigr\}\,d\bar v_{n-1}}$$

$$\times\frac{\int_M \exp\Bigl\{-\frac{|v_n-v_{n-1}-u_n+u_{n-1}|^2}{2\Delta s_k\Delta t_n}\Bigr\}p(v_n)\,dv_n}{\int_M \exp\Bigl\{-\frac{|\bar v_n-v_{n-1}-u_n+u_{n-1}|^2}{2\Delta s_k\Delta t_n}\Bigr\}\,d\bar v_n},$$

where $\Delta t_i = t_i - t_{i-1}$, $\Delta s_j = s_j - s_{j-1}$, and, to simplify the notation, instead of $\lambda_M(dz)$ we use dz. We have also assumed that $t_n = t$ and $s_k = s$. Let us denote this integral by $I(\mathcal{P}_1, \mathcal{P}_2, p)$.

Lemma 3. *The integral $I(\mathcal{P}_1, \mathcal{P}_2, p)$ converges to $\exp\bigl\{-\frac{st}{2}\Delta_M\bigr\}p$ if the meshes of $|\mathcal{P}_1|$ and $|\mathcal{P}_2|$ tend to zero.*

Sketch of the proof. Using Corollaries 1 and 2 of Proposition 1, we obtain the following asymptotics for the integral $I(\mathcal{P}_1, \mathcal{P}_2, p)$:

$$I(\mathcal{P}_1, \mathcal{P}_2, p)(x) = p(x) - \frac{st}{2}\,\Delta_M p(x) + O(s^2t^2) + O\bigl(|\mathcal{P}_2|, |\mathcal{P}_1|\bigr).$$

Hence we see that the following limit exists:

$$\lim_{|\mathcal{P}_1|\to 0,\,|\mathcal{P}_2|\to 0} I(\mathcal{P}_1, \mathcal{P}_2, p)(x) = p(x) - \frac{st}{2}\,\Delta_M p(x) + O(s^2t^2)\bigl(= (Q_{st}p)(x)\bigr).$$

Further, we verify that $Q_{\tau+\Delta\tau} = Q_\tau Q_{\Delta\tau}$. Let s and t satisfy $\tau = st$. Let us find a Δs and Δt such that $\tau + \Delta\tau = (s+\Delta s)(t+\Delta t)$, and consider the integral $I(\mathcal{P}_1^{[0,\,t+\Delta t]}, \mathcal{P}_2^{[0,\,s+\Delta s]}, p)$ for partitions of the closed intervals $[0, s+\Delta s]$ and $[0, t+\Delta t]$ resulting from supplementing the corresponding partitions of the closed intervals $[0, s]$ and $[0, t]$ by points of partitions of the closed intervals $[t, t+\Delta t]$, $[s, s+\Delta s]$. We can easily see from the structure of the integral $I(\mathcal{P}_1, \mathcal{P}_2, p)$ that

$$I(\mathcal{P}_1^{[0,\,t+\Delta t]}, \mathcal{P}_2^{[0,\,s+\Delta s]}, p) = I\bigl(\mathcal{P}_1^{[0,\,t]}, \mathcal{P}_2^{[0,\,s]}, I(\mathcal{P}_1^{[t,\,t+\Delta t]}, \mathcal{P}_2^{[s,\,s+\Delta s]}, p)\bigr).$$

Passing to the limit in this expression and taking $\Delta\tau = s\Delta t + t\Delta s + \Delta s\Delta t$ into account, we find that Q_τ is a semigroup which, by the proof above, satisfies

$$(Q_\tau p)(x) = p(x) - \frac{\tau}{2}\,\Delta_M p(x) + O(\tau^2).$$

Thus, we find that

$$Q_\tau = \exp\Big\{-\frac{\tau}{2}\,\Delta_M\Big\}. \quad \square$$

For a function f depending on ω at several points, such as at points $\xi_i \in [0, s]$ and $\tau_j \in [0, t]$, the integral $I(\mathcal{P}_1, \mathcal{P}_2, p)$ will be of the same form. The convergence examined above occurs on each square $[\xi_{i-1}, \xi_i] \times [\tau_{j-1}, \tau_j]$. Each of operators of the form

$$\exp\Big\{-\frac{\Delta\xi_i\Delta\tau_j}{2}\,\Delta_M\Big\}$$

acts on the corresponding variable of the function p defined as

$$f(\omega) = p(\omega_{11}(\xi_1, \tau_1), \dots, \omega_{kl}(\xi_l, \tau_l)),$$

where ω_{ij} is defined on $[0, \xi_i - \xi_{i-1}] \times [0, \tau_j - \tau_{j-1}]$ by the formula

$$\omega_{ij}(s, t) = \omega(\xi_{i-1} + s, \tau_{j-1} + t). \quad \square$$

Corollary 3. *Suppose that M is a compact Lie group. Then the random field $\mathbf{W}_M^x$ regarded as a process with values in $C([0,1], M)$, coincides with the Brownian motion constructed in* [1].

Sketch of the proof. The proof follows from Theorem 1 and Theorem 2.15 from [4] (Theorem 2.15 from [4] was also proved in Lemma 3.3 from [5]). $\square$

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