CONSTRUCTING A LIE GROUP ANALOG FOR THE MONSTER LIE ALGEBRA

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ABSTRACT. Let m be the Monster Lie algebra. We summarize several interrelated constructions of Lie group analogs for m. Our constructions are analogs for m of Chevalley and Kac–Moody groups and their generators and relations.

1. Introduction

Let m be the Monster Lie algebra, constructed by Borcherds ([B1, B2, B3]). Borcherds discovered m as an example of a new class of Lie algebras which became known as *generalized Kac–Moody algebras*, or *Borcherds algebras*. The invariant bilinear form on m gives rise to the following generalized Cartan matrix ([Jur1]):

$$A = \begin{pmatrix} c(-1) & c(1) & c(2) & c(2) & c(3) &$$

where c(i) is the coefficient of q^i in the modular function

$$J(q) = j(q) - 744 = \sum_{i \ge -1} c(i)q^i = \frac{1}{q} + 196884q + 21493760q^2 + 864299970q^3 + \cdots,$$

and j(q) is the well-known j-function.

Borcherds constructed \mathfrak{m} ([B2]) to prove part of the Conway–Norton conjectures ([CN]). A fundamental component of Borcherds' construction was Frenkel, Lepowsky and Meurman's representation ([FLM1], [FLM2]) of the Fischer–Griess Monster finite simple group \mathbf{M} . Borcherds constructed \mathfrak{m} as a certain quotient of the 'physical space' of the vertex operator algebra $V = V^{\natural} \otimes V_{1,1}$, where V^{\natural} the moonshine module of [FLM1], [FLM2], a graded \mathbf{M} -module with $\mathrm{Aut}(V^{\natural}) = \mathbf{M}$, and $V_{1,1}$ is a vertex operator algebra for the even unimodular 2-dimensional Lorentzian lattice $II_{1,1}$.

The Monster Lie algebra has an equivalent characterization as the Lie algebra $\mathfrak{m}=\mathfrak{g}(A)/\mathfrak{z}$ given by generators and relations associated to the matrix A, where \mathfrak{z} is the center of $\mathfrak{g}(A)$ ([B3], [Jur1] and Section 2). We use this latter characterization of \mathfrak{m} for our purposes.

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The Monster Lie algebra has the triangular decomposition $\mathfrak{m} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. We also use the following decomposition

$$\mathfrak{m} \cong \mathfrak{u}^- \oplus \mathfrak{gl}_2(-1) \oplus \mathfrak{u}^+$$

where $\mathfrak{gl}_2(-1) \cong \mathfrak{gl}_2$, and \mathfrak{u}^+ (resp. \mathfrak{u}^-) are subalgebras freely generated by certain positive (respectively negative) imaginary root vectors ([Jur1], [Jur2] and Theorem 2.1).

The class of Borcherds algebras has been widely studied. For example, the appearance of m and other Borcherds algebras as symmetries in heterotic string theory have been noted ([Ca], [HM], [PPV]). However, there have been no constructions of Lie group analogs for Borcherds algebras.

There are many methods for constructing Kac–Moody groups, the Lie group analogs for infinite dimensional Kac–Moody algebras (see, for example, [CG, Ga1, GW, KP, Ku, Ma, Mar, Ma84, MT, Rou, Sl, Ti]). For Borcherds algebras, the situation is quite different, due in part to the absence of suitable representation theoretic, geometric, algebraic geometric and analytic methods in this setting.

In particular, the root vectors associated with real roots of a Kac–Moody or Borcherds algebra are locally adnilpotent. In the Kac–Moody case, the simple roots are all real and associated choices of root vectors generate the whole Kac–Moody algebra. For Borcherds algebras, this is no longer the case. There are standard Chevalley generators of \mathfrak{n}^\pm associated to simple imaginary roots, but these do not act locally nilpotently on the adjoint representation, or on the highest weight modules of interest.

Here we report on several works in progress where we construct groups associated to m. We give several interrelated constructions which serve different purposes.

We construct a complete pro-unipotent group of automorphisms of a completion $\widehat{\mathfrak{m}}$ of \mathfrak{m} (Section 5 and [CJM]) by proving convergence of these maps on finite dimensional subspaces. For this purpose, we introduce the notion of *pro-summability* (Section 4 and [CJM]).

We also construct a group $G(\mathfrak{m})$ given by generators and relations, and show that it has an analog of a unipotent subgroup which acts as automorphisms of $\widehat{\mathfrak{m}}$ (Section 6 and [CJM], [ACJM]).

To construct the analog of a simply connected Kac–Moody Chevalley group for $\mathfrak m$ would require the use of an integrable highest weight representation of $\mathfrak m$. Since the simple imaginary root vectors which generate $\mathfrak m$ are not locally nilpotent on any of the highest weight modules that we encounter (including generalized Verma modules), we need a different approach. It was shown in [JLW] that $\mathfrak m$ has a representation on a certain tensor algebra $T(\mathcal V)$ analogous to standard irreducible modules for semisimple and Kac–Moody algebras. The parabolic subalgebra $\mathfrak{gl}_2(-1)\oplus\mathfrak u^+$ of $\mathfrak m$ is locally nilpotent on the $\mathfrak m$ -module $T(\mathcal V)$ ([JLW]). Using this representation, we construct the analog of a simply connected Kac–Moody Chevalley group associated to the parabolic subalgebra $\mathfrak{gl}_2(-1)\oplus\mathfrak u^+$ of $\mathfrak m$ (Section 7 and [CGJM]).

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2. The Monster Lie algebra

For $j \in \mathbb{Z}$, we recall the definition of c(j) above. Define index sets

$$I^{\text{re}} = \{(-1, 1)\}, \quad I^{\text{im}} = \{(j, k) \mid j \in \mathbb{N}, \ 1 \leqslant k \leqslant c(j)\},$$
$$I = \{(j, k) \mid j, k \in \mathbb{Z}, \ 1 \leqslant k \leqslant c(j)\} = I^{\text{re}} \sqcup I^{\text{im}}$$

and generalized Cartan matrix

$$A = (a_{jk,pq})_{(j,k),(p,q)\in I}$$
, where $a_{jk,pq} = -(j+p)$.

The Lie algebra $\mathfrak{g}(A)$ has generating set $\{e_{jk}, f_{jk}, h_{jk} \mid (j,k) \in I\}$ and defining relations

$$[h_{jk}, h_{pq}] = 0,$$

$$[h_{jk}, e_{pq}] = -(j+p)e_{pq},$$

$$[h_{jk}, f_{pq}] = (j+p)f_{pq},$$

$$[e_{jk}, f_{pq}] = \delta_{jp}\delta_{kq}h_{jk},$$

$$(ad e_{-11})^j e_{jk} = (ad f_{-11})^j f_{jk} = 0,$$

for all (j,k), $(p,q) \in I$. The generators are the usual Chevalley generators of the Lie algebra associated with the matrix A, where the double index is chosen to reflect the block structure of A (see also [JLW], page 10). The Cartan subalgebra is $\mathfrak{h}_A := \sum_{(j,k) \in I} \mathbb{C} h_{jk} \subseteq \mathfrak{g}(A)$.

As in [Jur1], [Jur2], we define the Monster Lie algebra \mathfrak{m} to be $\mathfrak{m}:=\mathfrak{g}(A)/\mathfrak{z}$ where \mathfrak{z} is the center of the Lie algebra. Define the following elements of \mathfrak{m} :

$$h_1 := \frac{1}{2}(h_{-11} - h_{11}) + \mathfrak{z}, \qquad h_2 := -\frac{1}{2}(h_{-11} + h_{11}) + \mathfrak{z}, e_{-1} := e_{-11} + \mathfrak{z}, \qquad f_{-1} := f_{-11} + \mathfrak{z}.$$

We will write e_{jk} for $e_{jk} + \mathfrak{z}$ and f_{jk} for $f_{jk} + \mathfrak{z}$, for all $(j, k) \in I^{im}$.

Then m is the Borcherds (generalized Kac-Moody) algebra with generating set

$${e_{-1}, f_{-1}, h_1, h_2} \cup {e_{jk}, f_{jk} \mid (j, k) \in I^{im}}.$$

These generators are subject to the following defining relations ([Jur1], [Jur2]):

$$[h_{1}, h_{2}] = 0,$$

$$[h_{1}, e_{-1}] = e_{-1},$$

$$[h_{1}, e_{jk}] = e_{jk},$$

$$[h_{1}, f_{-1}] = -f_{-1},$$

$$[h_{2}, e_{jk}] = je_{jk},$$

$$[h_{2}, e_{jk}] = je_{jk},$$

$$[h_{2}, f_{-1}] = f_{-1},$$

$$[h_{2}$$

for all $(j,k),\ (p,q)\in I^{\mathrm{im}}.$ The Cartan subalgebra of \mathfrak{m} is $\mathfrak{h}:=\mathfrak{h}_A/\mathfrak{z}=\mathbb{C}h_1\oplus\mathbb{C}h_2\subseteq\mathfrak{m}.$ Note that A has rank 2.

Now e_{-1} (resp. f_{-1}) is the positive (respectively negative) simple real root vector, and the generators e_{jk} (resp. f_{jk}) are the positive (respectively negative) simple imaginary root vectors.

We define the extended index set

$$E = \{(\ell, j, k) \mid (j, k) \in I^{\mathrm{im}}, \ 0 \leqslant \ell < j\} = \{(\ell, j, k) \mid j \in \mathbb{N}, \ 1 \leqslant k \leqslant c(j), \ 0 \leqslant \ell < j\}.$$

and set

$$e_{\ell,jk} := \frac{(\operatorname{ad} e_{-1})^{\ell} e_{jk}}{\ell!}$$
 and $f_{\ell,jk} := \frac{(\operatorname{ad} f_{-1})^{\ell} f_{jk}}{\ell!}$,

for $(\ell, j, k) \in E$.

The following non-trivial result gives an additional non-standard generating set which we will use for our group constructions.

Theorem 2.1. ([Jur1], [Jur2]) Let $\mathfrak{gl}_2(-1)$ be the subalgebra of \mathfrak{m} with basis $\{e_{-1}, f_{-1}, h_1, h_2\}$. Then

$$\mathfrak{m} = \mathfrak{u}^- \oplus \mathfrak{gl}_2(-1) \oplus \mathfrak{u}^+$$

where $\mathfrak{gl}_2(-1) := \langle e_{-1}, f_{-1}, h_1, h_2 \rangle \cong \mathfrak{gl}_2$, \mathfrak{u}^+ is a subalgebra freely generated by $\{e_{\ell,jk} \mid (\ell,j,k) \in E\}$ and \mathfrak{u}^- is a subalgebra freely generated by $\{f_{\ell,jk} \mid (\ell,j,k) \in E\}$.

3. exp and ad for infinite dimensional Lie algebras

For a finite dimensional semisimple Lie algebra \mathfrak{g} , the simple root vectors e_i and f_i act nilpotently on the adjoint representation, that is, e_i and f_i are ad-nilpotent. Thus there are automorphisms of the form

$$\exp(u \operatorname{ad}(x))(y) = y + u[x, y] + \frac{u^2}{2!}[x, [x, y]] + \cdots$$

For $x = e_i$ or $x = f_i$, these are finite sums, thus are well defined automorphisms of \mathfrak{g} .

For a Kac–Moody algebra \mathfrak{g} , the simple root vectors e_i and f_i are real and hence they act *locally nilpotently* on \mathfrak{g} . That is, for all $y \in \mathfrak{g}$, we have $(\operatorname{ad}(e_i))^n(y) = 0$ and $(\operatorname{ad}(f_i))^m(y) = 0$ for some $n, m \gg 0$. This means that $\exp(u \operatorname{ad}(e_i))$ and $\exp(v \operatorname{ad}(f_i))$, $u, v \in \mathbb{C}$, are summable. Recall that an infinite sum $\sum_n E_n$ of operators is called *summable* if $\sum_n E_n(y)$ reduces to a finite sum for all y ([LL]). Again this means that $\exp(u \operatorname{ad}(e_i))$ and $\exp(v \operatorname{ad}(f_i))$ can be viewed as elements of $\operatorname{Aut}(\mathfrak{g})$. Replacing ad with a representation on an integrable highest weight module V, a similar method gives an analog of a simply connected Chevalley group in the Kac–Moody case ([CG], [CLM]).

For the Monster Lie algebra m, the real simple root vectors e_{-1} and f_{-1} act locally nilpotently on the full adjoint representation. However, the imaginary simple root vectors e_{jk} and f_{jk} do not necessarily act locally adnilpotently, and so this approach no longer works for the Monster Lie algebra (or other Borcherds algebras with imaginary simple roots). The same is true when we replace the adjoint representation by faithful highest weight modules, including generalized Verma modules.

4. Completion $\widehat{\mathfrak{m}}$ of \mathfrak{m} and pro-summability

The algebra m has the usual triangular decomposition

$$\mathfrak{m}=\mathfrak{n}^-\oplus\mathfrak{h}\oplus\mathfrak{n}^+$$

where $\mathfrak{n}^{\pm}=\bigoplus_{\alpha\in\Delta^{\pm}}\mathfrak{m}_{\alpha}$, \mathfrak{h} is the Cartan subalgebra, Δ^{\pm} are the sets of positive (respectively negative) roots and \mathfrak{m}_{α} are the root spaces. Let $\Delta=\Delta^{+}\sqcup\Delta^{-}$ and let Q denote the root lattice, that is, the \mathbb{Z} -span of Δ .

The algebra \mathfrak{m} is Q-graded. We can define a \mathbb{Z} -grading on \mathfrak{m} by defining a map

$$\lambda: Q \to \mathbb{Z},$$

 $(m,n) \mapsto m+n.$

This gives compatible \mathbb{Z} -grading whose homogeneous components are of the form

$$\mathfrak{m}_k = \bigoplus_{\alpha \in \lambda^{-1}(k) \cap \Delta} \mathfrak{m}_\alpha,$$

for $k \neq 0$, and $\mathfrak{m}_0 = \mathfrak{gl}_2(-1)$. The set $\Delta_k := \lambda^{-1}(k) \cap \Delta$ is finite, which ensures that \mathfrak{m}_k is finite dimensional for each $k \in \mathbb{Z}$. For i > 0, let

$$\mathfrak{n}_i = \bigoplus_{\substack{i' \geqslant i \\ 4}} \mathfrak{m}_{i'}.$$

We have a descending chain of ideals ([CJM])

$$\mathfrak{n}^+ = \mathfrak{n}_0 \geqslant \mathfrak{n}_1 \geqslant \cdots \geqslant \mathfrak{n}_i \geqslant \cdots$$

with the property that $\mathfrak{n}^+/\mathfrak{n}_i$ is a finite dimensional nilpotent Lie algebra for each $i \ge 0$. Thus \mathfrak{n}^+ is *pro-nilpotent*. As in [Ku], Chapter IV, Section 4, we define the *(positive) formal completion* of \mathfrak{m} to be

$$\widehat{\mathfrak{m}} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \widehat{\mathfrak{n}}^+$$

where

$$\widehat{\mathfrak{n}}^+ := \prod_{\alpha \in \Delta^+} \mathfrak{m}_\alpha = \prod_{i \in \mathbb{N}} \mathfrak{m}_i$$

is the pro-nilpotent completion of n^+ . Let

$$\widehat{\mathfrak{n}}_i = \prod_{i' \geqslant i} \mathfrak{m}_{i'}.$$

Then

$$\widehat{\mathfrak{n}}^+ \cong \varprojlim_{i \geq 1} \widehat{\mathfrak{n}}^+ / \widehat{\mathfrak{n}}_i.$$

Let $e_{\ell,jk}$ be the imaginary root vector as above. Then $\exp(u \operatorname{ad}(e_{\ell,jk}))(y)$ is generally an infinite sum for $u \in \mathbb{C}$, $y \in \mathfrak{m}$. However, we call $\exp(u \operatorname{ad}(e_{\ell,jk}))$ pro-summable since $\exp(u \operatorname{ad}(e_{\ell,jk}))(y)$ reduces to a finite sum for all y in

$$\mathfrak{n}^- \oplus \mathfrak{h} \oplus (\widehat{\mathfrak{n}}^+/\widehat{\mathfrak{n}}_i)$$

and for all i > 0 ([CJM]). By taking the inverse limit, $\exp(u \operatorname{ad}(e_{\ell,jk}))$ is a well-defined automorphism of $\widehat{\mathfrak{m}}$.

Note, however, that $\exp(u \operatorname{ad}(e_{\ell,jk}))$ is *not* a well-defined automorphism of \mathfrak{m} . Nor can $\exp(u \operatorname{ad}(f_{\ell,jk}))$ be defined as an automorphism of \mathfrak{m} or $\widehat{\mathfrak{m}}$, although it can be defined as an automorphism of the corresponding *negative* formal completion of \mathfrak{m} , defined analogously to $\widehat{\mathfrak{m}}$ as above.

5. The complete pro-unipotent group

The Lie algebra $\widehat{\mathfrak{m}}$ has an induced topology as a subset of $\mathfrak{X} := \prod_{k \in \mathbb{Z}} \mathfrak{m}_k$ with the product topology. The group $\operatorname{Aut}(\widehat{\mathfrak{m}}) \subseteq \operatorname{End}(\mathfrak{X})$ inherits a natural topology from $\widehat{\mathfrak{m}}$ as a subspace of the space $\operatorname{End}(\mathfrak{X})$ of linear maps with the pointwise topology.

For $y \in \widehat{\mathfrak{m}}$, write the \mathfrak{m}_k component of y as y_k and let N be the smallest integer with $y_N \neq 0$. Then $y \in \widehat{\mathfrak{m}}$ has the expression

$$y = \sum_{k=N}^{\infty} y_k.$$

For $k \in \mathbb{Z}$, set

$$\widehat{\mathfrak{m}}_k := \begin{cases} \widehat{\mathfrak{n}}_k & \text{if } k \geqslant 1 \\ \bigoplus_{k \leqslant j \leqslant 0} \mathfrak{m}_j \oplus \widehat{\mathfrak{n}}^+ & \text{if } k \leqslant 0. \end{cases}$$

where $\mathfrak{m}_0 = \mathfrak{gl}_2(-1)$. For $n \in \mathbb{N}$, we define

$$\widehat{U}_n := \{ \varphi \in \operatorname{Aut}(\widehat{\mathfrak{m}}) \mid \varphi(y) \in \mathcal{Y} + \widehat{\mathfrak{m}}_{k+n} \text{ whenever } y \in \mathfrak{m}_k, \text{ for some } k \in \mathbb{Z} \}$$

and let

$$\widehat{U} := U_{-1}\widehat{U}_1,$$

where $U_{-1} = \{\exp(t \operatorname{ad}(e_{-1})) \mid t \in \mathbb{C}^{\times}\}$ is the root group corresponding to the real root α_{-1} .

Then \widehat{U} is a closed pro-unipotent subgroup of $\operatorname{Aut}(\widehat{\mathfrak{m}})$ with $\widehat{U} \geqslant \widehat{U}_1 \geqslant \widehat{U}_2 \geqslant \cdots$ and each $\widehat{U}/\widehat{U}_i$ is a finite dimensional unipotent algebraic group of automorphisms of $\widehat{\mathfrak{n}}/\widehat{\mathfrak{n}}_i$.

Every element $g \in \hat{U}$ can be shown to have the form

$$g = \prod_{i=1}^{\infty} \exp(\operatorname{ad}(x_i))$$

for some $x_i \in \widehat{\mathfrak{n}}_i$ ([CJM]). Any element $y \in \widehat{\mathfrak{m}}$ is an infinite (formal) sum but, for $g \in \widehat{U}$, $g \cdot y$ is finite when y is restricted to $\mathfrak{n}^- \oplus \mathfrak{h} \oplus (\widehat{\mathfrak{n}}^+/\widehat{\mathfrak{n}}_i)$ for all i > 0. Thus every element $g \in \widehat{U}$ is pro-summable when expanded as an infinite sum of endomorphisms.

In particular, \hat{U} is generated (as a topological group) by the summable series

$$\exp(u \operatorname{ad}(e_{-1})) = 1 + u \operatorname{ad}(e_{-1}) + \frac{u^2}{2!} \operatorname{ad}(e_{-1})^2 + \cdots$$

and the pro-summable series

$$\exp(u \operatorname{ad}(e_{jk})) = 1 + u \operatorname{ad}(e_{jk}) + \frac{u^2}{2!} \operatorname{ad}(e_{jk})^2 + \cdots,$$

for $(j, k) \in I^{im}$, all of which are power series in u with constant term 1.

The group \hat{U} is the analog of a completion of the unipotent subgroup of the adjoint form of a Chevalley or Kac–Moody group (as in [CLM]). For Kac–Moody algebras, similar complete pro-unipotent groups have been constructed by Kumar ([Ku]) and Rousseau ([Rou]).

Our construction gives rise to the following analog of the group adjoint representation. Let

$$\operatorname{Ad}_{\widehat{U}/\widehat{U}_i}:\widehat{U}/\widehat{U}_i\to\operatorname{Aut}(\widehat{\mathfrak{n}}/\widehat{\mathfrak{n}}_i)$$

be the adjoint representation of the finite dimensional linear algebraic group \hat{U}/\hat{U}_i over $\mathbb C$ and let

$$\operatorname{Exp}_i: \widehat{\mathfrak{n}}/\widehat{\mathfrak{n}}_i \to \widehat{U}/\widehat{U}_i$$

be the exponential map. Taking inverse limits allows us to define Ad and Exp for \widehat{U} and $\widehat{\mathfrak{n}}^+$ (as in Section 4.4.25 of [Ku]).

For $x \in \widehat{\mathfrak{n}}$ and $g \in \widehat{U}$ we have

$$\operatorname{Exp}(\operatorname{Ad}(g))(x) = g\operatorname{Exp}(x)g^{-1}$$

where $\operatorname{Exp}:\widehat{\mathfrak{n}}^+ \to \widehat{U}$ is the unique map making the following diagram commute

$$\hat{\mathfrak{n}}^{+} \longrightarrow \hat{\mathfrak{n}}/\hat{\mathfrak{n}}_{i}$$

$$\text{Exp} \qquad \qquad \downarrow \text{Exp}_{i}.$$

$$\hat{U} \longrightarrow \hat{U}/\hat{U}_{i}$$

6. Generators and relations

In the finite dimensional case, Steinberg ([St]) gave a defining presentation for (adjoint and simply connected) Chevalley groups over commutative rings. Tits ([Ti]) gave generators and relations for Kac–Moody groups, generalizing the Steinberg presentation. A suitable generalization of these methods gives us a group $G(\mathfrak{m})$ for \mathfrak{m} in terms of generators and relations.

To motivate our method for constructing the group $G(\mathfrak{m})$, let G be an adjoint Kac–Moody group corresponding to a symmetrizable Kac–Moody algebra \mathfrak{g} . Then all elements of G can be constructed as automorphisms of \mathfrak{g} .

As we have seen, it is not possible construct all the analogous group elements as automorphisms of \mathfrak{m} . For example, we may construct automorphisms of $\widehat{\mathfrak{m}}$ corresponding to positive roots of \mathfrak{m} but not negative roots (as in the construction of \widehat{U}). We may also consider groups $\mathrm{GL}_2(-1)$ and $\mathrm{GL}_2(\ell,j,k)$ (defined below) which are automorphisms of \mathfrak{gl}_2 subalgebras of \mathfrak{m} , corresponding to the real root α_{-1} and imaginary roots $\alpha_{\ell,jk}$ respectively. Using pairwise intersections of these subalgebras allows us to construct $G(\mathfrak{m})$ as an amalgam of all these groups. This gives rise to the following presentation. Define the set of symbols

$$\mathcal{X} = \{ H_1(s), H_2(s), X_{-1}(u), Y_{-1}(u), X_{\ell, ik}(u), Y_{\ell, ik}(u) \mid s \in \mathbb{C}^{\times}, u \in \mathbb{C}, (\ell, j, k) \in E \}.$$

Define the constant

$$c_{\ell j} := (-1)^{\ell+1} {j-1 \choose \ell} (\ell+1)(j-\ell)$$

and additional symbols:

$$\widetilde{w}_{-1}(s) := X_{-1}(s)Y_{-1}(-s^{-1})X_{-1}(s), \qquad \widetilde{w}_{-1} := \widetilde{w}_{-1}(1),$$

$$\widetilde{w}_{\ell,jk}(s) := X_{\ell,jk}(s)Y_{\ell,jk}\left(\frac{-s^{-1}}{c_{\ell j}}\right)X_{\ell,jk}(s), \qquad \widetilde{w}_{\ell,jk} := \widetilde{w}_{\ell,jk}(1).$$

Let $(g,h) = ghg^{-1}h^{-1}$ denote the usual group commutator. We define the set of relations \mathcal{R} as follows, for all $s,t\in\mathbb{C}^\times$, $u,v\in\mathbb{C}$, $(\ell,j,k),(m,p,q)\in E$.

Relations in the $GL_2(\mathbb{C})$ -subgroup associated with the real simple root:

$$\begin{split} X_{-1}(u)X_{-1}(v) &= X_{-1}(u+v), \\ Y_{-1}(u)Y_{-1}(v) &= Y_{-1}(u+v), \\ H_{1}(s)H_{1}(t) &= H_{1}(st), \\ H_{2}(s)H_{2}(t) &= H_{2}(st), \\ H_{1}(s)H_{2}(t) &= H_{2}(t)H_{1}(s), \\ \widetilde{w}_{-1}X_{-1}(u)\widetilde{w}_{-1}^{-1} &= Y_{-1}(-u), \\ \widetilde{w}_{-1}Y_{-1}(u)\widetilde{w}_{-1}^{-1} &= X_{-1}(-u), \\ Y_{-1}(-t)X_{-1}(s)Y_{-1}(t) &= X_{-1}(-t^{-1})Y_{-1}(-t^{2}s)X_{-1}(t^{-1}), \\ \widetilde{w}_{-1}(s)\widetilde{w}_{-1} &= H_{1}(-s)H_{2}(-s^{-1}), \\ \widetilde{w}_{-1}H_{1}(s)\widetilde{w}_{-1}^{-1} &= H_{2}(s), \\ \widetilde{w}_{-1}H_{2}(s)\widetilde{w}_{-1}^{-1} &= H_{1}(s), \\ H_{1}(s)X_{-1}(u)H_{1}(s)^{-1} &= X_{-1}(su), \\ H_{2}(s)X_{-1}(u)H_{2}(s)^{-1} &= X_{-1}(s^{-1}u), \\ H_{1}(s)Y_{-1}(u)H_{1}(s)^{-1} &= Y_{-1}(s^{-1}u), \\ H_{2}(s)Y_{-1}(u)H_{2}(s)^{-1} &= Y_{-1}(su), \end{split}$$

Relations between generators $X_{-1}(u), Y_{-1}(u), X_{\ell,jk}(u), Y_{\ell,jk}(u)$:

$$\begin{split} (X_{\ell,jk}(u),Y_{m,pq}(v)) &= 1 \qquad \text{for } j \neq p, \, k \neq q, \, \text{or } |\ell-m| > 1, \\ X_{\ell,jk}(u+v) &= X_{\ell,jk}(u)X_{\ell,jk}(v), \\ Y_{\ell,jk}(u+v) &= Y_{\ell,jk}(u)Y_{\ell,jk}(v), \\ (X_{-1}(s),X_{j-1,jk}(t)) &= 1, \\ (Y_{-1}(s),X_{0,jk}(t)) &= 1, \\ (X_{-1}(s),Y_{0,jk}(t)) &= 1, \\ (Y_{-1}(s),Y_{j-1,jk}(t)) &= 1. \end{split}$$

Action of the element \widetilde{w}_{-1} corresponding to the real root α_{-1} :

$$\widetilde{w}_{-1}X_{\ell,jk}(u)\widetilde{w}_{-1}^{-1} = X_{j-1-\ell,jk}((-1)^{j-\ell-1}u),$$

$$\widetilde{w}_{-1}Y_{\ell,jk}(u)\widetilde{w}_{-1}^{-1} = Y_{j-1-\ell,jk}((-1)^{j-\ell-1}u),$$

Relations in the $GL_2(\mathbb{C})$ -subgroup associated with imaginary roots:

$$H_{1}(s)X_{\ell,jk}(u)H_{1}(s)^{-1} = X_{\ell,jk}(s^{\ell+1}u),$$

$$H_{1}(s)Y_{\ell,jk}(u)H_{1}(s)^{-1} = Y_{\ell,jk}(s^{-(\ell+1)}u),$$

$$H_{2}(s)X_{\ell,jk}(u)H_{2}(s)^{-1} = X_{\ell,jk}(s^{j-\ell}u),$$

$$H_{2}(s)Y_{\ell,jk}(u)H_{2}(s)^{-1} = Y_{\ell,jk}(s^{-(j-\ell)}u),$$

$$\tilde{w}_{\ell,jk}(s)\tilde{w}_{\ell,jk} = H_{1}\left((-s)^{1/(\ell+1)}\right)H_{2}\left((-s)^{1/(j-\ell)}\right),$$

$$Y_{\ell,jk}(-t)X_{\ell,jk}(s)Y_{\ell,jk}(t) = X_{\ell,jk}\left(\frac{-t^{-1}}{c_{\ell j}}\right)Y_{\ell,jk}\left(-c_{\ell j}t^{2}s\right)X_{\ell,jk}\left(\frac{t^{-1}}{c_{\ell j}}\right),$$

$$\tilde{w}_{\ell,jk}H_{1}(s)\tilde{w}_{\ell,jk}^{-1} = H_{2}\left(s^{-(\ell+1)/(j-\ell)}\right),$$

$$\tilde{w}_{\ell,jk}H_{2}(s)\tilde{w}_{\ell,jk}^{-1} = H_{1}\left(s^{-(j-\ell)/(\ell+1)}\right),$$

$$\tilde{w}_{\ell,jk}X_{\ell,jk}(u)\tilde{w}_{\ell,jk}^{-1} = Y_{\ell,jk}\left(\frac{-u}{c_{\ell j}}\right),$$

$$\tilde{w}_{\ell,jk}Y_{\ell,jk}(u)\tilde{w}_{\ell,jk}^{-1} = X_{\ell,jk}\left(-c_{\ell j}u\right),$$

$$s) = X_{\ell,jk}\left(\frac{s^{-1}}{c_{\ell,i}}\right)H_{1}\left([-c_{\ell j}s]^{-1/(\ell+1)}\right)H_{2}\left([-c_{\ell j}s]^{-1/(j-\ell)}\right)\tilde{w}_{\ell,jk}X_{\ell,jk}\left(\frac{s^{-1}}{c_{\ell j}}\right)$$

$$Y_{\ell,jk}(s) = X_{\ell,jk} \left(\frac{s^{-1}}{c_{\ell j}} \right) H_1 \left([-c_{\ell j} s]^{-1/(\ell+1)} \right) H_2 \left([-c_{\ell j} s]^{-1/(j-\ell)} \right) \widetilde{w}_{\ell,jk} X_{\ell,jk} \left(\frac{s^{-1}}{c_{\ell j}} \right).$$

We now define $G(\mathfrak{m})$ as the group given by this presentation, that is,

$$G(\mathfrak{m}) = \langle \mathcal{X} \mid \mathcal{R} \rangle = F(\mathcal{X})/N_{\mathcal{R}}$$

where $F(\mathcal{X})$ denotes the free group on \mathcal{X} and $N_{\mathcal{R}}$ denotes the normal closure of the relations \mathcal{R} . We define the following subgroups of $G(\mathfrak{m})$:

$$U^{+}(\mathbb{C}) = \langle X_{-1}(u), X_{\ell,jk}(u) \mid u \in \mathbb{C}, (\ell, j, k) \in E \rangle,$$

$$U^{+}_{\mathrm{im}}(\mathbb{C}) = \langle X_{\ell,jk}(u) \mid u \in \mathbb{C}, (\ell, j, k) \in E \rangle,$$

$$U^{+}_{\mathrm{im}}(\mathbb{Z}) = \langle X_{\ell,jk}(1) \mid (\ell, j, k) \in E \rangle,$$

$$H = \langle H_{1}(s), H_{2}(s) \mid s \in \mathbb{C}^{\times} \rangle.$$

We call H the *toral* subgroup of $G(\mathfrak{m})$.

Proposition 6.1. ([CJM]) The subgroup $U^+_{im}(\mathbb{C})$ is a free product of additive abelian groups isomorphic to \mathbb{C} , indexed over E. The subgroup $U^+_{im}(\mathbb{Z})$ is a countably-generated free group. For fixed (ℓ, j, k) , the group $\langle X_{\ell, jk}(u) \mid a \in \mathbb{Z} \rangle \cong \mathbb{Z}$ is an infinite cyclic subgroup.

The following theorem gives the relationship between the groups $G(\mathfrak{m})$ and \widehat{U} .

Theorem 6.2. The map

$$X_{\ell,jk}(u) \mapsto \exp(u \operatorname{ad}(e_{\ell,jk}))$$

embeds $U_{\mathrm{im}}^+(\mathbb{C})$ as a dense subgroup of \widehat{U} .

Theorem 6.2 shows that an element of the subgroup $U^+_{\mathrm{im}}(\mathbb{C})$ of $G(\mathfrak{m})$ can be identified with an automorphism of $\widehat{\mathfrak{m}}$, and every automorphism in \widehat{U} can be approximated by elements of $U^+_{\mathrm{im}}(\mathbb{C})$.

Our group $G(\mathfrak{m})$ does not act as an automorphism group of \mathfrak{m} . However, the action of the $\mathfrak{gl}_2(-1)$ subalgebra on \mathfrak{m} is locally nilpotent, and thus elements of the $GL_2(\mathbb{C})$ subgroup

$$GL_2(-1) = \langle X_{-1}(u), Y_{-1}(u), H_1(s), H_2(s) \mid u \in \mathbb{C}, s \in \mathbb{C}^{\times} \rangle$$

can be identified with automorphisms of m. The groups

$$\operatorname{GL}_2(\ell,j,k) = \langle X_{\ell,jk}(u), Y_{\ell,jk}(u), H_1(s), H_2(s) \mid u \in \mathbb{C}, s \in \mathbb{C}^{\times}, (\ell,j,k) \in E \rangle$$

act as automorphisms on subalgebras $\mathfrak{gl}_2(\ell, j, k)$ but not on all of \mathfrak{m} .

7. Analog of a simply connected Kac–Moody Chevalley group for m

An analog of a simply connected Chevalley group can be constructed for Kac–Moody algebras $\mathfrak g$ over commutative rings ([CG], [CLM], [Rou], [Ti]). Constructing these groups requires a significant amount of additional data such as a $\mathbb Z$ -form of the universal enveloping algebra of $\mathfrak g$, as well as an integrable highest weight representation of $\mathfrak g$.

To construct an analog of a simply connected Kac-Moody Chevalley group for m would require the use of an integrable highest weight representation of m. Recall that the simple imaginary root vectors which generate m are not locally nilpotent on non-trivial highest weight m-modules.

In this section, we outline the construction of a group ([CGJM]) associated to the parabolic subalgebra $\mathfrak{gl}_2(-1)\oplus\mathfrak{u}^+$ of \mathfrak{m} which acts locally nilpotently on a certain tensor algebra $T(\mathcal{V})$, constructed in [JLW]. The module $T(\mathcal{V})$ is analogous to standard irreducible modules for semisimple and Kac–Moody algebras.

Using the notation of Section 2, for j > 0, we define

$$\mathcal{V}_{j}^{-} = \coprod_{1 \leq k \leq c(j)} \mathcal{U}(\mathbb{C}f_{-1}) \cdot f_{jk}$$
$$\mathcal{V}_{j}^{+} = \coprod_{1 \leq k \leq c(j)} \mathcal{U}(\mathbb{C}e_{-1}) \cdot e_{jk},$$

where \mathcal{U} denotes the universal enveloping algebra. Set

$$\mathcal{V}^- = \coprod_{j>0} \mathcal{V}_j^-, \qquad \mathcal{V}^+ = \coprod_{j>0} \mathcal{V}_j^+.$$

Then we have $\mathfrak{u}^- = L(\mathcal{V}^-)$, $\mathfrak{u}^+ = L(\mathcal{V}^+)$, where L(V) denotes the free Lie algebra on a vector space V. Let V denote V^- . Let T(V) denote the tensor algebra of V.

In [JLW], the authors showed that the Lie algebra \mathfrak{m} can be realized as a Lie algebra of operators on the irreducible \mathfrak{m} -module $T(\mathcal{V})$, identified with a generalized Verma module induced from a one dimensional \mathfrak{gl}_2 -representation $L(\lambda)$ for any $\lambda \in (\mathfrak{h}_A)^*$ satisfying

(1)
$$\lambda(h_{-1}) = 0, \quad \lambda(h_{ik}) = aj + b$$

for each j > 0, $1 \le k \le c(j)$, for some fixed real numbers a, b such that a > 0 and a + b > 0.

This representation has the property that the parabolic subalgebra $\mathfrak{gl}_2(-1) \oplus \mathfrak{u}^+$ of \mathfrak{m} is locally nilpotent on the \mathfrak{m} -module $T(\mathcal{V})$ ([JLW]).

The module $T(\mathcal{V})$ has a filtration

$$\mathbb{C} = T_0(\mathcal{V}) \subseteq T_1(\mathcal{V}) \subseteq T_2(\mathcal{V}) \subseteq \cdots \subseteq T_n(\mathcal{V}) \subseteq \cdots \subseteq T(\mathcal{V}),$$

where $T_n(\mathcal{V}) = T\left(\coprod_{j \leq n} \mathcal{V}_j\right)$, that is invariant under the action of $\mathfrak{gl}_2(-1) \oplus \mathfrak{u}^+$. In particular, every element of $T(\mathcal{V})$ lies in a proper $\mathfrak{gl}_2(-1) \oplus \mathfrak{u}^+$ -submodule that is a finitely generated tensor algebra.

Combining the degree grading on the tensor algebra T(V) of any vector space V given by

$$T(V) = \bigoplus_{s \geqslant 0} T^{(s)}(V) \text{ where}$$

$$T^{(s)}(V) = \operatorname{span}_{\mathbb{C}} \{ v_1 \otimes v_2 \otimes \cdots \otimes v_s \mid v_1, \dots, v_s \in V \},$$

with the filtration on $T(\mathcal{V})$, for any $n \ge 1$ we get the grading

$$T_n(\mathcal{V}) = \bigoplus_{s \geqslant 0} T_n^{(s)}(\mathcal{V})$$
 where $T_n^{(s)}(\mathcal{V}) = T_n(\mathcal{V}) \cap T^{(s)}(\mathcal{V}).$

The subspaces $T_n^{(s)}(\mathcal{V})$ are finite dimensional. The associated filtration on $T_n(\mathcal{V})$ is

$$T_n^{(\leqslant 0)}(\mathcal{V}) \subseteq T_n^{(\leqslant 1)}(\mathcal{V}) \subseteq \cdots \subseteq T_n^{(\leqslant s)}(\mathcal{V}) \subseteq \cdots \subseteq T_n(\mathcal{V}), \text{ where}$$

$$T_n^{(\leqslant s)}(\mathcal{V}) := \coprod_{i=0}^s T_n^{(i)}(\mathcal{V}).$$

We let $\rho_{n,s} = \rho|_{T_n^{(\leqslant s)}(\mathcal{V})}$. For n = 0, define $T_0^{(0)}(\mathcal{V}) := \mathbb{C}$ and $T_0^{(i)}(\mathcal{V}) := 0$ for i > 0.

Proposition 7.1. For any $n, s \ge 0$, the subspace $T_n^{(\leqslant s)}(\mathcal{V})$ is a $\mathfrak{gl}_2(-1) \oplus \mathfrak{u}^+$ -submodule, such that for $s \ge 1$,

$$\rho_{n,s}(\mathfrak{u}^+)T_n^{(s)}(\mathcal{V}) \subseteq T_n^{(s-1)}(\mathcal{V}),$$

$$\rho_{n,s}(\mathfrak{gl}_2(-1))T_n^{(s)}(\mathcal{V}) \subseteq T_n^{(s)}(\mathcal{V}).$$

In particular, $\rho_{n,s}(\mathfrak{u}^+)$ is a finite dimensional nilpotent Lie algebra and $\rho_{n,s}(\mathfrak{gl}_2(-1))$ is finite dimensional.

Choosing λ subject to the conditions (1) as above, for all $n, s \ge 0$, we may define a family of groups $G_{n,s}^{\lambda}(\mathfrak{gl}_2(-1) \oplus \mathfrak{u}^+)$ associated to $\rho_{n,s}(\mathfrak{gl}_2(-1) \oplus \mathfrak{u}^+)$ as follows ([CGJM])

$$G_{n,s}^{\lambda}(\mathfrak{gl}_2(-1) \oplus \mathfrak{u}^+) = \langle \exp(c\rho_{n,s}(e_{jk})), \exp(c\rho_{n,s}(e_{-1})), \exp(c\rho_{n,s}(f_{-1})), \exp(t\rho_{n,s}(h_{jk})) \exp(t\rho_{n,s}(h_1)) \mid (j,k) \in I, \ c \in \mathbb{C}, \ t \in \mathbb{C}^{\times} \rangle.$$

Since each $\rho_{n,s}(\mathfrak{u}^+)$ is finite dimensional, $G_{n,s}^{\lambda}(\mathfrak{gl}_2(-1)\oplus\mathfrak{u}^+)$ is a well defined subgroup of $\operatorname{Aut}(T_n^{\leqslant s}(\mathcal{V}))$. This is an analog of the parabolic subgroup of a simply connected Chevalley group associated to the subalgebra $\mathfrak{gl}_2(-1)\oplus\mathfrak{u}^+$ of \mathfrak{m} .

We define subgroups of $G_{n,s}^{\lambda}(\mathfrak{gl}_2(-1)\oplus\mathfrak{u}^+)$ as follows. Let $\mathfrak{n}^+=\mathfrak{u}^+\oplus\mathbb{C} e_{-1}$ as in Section 2 and define

$$\begin{split} U_{n,s}^{\lambda}(\mathfrak{n}^{+}) &= \langle \exp(c\rho_{n,s}(e_{jk})), \exp(c\rho_{n,s}(e_{-1})) \mid (j,k) \in I, \ c \in \mathbb{C} \rangle, \\ U_{n,s}^{\lambda}(\mathfrak{u}^{+}) &= \langle \exp(c\rho_{n,s}(e_{jk})) \mid (j,k) \in I, \ c \in \mathbb{C} \rangle, \\ H_{n,s}^{\lambda} &= \langle \exp(t\rho_{n,s}(h_{1})), \exp(t\rho_{n,s}(h_{2})) \mid t \in \mathbb{C}^{\times} \rangle, \\ L_{n,s}^{\lambda} &= \langle \exp(c\rho_{n,s}(e_{-1})), \exp(c\rho_{n,s}(f_{-1})), \exp(t\rho_{n,s}(h_{1})), \exp(t\rho_{n,s}(h_{2})) \mid c \in \mathbb{C}, \ t \in \mathbb{C}^{\times} \rangle. \end{split}$$

Theorem 7.2. ([CGJM]) For λ subject to conditions (1) and for all $n, k \ge 0$ we have the following.

- The groups $U_{n,s}^{\lambda}(\mathfrak{n}^+)$ are pro-unipotent.
- The groups $U_{n,s}^{\lambda}(\mathfrak{u}^+)$ are free products of additive groups isomorphic to $(\mathbb{C},+)$ and are pro-unipotent.
- The groups $H_{n,s}^{\lambda}$ are isomorphic to the toral subgroup of $GL_2(\mathbb{C})$.
- The groups $L_{n,s}^{\lambda}$ are isomorphic to $\mathrm{GL}_2(\mathbb{C})$.
- $G_{n,s}^{\lambda}(\mathfrak{gl}_2(-1) \oplus \mathfrak{u}^+) = L_{n,s}^{\lambda} \ltimes U_{n,s}^{\lambda}(\mathfrak{n}^+).$

8. Conflicts of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

9. Data availability

This manuscript has no associated data.

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