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# CONSTRUCTING BEZOUT DOMAINS

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**Introduction.** Techniques for constructing Bezout domains are described and recent examples are given.

If $R$ is an integral domain, $Q$ the quotient field of $R$, and $U$ the units of $R$, then $(Q - \{0\})/U$ is a partially ordered group called the divisibility group of $R$. The emphasis is on constructing a desirable divisibility group and then the integral domain from this ordered Abelian group. Given a totally ordered Abelian group, W. Krull used a group algebra to construct a valuation ring with that divisibility group (also known as the value group). I. Kaplansky and P. Jaffard generalized this so that given a lattice ordered Abelian group there exists an integral domain with that divisibility group. J. Ohm showed that this integral domain is a Bezout domain, and he popularized the use of this construction for generating examples. Similarly, given a totally ordered Abelian group there exists a long power series ring with that divisibility group, and this long power series ring is a maximally complete valuation ring which can be used to generate examples.

In section one the preliminaries are given. This includes the development of the Krull-Kaplansky-Jaffard-Ohm construction of Bezout domains from lattice ordered Abelian groups and the transferring of the properties between a Bezout domain and its divisibility group. In section two there is given a brief discussion of long power series rings and how they can be used to generate examples. In section three many examples of Bezout domains obtained by the Krull-Kaplansky-Jaffard-Ohm construction appear. Finally in section four related approaches to the subject are discussed, and suggestions for future study are indicated.

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1. **Definitions and basic results.** All groups will be assumed to be Abelian. $R$ will always denote an integral domain, $Q$ the field of fractions of $R$, and $U$ the units of $R$. $A^*$ will denote the non-zero elements of $A$, whenever $A$ has an additive algebraic structure. $Z$ will denote the additive group of integers with the usual ordering and $N = \{1, 2, 3, \cdots\}$ the natural numbers.

The first few definitions and results are well-known. For more details on ordered groups see [6] or [13], and for more details on valuation rings see [36], [1], [29], [6] or [13]. *$G$ is a partially ordered group* if $G$ is a group with a partial ordering $\leqq$ such that $g \leqq h$ implies $g + i \leqq h + i$ for all $g, h, i \in G$. If $G$ is a partially ordered group $G_+$ will denote $\{g \in G : g \geqq 0\}$. If $G$ and $G'$ are partially ordered groups and $f: G \to G'$, then $f$ is an *order homomorphism* if $f$ is a group homomorphism and $g \leqq h$ implies $f(g) \leqq f(h)$ for all $g, h \in G$. With the same notation $f$ is an *order isomorphism* if $f$ is a group isomorphism and both $f$ and $f^{-1}$ are order homomorphisms. If $G$ is a group with $G_1$ and $G_2$ subsets of $G$, then let $G_1 + G_2 = \{g_1 + g_2 \in G : g_1 \in G_1$ and $g_2 \in G_2\}$, and let $-G_1 = \{-g_1 \in G : g_1 \in G_1\}$. If $G$ is a partially ordered group, then (i) $0 \in G_+$, (ii) $G_+ + G_+ \subset G_+$, and (iii) $G_+ \cap (-G_+) = \{0\}$. Conversely, if $G$ is a group and $P$ is a subset of $G$ satisfying (i), (ii), and (iii) of the last statement, then $G$ is a partially ordered group with the ordering $\leqq$ given by $g \leqq h$ if $h - g \in P$.

Let $G = Q^*/U$ and define $aU \leqq bU$ for $a, b \in Q^*$ if $a^{-1}b \in R$. Then $G$ is a partially ordered group, called the *divisibility group of* $R$. If $G$ is the divisibility group of $R$, then $G_+ = R^*/U$. $G$ is a *totally ordered group,* or linearly ordered group, if $G$ is a partially ordered group whose partial ordering is a total ordering ($g \leqq h$ or $h \leqq g$ for all $g, h \in G$). $R$ is a valuation ring if $a \mid b$ or $b \mid a$ for all $a, b \in R$. If $G$ is a totally ordered group and $Q$ is a field, then $v : Q^* \to G$ is a *valuation* if (i) $v(xy) = v(x) + v(y)$ for all $x, y \in Q^*$, and (ii) $v(x + y) \geqq \inf(v(x), v(y))$ for all $x, y \in Q^*$ such that $x + y \neq 0$. It will always be assumed that if $v : Q^* \to G$ is a valuation, then $v$ is surjective, i.e., $v(Q^*) = G$. If $R$ is a valuation ring, then the canonical map of $Q^*$ onto the divisibility group of $R$ is a valuation. On the other hand, if $v : Q^* \to G$ is a valuation and we let $R = \{0\} \cup \{x \in Q^* : v(x) \geqq 0\}$, then $R$ is a valuation ring, $Q$ is the field of fractions of $R$, and $G$ is order isomorphic to the divisibility group of $R$. $R$ is a valuation ring if and only if the divisibility group of $R$ is a totally ordered group. Thus if $R$ is a valuation ring, the divisibility group of $R$ is the same as the value group of the corresponding valuation. If $G$ is a totally ordered group, then $G$ is a torsion-free group.

THEOREM 1.1 (W. KRULL [17], p. 164). *If $G$ is a totally ordered group, then there exists a valuation ring whose divisibility group is order isomorphic to $G$.*

PROOF. Let $k$ be a field and let $S$ be the group algebra $k[G]$. Let $Q$ be the field of fractions of $S$, and define $v : Q^* \to G$ by

$$v\left( \sum_{i=1}^{m} c_i X_{g_i} \Big/ \sum_{j=1}^{n} c_j' X_{g_j'} \right)$$
$$= \inf(g_i : i = 1, 2, \cdots, m) - \inf(g_j' : j = 1, 2, \cdots, n),$$

where it is assumed that $c_i \in k^*$ for all $i = 1, 2, \cdots, m$ and $g_i \neq g_j$ if $i \neq j$, and similarly for the expression in the denominator. $v$ is a valuation. If $R$ is the valuation ring associated to $v$ (see the last paragraph), then $R$ is the desired valuation ring.

Alternate proofs use $S = k[G_+]$ as in [1], p. 107, or $S = k[X_g : g \in G]$, where $X_g$ are indeterminants over $k$, as in [6], 18.5.

$G$ is a *lattice ordered group* if $G$ is a partially ordered group such that $\inf(g, h)$ and $\sup(g, h)$ exist in $G$ for all $g, h \in G$. If $G$ is a partially ordered group and $\sup(g, h)$ exist in $G$ for all $g, h \in G$, then $G$ is a lattice ordered group because $\inf(g, h) = -\sup(-g, -h)$. Similarly a partially ordered group in which infs always exist is a lattice ordered group. If $G$ is a lattice ordered group and $X$ is a subset of $G$, then $X$ is a *sublattice of* $G$ if $\inf_X(x, y) = \inf_G(x, y)$ and $\sup_X(x, y) = \sup_G(x, y)$ for all $x, y \in X$. If $G$ and $G'$ are lattice ordered groups and $f: G \to G'$, then $f$ is a *lattice homomorphism* if $f$ is a group homomorphism and $f(\inf(g, h)) = \inf(f(g), f(h))$ for all $g, h \in G$. Clearly such an $f$ will also satisfy $f(\sup(g, h)) = \sup(f(g), f(h))$ for all $g, h \in G$, $f$ is an order homomorphism, and $f(G)$ is a sublattice of $G'$. In an obvious manner one defines *lattice isomorphism, lattice embedding*, etc. If $G$ is a lattice ordered group, then $G$ is a torsion-free group [6], 15.7 or [13], Corollary p. 10.

Two types of orderings on the product of partially ordered groups appear often. First, if $G_\alpha$ is a partially ordered group for $\alpha \in \Gamma$, then the product $G = \prod G_\alpha$ can be ordered as follows: for $(x_\alpha), (y_\alpha) \in G$, then $(x_\alpha) \geqq (y_\alpha)$ if $x_\alpha \geqq y_\alpha$ for all $\alpha \in \Gamma$. This is called the *product ordering* on $G$ and makes $G$ a partially ordered group. If each of the $G_\alpha$ is a lattice ordered group, then the product is also a lattice ordered group. Secondly, suppose $G_\alpha$ is a partially ordered group for $\alpha \in \Gamma$ and suppose the index set $\Gamma$ is well ordered. Then $G = \prod_{\alpha \in \Gamma} G_\alpha$ can be ordered as follows: for $(x_\alpha), (y_\alpha) \in G$, then $(x_\alpha) \geqq (y_\alpha)$ if $(x_\alpha) = (y_\alpha)$ or

$\alpha_0 = \inf\{\alpha \in \Gamma : x_\alpha \neq y_\alpha\}$ implies $x_{\alpha_0} > y_{\alpha_0}$. This is called the *lexicographic ordering* on $G$ and makes $G$ a partially ordered group. If each of the $G_\alpha$ is a totally ordered group, then the product with the lexicographic ordering is also a totally ordered group. However, if each of the $G_\alpha$ is a lattice ordered group, then the product with the lexicographic ordering need not be a lattice ordered group. As a special case of the lexicographic ordering, take $Z^2 = Z \oplus Z$ with $(a, b) \geqq (c, d)$ if $a > c$ or ($a = c$ and $b \geqq d$). Any subgroup of a partially ordered group is a partially ordered group. Thus the product ordering on a direct sum of partially ordered groups is the ordering obtained by considering the direct sum as a subgroup of the product with the product ordering.

$R$ is a *Bezout domain* if every finitely generated ideal of $R$ is a principal ideal of $R$. Thus $R$ is a Bezout domain if and only if for all $a, b \in R$ there exists $g \in R$ such that $Rg = Ra + Rb$. In this case $g$ is a greatest common divisor of $a$ and $b$, so a Bezout domain is an integral domain in which such a *gcd* $g$ always exists and $g$ is an $R$-linear combination of $a$ and $b$.

It has been noticed that valuation rings are characterized by their divisibility group being a totally ordered group. Similarly $R$ is a unique factorization domain if and only if the divisibility group of $R$ is order isomorphic to a direct sum of copies of $Z$ with the product ordering. Also the divisibility group of $R$ is a lattice ordered group if and only if the intersection of two principal ideals of $R$ is a principal ideal of $R$, i.e., for all $a, b \in R$ there exists $\ell \in R$ such that $R\ell = Ra \cap Rb$. In this case $\ell$ is a least common multiple of $a$ and $b$, so these are integral domains in which lcm's always exist.

If $R$ is a Bezout domain, then the divisibility group of $R$ is a lattice ordered group. For the existence of *gcd*'s in $R$ corresponds to the existence of infs in the divisibility group of $R$, and a partially ordered group in which infs always exist in a lattice ordered group. Consequently, if $R$ is a Bezout domain, then the intersection of two principal ideals of $R$ is a principal ideal of $R$. The converse is not true. For if $R = k[X, Y]$, a polynomial ring in two variables over a field, then $R$ is not a Bezout domain, yet the intersection of two principal ideals of $R$ is a principal ideal of $R$.

The relationship between ideals of $R$ and subsets of the divisibility group of $R$ will now be discussed. Given a lattice ordered group $G$, $I$ is an *ideal of* $G$ if (i) $I \subset G_+{}^*$, (ii) $x \in I, y \in G$, and $y > x$ implies $y \in I$, and (iii) $x, y \in I$ implies $\inf(x, y) \in I$. $I$ is a *prime ideal of* $G$ if $I$ is an ideal of $G$ and $G_+ - I$ is a semigroup, i.e., $x, y \in G_+ - I$ implies $x + y \in G_+ - I$. Note that $\emptyset$ is always a prime ideal of $G$. $I$ is *a maximal ideal of G* if $I$ is a maximal element of the set of all ideals of $G$ under the operation of set inclusion.

PROPOSITION 1.2. *Let $G$ be the divisibility group of $R$, let $\pi : Q^* \to Q^*/U = G$ be the canonical map, and suppose $R$ is a Bezout domain. Then there is a one-to-one order preserving correspondence between the set of all proper ideals of $R$ and the set of all ideals of $G$. A proper ideal $J$ of $R$ corresponds to the ideal $\pi(J^*)$ of $G$. Under this correspondence prime ideals of $R$ correspond to prime ideals of $G$ and maximal ideals of $R$ correspond to maximal ideals of $G$.*

PROOF. Straightforward, or see [31], 2.2.

J. Mott [26] 2.1. generalized the one-to-one correspondence for the prime ideals. If $G$ is a lattice ordered group with only one maximal ideal, then $G$ is a totally ordered group. A consequence is the well-known result that a Bezout domain with only one maximal ideal is a valuation ring.

PROPOSITION 1.3. *Let $G$ be the divisibility group of $R$, let $\pi : Q^* \to Q^*/U = G$ be the canonical map, and suppose $R$ is a Bezout domain. Let $P$ be a prime ideal of $R$. Define $H$ to be the subgroup of $G$ generated by $G_+ - \pi(P^*)$, i.e.,*

$$H = \left\{ \sum_{i=1}^{k} n_i g_i \in G : g_i \in G_+ - \pi(P^*) \text{ and } n_i \in Z \right\}$$

$$= \{g_1 - g_2 \in G : g_i, g_2 \in G_+ - \pi(P^*)\}.$$

*Then $G/H$ is a lattice ordered group with the ordering given by $g_1 + H \geqq g_2 + H$ if there exists $h \in H$ such that $g_1 - g_2 + h \geqq 0$. Moreover, the divisibility group of $R_P$ is lattice isomorphic to $G/H$.*

PROOF. If $h \in H_+$, then $h \notin \pi(P^*)$. If $h \in H_+$, $g \in G$, and $h \geqq g \geqq -h$, then $g \in H$, i.e., $H$ is a convex subgroup of $G$. To show that $G/H$ is a lattice ordered group the only non-trivial part is the antisymmetry. Suppose $g_1 + H \geqq g_2 + H$ and $g_2 + H \geqq g_1 + H$. Then $g_1 - g_2 + h \geqq 0$ and $g_2 - g_1 + h' \geqq 0$ for some $h, h' \in H$. By adding to $h$ and $h'$ terms of the form $n_i g_i$ where $g \in G_+ - \pi(P^*)$ and $n_i \in Z_+$, we may assume $h = h' \geqq 0$. Hence $h \geqq g_1 - g_2 \geqq -h$, $g_1 - g_2 \in H$, and so $g_1 + H = g_2 + H$, showing $G/H$ is a lattice ordered group. If $f : G \to G/H$ is the canonical homomorphism, then $f \circ \pi$ is a valuation with corresponding valuation ring $R_P$. Thus $G/H$ is lattice isomorphic to the divisibility group of $R_P$.

PROPOSITION 1.4. *Let $G$ be a lattice ordered group, let $S$ be a subsemigroup of $G_+$(i.e., $s, s' \in S$ implies $s + s' \in S$), and let $I$ be an ideal of $G$ such that $I \cap S = \emptyset$. Then there exists a maximal element under the operation of set inclusion of $\{J \subset G : J$ is an ideal of $G$, $J \supset I$, and $J \cap S = \emptyset\}$, and every such maximal element is a prime ideal of $G$.*

PROOF. Zorn's Lemma gives the existence of a maximal element, and a straightforward computation shows that every such maximal element is a prime ideal of $G$.

PROPOSITION 1.5. *Let $\{G_\alpha\}_{\alpha\in\Gamma}$ be a family of lattice ordered groups, let $G = \oplus \sum_{\alpha\in\Gamma} G_\alpha$ have the product ordering, and let $\pi_\alpha : G \to G_\alpha$ be the projection homomorphism for $\alpha \in \Gamma$. If $P$ is a prime ideal of $G$, then $P = \emptyset$ or there exists $\alpha \in \Gamma$ and a prime ideal $P_\alpha$ of $G_\alpha$ such that $P = G_+ \cap \pi_\alpha{}^{-1}(P_\alpha)$.*

PROOF. Straightforward [8], Lemma 1.

THEOREM 1.6 (P. LORENZEN [19]). *Every lattice ordered group can be lattice embedded into a direct product of totally ordered groups with the product ordering.*

PROOF ([13], Theorem 2, p. 37). Let $G$ be a lattice ordered group and let $\Gamma$ be the set of maximal ideals of $G$. For $M \in \Gamma$ let $H_M = \{g_1 - g_2 \in G : g_1, g_2 \in G_+ - M\}$, and define $G_M = G/H_M$ where $G_M$ is given the ordering $g_1 + H_M \geqq g_2 + H_M$ if $g_1 - g_2 + h \geqq 0$ for some $h \in H_M$(see 1.3). Define $f_M : G \to G_M$ to be the projection homomorphism. $f_M$ is a lattice homomorphism and $f_M(M)$ is the only maximal ideal of $G_M$, so $G_M$ is a totally ordered group. Then $\prod_{M\in\Gamma} f_M : G \to \prod_{M\in\Gamma} G_M$ is the required embedding, where of course the last group has the product ordering.

The reader is referred to [5] for a discussion of related embeddings.

THEOREM 1.7 (I. KAPLANSKY AND P. JAFFARD). *If $G$ is a lattice ordered group, then there exists an integral domain whose divisibility group is lattice isomorphic to $G$.*

PROOF (J. OHM [28], p. 589). By 1.6 there exists a lattice embedding $f: G \to G' = \prod_{M\in\Gamma} G_M$ where $G_M$ is a totally ordered group for all $M \in \Gamma$ and $G'$ has the product ordering. Let $\pi_M : G' \to G_M$ be the canonical projection for $M \in \Gamma$. Let $k$ be a field and let $\{Y_g : g \in G\}$ be a set of indeterminants over $k$ indexed by $G$. Let $Q = k(\{Y_g : g \in G\})$. To define $\phi_M : Q^* \to G_M$, first consider monomials in $k[\{Y_g : g \in G\}]^*$. Let $\phi_M(cY_{g_1}^{n_1}Y_{g_2}^{n_2} \cdots Y_{g_r}^{n_r}) = \sum_{i=1}^{r} n_i\pi_M \circ f(g_i)$ where $c \in k^*$,

$g_i \in G$, and $n_i \in Z_+$. For $p \in k[\{Y_g : g \in G\}]^*$, let $\phi_M(p) = \inf\{\phi_M(m_i) : m_i$ are the distinct monomials appearing in $p\}$. Then for $p, p' \in k[\{Y_g : g \in G\}]^*$ let $\phi_M(p/p') = \phi_M(p) - \phi_M(p')$. This defines $\phi_M : Q^* \to G_M$ and by [1], Lemma 1 and obvious generalization, p. 160, $\phi_M$ is a valuation. Define $\phi : Q^* \to G'$ by $\phi = \prod_{M \in \Gamma} \phi_M$. Then $\phi$ satisfies for $q, q' \in Q^*, \phi(qq') = \phi(q) + \phi(q')$ and $\phi(q + q') \geqq \inf(\phi(q), \phi(q'))$ if $q + q' \neq 0$. Let $R = \{0\} \cup \{x \in Q^* : \phi(x) \geqq 0\}$. Then $R$ is an integral domain with quotient field $Q$ and divisibility group $\phi(Q^*)$. For $g \in G, \phi(Y_g) = f(g)$ so $f(G) \subset \phi(Q^*)$. By the definition of $\phi$, $\phi(Q^*)$ is the sublattice of $G'$ generated by $f(G)$. $f(G)$ is a sublattice of $G'$ since $f$ is a lattice homomorphism, and so $\phi(Q^*) = f(G)$. Thus the divisibility group of $R$ is lattice isomorphic to $G$, as desired.

Proposition 1.8 (J. Ohm). *If G is a lattice ordered group and R is an integral domain whose divisibility group is lattice isomorphic to G and is obtained as in the proof of* 1.7, *then R is a Bezout domain.*

Proof. Slight modification of proof given in [6], 18.6 or [8], p. 1370.

W. Krull first proved 1.7 for $G$ a totally ordered group (1.1), and P. Jaffard first published 1.7 as stated. I. Kaplansky also obtained 1.7 in his thesis at Harvard University, 1941, although it was never published. His proof uses the theorem of Lorenzen (1.6), so presumably it is the same as the one given above. P. Jaffard's proof of 1.7 is not as elegant as the proof given above [6] 18.6 or [13] Theorem 3, p. 78. J. Ohm noticed that the integral domain obtained in 1.7 is a Bezout domain (1.8). This is summarized in the following theorem.

Theorem 1.9 (Krull-Kaplansky-Jaffard-Ohm). *If G is a lattice ordered group, then there exists a Bezout domain whose divisibility group is lattice isomorphic to G.*

2. **Constructing valuation rings.** In section one, the Krull Theorem (1.1) was generalized to 1.9 by allowing the group to be lattice ordered instead of totally ordered. In this section another approach is briefly explored. Some of the following appears in 0. Schilling's text [29], although it is quite difficult to read.

Let $k$ be a field and let $G$ be a totally ordered group. Define $Q$ to be the set of all elements $\sum_{\alpha \in \Gamma} c_{g_\alpha} X_{g_\alpha}$ where $\Gamma$ is the set of all ordinals less than some fixed ordinal (the fixed ordinal varying with different elements of $Q$), $g_\alpha \in G$, $\alpha < \beta$ implies $g_\alpha < g_\beta$, and $c_{g_\alpha} \in k$. Let $R$ be the subset of $Q$ consisting of all the elements with the added restriction that $g_\alpha \geqq 0$ for all $\alpha \in \Gamma$. $R$ is called the *long power series ring relative* *to k and G.* The name long power series ring was suggested by L. S. Levy. With the standard operations $R$ is a valuation ring and $Q$ is its field of fractions.

A valuation ring $R$ is *maximally complete* if whenever $R$ is embedded into another valuation ring $R'$, then this embedding is an isomorphism or either the divisibility group of $R'$ or the residue field of $R'$ is strictly larger than that of $R$ via the induced embeddings. According to [14] this definition is due to F. K. Schmidt but first published by W. Krull in 1932. By Zorn's Lemma it follows that any valuation ring can be embedded into a maximally complete valuation ring with isomorphic divisibility group and residue fields via the induced embeddings [29], Theorem 5, p. 38. I. Kaplansky [14] showed that if $R$ is a valuation ring this maximal completion is unique up to isomorphism if the characteristic of the residue field is zero, and by an example this maximal completion need not be unique up to isomorphism if the characteristic of the residue field of $R$ is not zero. Also studied is the question of when a maximally complete valuation ring is a long power series ring.

An $R$-module $M$ is *linearly compact* if given $\{M_i\}_{i \in I}$ a set of submodules of $M$, and $\{x_i\}_{i \in I} \subset M$, then the whole family of congruences $\{x \equiv x_i \bmod M_i\}_{i \in I}$ is solvable whenever every finite subfamily of the congruences is solvable. $R$ is a *maximal* integral domain if $R$ is a linearly compact $R$-module. Then for $R$ a valuation ring, $R$ is maximally complete if and only if $R$ is maximal [29]. It is reasonably straightforward to show that every long power series ring is maximal, and so is maximally complete. This is the property that makes long power series rings so desirable.

A long power series ring is now used to construct an example of a ring of type I due to B. Osofsky [22], p. 76 or [23], p. 119–120. $R$ is a *ring of type I* if (i) $R$ has exactly two maximal ideals $M_1$ and $M_2$, (ii) $M_1 \cap M_2$ does not contain a non-zero prime ideal of $R$, and (iii) $R_{M_1}$ and $R_{M_2}$ are maximal valuation rings. This definition is due to E. Matlis [22], p. 76 or [23], p. 119. If $R$ is a ring of type I, $R = R_{M_1} \cap R_{M_2}$ and so $R$ is a Prüfer domain [6], 22.8, and a Prüfer domain with only finitely many maximal ideals is a Bezout domain [12], Corollary 5. Rings of type I are used to characterize another type of ring, called rings with property $D$ [22] or [23].

To construct a ring of type I, begin with $R$ the long power series ring relative to $C$ and $G$, where $C$ is the field of complex numbers and $G$ is the additive group of rationals with the standard total ordering. Let $Q$ be the field of fractions of $R$. One can show that $Q$ is an algebraically closed field. Hence one can construct an automorphism $f$:

$Q \to Q$ fixing the coefficients $C$, and such that $f(X) = 1 - X$ where $X$ is shortened notation for $1X_1$, the element of $R$. One sees that $R \cap f(R)$ is the desired ring of type I.

A generalization of this appears in the paper by T. Shores and R. Wiegand [32], Section 4, namely if $n \geqq 2$, $n \in N$, then there exists a Bezout domain $R$ with exactly $n$ maximal ideals and every localization at a maximal ideal of $R$ is a maximal valuation ring of Krull dimension one. These are also examples of what G. Klatt and L. S. Levy call pre-self-injective rings [16], 3.5. P. Vamos has recently generalized these examples [34], Proposition 12, namely there exists a Bezout domain $R$ with infinitely many maximal ideals such that every localization at a maximal ideal of $R$ is a maximal valuation ring of Krull dimension one and every non-zero element of $R$ is an element of only a finite number of maximal ideals of $R$.

3. **Examples of Bezout domains.** This section is devoted to giving examples of Bezout domains by the Krull-Kaplansky-Jaffard-Ohm construction, 1.9. Thus appropriate lattice ordered groups will be constructed, and using 1.9 a Bezout domain with that divisibility group exists. Then one translates the built-in properties of the lattice ordered group to the appropriate desired properties of the Bezout domain. Most of the examples given are counterexamples to disprove conjectures made in the literature, although a few are used in existence proofs to indicate that a ring with desired properties does exist, as in Lewis' example 3.4. Often the examples desired are not necessarily Bezout domains, it is just noted that the construction 1.9 gives Bezout domains. Much of the credit for popularizing this technique goes to J. Ohm, through his papers and the students who studied under him. The early examples appeared in the middle 1960's and had to do with the complete integral closure of an integral domain. These are discussed first.

If $x \in Q$, then $x$ is *almost integral over* $R$ if there exists $y \in R^*$ such that $yx^n \in R$ for all $n \in N$. The set of all elements of $Q$ almost integral over $R$ is called the *complete integral closure of* $R$ and is denoted $R_c$. It is trivial to check that $R_c$ is a subring of $Q$ containing $R$. $R_{c^2}$ will be used for $(R_c)_c$ and inductively $R_{c^{n+1}}$ for $(R_{c^n})_c$ where $n \in N$. $R$ is said to be *completely integrally closed* if $R = R_c$. If $x \in Q$ and $x$ is integral over $R$, then $x$ is almost integral over $R$. The converse is not true. For if $R$ is a valuation ring with divisibility group order isomorphic to $Z^2$ ordered lexicographically, and $x \in Q$ whose image in $Z^2$ via the canonical map is $(0, -1)$, then $x$ is almost integral over $R$, yet $x$ is not integral over $R$. If $R$ is a valuation ring, then $R_c$ is easily seen to be $Q$ if $R$ has no minimal non-zero prime ideal, and $R_c = R_P$ if $P$ is the minimal non-zero prime ideal of $R$. Thus $R_c$ is completely integrally closed, i.e., $R_{c^2} = R_c$ if $R$ is a valuation ring.

The problem of characterizing the completely integrally closed domains led to several questions. One question that arose: is $R_c$ completely integrally closed for all $R$? This is not true, with the first example of an integral domain $R$ such that $R_{c^2} \neq R_c$ being given in [7], Example 1, p. 354, although this example is not a Prüfer domain, and hence not a Bezout domain. This naturally led to the question: if $R$ is a Prüfer domain, is $R_c$ completely integrally closed? Again the answer is no, with the counterexample due to W. Heinzer [9] or [6], 19.13. The construction uses 1.9, and gives a Bezout domain with infinite Krull dimension. This led to the question: if $R$ is a Prüfer domain of finite Krull dimension, is $R_c$ completely integrally closed? The answer is still no, with the counterexample due to P. Sheldon [31], Example 1. The construction uses 1.9, and gives a Bezout domain $R$ of Krull dimension two, the smallest possible, for which $R_c$ is not completely integrally closed.

The last two examples have the property that $R \subsetneq R_c \subsetneq R_{c^2} = R_{c^3}$, i.e., $R_{c^2}$ is completely integrally closed. This led to the question: is $R_{c^2}$ always integrally closed? The answer is no with the counterexample due to P. Hill [11], Theorem 1. Using the construction of 1.9, he shows that for $n \in N$ there exists a Bezout domain $R$ such that $R \subsetneq R_c \subsetneq \cdots \subsetneq R_{cn} = R_{c^{n+1}}$. This leads easily to another example by P. Hill that there exists a Bezout domain $R$ such that $R_{c^n} \subsetneq R_{c^{n+1}}$ for all $n \in N$.

Before giving these examples, a method for determining when $R$ is completely integrally closed in terms of its divisibility group is discussed. If $G$ is a lattice ordered group and $g \in G$, then $g$ is *bounded* if $g \in G_+$ and there exists $h \in G$ such that $ng \leqq h$ for all $n \in N$. Let $B(G)$ denote the subgroup of $G$ generated by the bounded elements of $G$, i.e., $B(G) = \{g_1 - g_2 \in G : g_1$ and $g_2$ are bounded elements of $G\}$. If $R$ is a Bezout domain with divisibility group $G$, then it is easily seen that the divisibility group of $R_c$ is order isomorphic to $G/B(G)$ where $G/B(G)$ is ordered by $g_1 + B(G) \geqq g_2 + B(G)$ if there exists $b \in B(G)$ such that $g_1 - g_2 + b \geqq 0$. Thus if $R$ is a Bezout domain with divisibility group $G$, then $R \subsetneq R_c$ if and only if $B(G) \neq \{0\}$. This generalizes to a criterion for deciding whether $R_{cn} \subsetneq R_{c^{n+1}}$ for $n \in N$ by looking at $B(G)$, $B(G/B(G))$, $B((G/B(G))/B(G/B(G)))$, etc.

*Example* 3.1 (P. Hill [11], Theorem 2). There exists a Bezout domain $R$ such that $R_{c^n} \subsetneq R_{c^{n+1}}$ for all $n \in N$.

Let $H = Z^2$ be ordered lexicographically. View an element $H^{Z^*}$ as

$(\cdots, (a_{-1}, b_{-1}), (a_1, b_1), \cdots)$ or simply $((a_n, b_n))$ where $a_n, b_n \in Z$ for $n \in Z^*$. Define $G_1 = \{((a_n, b_n)) \in H^{Z^*} : a_n = 0$ if $n < 0$, $a_n = 0$ for all but finitely many $n \in Z^*$, and $b_n = 0$ for all but finitely many $n < 0$, $n \in Z^*\}$. Giving $H^{Z^*}$ the product ordering and viewing $G_1$ as a subgroup, $G_1$ becomes a lattice ordered group. Let $\{N_k\}_{k \in N}$ be a countably infinite partition of $N$ such that $N_k$ is infinite for all $k \in N$. Define $G_2 = \{((a_n, b_n)) \in G_1 : b_n$ is constant on $N_k$ for all but finitely many $k \in N$; and for all $k \in N$ there exists $a, b \in Z$ such that $b_n = na + b$ for all but finitely many $n \in N_k\}$. Let $K = \{((a_n, b_n)) \in G_1 : a_n = 0$ for all $n \in Z^*$, $b_n = 0$ for all $n < 0$, $n \in Z^*$, and $b_n = 0$ for all but finitely many $n \in Z^*\}$. Then $G_2 \supset K$ are subgroups of $G_1$ and $G_2/K$ is ordered by $g + K \geqq g' + K$ if there exists $k \in K$ such that $g - g' + k \geqq 0$. Although not obvious, $G_2/K$ turns out to be order isomorphic to $G_1$. Let $\beta : G_2 \to G_1$ be the composition of the natural projection $G_2 \to G_2/K$ and the isomorphism $G_2/K \to G_1$. Inductively for $n \in N$, $n \geqq 2$, define $G_{n+1} = \beta^{-1}(G_n)$. One sees that $G_{n+1}/K \cong G_n$ and $B(G_{n+1}) = K$ for all $n \in N$, and so $G_{n+1}/B(G_{n+1}) \cong G_n$. Also $B(G_1/B(G_1)) = B(B_1/K) \neq \{0\}$ and $B((G_1/K)/B(G_1/K)) = \{0\}$.

For $n \in N$, $n > 2$, if one uses 1.9 to construct a Bezout domain $R$ with divisibility group lattice isomorphic to $G_{n-1}$, then by induction one easily sees that $R \subsetneq R_c \subsetneq \cdots \subsetneq R_{c^n} = R_{c^{n+1}}$ getting the earlier example of P. Hill mentioned. Let $G = \oplus \sum_{n \in N} G_n$ with the product ordering. Using 1.9, if $R$ is a Bezout domain with divisibility group lattice isomorphic to $G$, then it easily follows that $R_{c^n} \subsetneq R_{c^{n+1}}$ for all $n \in N$, and this is the desired example 3.1.

The problem of characterizing the completely integrally closed domains led to other questions. W. Krull conjectured that $R$ is a completely integrally closed domain if and only if $R$ is the intersection of valuation overrings of Krull dimension one or less, i.e., $R = \bigcap \{V : V$ is a valuation ring, $K \dim V \leqq 1$, and $R \subset V \subset Q\}$. T. Nakayama showed this conjecture to be false. An easier example is due to J. Ohm [6], 19.12, namely there exists a Bezout domain $R$, $R \neq Q$, which is completely integrally closed and does not admit a Krull dimension one valuation overring. Ohm's example uses the construction of 1.9 and yields a Bezout domain of infinite Krull dimension. This led to the question of whether this is possible for a domain with finite Krull dimension. The answer is yes, with the following being a minimal example in the sense that the Krull dimension is as small as possible, two.

*Example* 3.2 (P. Sheldon [31] Example II). There exists a Bezout domain $R$ of Krull dimension two, completely integrally closed, but $R$ is not the intersection of Krull dimension one valuation overrings.

Let $G_0$ be the set of all functions $f$: $[0, 1] \to Z$ where we identify two functions if they agree on all but finitely many points. Actually $G_0$ is $Z^{[0,1]}$ modulo an equivalence relation, but an element of $G_0$ will be written as a function $f$: $[0, 1] \to Z$, the function being any representative of the equivalence class. Let $F$ be the set of all step functions of $G_0$, i.e., $F = \{f \in G_0 :$ there exists $0 = a_0 < a_1 < \cdots < a_n = 1$ such that for all but finitely many $x, y \in (a_i, a_{i+1})$ one has $f(x) = f(y)\}$. For $b \in [0, 1]$ define $h_b : [0, 1] \to Z$ by $h_b(b) = 1$ and $h_b(x) = \|1/(x - b)^2\|$ for all $x \in [0, 1] - \{b\}$, where $\| \cdots \|$ denotes the greatest integer function. Let $G$ be the subgroup of $G_0$ generated by $F \cup \{h_b\}_{b \in [0,1]}$. Thus a typical element of $G$ is of the form $f + n_1 h_{b_1} + \cdots + n_r h_{b_r}$ *for* $f \in F$, $n_i \in Z$, and $b_i \in [0, 1]$. $G$ is ordered by $g \geqq g'$ if $g(x) \geqq g'(x)$ for all but finitely many $x \in [0, 1]$. Then $G$ is a lattice ordered group. Use 1.9 to obtain a Bezout domain $R$ whose divisibility group is lattice isomorphic to $G$. $\{g \in G_+ : h_b$ appears with a non-zero coefficient in the expression of $g\}$. For $b \in (0, 1]$ define $P_b' = \{g \in G_+ :$ there exists $c \in [0, b)$ such that $g(x) > 0$ for all but finitely many $x \in (c, b)\}$. For $b \in [0, 1)$ define $P_b'' = \{g \in G_+$: there exists $c \in (b, 1]$ such that $g(x) > 0$ for all but finitely many $x \in (b, c)\}$. Then the complete set of prime ideals of $G$ is $\{\emptyset\} \cup \{P_b : b \in [0, 1]\} \cup \{P_b' : b \in (0, 1]\} \cup \{P_B'' : b \in [0, 1))$ and the only containment relation amongst the primes are $\emptyset \subsetneq P_b \subsetneq P_b'$, $P_b \subsetneq P_b''$ [31], Example II. Thus $K \dim R = 2$ by 1.2. Let $f_0 \in F \cap G_+^*$ and choose $x_0 \in Q^*$ such that $\pi(x_0) = -f_0$. If $V$ is a valuation ring, $K \dim V = 1$, $R \subset V \subset Q$, and $m(V)$ is the maximal ideal of $V$, then $\pi(m(V) \cap R^*) = P_b$ for some $b \in [0, 1]$ [6], p. 334. Thus $x_0 \in V$. Hence $x_0 \in \cap \{V : V$ is a valuation ring, $K \dim V = 1$, and $R \subset V \subset Q\} - R$, and so $R$ is not an intersection of Krull dimension one valuation overrings, as desired.

Another example involving the complete integral closure has to do with power series rings. $X$ will denote an indeterminant over the appropriate ring. If $R$ and $S$ are valuation rings with the same quotient field and such that $R_c = S_c$, then the quotient fields of $R[[X]]$, $S[[X]]$, and $R_c[[X]]$ are all the same [30], 4.1. In other words the complete integral closure of $R$ determines the quotient field of $R[[X]]$ if $R$ is a valuation ring. Is this true for any domain $R$? The answer is no, with the counterexample due to P. Sheldon [30], 4.8, namely there exists a Bezout domain $R$ such that $R \subsetneq R_c = R_{c^2} \subsetneq Q$, i.e., $R$ and $R_c$ have the same complete integral closure, yet the quotient fields of $R[[X]]$ and $R_c[[X]]$ are not equal. 1.9 is used to construct this example.

Two other examples are briefly mentioned without elaborating on the history or the definitions. Let it suffice to say that each answers an earlier conjecture and each uses the construction of 1.9. The first, due to W. Heinzer [8], shows that for each $n \in N$ there exists a $J$-Noetherian domain of $J$-dimension $n$ with 1 in the stable range. The second, due to J. Brewer, P. Conrad, and P. Montgomery [4], shows that there exists an elementary divisor ring $R$ such that $R$ is not adequate, yet every non-zero prime ideal of $R$ is a subset of only one maximal ideal of $R$.

A commutative ring with identity is *locally Noetherian* if all its localizations at maximal ideals are Noetherian. W. Heinzer and J. Ohm studied locally Noetherian rings in [10]. The obvious conjecture that a locally Noetherian ring is Noetherian is false. Although quite involved, the standard example of a locally Noetherian ring which is not Noetherian is given in [27] and involves the integral closure of the ring of integers in the field gotten by adjoining for all prime integers $p$ a primitive $p^{\text{th}}$ root of unity to the rationals. The following is a much simpler example of the same property.

*Example* 3.3 (W. Heinzer and J. Ohm [10], 2.2). There exists a domain which is not Noetherian, yet all its localizations at maximal ideals are Noetherian valuation rings.

View an element of $Z^N$ as $(z_1, z_2, \cdots)$ or simply $(z_n)$ where $z_n \in Z$ for all $n \in N$. Let $G = \{(z_n) \in Z^n : \text{there exists } k \in Z \text{ such that } z_n = k$ for all but finitely many $n \in N\}$. Let $Z^N$ have the product ordering and $G$ the induced ordering as a subgroup of $Z^N$. Then $G$ is a lattice ordered group. Using 1.9, there exists a Bezout domain $R$ whose divisibility group is lattice isomorphic to $G$. Let $\pi : Q^* \rightarrow G$ be the canonical map. For $i \in N$ let $M_i = \{0\} \cup \{r \in R^* : \text{ if } \pi(r) = (z_n),$ then $z_i > 0\}$. Let $M_\infty = \{0\} \cup \{r \in R^* : \text{if } \pi(r) = (z_n), \text{ then } z_n > 0$ for all but finitely many $n \in N\}$. Then the complete set of maximal ideals of $R$ is $\{M_i\}_{i \in N \cup \{\infty\}}$. Using 1.3, it follows that the divisibility group of $R_{M_i}$ is order isomorphic to $Z$, for all $i \in N \cup \{\infty\}$, and thus $R_{M_i}$ is a Noetherian valuation ring (discrete rank one valuation ring). $M_\infty$ is an ideal of $R$ which is not finitely generated, so $R$ is not Noetherian, as desired.

In trying to characterize the integral domains $R$ with the property that all finitely generated $R$-modules are a direct sum of cyclic submodules, partial results led to Bezout domains with the property that one maximal ideal is contained in the union of the rest of the maximal ideals [3]. The question arose as to whether this was related to the Krull dimension of the domain. The following two examples indicate that this is not the case. The first, [3], Example 17, shows that there exists a Bezout domain $R$ with the complete set of maximal ideals of $R$ being $\{M_n\}_{n \in N \cup \{\infty\}}$, $M_\infty \subset \bigcup_{n \in N} M_n$, and $K \dim R_{M_n} = 1$ for all $n \in N \cup \{\infty\}$. The $R$ constructed is the same as the $R$ in example 3.3, which of course uses 1.9. The second example, [3], Example 18, shows that there exists a Bezout domain $R$ with maximal ideals $M_\infty$, $M_1$, $M_2, \cdots$ such that $M_\infty \subset \bigcup_{n \in N} M_n$, $K \dim R_{M_n} = 1$ for all $n \in N$, and $K \dim R_{M_\infty} = \infty$. This example is obtained using 1.9 by letting $R$ be a Bezout domain whose divisibility group is lattice isomorphic to $Z^N$ with the product ordering.

The final example to be presented is perhaps the best example, due to W. J. Lewis. By this example the spectrum of Bezout domains and Prüfer domains are characterized. If S is a commutative ring with identity, let spec S denote the set of all prime ideals of S considered as a partially ordered set under set inclusion. If $X$ is a partially ordered set, the following two conditions are of interest:

($K1$) Every chain of $X$ has a supremum and an infimum.

($K2$) If $x, y \in X$, $x < y$, then there exist $x_1, y_1 \in X$ such that $x \leqq x_1 \leqq y_1 \leqq y$ and there does not exist an element of $X$ properly between $x_1$ and $y_1$. In condition ($K2$) $x_1$ and $y_1$ are called *immediate neighbors* and this is donated $x_1 \ll y_1$. If S is a commutative ring with identity, then spec S satisfies ($K1$) and ($K2$) [15], Theorems 9 and 11. A partially ordered set $X$ is a *tree* if $x, y, z \in X, x \leqq z$, and $y \leqq z$ implies $x \leqq y$ or $y \leqq x$. If $R$ is a Prüfer domain or a Bezout domain, then spec $R$ is a tree, since $P \in$ spec $R$ implies $R_P$ is a valuation ring. Thus if $R$ is a Prüfer domain or a Bezout domain, then spec $R$ is a tree satisfying ($K1$), ($K2$), and spec $R$ has a unique minimal element.

*Example* 3.4 (W. J. Lewis [18], 3.1). If $X$ is a tree satisfying ($K1$), ($K2$), and $X$ has a unique minimal element, then there exists a Bezout domain $R$ such that spec $R$ is order isomorphic to $X$.

Let the $X$ be given. Define $Y = \{y \in X:$ there exists $z \in X$ such that $z \ll y\}$, with $Y$ having the induced ordering from $X$. Let $G = \{f: Y \to Z : f(y) = 0$ for all but finitely many $y \in Y\}$. With pointwise addition $G$ is a group. For $f \in G$ let $MS(f) = \{y \in Y : f(y) \neq 0$ and $f(s) = 0$ for all $s \in Y$, $s < y\}$. Let $G_+ = \{f \in G : f(y) > 0$ for all $y \in MS(f)\}$. Then $G_+$ is the set of positive elements for a partial ordering of $G$, and in fact this makes $G$ a lattice ordered group. Using 1.9, let $R$ be a Bezout domain whose divisibility group is lattice isomorphic to $G$. That $R$ has the required properties is proved in [18], p. 431–33. Since every Bezout domain is a Prüfer domain, it follows from the above that the spectrum of a Bezout domain and a Prüfer domain is characterized by these conditions.

4. **Additional comments.** In this section related approaches to the subject are discussed and suggestions for future study are indicated. The first and most obvious suggestion is to find more examples of Bezout domains using 1.9.

The second suggestion involves generalizing the long power series ring. By Krull's construction, 1.1, given a totally ordered group one can construct a valuation ring whose divisibility group is the given one. However, one can also construct a long power series ring whose divisibility group is the given one, and this long power series ring has desirable properties, namely it is maximally complete. Is it possible to generalize this construction of long power series rings to get a Bezout domain when the given group is lattice ordered? Is there an appropriate generalization of maximally complete for Bezout domains? It is known that if $R$ is a maximal integral domain, then $R$ has only one maximal ideal [37], Proposition 14. Thus a useful generalization of maximally complete will not be equivalent to being maximal. A partial result along these lines and related to 3.4 is due to S. Wiegand [35], namely if $X$ is a finite tree with a unique minimal element, then $X$ is order isomorphic to spec $R$ for some Bezout domain $R$ such that every localization of $R$ at a prime ideal is a maximal valuation ring.

A more homological approach might be preferable. $R$ is a maximal valuation ring if and only if $R$ is a valuation ring complete in the valuation topology and $Q/R$ is an injective $R$-module [20], Theorem 9. The $R$-topology of an integral domain generalizes the valuation topology for a valuation ring, and if $R$ is an integral domain, then $R$ is complete in the $R$-topology if and only if $R \cong \mathrm{Hom}_R(Q/R, Q/R)$ [23], Theorem 10 or [21], 6.4. Thus $R$ is a maximal valuation ring if and only if $R$ is a valuation ring such that $R \cong \mathrm{Hom}_R(Q/R, Q/R)$ and $Q/R$ is an injective $R$-module. How do these concepts carry over to arbitrary integral domains?

Related to this discussion is the following definition: $R$ is *almost maximal* if every proper homomorphic image of $R$ is a linearly compact $R$-module, i.e., cyclic torsion modules are linearly compact. If $R$ is a valuation ring, then $R$ is almost maximal if and only if $Q/R$ is an injective $R$-module if and only if every finitely generated $R$-module is a direct sum of cyclic submodules. See [2] for references, more equivalences, and generalizations of the last statement.

Starting with lattice ordered groups, 1.9 is used to construct Bezout domains. In other words, using properties of groups one derives properties of integral domains. Turning this around, properties of integral domains can be used to derive properties of ordered groups. This is the interest in [24], where also a good historical description of the group of divisibility is given. In [25] there is an example of how 1.9 is used to derive a fact about ordered groups.

The Krull-Kaplansky-Jaffard-Ohm Theorem, 1.9, is very useful for generating examples. Can this be generalized? For example, is there a larger class of partially ordered groups than the lattice ordered groups for which this construction can be performed? If so, what kind of domain is constructed? This was studied by J. Ohm [28] with several open questions remaining, and generalized by his student D. Spikes [33] to commutative rings possibly with zero divisors. Related is the question of extension of semi-valuations. If one has a valuation $v : Q^* \to G$ and $F$ is a field extension of $Q$, then in some cases one can extend $v$ to a valuation $v' : F^* \to G$ [36] or [1]. If $R$ is an integral domain with divisibility group $G$, $\pi : Q^* \to G$ the canonical map ($\pi$ is then a semi-valuation), and $F$ is a field extension of $Q$, then can $\pi$ be extended to an appropriate $\pi' : F^* \to G$?

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