Constructing dependent random probability measures from completely random measures

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presented by **Sinead Williamson**[4]

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ICML 2013

Introduction

Two strands of research in NPBayes modelling of random probability measures (RPMs):

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priors that are more expressive than the Dirichlet Process

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- e.g. power-law behaviour or more uncertainty on number of clusters:
  Normalized Random Measures [James et al., 2005, Kingman, 1975]
  (e.g. normalized generalized Gamma process)
  Poisson-Kingman processes [Pitman, 2003]
  (e.g. Pitman-Yor process [Pitman and Yor, 1997])

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- for data violating the assumption of exchangeability:
  Time-series, spatial data, conditional density modelling
- Research traces back to work of [MacEachern, 1999] on *dependent RPMs*

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This talk:

Flexible constructions for dependent RPMs with flexible marginals

Relevant work

There is a rich literature on dependent RPMs

- the seminal work of [MacEachern, 1999] on dependent DPs
- Existing work that is directly relevant [Rao and Teh, 2009, Nipoti, 2010, Lijoi et al., 2012, Foti et al., 2012, Lin and Fisher, 2012, Griffin et al., 2013]

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- C. Chen, V. Rao, W. Buntine and Y.W. Teh (2013)
  Dependent Normalized Random Measures

Completely random measures (CRMs)

- A random measure μ on some space $(\mathbb{X}, \Sigma_X)$ such that

$$\mu(A) \perp\!\!\!\perp \mu(B) \text{ if } A \text{ and } B \text{ are disjoint}$$

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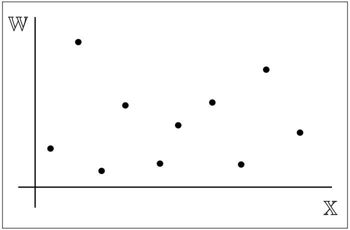
- (x_i, w_i) : events of a Poisson process on the space $\mathbb{X} \times \mathbb{W}$, where $\mathbb{W} = [0, \infty)$.
- The Poisson process has intensity $\nu(w, x) = \rho(w)h(x)$, where
  $\rho(w)$ is the Lévy intensity of the CRM, and
  $h(x)$ is the base probability density.

Normalized random measures



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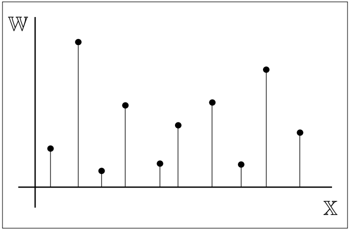

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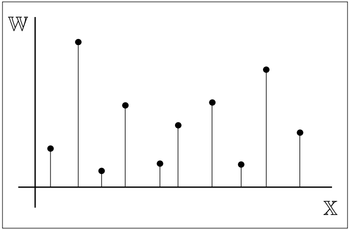

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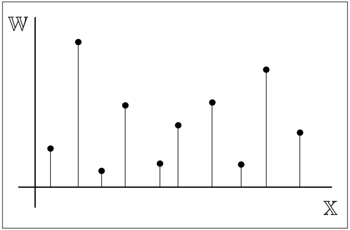

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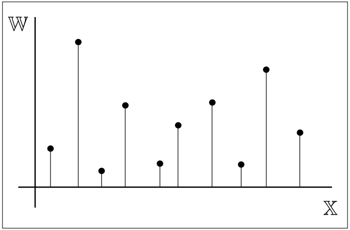

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We want: Dependent normalized random measures, G_t

Dependent normalized random measures

- Define a common latent CRM/Poisson process.
- Define dependent measures via transformations of this process.

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  - Superposition [Rao and Teh, 2009, Griffin et al., 2013]
  - Rescaling
  - Thinning [Lin et al., 2010, Lin and Fisher, 2012]

Normalize these dependent CRMs to produce dependent NRMs.

Superposition theorem

- The superposition of two independent Poisson processes with intensity $\nu_i(\cdot),\ i = 1, 2$ is a Poisson process with intensity $\nu_1(\cdot) + \nu_2(\cdot)$
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- If $\nu(\cdot)$ factors as $\rho(w)h(x)\nu_a(a)$, then the resulting CRM has Lévy intensity $\left(\int_{\mathbb{A}} \nu_a(a)\mathrm{d}a\right) \rho(w)h(x)$.

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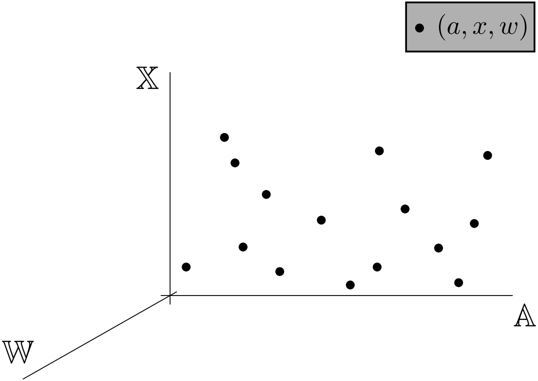

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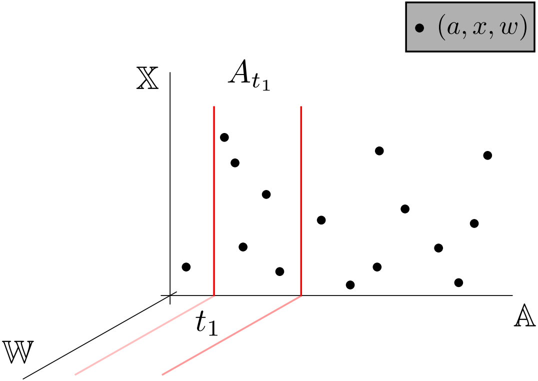

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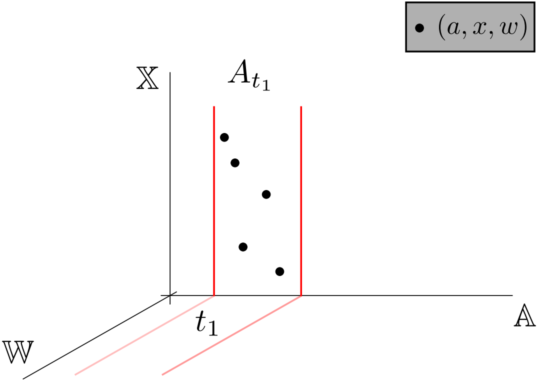

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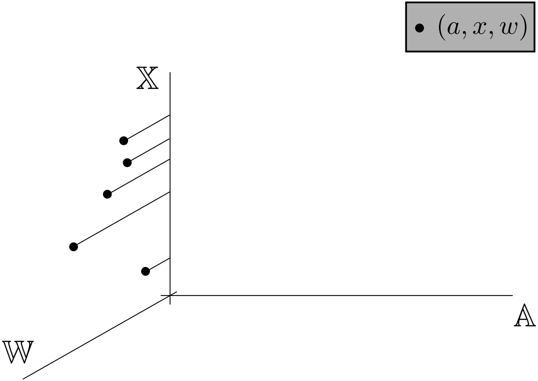

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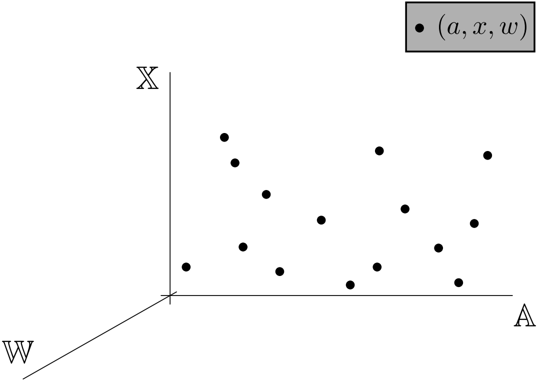

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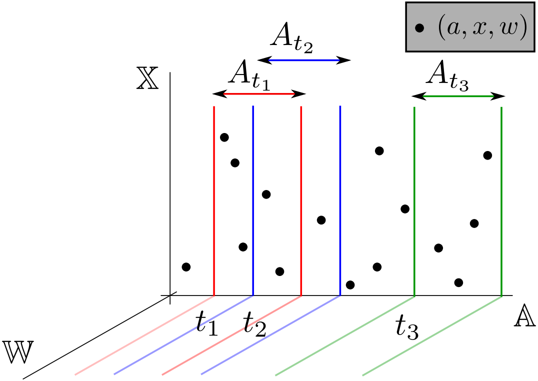

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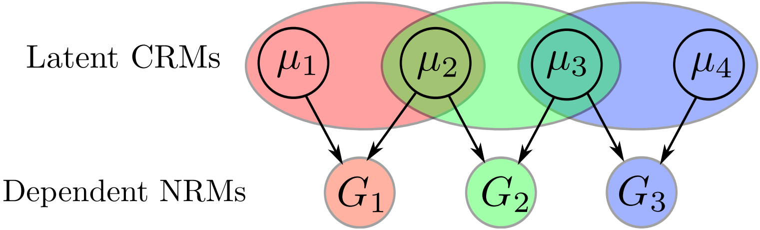
- Associate each t with a subset $\mathbb{X} \times \mathbb{W} \times A_t$
- Restrict to A_t, and project onto the original space, defining an NRM
- Dependency across NRMs is controlled by amount of overlap of A_t's

Dependent normalized random measures ([Griffin et al., 2013])



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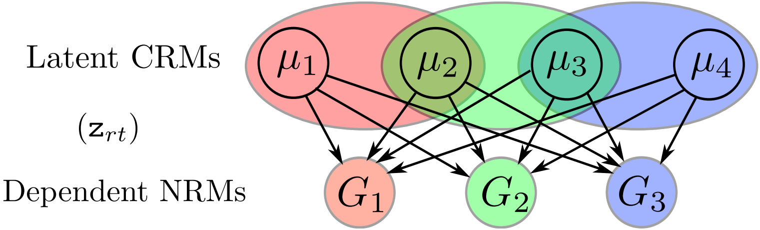
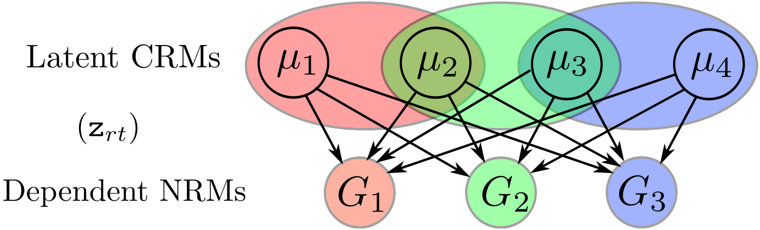


$$G_t \propto \sum_{r=1}^{R} z_{tr}\mu_r, \quad z_{tr} \in \{0, 1\} \tag{1}$$

Mixed normalized random measures



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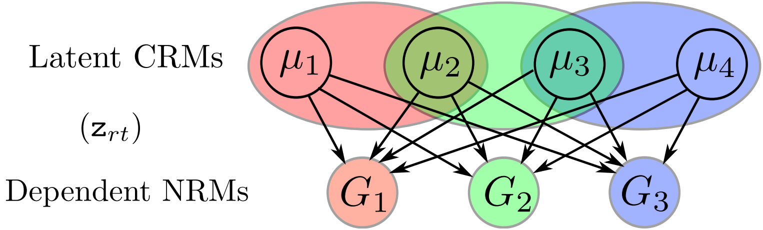

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- Poisson mapping theorem:
  - If $\{w_i\}$ is a sample from a Poisson process with intensity $\nu(w)$, then $\{\mathbf{z}w_i\}$ is a Poisson process with intensity $\mathbf{z}^{-1}\nu(w/\mathbf{z})$.

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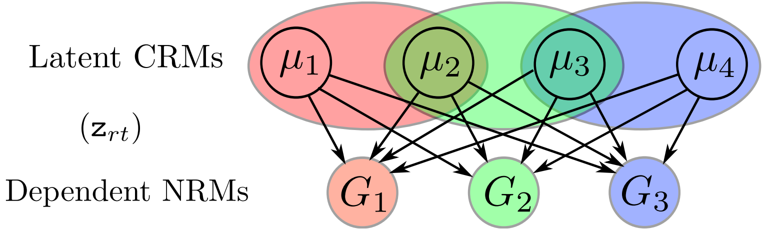

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- $\mathbf{z}_{tr}$ governs how strongly atoms of CRM r contribute to covariate t.
- Given a set $\{\mathbf{z}_{tr}\}$, for each t, G_t is an NRM.

Thinning theorem

- If $\{w_i\}$ is a sample from a Poisson process with intensity $\nu(w)$, then $\{z_i w_i\}$, where $z_i \stackrel{i.i.d.}{\sim}$ Bernoulli(p) is Poisson with intensity $p\nu(w)$.
- Suggests independently thinning atoms of a CRM to form a new CRM ([Lin et al., 2010]).
- Corresponds to SNRM with an exponential number of CRM.

Thinned Normalized Random Measures

- Spatial NRM characterized by a set $\{\mathrm{z}_{tr} \in \{0,1\} \quad \forall t \in \mathcal{T}, r \in \mathcal{R}\}$
- z_{tr} specifies whether or not *all* atoms of CRM r are present at covariate t.

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- Thinned NRM: introduce indicator variables $z_{trw} \in \{0, 1\}$ for each atom.

$$z_{rtk} \sim \text{Bernoulli}(q_{rt}) \quad k = 1, 2, \cdots$$

Then, the probability measure at covariate t is given by

$$\mu_t(\mathrm{d}\theta) = \frac{1}{\hat{\mu}_t(\Theta)} \hat{\mu}_t(\mathrm{d}\theta), \text{ where } \hat{\mu}_t(\mathrm{d}\theta) = \sum_{r=1}^{R} \sum_{k=1}^{\infty} z_{rtk} w_{rk} \tag{2}$$

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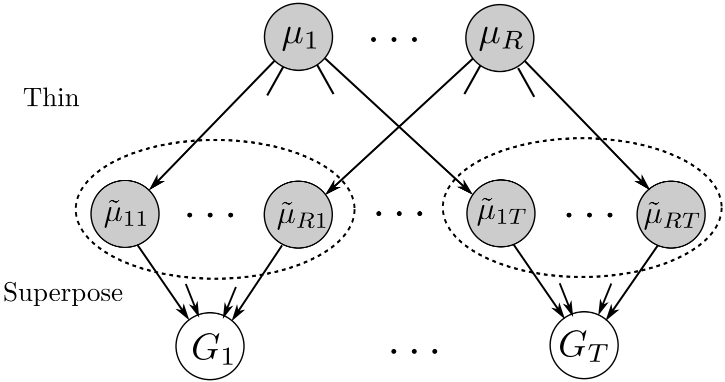
Proposition

Conditioned on the set of q_{rt}'s, each random probability measure μ_t defined in (2) is marginally distributed as a normalized random measure with Lévy measure $\sum_r z_{rt} \nu_r(\mathrm{d}w, \mathrm{d}\theta)$.

Dependent normalized random measures



Inference

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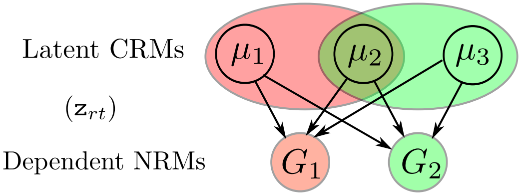
- At a high level:



  - One can superimpose 3 CRMs to construct 2 dependent RPMs, each marginally an NRM.
  - However, given observations from one , the other is *no longer* an NRM.
  - It becomes a *mixture* of NRMs.

Inference (a marginal sampling approach)

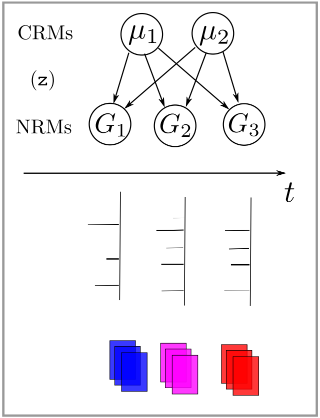
- Following [James et al., 2005], introduce auxiliary variables u_t $\forall t \in \mathcal{T}$
- Conditionally marginalize the CRMs μ_r to obtain a generalized CRP
- Alternately resample partition given $\{u_t\}$, and then $\{u_t\}$ given partition.
- Works for SNRM and MNRM, but impractical for TNRM.

Inference (a slice sampling approach)

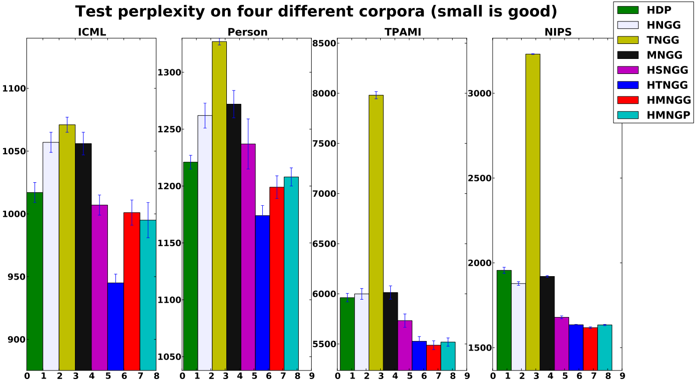
- Following [Walker, 2007], introduce auxiliary variables $s_r \; \forall r \in \mathcal{R}$
- Instantiate atoms of the CRM μ_r larger than s_r
- Conditionally sample partition of observations, and associated $z's$
- Alternately resample μ_r given $\{s_r\}$, and then $\{s_r\}$ given μ_r.

Application: Document modelling

- Four corpora of documents (ICML, Person, TPAMI, NIPS)
- Documents organized by year.
- Largest corpus: NIPS
  - 17 years, 2483 documents, 3.28M words and a vocabulary of 14K
- Use a nonparametric topic model.



Application to document modelling



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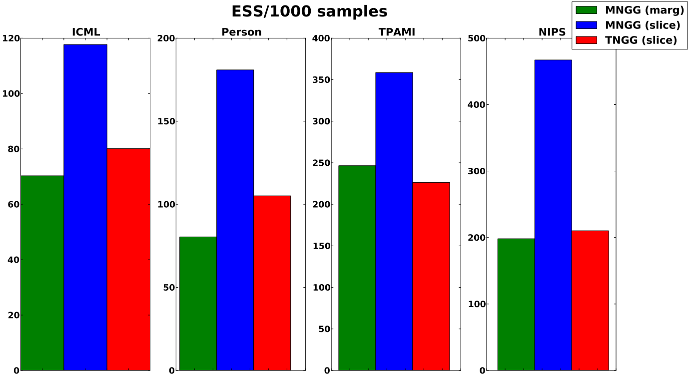

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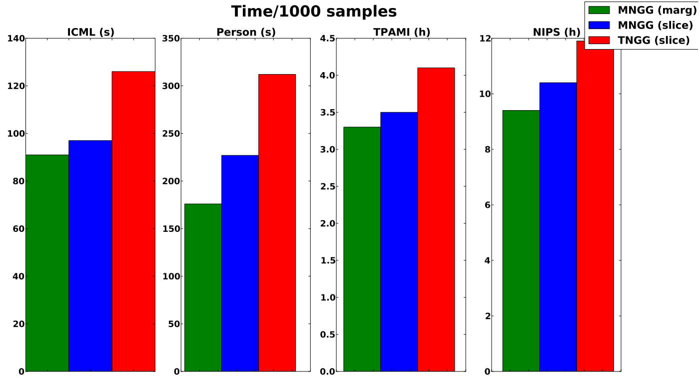

Thank you!

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