CONSTRUCTING MEANING FOR THE CONCEPT OF EQUATION

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ABSTRACT

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This study addresses itself to the development of a teaching-learning scheme for first degree equations in one unknown and to the examination of the way in which students think about and understand the concepts involved, during the actual teaching process.

The teaching-learning scheme, based on learning theories of Piaget, Steffe, Herscovics, and Bruner, attempts to guide the student in constructing meaning for the new algebraic form (an equation) on the basis of his existing arithmetic knowledge by introducing "arithmetic identities" and transforming them (by hiding a number) into equations. It focuses on three main constructions: extending the notion of the equal sign, constructing meaning for the concept of equation, constructing meaning for the rule, "Do the same thing to both sides." (Solution processes are beyond the scope of this study).

The methodology used, a version of the Soviet "teaching experiment," studies teaching and learning processes simultaneously. Six students of various abilities from grades 7 and 8 are interviewed individually. The two sessions with each subject are audio-taped and transcribed completely for purposes of analysis. The analysis, which comprises the major part of this thesis, examines the subjects'

thinking about the concepts, focuses on their reactions to the presentation, compares their responses (both written and verbal), and presents common thinking patterns.

The analysis shows that all subjects are able to construct meaning for the concepts involved. Unexpected results lead to a new perspective of the operational nature of the students' thinking.

ACKNOWLEDGEMENTS

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I acknowledge the generosity of Concordia University and the Ministère de l'Education du Quebec in granting me fellowships. I am also extremely thankful to have had my research expenses covered by an FCAC grant to Professors V. Byers and N. Herscovics (#EQ-1286; 1978-1979) from the Ministère de l'Education du Quebec.

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INTRODUCTION

Having been a mathematics teacher at the secondary level for eight years, a department head, and also a consultant at the elementary level, I thought, as I was beginning this study, that I knew many of the pedagogical problems associated with both the teaching and the learning of school mathematics. However, having spent a year and a half on this project, I can now say that I am only just beginning to become aware of the enormous dimensions of these problems.

The difficulties that junior high school children can experience in trying to understand certain algebraic concepts, which I have in the past sometimes taken for granted as being simple or even trivial, proved not only to be greater than expected but also included problem areas which had never occurred to me before. One of the more glaring examples of this was the demand that I had often placed on my former students to manipulate algebraic expressions without having provided them with the means to construct sufficient underlying meaning not only for the formal language but also for the concepts involved.

Having acquired, as a result of doing this research, a new awareness of the cognitive problems faced by junior high school students with respect to algebra, I feel strongly about sharing it with others. Thus, it is hoped that this thesis may shed some light on both the teaching and learning of equations and will help other teachers, as I have been helped, so that together we may work towards making algebra more meaningful for students.

Following is an overview of each chapter of this thesis. (N.B.: Whenever the pronoun "we" is used in the thesis, it refers to the author).

Chapter I points out the widespread learning difficulties experienced by many high school students with algebra. Recent research is cited which shows that they have problems understanding both the concept of equation and the use of algebraic symbolism. An evaluation of some factors (i.e., textbook approaches and teachers' notions) contributing to these difficulties is included.

Chapter II introduces the theoretical framework used in developing an alternate approach to the teaching of equations. These theoretical considerations relate the construction of meaning to Piaget's theory of equilibration, describe the various modes of understanding involved, relate these modes of understanding to Bruner's modes of representation, and describe the teaching-learning model "Didactic Reversal."

Chapter III suggests a teaching-learning scheme for first-degree equations in one unknown based on the theoretical notions described in the previous chapter. The teaching-learning scheme focuses on three basic constructions: extending the student's notion of the equal sign, constructing meaning for the concept of equation, constructing meaning for the rule, "Do the same thing to both sides."

Chapter IV describes the methodology and procedures used in the study. This includes an examination of several research methodologies and the justification for choosing the Soviet "teaching experiment" as the methodology most appropriate for this investigation. The chapter contains also a description of the procedures followed in selecting the subjects and in carrying out the experiment, including a list of the questions which were presented to the subjects during the study.

Chapter V analyzes the protocols of the individual interviews with the six subjects. It presents an examination of the way in which each learner was thinking about the new material during the actual teaching process. It also provides a comparison of the written and verbal responses of the students involved, including a description of the common thinking patterns which emerged.

Chapter VI presents a summary (of the first four chapters), draws conclusions (both experimental and general), and suggests some pedagogical implications and areas for further research.

CHAPTER I

STATEMENT OF THE PROBLEM

This chapter attempts to identify the pedagogical problems involved in the teaching and learning of the concept of equation at the junior high school level by looking at some recent research studies, teachers' notions and teaching methods, and current textbook presentations.

Some Research Findings

Leading mathematics educators have drawn our attention to the existence of serious pedagogical problems involved in the introduction of algebra. Easley has pointed out that "there are many high school students for whom the study of algebra presents immense learning problems ... many children don't understand the meaning of equations."[1] Davis has also referred to the great cognitive demands involved in the understanding of equations.[2] However, little research has been done in this area. As Weaver and Suydam have pointed out, "Rarely was there related research (on meaning and understanding) within the context of

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[1] Jack Easley, personal communication, February 20, 1979.

[2] Robert B. Davis, "Cognitive Processes Involved in Solving Simple Algebraic Equations," The Journal of Children's Mathematical Behavior, Vol. 1, No. 3 (Summer, 1975), p. 27.

secondary-school mathematics."[1] This has been confirmed by our computer search of the research literature.

Among the few studies done on the subject is one by Wagner which points out the widespread learning difficulties experienced by high school students in understanding the concept of equation.[2] As part of the research for her doctoral dissertation, Wagner presented 72 individual students with two equations,

$$7 \times w + 22 = 109 \text{ and } 7 \times n + 22 = 109,$$

and asked which solution would be larger, w or n. Some of the answers she received included, "the solution of the first equation is greater than the second one because w comes later than n in the alphabet"; "can't tell until both equations are solved"; "of course, the solution is the same." A student who responded that both solutions would necessarily be the same was said to "conserve equation," whereas a student who believed that either w or n would be larger was classified as a "nonconserver of equation." Fifty percent of the 12-year-olds and twenty percent of the 17-year-olds interviewed did not "conserve equation." These results point out the difficulties that many students have with one of the very basic notions in algebra.

In another study done in Nottingham in 1975, D. E. Firth tested 17 pupils, aged 14 and 15 years, on their ability to handle the symbolism

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[1] J. F. Weaver and M. N. Suydam, Meaningful Instruction in Mathematics Education (Columbus, Ohio: ERIC Center for Science, Mathematics, and Environmental Education), 1972, ED068329, p. 2.

[2] Sigrid Wagner, "Conservation of Equation, Conservation of Function, and Their Relationship to Formal Operational Thinking," Unpublished doctoral dissertation, New York University, 1977.

of algebra.[1] One of the questions he asked was:

x is any number

a) Write the number which is 3 more than x ...
b) Write the number which is 5 less than x ...
c) Write the number which is twice as big as x ...
d) Write the number which is 50% bigger than x ...

The percentages of wrong answers were 41%, 35%, 47% and 59% respectively. It is interesting to note that in part a) five of the seventeen pupils had each chosen a particular value for x and had given an answer three bigger than the value chosen. All five continued to use the same value for the remaining three parts of the question. The results of this study would seem to indicate that many students have very little grasp of algebraic symbolism and that they also lack confidence in the use of algebraic terms.

Because there have been so few studies done which focus on the student's understanding of algebraic concepts and his meaning for symbols, very little is known about how students learn, do, and understand algebra. As a result, teachers very often underestimate the learning difficulties encountered by students beginning the study of algebra. They also have some misconceptions about the thinking and understandings which students bring with them into high school.

Some Common Misconceptions

One such misconception centers on the student's notion of the meaning of the equal sign. That pupils interpret symbols differently

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[1] D. E. Firth, "A Study of Rule Dependence in Algebra", Unpublished M. Phil. thesis, University of Nottingham, 1975.

from adults has been shown by Ginsburg.[1] From his work with younger children, he points out that children's understanding of symbols refers to actions: Children do not view 3 + 5 = 8 as an arithmetic equivalence but rather operationally as their reading "3 and 5 make 8" indicates. A carry-over from this is evidenced among many seventh and eighth graders who consider that the right hand side of such arithmetic statements is for the answer only.[2]

Another study done in Grenoble, France, by Laborde (1978) also bears witness to the operational nature of the thinking of 12 and 13 year olds. Laborde found that students preferred to express the relation "n est le nombre de chiffres de a" in the form "Quand on compte les chiffres, qui forment le nombre a, on obtient n."[3] This obvious preference of the pupils for a "dynamic" rather than a "static" expression of the relation is explained in the following way by Laborde: "a et n ne sont plus indépendents du temps; n n'existe qu'après a, car il faut compter le nombre de chiffres de a pour trouver n."[4] This preference, even among high school students, for the dynamic rather than the static approach is something many teachers are not aware of.

Another misconception some teachers have regarding the previous

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[1]Herbert Ginsburg, Children's Arithmetic (New York: D. Van Nostrand Co., 1977), p. 90.

[2]N. Herscovics and C. Kieran, "Constructing Meaning for the Concept of Equation," The Mathematics Teacher, accepted subject to minor revisions.

[3]C. Laborde, "Relations Arithmétiques -- Aspect Statique -- Aspect Dynamique," Educational Studies in Mathematics 9 (1978), p. 41.

[4]Ibid.

knowledge of their new high school students is the presumption that these children have a good grasp of the concept of an unknown. Even though these children have learned to handle in elementary school such problems as 2 + ☐ = 7 or ☐ x 3 = 6, their view of the empty box is nothing more than a blank -- a space in which to write the solution. The box acts as a placeholder, and the student writes the solution to the equation in the box, rather than equal to the box.[1]

Some teachers also mistakenly interpret the pupil's ability to solve 2·☐ = 18 as an indication of an algebraic process skill. In fact, the student is merely plugging in the correct number by using the arithmetic facts stored in his memory. That no algebraic process is involved shows up later when the same child cannot solve similar equations containing larger whole numbers, integers, or rational numbers. Any solution process which is dependent on the numbers used cannot be considered an algebraic process, for according to Petitto, "in contrast to arithmetic, algebra is not a matter of learning new manipulations of numbers but involves, rather, the formulation and manipulation of formal statements whose numerical content is relatively incidental."[2]

However, it is precisely this "manipulation of formal statements" referred to by Petitto which causes many of the cognitive problems of

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[1] Sigrid Wagner, "Conservation of Equation and Function and Its Relationship to Formal Operational Thought," Paper presented at the annual meeting of the American Educational Research Association, New York City, 1977, p. 4.

[2] Andrea Petitto, "The Role of Formal and Non-Formal Thinking in Doing Algebra," Paper presented at the annual meeting of the American Psychological Association, Toronto, 1978, p. 1.

high school students who are just being introduced to algebra. For the most part, algebra is taught formally in the sense that students are expected to gain understanding by manipulating algebraic forms which are still meaningless to them. Gertrude Hendrix has taken issue with "the current practice that the way to develop a concept is to show example after example of the thing, at the same time repeating a word for the thing."[1] One has only to look at some of the textbooks currently in use to see that this is so.

Some Current Textbook Approaches

One common approach to the teaching of equations is based on the assumption that through the manipulations involved in solving equations, the student will be able to construct some meaning for the algebraic form. In a widely used textbook, Modern Algebra, Book 1, which we view as representative of the ones in current use, the authors, after defining "variable" and "open expression" (i.e., 7 x n), approach equations in the following way:

> An equation, such as x + 4 = 6, which contains one or more variables is called an open sentence. An open sentence is a pattern for the different statements -- some true, some false -- which you obtain by replacing each variable by the names for the different values of the variable .... The set that consists of the members of the domain of the variable for which an open sentence is true is called the truth set or the solution set of the open sentence over that domain.[2]

These definitions are followed by several exercises, such as: "Solve

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[1]Gertrude Hendrix, "Prerequisite to Meaning," The Mathematics Teacher (November 1950), p. 335.

[2]M. P. Dolciani and W. Wooton, Modern Algebra, Book 1 (Boston: Houghton Mifflin Co., 1970), p. 44.

each open sentence over the given set," or "Substitute the members of the given replacement sets in the open sentence and tell whether the resulting statements are true or false."[1]

Is it possible for any but a few students to derive any meaning from such a presentation? Not only is the concept defined and given a name before the student has developed any awareness of it, but the entire presentation is excessively formalistic and unnecessarily overloaded with unfamiliar and premature terminology. Rather than trying to create in the student some meaning for the new mathematical form of $x + 4 = 6$, the authors seem to hope that, by giving the student the name for and several examples of equations, they can create understanding through practice. Yet Ausubel has stated that "much more can obviously be apprehended and retained if the learner is required to assimilate only the substance of ideas rather than the verbatim language used in expressing them."[2] This technique of repetition and practice in order to teach a concept, while it may be effective with some students, with others, will amount to meaningless manipulation of meaningless symbols.

Another common approach to making algebra meaningful is the "word problem" approach whereby the student is expected to gain understanding of equations by working with "word problems." But here we must distinguish between "meaning" and "relevance." Word problems may give relevance to equations, but they don't necessarily give meaning. In a recent

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[1] Ibid., pp. 45-47.

[2] David P. Ausubel, "Facilitating Meaningful Verbal Learning in the Classroom," Mathematics Teaching and Learning, ed. Jon L. Higgins (Worthington, Ohio: Charles A. Jones Publishing Co., 1973), p. 203.

study done at the University of Massachusetts, Clement, Lochhead, and Soloway found that even science-oriented college freshmen have a great deal of difficulty translating word problems into equations.[1] In one of their test questions: "Write an equation using the variables S and P to represent the following statement: 'There are six times as many students as professors at this university'," 37% answered incorrectly. Since such college freshmen, whom we can presume know what an equation is, have difficulty translating simple word problems into equations, this implies the existence of cognitive problems particular to the translation process. Thus to try and construct meaning for the concept of equation through word problems may create relevance, but is also likely to compound the difficulties encountered by the high school student in the acquisition of the concept of equation.

Another difficulty which arises with the "word problem" approach is the "Let's use the letter x to represent the number" aspect. Some students seem to have difficulty thinking this way, as has already been pointed out by Firth. Further evidence has been provided by Davis in his recounting of an interview with a very bright seventh-grader who was not recognizing that "x was some number."[2] Van Engen also has pointed out that "the symbol x represents a class of numbers which may cause

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[1]John Clement, Jack Lochhead, and Elliot Soloway, "Translating Between Symbol Systems: Isolating a Common Difficulty in Solving Algebra Word Problems" (Cognitive Development Project), Unpublished manuscript, University of Massachusetts, 1979.

[2]Davis, "Cognitive Processes Involved in Solving Simple Algebraic Equations," p. 17.

considerable difficulty"[1] to many algebra students. When we say, "let's use the letter x to represent the number," we are confronting the students with a mathematical form totally new to them. Stanley Bezuszka has recounted the experience of one teacher saying to his class, "Let x be any number," and a student responding with, "Sir, is that the same as saying, 'let 3 be any letter'?"[2]

Another method of presentation suggested by some textbooks is the teaching of 1st degree equations within the framework of functions.

For example: Given the function $\alpha : \mathbb{Z} \longrightarrow \mathbb{Z}$

$$x \longmapsto \frac{x}{2} + 3$$

If the image of this function $\frac{x}{2} + 3$ is 0, find the solution of this equation.[3]

Perhaps this method is attractive to some teachers because of its mathematical elegance, but it can be disastrous for the students. To follow this approach, the junior high school student must first understand functions before learning about equations. Because the concept of function involves a higher level of abstraction than equation,[4] this

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[1] Henry Van Engen, "The Formation of Concepts," The Learning of Mathematics, Its Theory and Practice, Twenty-First Yearbook of the National Council of Teachers of Mathematics (Washington: National Council of Teachers of Mathematics, 1953), p. 95.

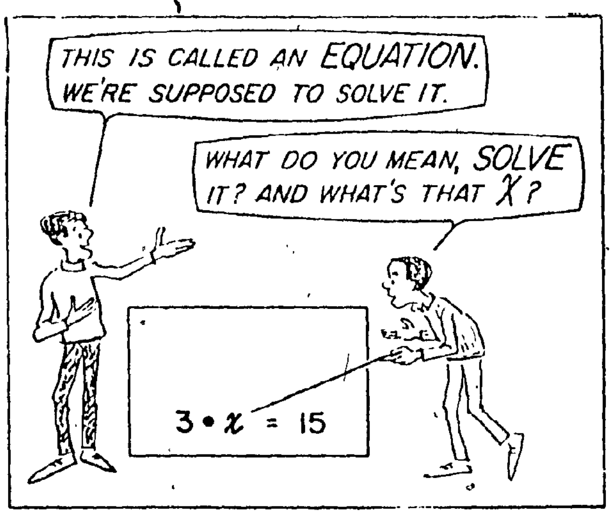
[2] Stanley Bezuszka, "Mathematics Literacy -- A Must or a Myth?" Communication, Mathbec '77, Montreal, Concordia University, May 1977.

[3] Denis LaBoissonnière, Cécile Lévesque and Reynaldo Rivard, Equation du 1er Degré à une Inconnue, Book in preparation, 1978.

[4] Sigrid Wagner, communication to N. Herscovics, December 1977.

approach may be unnecessarily detrimental for all but those who can follow such a formal approach.

Not every textbook fails to present equations meaningfully. There are some exceptions which have addressed themselves to some very basic questions. One such text which seems to recognize some of the cognitive problems which students face with algebra is Stretchers and Shrinkers, written by Braunfeld and the University of Illinois Committee on School Mathematics and mentioned in the NACOME report for its "pedagogical innovations."[1]



In the above illustration from Stretchers and Shrinkers,[2] we can see that the authors of this grade 7 course recognize that many children have difficulties with letters and even with the word "solve." But even

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[1] Overview and Analysis of School Mathematics Grades K-12, National Advisory Committee on Mathematical Education (Washington: Conference Board of the Mathematical Sciences, 1975), p. 32.

[2] Peter G. Braunfeld, Stretchers and Shrinkers (New York: Harper and Row, 1969), p. 71.

this text sidesteps asking or answering the question, "But what is an equation?".

Most textbooks, by and large, fail to come to terms with the cognitive difficulties encountered by students when they are faced with a new mathematical form. Thomas Kieren points out that:

> Most curriculums in mathematics ... were not developed using an analysis of how children or adolescents 'thought' about the subjects at hand or how they could go about building up systematic mechanisms for developing desired skills, concepts or abilities.[1]

However, it is not only the textbooks which are deficient.

Some Teacher-Related Aspects of the Problem

We asked some high school teachers (80) attending a workshop[2] on the teaching of equations how they would respond to a student asking the question, "But what is an equation?" Here are some of their answers:

1. C'est la symbolisation de la comparaison de termes équivalents.
2. C'est une formule qui permet de résoudre des problèmes concrets en mathématiques.
3. C'est l'affirmation de l'égalité de la valeur de deux énoncés mathématiques contenant des lettres.
4. C'est un système où il y a une inconnue dont on doit trouver une valeur numérique pour rendre véridique l'égalité.
5. C'est une façon de représenter une situation en état d'équilibre.

One cannot help but notice the formalistic expressions that most used for their answers. The same question was posed to a class of

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[1] Thomas E. Kieren, "The Rational Number Construct -- Its Elements and Mechanisms," Working Paper, p. 11.

[2] GRMS Convention (Groupe des Responsables en Mathématique au Secondaire), Sherbrooke, Quebec, June 1978.

mathematics teachers (22) at a local university who were taking courses in order to obtain a postgraduate degree in the teaching of mathematics. Here are some of the typical responses:

1. There are two kinds of sentences one can encounter -- open and closed. A closed sentence ($4 + 3 = 7$, $4 + 3 = 43$) may be true or false. It is an example of an equation. An open sentence ($n + 1 = 4$) leaves the individual to determine what values he may use to make the open sentence either true or false. An open sentence is also an equation.

2. An equation is a puzzle.

3. An equation is a sentence in the language of mathematics in which you have to fill in the blank [i.e., an equation in one variable].

4. I try to compare an equation with a see-saw and try to describe the balance structure in an equation. After introducing variables I show the equation as a statement which involves a combination of variables and constants that make a balance. I also compare an equation as a sentence structure with an expression, and point out that the main difference is a verb.

5. An equation is an algebraic statement involving known and unknown quantities. The known quantities are called constants; the unknown quantities are called variables. Our task is to find a value for the variable which will make the statement true.

The above two samples of teachers' answers to the student's question: "But what is an equation?" give further indication of the scope of the problem. High school mathematics teachers have mathematical abilities enabling them to achieve relatively easily a formal level of understanding. However, having acquired this level, it seems to be quite difficult for them to shift to a non-formal method of teaching, which may explain why so many topics are taught formally, even at the junior high school level. There is the added problem that once a mathematical concept has been understood, it has a tendency to seem trivially simple. This can prevent the teacher from seeing why his pupils don't understand something which he sees as being obvious. The above two problems explain why teachers cannot always evaluate properly the obstacles confronting the learner nor communicate at a level accessible to the learner.

Summary

Though there has been little research thus far on the high school student's understanding of algebraic concepts, those studies which have been done indicate that the conceptual difficulties involved in learning algebra are greater and more widespread than is commonly believed.

The problem is not simply a reflection of students' intellectual abilities, for as Skemp points out, there are many "pupils who, though intelligent and hard-working, 'couldn't do mathematics'."[1] It is rather a reflection of the way they are taught. Teachers and textbook writers are, on the whole, simply not aware of the cognitive obstacles confronting the learner. They have not addressed themselves to the question of what it means to understand an equation. Many high school mathematics teachers, to whom conceptual understanding comes fairly easily, fail to see why their students don't understand. They insist on very formal presentations, even at the junior high school level. They presume that the student entering high school has a good grasp of mathematical symbolism and notation, and fail to take into consideration the operational nature of many students' thinking.

Most of the textbooks currently in use simply don't deal with the problem. Their presentations, though they may be mathematically elegant, are lost on very many students because of their excessive formalism and

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[1]Richard R. Skemp, The Psychology of Learning Mathematics (Harmondsworth, England: Penguin Books, 1971), p. 15.

unwarranted mathematical jargon. The authors' needs to "state things correctly the first time around" and to be as general as possible in order to cover all future situations causes them to introduce prematurely terms such as "variable," when "unknown" would be more cognitively appropriate.[1] They also define a concept and give it a name, before creating an awareness for it in the student.[2] It does not take very long for some students to "turn off" mathematics, for they feel they are pushing around meaningless symbols. The approach of most texts that, through manipulations, the student will be able to construct some meaning for the algebraic form simply does not work for many students. It is our contention that a formal approach to the teaching of algebra leads to an "instrumental understanding," which Skemp defines as "rules without reasons,"[3] and will be meaningful to only a small number of students.

In this study we propose an alternate approach to the teaching of first degree equations in one unknown based on the student's need to construct meaning for the concept of equation.

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[1] D. E. Kuchemann, "The Understanding of Numerical Variables by Children Aged 12-15," Unpublished manuscript, March 1977.

[2] Hendrix, "Prerequisite to Meaning," p. 335.

[3] Richard R. Skemp, "Relational Understanding and Instrumental Understanding," Mathematics Teaching No. 77, December 1976.

CHAPTER II

THEORETICAL FRAMEWORK

This chapter will describe the theoretical framework of the teaching-learning scheme which we propose as an alternate approach to the teaching of first-degree equations in one unknown. These theoretical considerations will 1) relate the construction of meaning to Piaget's theory of equilibration, 2) describe the various modes of understanding involved using an expanded version of the Tetrahedral Model of understanding, 3) relate these modes of understanding to Bruner's modes of representation, and 4) introduce "Didactic Reversal," a teaching-learning model which integrates these theories.

How Does One Construct Meaning?

Thomas Kieren distinguishes between meaning and understanding in the following way: "meaning" applies to the process of building up or developing concepts, whereas "understanding" applies to the development and maintenance of interconnections and applications of ideas back to the realm of facts in which they are rooted, in order to be constantly tested.[1] However, the construction of meaning for algebraic concepts rests on a fair amount of previously acquired mathematics and requires the constant going back and forth between the new concepts and the facts (the arithmetic constructs) upon which they are built. Thus, in the

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[1] Kieren, "The Rational Number Construct -- Its Elements and Mechanisms," p. 3.

context of our work, the "construction of meaning" involves both the development of concepts and their interconnections or, in Kieren's words, both "meaning" and "understanding."

Piaget wrote in To Understand is to Invent that "every new truth to be learned must be rediscovered or at least reconstructed by the student, and not simply imparted to him."[1] But how does one construct meaning for a concept? This question must be answered within the framework of cognitive growth. Piaget has identified the major factors contributing to the development of cognitive growth of children as including: 1) maturation, 2) experience, 3) language, and 4) equilibration. Of these four factors, "Piaget considers equilibration to be the most fundamental to the growth of mathematical concepts."[2] According to Flavell, equilibration is the process of bringing assimilation and accommodation into balanced coordination. Assimilation refers to the process by which new events are integrated into existing mental structures. The complementary process of accommodation concerns the resulting changes in these mental structures.

Thus Steffe has defined learning, in this context, as the process by which new information is assimilated into available cognitive structures and to the modification of those structures. "In fact learning is conceived of as being possible only when there is active assimilation

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[1]Jean Piaget, To Understand is to Invent (New York: The Viking Press, 1973), p. 15.

[2]Leslie P. Steffe and Charles D. Smock, "On a Model for Learning and Teaching Mathematics," Research on Mathematical Thinking of Young Children, ed. Leslie P. Steffe (Reston, Virginia: National Council of Teachers of Mathematics, 1975), p. 10.

and accommodation."[1] Emphasis is on the activity of the learner. However, one factor to bear in mind is that the learner can assimilate only those things which past experiences have prepared him to assimilate. Flavell states further that if there is a gap between the old and the new, the new object cannot be assimilated.[2] "In this sense learning is possible whenever the more complex structure to be learned is based on available, simpler structures."[3]

If a learner is confronted with a new event which he cannot link up with what he knows, a certain number of modifications will have to be made to the existing mental structures in order for the learner to assimilate the new information.

> When a subject says, 'I do not understand', he means that the new object in front of him is too complex for him to adjust, or is presented to him in such a manner as not to enable him to proceed easily to an adjustment while the expression, 'Now I understand', means that the appropriate modifications have been made and the subject is able to integrate the unfamiliar object.[4]

As has been pointed out above, a learner may sometimes have difficulty in linking up a new event with his existing knowledge because the new material has not been presented to the learner in a manner easy

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[1]Leslie P. Steffe, "Constructivist Models for Children's Learning in Arithmetic," Paper given at the Research Workshop on Learning Models, Durham, New Hampshire, 1977, p. 5.

[2]John H. Flavell, The Developmental Psychology of Jean Piaget (New York: D. Van Nostrand Co., 1963), p. 50.

[3]Steffe, "Constructivist Models for Children's Learning in Arithmetic," p. 5.

[4]Gerald Noelting, "Constructivism as a Model for Cognitive Development and (Eventually) Learning," Paper presented at the Second International Conference for the Psychology of Mathematics Education, Osnabruck, September 1978, pp. 205-206.

to assimilate. In this connection, Herscovics has distinguished between two possible approaches for introducing new material: "Starting from the new topic, one can transform it to reach the student's cognition, or starting from the student's cognition, one can transform it to reach the new topic."[1] As an example of "transforming the new topic to reach the student's cognition," we can look at the way equations are treated in most textbooks: the student just beginning the study of equations is confronted with a new mathematical form, such as $3x + 2 = 17$, which he must manipulate in order to find the solution $x = 5$. It is only after the verification process when he sees $3 \cdot 5 + 2 = 17$ that there is any attempt made to connect equations ($3x + 2 = 17$), the new topic, with the student's old knowledge, arithmetic identities ($3 \cdot 5 + 2 = 17$). Such an approach seems "heavily weighted towards accommodation"[2] and, as such, is often the cause of the student's inability to construct meaning.

On the other hand, one can choose the alternate approach of "transforming the student's cognition" by beginning with the arithmetic identity, $3 \cdot 5 + 2 = 17$, a concept which already exists in his cognition, and constructing from it, by means of gradual transformations performed on the student's cognition, the concept of equation. This approach, which will be described in detail in the next chapter, may be seen as showing "a relative preponderance of the assimilative component (of

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[1] Nicolas Herscovics, "A Learning Model for Some Algebraic Concepts," Explorations in the Modeling of the Learning of Mathematics, ed. K. Fuson and W. Geeslin (Columbus: ERIC Clearing House for Science, Mathematics, and Environmental Education, 1979), in print.

[2] Flavell, The Developmental Psychology of Jean Piaget, p. 49.

equilibration)."[1] It allows the student to construct meaning for the new concept because it builds upon a simpler or equivalent idea in the student's existing mental structures and transforms them in such a way that there are no gaps between the old and the new material.

As has already been mentioned by Flavell, it is the presence of gaps which prevents new material from being assimilated. Sometimes these gaps are content-related, sometimes they are form-related, sometimes both. By the process of constructing meaning for a concept, we are attempting to bridge the gaps in content and at the same time give meaning to the form (notation and symbolism). However, the order in which this is done can be of crucial importance for the acquisition of learning.

Modes of Understanding and Modes of Representation

The pedagogical implications of distinguishing between content and form have been pointed out by Byers: "Teaching for understanding requires that the continuity of mathematical content be demonstrated to the student during, and prior to, the introduction of new mathematical forms."[2]

Bruner has also discussed the futility of premature formalism:

> But it is futile to attempt ... formal explanations distant from the child's thinking ... or the child learns not to understand

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[1] Ibid.

[2] Victor Byers, "Essays in Mathematics Education, Part 2," Unpublished manuscript.

> ... but rather to apply certain devices or recipes without understanding their significance or connectedness.[1]

John Biggs too refers to the same problem:

> If children are presented with symbols before they have abstracted the concepts that the symbols, or the operations upon the symbols, represent, the only way they can deal with them is by rote.[2]

Biggs further states that:

> In algebra the child is dealing with symbols that are one or several stages removed from the facts to which they are related ... and ... unless he knows what concepts the symbols stand for ... and understands the logical structure of his operations, and is aware of their conventional notation, there is clearly no connection between the top row of squiggles and the bottom row. No amount of symbol manipulation can establish the linkage between notational and logical relationships.[3]

In this regard, Dienes has suggested that, since it is very easy to forget what the symbols stand for, we should be going back and forth more often between symbol and symbolized. He further recommends that, in order to avoid situations where symbol manipulating takes the place of understanding, "we need to make sure that at every stage there is a possibility of feedback into the original experiences from which these structures were abstracted."[4]

This going back and forth between the symbol and the symbolized

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[1]Jerome S. Bruner, The Process of Education (New York: Random House, 1963), p. 38.

[2]John Biggs, "The Psychopathology of Arithmetic," New Approaches to Mathematics Teaching, ed. F. W. Land (London: Macmillan and Co., 1963), p. 61.

[3]Ibid., p. 63.

[4]Z. P. Dienes, "Research in Progress," New Approaches to Mathematics Teaching, ed. F. W. Land (London: Macmillan and Co., 1963), p. 49.

can be considered in terms of seeing the relationship between the content and the form, between the idea and the notation. The relationship between the two has been described in some detail by Byers and Herscovics in their Tetrahedral Model of understanding,[1] which is based on Bruner's distinction between intuitive and analytic thinking and on Skemp's definition of instrumental and relational understanding. Byers and Herscovics define "formal" understanding as the ability 1) to connect mathematical symbolism and notation with relevant mathematical ideas, and 2) to combine these ideas into chains of logical reasoning. However, only the first part of their definition is relevant to our work, since the second part applies mainly to proofs. Thus when we speak of formal understanding, we mean understanding of form, i.e., the connection of the symbol with the idea.

In addition, the authors define three other modes of understanding, "instrumental," "relational," and "intuitive," all of which are relevant to this study. Their definition of "instrumental" understanding: the ability to apply an appropriate remembered rule to the solution of a problem without knowing why the rule works, is a restatement of Skemp's definition, "rules without reasons."[2] An example of this type of understanding would be the child who can manipulate symbols but who lacks any awareness of the underlying ideas or relationships.

The third mode of understanding, "relational" understanding, an

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[1]Victor Byers and Nicolas Herscovics, "Understanding School Mathematics," Mathematics Teaching, 81 (December 1977), pp. 24-27.

[2]Skemp, "Relational Understanding and Instrumental Understanding."

expansion of Skemp's "knowing what to do and why,"[1] is defined by Byers and Herscovics as the ability to deduce specific rules or procedures from more general mathematical relationships. However, we feel that there is a need, for the purposes of our study, to extend this definition slightly. We believe that it ought to include the ability to establish relationships between concepts and to make connections.

The fourth mode of understanding defined in the Tetrahedral Model is "intuitive" understanding: the ability to solve a problem without prior analysis of the problem. This definition is restricted to problem-solving and we wish to extend it to concept formation. Just as in problem-solving, the intuitive understanding involved in the construction of a concept involves prior experience since such a construction is based on previously acquired knowledge. Intuitive understanding of a concept implies the ability to visualize and leads to a global perception. However, as has been pointed out, a premature use of symbolism may create obstacles to a global perception since it may detract from the "continuity of content" by focusing on the symbolism instead of the concept it represents. This continuity of content thus involves non-symbolic modes of representation.

Bruner has described three modes of representation: enactive (acting out through concrete operations), iconic (involving images), and symbolic.[2] When related to modes of understanding, the enactive and

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[1] Ibid.

[2] Jerome S. Bruner, Toward a Theory of Instruction (Cambridge: Harvard University Press, 1966), p. 11.

iconic modes can be associated with intuitive understanding and the symbolic with formal understanding. In the usual course of a young child's development, he moves through the three representations in this order. However, it has been our experience that at the junior high school level, many students need to go through these three stages in order to develop meaning for the symbolic mode of representation.

We now introduce a teaching-learning model, "Didactic Reversal," which attempts to achieve accommodation through assimilation by integrating the various modes of understanding and representation.

"Didactic Reversal"

"Didactic Reversal" is a teaching-learning model originated by Herscovics[1] which incorporates the preceding theoretical considerations and it will be used to a large extent as a model for our study.

According to this model a teaching scheme must identify first and foremost the essence of whatever concept is to be acquired and then determine the level of the student's cognition, since it must always be the starting point of any construction.

This is then followed by determining a sequence of intermediate subconcepts which will expand the student's cognition and eventually link up with the new concept. In order to help the learner in this construction, each intermediate subconcept must be analyzed in terms of its relative assimilative nature. These assimilative questions may involve an extension of content or a problem of representation. However, at

[1]Herscovics, "A Learning Model for Some Algebraic Concepts."

each intermediate step, the intuitive modes of representation precede the symbolic one, thus encouraging intuitive and relational modes of understanding and leading to the formal mode. Thus the formal mode acquires a representative character since every intermediate subconcept is first expressed in other forms. Of course, this can apply to the main new concept if it involves a new mathematical form. It is only when this new mathematical form has acquired meaning through this construction that the process is reversed (whence the name "Didactic Reversal") and that the learner is encouraged to find the form which existed previously, thus relating the new mathematical form back to the concept upon which it was built.

The principles of "Didactic Reversal" can be applied to the teaching of equations. As has already been pointed out, standard presentations of the teaching of equations begin with an equation, such as $3x + 5 = 17$, a new and possibly meaningless algebraic form, which through manipulation produces the solution, $x = 4$. Then through verification, which yields the arithmetic equality, $3 \cdot 4 + 5 = 17$, the student ends up with a simpler known form, the arithmetic equality. This beginning with the new form can cause a cognitive disequilibrium. Attempts to make the new form meaningful by relating it back, after the fact, to the old known form can produce a huge accommodation problem for some students. "Didactic Reversal" suggests starting the other way: beginning with the known form which exists in the student's cognition, the arithmetic equality ($3 \cdot 4 + 5 = 17$), and gradually building up to the new algebraic form ($3x + 5 = 17$) by means of a set of assimilative intermediate subconcepts. When the new algebraic form, the equation, has acquired meaning, the student can then work with it (i.e., solve it) and relate it back to its arithmetic form. The learning of equations is thus no longer an accommodation problem, but rather a process of "accommodation through assimilation."[1]

The next chapter will integrate all of the theoretical considerations, which have been herein described, into a teaching-learning scheme for first-degree equations in one unknown.

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[1] Ibid.

CHAPTER III

A TEACHING-LEARNING SCHEME FOR EQUATIONS

Following is a teaching-learning scheme which attempts, in Part 1, to construct meaning for the concept of equation, and, in Part 2, to lay the groundwork for the eventual justification of the algebraic operations used in the solution of equations. The development of both of these aims will be based on the theoretical considerations discussed in the previous chapter.

PART I: Constructing Meaning For the Concept of Equation

Preliminary Considerations

In order to construct meaning for the concept of equation, we begin by 1) identifying the essence of the concept to be acquired, and 2) determining the level of the student's cognition relative to the concept involved. Thus we first ask ourselves, "What is a first-degree equation in one unknown?" and then, "What is the student's level of cognition in this area?"

An answer to the first question might be the following: An algebraic equation is a mathematical form which involves the notions of equality and unknown. However, it is a new mathematical form for most junior high school students -- new because of the presence of an unknown, and new because of the extended sense of the equal sign. Even if the students have seen letters in equalities before at the elementary level, their meaning of the letter (as placeholder) may not at all correspond to the algebraic interpretation of the unknown, as pointed out by Wagner[1] and Davis.[2] Secondly, their sense of the equal sign is rather primitive, according to Ginsburg,[3] Davis,[4] Herscovics and Kieran,[5] and as such may be insufficient to allow them to attach any meaning to equations such as 3x + 4 = 6 - 10x, where the right hand side doesn't give merely a single-number result.

Thus one must find a way to link up the students' existing mental structures with the new concept of equation. They are familiar with arithmetic equalities of the form, 3 x 9 = 27 or 10 + 2 = 12, whereas they are not familiar with algebraic equalities of the form 3x + 4 = 6 - 10x. However, there is a relationship between the two: the algebraic form can be viewed as another representation of the arithmetic form. Thus by means of appropriate transformations performed on the arithmetic form, the algebraic form can be integrated into the student's mental structures.

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[1] Wagner, "Conservation of Equation and Function and Its Relationship to Formal Operational Thought," p. 4.

[2] Davis, "Cognitive Processes Involved in Solving Simple Algebraic Equations," p. 17.

[3] Ginsburg, Children's Arithmetic, p. 90.

[4] Davis, "Cognitive Processes Involved in Solving Simple Algebraic Equations," pp. 18-19.

[5] Herscovics and Kieran, "Constructing Meaning for the Concept of Equation."

Extending the Notion of the Equal Sign

The first transformation that must be made is the expansion of the student's notion of the equal sign. Robert Davis has pointed out that "the ability to use the equal sign in several different ways is one of the cognitive demands"[1] of working with equations at the high school level.

For most students entering junior high school, their notion of the equal sign is fairly primitive. As has already been mentioned, if students are asked to give an example of an equality, several will give one with an operation on one side and the result on the other. This is a carry-over from the way they viewed the equal sign in elementary school. From his work with younger children, Ginsburg points out that children's understanding of symbols refers to actions. As stated in Chapter I, they do not view 3 + 5 = 8 as an arithmetic equivalence but rather operationally, as their reading, "3 and 5 make 8", indicates. Ginsburg also found that many elementary school children, when asked how to read, □ = 3 + 4, would answer: "Blank equals 3 plus 4," but then add: "It's backwards!". They would then change it to 4 + 3 = □ and say: "You can't go, 7 equals 3 plus 4."[2] Ginsburg also found that many children "cannot read sentences that express relationships like 3 = 3 or 4 = 4."[3]

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[1] Davis, "Cognitive Processes Involved in Solving Simple Algebraic Equations," p. 27.

[2] Ginsburg, Children's Arithmetic, p. 84.

[3] Ibid., p. 85.

Davis also refers to the same problem when he says that elementary school language considers 3 + 5 to be a problem or a question and 8 to be the answer. By contrast, standard high school algebra language considers that 3 + 5 is "both an indication of a process" and also "a name of the answer." He states that many students who are still using elementary school language in high school are not prepared to accept 6x as "a name for the answer of what you get when you multiply 6 by x," but only as a statement of the task that "6 is supposed to be multiplied by x."[1] He comments further that, looked at cognitively, the equal sign changes meaning as children pass from kindergarten to grade 9. Both Ginsburg and Davis state that, at first, the equal sign is not symmetric, but is part of the statement of a question, so that primary grade children feel comfortable with 3 + 5 = 8 but not with 8 = 3 + 5 nor with 3 + 5 = 3 + 5. According to Davis, "what is necessary is reinterpreting 8 as 'a name of a number' and 3 + 5 as 'a name of a number' and 3 + 5 = 8 as saying that 'both 3 + 5 = 8 name the same number'."[2]

Although it is necessary to extend the meaning of the equal sign, this "renaming" of "3 + 5," within the trivial context of "3 + 5 = 8," may seem somewhat artificial to a student who is focusing on the arithmetic operation, especially when the result is staring him in the face. Our experience indicates that junior high school students do not look at 6 x 4 + 3 as "another name for 27."

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[1]Davis, "Cognitive Processes Involved in Solving Simple Algebraic Equations," p. 18.

[2]Ibid., p. 19.

We suggest that any first extension of the meaning of the equal sign must take into account the operational nature of the student's thinking and needs to be introduced in a non-trivial context, that is, with arithmetic equalities which do not contain the "answer" on either side. With at least one operation on each side, this allows for the interpretation of the equal sign to mean

BOTH SIDES YIELD THE SAME VALUE.

Such an approach will also allow for the construction of meaning for a much broader class of equations than those limited to a single operation. In fact, by extending the meaning of the equal sign to include multiple operations on each side, the students will be able to give meaning to equations involving multiple operations, e.g., $ax \pm b = c \pm dx$.

Thus this extension of the student's notion of the equal sign would begin by asking him (her) to give an equality with an operation on both sides. The possible cognitive disequilibrium caused by this question would be resolved when the student notices that both operations on the left and right side yielded the same number. Further extensions of this notion would be achieved by having the student construct equalities with a different operation on each side (if the first answer had involved the same operation on each side), with two operations on each side, and so on, leading up to multiple and different operations on each side, e.g., $6 \times 3 + 10 - 4 = 48 \div 2$.

After the students have constructed several of these multi-operation equalities and have thus extended their concept of the equal sign, the name "arithmetic identity" would be given to these arithmetic equalities in order to distinguish them from those equalities containing an unknown which will be called "equations." Some texts use the term "equation" interchangeably for both algebraic and arithmetic equalities, but this can lead to some confusion for the students. Even though we do not want to introduce any "unnecessary" new vocabulary, we feel that these arithmetic equalities should be given a specific name for they are such an integral part of the process of constructing meaning for the concept of equation. Giving a specific name will also help anchor the concept. The name "arithmetic identity" reflects the arithmetic nature of the equalities and the identical value borne by both sides.

However, we don't feel that the term "arithmetic identity" or any other new term which represents a concept should be introduced until after the concept has been acquired. This principle will also be adhered to later when introducing "equation" and "unknown," for as Ginsburg pointed out, "The child needs first ... to develop basic mathematical concepts; only after this has been done is there a need to introduce the relevant vocabulary, notation, etc."[1] Max Beberman also spoke of the same need "that the student become aware of a concept before a name has been assigned to the concept."[2]

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[1] Herbert Ginsburg, "The Case of Peter: Introduction and Part 1," The Journal of Children's Mathematical Behavior, Vol. 1, No. 1 (Winter, 1971-72), p. 67.

[2] Jerome S. Bruner, On Knowing (New York: Atheneum, 1965), p. 102.

Introducing the Concept of Equation

Having extended the student's notion of the equal sign by means of arithmetic identities involving multiple operations, we now use this new knowledge as a basis for introducing the concept of equation. The concept of equation is introduced as

AN ARITHMETIC IDENTITY WITH A HIDDEN NUMBER.

We must at this point take one of the student's arithmetic identities and hide a number. This hiding will be done in three stages which approximate Bruner's three modes of representation. By building up to the concept of equation from arithmetic identities, we are maintaining "continuity in the mathematical content" while constructing meaning for the new algebraic form.

The first stage in the process of hiding a number of an arithmetic identity involves covering a number with one's finger. At this stage the learner is actively involved; he is acting out the hiding of the number in accordance with Bruner's "enactive" mode of representation. This level may also be compared with the first step of Van Engen's suggested sequence in concept formation: "action-picture--symbol."[1] The name "equation" is now used to describe the arithmetic identity with a hidden number. However, the student's understanding of the concept of equation is still at the intuitive level in view of the mode of representation.

The second stage of the hiding process involves hiding the number

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[1]Van Engen, "The Formation of Concepts," p. 86.

with a box instead of one's finger. This second mode of representation may be compared with Bruner's "iconic" phase or Van Engen's "picture" stage in concept formation. Even though students are familiar with the box idea, it is not as a placeholder or blank that they are seeing it now, but rather as something which is hiding a number. Their understanding of the concept of equation is still intuitive; however, it is at a higher level because the mode of representation has moved a little closer to the symbolic.

The third hiding stage in the process of constructing meaning for the concept of equation entails hiding a number by a letter of the alphabet rather than a box. This symbolic representation of the hidden number brings the learner to a level of formal understanding, for he has constructed meaning for a new mathematical form. His understanding of the algebraic form is anchored in his arithmetic, for equations are conceived of as being arithmetic identities with a hidden number.

That the letter is hiding some number is obvious to the learner when this approach is followed. Students who are subjected to more standard presentations do not always have this awareness, as an excerpt from one of Davis' interviews will indicate:

"I: Now we multiply the right hand side by x. What do we get?

Henry: How can we multiply by x when we don't know what x is?"[1]

According to Davis, "It seemed clear that Henry was not recognizing that x was, in fact, some <u>number</u>."[2]

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[1]Davis, "Cognitive Processes Involved in Solving Simple Algebraic Equations," p. 17.

[2]Ibid., p. 22.

In our construction one begins with an arithmetic identity, hides a number, and thus obtains an equation. Since the letter is hiding some number, this construction of meaning carries inherently with it the notion of solution, which is obtained by uncovering the number hidden by the letter. When the letter is replaced by the number it is hiding, we have recovered the arithmetic identity. There is a reversal possible, that is, a going to and fro between arithmetic identity and equation and between equation and arithmetic identity. Thus the learner acquires a relational understanding of the concept of equation in the sense that he has established a relationship between two mathematical forms. He can now conceive of some kind of "equivalence" between these different mathematical representations.

Our approach which gradually formalizes the student's intuition allows him to confront mathematical concepts before they are hidden in a new symbolism. Bruner has stressed the need of understanding ideas intuitively first:

> It is only when such basic ideas are put in formalized terms as equations or elaborated verbal concepts that they are out of reach of the young child, if he has not first understood them intuitively .... Unfortunately the formalism of school learning has somehow devalued intuition .... It may be of the first importance to establish an intuitive understanding of materials before we expose our students to the more formal methods.[1]

Another aspect of our constructivist approach is its operational flavor. This is shown in the way that the meaning of the equal sign has been extended in accordance with the student's way of thinking. In addition, our definitions are dynamic rather than static (in the sense

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[1] Bruner, The Process of Education, pp. 13, 58, 59.

described by Laborde), that is, rather than stating "An equation is an arithmetic identity with a hidden number" (static), we define an equation: "When we hide a number in an arithmetic identity, we get an equation" (dynamic).

Giving a Name to the Letter

At this point we can give a name to the letter used in an equation. We question, however, the need at this stage of introducing the term "variable." Though it seems to be the current trend in most textbooks to use "variable" even for first-degree equations in one unknown, this usage may be premature and thus cognitively unsound. The concept of "variable" is of a higher order of abstraction than the concept of "unknown" and may be an extremely difficult concept for some junior high school students to grasp. Wagner has said that "a developmental factor may account for some students' failure to understand the full meaning of variables."[1]

Further research in this area has been done by Kuchemann who gave a 51-item algebra test to 3000 English secondary school children in order to identify the way children interpret letters in mathematics.[2] From the results of this test, he has identified five different levels of interpretation at which students function when working with letters:

a) Letter Ignored.

b) Letter as Object (at this level, letters can be manipulated without

[1]Wagner, communication to N. Herscovics.

[2]Kuchemann, "The Understanding of Numerical Variables by Children Aged 12-15."

being first evaluated, but are thought of as representing objects rather than numbers).

c) Letter as Specific Unknown (at this level the letter is regarded as a specific, albeit unknown, number which can be operated upon without first needing to be evaluated).

d) Letter as Generalized Number (at this level, the letter is seen as being able to take, and as representing a group of values rather than one value only).

e) Letter as Variable.

An interesting aspect of this description is that even when children see letters as representing numbers, there are three distinct levels of understanding. However, his most telling results describe the difficulties that the students had with those items where the letter was to be used as a variable. In an example of one such item where the students were asked, "Which is larger, 2n or n + 2?" the rate of success was as follows: 4% for 2nd year secondary students, 6% for 3rd year, 10% for 4th year and 30% for 5th year.[1]

Thus, in view of Wagner's and Kuchemann's findings, we feel that at the junior high school level the term "unknown" is far more conceptually adequate than "variable" and also corresponds more closely to the idea of a hidden number.

Examination of the Student's Interpretations

Having introduced the concept of equation, we must now examine the student's interpretation of this concept. This can be done within the context of having the student build several equations and at the same time asking him to explain how he is building them. This gives us

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[1] Ibid.

the opportunity to see how the learner is thinking about the new material.

This phase of the teaching-learning scheme must be relatively unstructured in order to give the student room to express his own ideas and, at the same time, to give us the flexibility to question each individual student on the particular examples he or she has presented. One should encourage the student to give a variety of examples in order to try to discover the student's thought processes and also to see whether or not he is placing any unnecessary restrictions on the concept of equation.

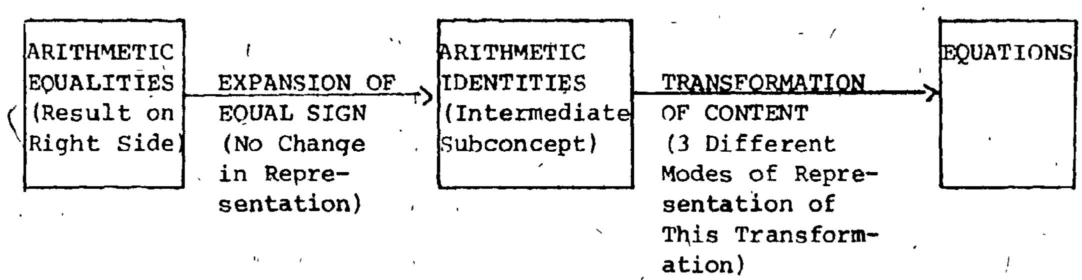
There are many possible unnecessary restrictions that should be watched for. Some students may give examples where the unknown is always on the left side (or right side), or always at the beginning (or end). Others may have a tendency to use always the same letter. This may only mean that the student is more comfortable with one particular letter. Nevertheless, it should be verified, in view of Wagner's findings, whether or not the student is "conserving equation" by asking him about an equation with a different letter hiding the same number.

One may also at this time ask the student if it's possible to make more than one equation from the one arithmetic identity. Some students may suggest hiding a different number. Others may ask if it's possible to hide more than one number (either the same number or different numbers). They should not be discouraged from any of these alternatives (although it would have to be pointed out that if one hides the same number twice, one uses the same letter, whereas if one hides different numbers, one uses different letters).

As a further indication of the student's interpretations of the concepts involved, one can ask the student to explain in his own words, what an equation is. This same question and others related to it can be asked again at a later time in order to determine what the student understands and to see if there has been any change in his thinking after a certain time lag.

Summary

Part 1 of this chapter involves constructing meaning for the concept of equation by starting with the existing mental structures of the learner and transforming these to link up with the new material. This is done by first expanding the student's notion of the equal sign. This achieves the integration of the intermediate subconcept of the arithmetic identity without requiring any change in representation. The next step, going from arithmetic identity to equation, includes both a transformation in content and in form. This transformation involves the change of the arithmetic identity into the equation, which is effected by following Bruner's three modes of representation. The diagram below outlines these transformations.



In Part 2 of this chapter, we will show how the rules used in the solution of equations can be induced from arithmetic identities.

PART 2: Laying the Groundwork for the Eventual Justification of the Algebraic Operations Used in the Solution of Equations

Introduction

Solving equations involves both the concept of reversibility of arithmetic operations and also the concept of "doing the same thing to both sides." We propose to lay the groundwork for the latter by having the student perform certain operations on arithmetic identities. The usefulness of arithmetic identities is not restricted to the construction of meaning for equations; they can also be used to induce the rules used in solving equations.

Many teachers introduce the idea of a scale to justify the algebraic operations used in solving equations. This may be a very good method to follow when one limits oneself to the simple operations of addition and subtraction of natural numbers. However, the scale does not lend itself readily to addition and subtraction of arbitrary rational numbers nor to the more complex operations of multiplication and division, for it is unlikely that high school students still think of these as repeated addition and subtraction.

The physical limitations involved with the scale can be avoided by the use of arithmetic identities. Arithmetic identities are a mathematical representation of the concept of equilibrium which is conveyed by the scale. Unlike the scale, however, arithmetic identities are not subject to physical restrictions. In addition, any operation performed on an arithmetic identity is immediately verifiable by the student. Furthermore, since our students now define equations as arithmetic identities with a hidden number, the operations performed on arithmetic identities can be transferred to operations on equations.

Inducing the Rules Used in Solving Equations

How can arithmetic identities be used to induce the rules involved in the solving of equations? We suggest beginning with one of the student's arithmetic identities and asking him what would happen if we added the number 2 to the right side. Is it still an arithmetic identity? How could he make it into an arithmetic identity again? The same questions are repeated with three different "add-ons." Following this, the student is asked if there seems to be some rule regarding the addition of numbers to arithmetic identities and if he thinks this rule would work with other different arithmetic identities.

We can then take the same arithmetic identity that we started with above (or a different one) and multiply the left side by some number. (It is understood that the student would have already learned about bracketing and order of operations). We then ask the student what has happened to the arithmetic identity and how we can make it into an arithmetic identity again. These questions would be repeated with three different multipliers, following which the student would be asked if there seemed to be some rule regarding the multiplication of arithmetic identities by different numbers. Again he would be asked if he thought that this rule applied to all arithmetic identities and to give an example.

The same line of questioning is used for the operations of subtraction and division. After the student has performed the above four operations on arithmetic identities and has come up with the rule for each different operation, we ask him if he can make up one rule which will cover the four separate rules. The student should be able to induce the rule:

WHATEVER YOU DO TO ONE SIDE, YOU HAVE TO DO TO THE OTHER SIDE ALSO.

This rule can then be applied to equations, since equations are seen as arithmetic identities with a hidden number. In this way the student can acquire a relational understanding of the operations used in solving equations, that is, he knows why he has to do the same thing to both sides. This is not meant to imply that the student must keep on using this method when solving equations and not take short cuts, for, as Skemp has pointed out, the student "does not ... have to derive (the rule) afresh everytime."[1] But having followed our approach, if the student later sees that transposing terms has the same effect, at least he will know why he is transposing terms and why this "short cut" works.

However, in this study, we do not go into the solution of equations. In this section we have only laid the groundwork for "doing the same thing to both sides" of an equation. The student must also learn eventually some strategy for solving equations. As with all problems whose solution involves multiple steps, a student may very well be able to justify each individual step without achieving the overview

[1] Skemp, "Relational Understanding and Instrumental Understanding."

needed to develop solution strategies. The development of solving strategies is beyond the scope of this dissertation.

In the chapter following, we shall describe the methodology of our study and include an outline of the questions (based on the teaching-learning scheme described in this chapter) which we asked our students.

CHAPTER IV

METHODOLOGY

The analysis of our teaching-learning scheme for equations requires that learning and instruction be considered simultaneously. This demands the use of a methodology that allows us to look at the way the learner understands and thinks about specific content during the actual teaching process. Thus in the first part of this chapter we review the various research methodologies available and justify our choice. The second part of this chapter describes the procedures used in our research.

PART I: Various Methodologies

The Psychometric Approach

According to Steffe, empirical studies in the tradition of psychometrics, research design, and statistics "have seldom dealt directly with the dynamics of human mathematical learning."[1] When mathematical behavior has been studied using these techniques, the focus has not been on the individual, but rather on the product of his learning, that is, on the written responses of the student.

Such educational research has centered on developing standard

[1] Steffe, "Constructivist Models for Children's Learning in Arithmetic," p. 1.

tests of mathematical achievement and on assessing the effects of various curricula on students' skills. Although such tests may be valuable for providing norms or in predicting fairly well students' performance on other similar tests, according to Ginsburg[1] and Opper,[2] their inherent inflexibility prevents them from providing any detailed information on children's mathematical learning or on their underlying mental structures.

In short, psychometric tests do not identify cognitive structures and processes. As pointed out by Easley, Piaget stated that such tests simply cannot provide "enough information to decide what structures are involved in a child's thinking."[3] Easley further mentions that Piaget and Inhelder later characterized such tests as giving only the "results of efficiency of mental activity without grasping the psychological operations in themselves."[4]

Since our aim is to investigate the learning processes involved in the acquisition of algebraic concepts, this particular methodology is quite obviously unsuitable.

The Analysis of Errors Approach

There is a common misconception that the right answer represents

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[1]Herbert Ginsburg, "The Case of Peter: Introduction and Part 1," p. 62.

[2]Sylvia Opper, "Piaget's Clinical Method," The Journal of Children's Mathematical Behavior, Vol. 1, No. 4 (Spring, 1977), p. 91.

[3]J. A. Easley, "The Structural Paradigm in Protocol Analysis," Journal of Research in Science Teaching, Vol. II, No. 3 (1974), p. 281.

[4]Ibid.

correct understanding and wrong answers represent either misunderstanding (if they occur often and consistently) or carelessness (if they occur sporadically). However, this is not necessarily so. Ginsburg has noted "that errors are seldom if ever trivial and meaningless; most often they reflect serious attempts to understand and are products of sensible approaches to a problem."[1] Thus errors should not be dismissed as merely the result of carelessness. On the other hand, right answers are not always indicative of correct understanding.

Erlwanger has investigated the nature of children's mathematical knowledge acquired in a program based on stressing "the right answer." In describing six case studies of children from grades 4, 5, and 6 in an I.P.I. (Individual Prescribed Instruction) program, he stated that "teachers' evaluation and diagnostic procedures, which focused largely on external mathematical behavior, were inadequate in revealing the children's underlying conceptions."[2] In addition, Erlwanger found that the teachers often misunderstood and misjudged the nature of the children's understanding and progress, and the adequacy of their learning experience. That one should view even right answers with caution has also been pointed out by Servais, who, as a result of working with students at the secondary level, remarked, "As long as a student does not alert us with a wrong answer, we have only a presumption of correct

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[1] Ginsburg, Children's Arithmetic, p. 68.

[2] Stanley H. Erlwanger, "Case Studies of Children's Conceptions of Mathematics -- Part I," The Journal of Children's Mathematical Behavior, Vol. 1, No. 3 (Summer, 1975), p. 158.

understanding."[1]

As evidence of this problem, we can look at the subject of one of Erlwanger's case studies, Benny, in Grade 6, who was judged by his teacher to be one of her best pupils in mathematics and who was making much better than average progress through the I.P.I. program. Benny would try several different methods in a trial-and-error fashion until he arrived at the one right answer of the I.P.I. key. Although he knew that an answer could be expressed in different ways, based on whichever method one used ($2 + .8 = 2\frac{8}{10}$, or $2 + .8 = 1.0$, or by the "picture" way $2 + .8 = 2.8$), he also knew that some of these might be wrong by the "key," if he hadn't been able to figure out which one was required for the "key." In any case, he firmly believed that all of his answers were really correct, "no matter what the key says."[2] Because he had been able to invent a multitude of techniques in order to get the right answer for the "key," he had in the process generalized many of these to form his own set of mathematical rules. Thus, although many of his own ideas were wrong, he was still able to be successful on his tests. Benny's case indicates that a seeming "mastery of content and skills does not imply understanding."[3]

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[1]W. Servais, "Humaniser l'Enseignement de la Mathématique," paper presented at Journée Internationale de l'Association des Professeurs de Mathématiques des Ecoles Publiques, Rennes, France, September 1976, p. 50.

[2]Stanley H. Erlwanger, "Benny's Conception of Rules and Answers in IPI Mathematics," The Journal of Children's Mathematical Behavior, Vol. 1, No. 2 (Autumn, 1973), p. 15.

[3]Ibid., p. 12.

As a result of Erlwanger's research, we can conclude that an analysis of the student's written work alone (both the right and the wrong answers) or the written work with only brief descriptions by the children will not give us any reliable indication of their understanding and as such is an inadequate methodology to be used for studying their mathematical understanding and thinking.

Piaget's Clinical Interview

Piaget developed the clinical interview as a method of exploring the thought processes of children of different ages. It has undergone some changes since its early days and has evolved to the partially standardized clinical method widely used by Piagetian researchers today.

The essential character of the clinical method is that of a dialogue or conversation held in an individual session between an adult, the interviewer, and a child, the subject of study. It is not a method that can be used in group-testing. According to Opper, it is a "hypothesis-testing situation, permitting the interviewer to infer rapidly the child's competence in a particular aspect of reasoning by means of observation of his performance at certain tasks."[1]

In order to allow for some comparability of results, the version of the clinical method commonly used today, particularly in replication studies of Piaget's work, is a partially standardized one. In this version, according to Opper, the subject is presented with a standard problem situation and is then asked a number of standardized questions.

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[1] Opper, "Piaget's Clinical Method," p. 92.

Once having presented these identical situations and questions, the interviewer may then conduct the experiment as he deems appropriate, thus retaining some of the freedom of the clinical method.

One of the advantages of the clinical interview is that it allows the interviewer to discover the learner's cognitive processes. Its flexibility of approach allows one to uncover patterns of thought which are inaccessible when one uses standardized testing procedures.

Though standard Piagetian experiments deal almost exclusively with informal mathematics rather than with the child's understanding of academic mathematics, this does not in any way detract from the value of the clinical interview as a research tool. It is an appropriate method for our study in the sense that it provides a way to discover how the student thinks and learns. However, since we wish to see how the child learns a particular mathematical topic within a specific teaching situation, we must use a methodology which incorporates the teaching component into the framework of the individual interview. Following is one such methodology.

The Soviet "Teaching Experiment"

In their introduction to Soviet Studies in the Psychology of Learning and Teaching Mathematics, Vol. I, Kilpatrick and Wirszup contrast the Piagetians who do not assign very much significance to the role of instruction in the development of the child but rather to the specific stages in the development of the child's thinking, with Soviet psychologists who ascribe a leading role to instruction.[1] The Soviet pedagogues study the development of thinking under the changing conditions of instruction. This philosophy carries over into their research which they constantly attempt to relate to the learning process as it occurs in school. Further to this, Easley has pointed out that "the curriculum, as it is, ... is accepted as the starting point of research. Improvements are sought within it, not by replacement of it."[2]

According to Menchinskaya, "Soviet instructional psychology concerns itself with the characteristic of mental activity <u>during instruction</u> and with the principles of the learning process itself."[3] In order to study the changes in mental activity under the influence of instruction, Soviet didacticians devised the methodology of the "teaching experiment."

Under one of its forms, "experiments of instruction," it involves the introduction of preliminary stages of study in which the initial information, abilities, and skills needed to master the new material are ascertained and arranged in a hierarchy. During the instruction which

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[1] Jeremy Kilpatrick and Izaak Wirszup (eds.), Soviet Studies in the Psychology of Learning and Teaching Mathematics, Vol. I: The Learning of Mathematical Concepts (Stanford, California: School Mathematics Study Group, 1969), p. v.

[2] J. A. Easley, "On Clinical Studies in Mathematics Education," Mathematics Education Information Report (Columbus: ERIC Science, Mathematics, and Environmental Education Clearinghouse, 1977), p. 22.

[3] N. A. Menchinskaya, "Fifty Years of Soviet Instructional Psychology," Soviet Studies in the Psychology of Learning and Teaching Mathematics, Vol. I, eds. Jeremy Kilpatrick and Izaak Wirszup (Stanford, California: School Mathematics Study Group, 1969), p. 5. (Emphasis added).

follows this planning stage, the experimenter investigates the learner's thinking and seeks to discover the changes within one or two stages of instruction. Menchinskava describes this form as being very advantageous in that it allows the experimenter to follow the particular changes in the mental processes of the same pupils on an individual basis.

Under another form, "experiments of assessment," they investigate children, who have already mastered certain concepts, on their ability to use them to solve problems. The same experimental task may be either given to students in different grades or given to students of one grade only, but at different stages of instruction.

These two forms are examples of the "experiencing" mode of the teaching experiment, where any one well-defined method of instruction is used. In both of these forms, the investigator "seeks to observe the 'dynamics' of the learning process."[1] Another mode of the teaching experiment is the "testing" mode where the investigator tries to use different methods to discover which of these promotes the most effective mastery of information.

When the Soviet researchers wish to look into the child's process of mastering and using concepts, they use the methodology of the individual experiment. On the other hand, when they wish to confirm the results of individual experiments on a wider quantitative basis, they use the collective experiment and written work.

However, according to Kantowski, the statistical analysis of

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[1] N. A. Menchinskaya, "The Psychology of Mastering Concepts: Fundamental Problems and Methods of Research," ibid., p. 89.

quantitative data is of less concern than the fairly subjective analysis of qualitative data.[1] Most studies deal with some aspect of the school situation, with the data often gathered from a sampling of "strong," "average," or "weak" students who are generally categorized and selected with the aid of the classroom teacher. The data collected are often qualitative, obtained in a clinical setting by recording verbal protocols for future analysis. In addition, the involvement with the same children can range over periods from about six weeks to the academic year.

The "dynamic" nature of the Soviet "teaching experiment" is one of its strongest points. Their method which is "primarily directed at disclosing and elucidating the very process of learning, as it takes place under the influence of pedagogy"[2] permits a researcher to study changes in mental activity as well as the effects of planned instruction on such activity and to "determine how instruction can optimally influence these processes."[3] Although the general "course" outline and content to be covered are determined in advance, the experimenter has the flexibility to plan a new instructional strategy for the following session on the basis of what occurred during the previous session. He

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[1]Mary Grace Kantowski, "The Teaching Experiment and Soviet Studies of Problem Solving," Mathematical Problem Solving: Papers from a Research Workshop, eds. L. Hatfield and D. Bradbard (Columbus: ERIC Science, Mathematics, and Environmental Education Clearinghouse, in press).

[2]Menchinskaya, "The Psychology of Mastering Concepts: Fundamental Problems and Methods of Research," p. 89.

[3]Kantowski, "The Teaching Experiment and Soviet Studies of Problem Solving."

also has the freedom to intervene, to introduce hints and to offer the right material at the right time. This methodology permits the researcher to observe how a subject is operating, to discover any erroneous concepts, and to determine levels of understanding rather than mere numbers of correct solutions.

The version of the Soviet "teaching experiment" which will be appropriate for our research is the "experiment of instruction," an example of the "experiencing form." According to this form, as has already been described, one ascertains which subconcepts are required in order to acquire the concept in question and arranges them in a hierarchy. During the instruction which follows, the interviewer, by means of individual interviews, seeks to discover the changes within the learner's thinking, and the effects of the planned instruction. Although the instructional theme is planned in advance, the interviewer has the flexibility, whenever he deems it necessary, to alter his plan or to pursue any unforeseen or interesting responses which may come out during the interview.

PART 2: Procedures

Selection of Subjects

The total number of students involved in this study is six: Barbara, Michel, Greg, Caroline, Piero and Patricia.

Barbara and Michel were the first two subjects we worked with, in November and early December, 1977. Barbara (Birth Date: March 10, 1965) was a grade 7 student in a large comprehensive high school in Brossard. Since she was a neighborhood girl known to the interviewer, permission to do the experimental work with her was sought directly from her parents. Similarly, permission to work with Michel (Birth Date: July 5, 1964), a neighborhood boy who was in grade 8 at a small private high school in Montreal West, was granted by his parents.

According to the methodology of the Soviet "teaching experiment," the subjects chosen are generally categorized by their classroom teacher. Since there was no contact between the interviewer and Barbara's and Michel's teachers, some other measure of their general mathematical ability had to be used. Thus the parents were asked how their children were doing in mathematics at school. As an added measure, the Henmon-Nelson Test of Mental Ability, Form B, 6-9, was administered to all six subjects.

Michel was judged by his parents to be generally quite weak in his school subjects, although they felt that he was a little better at mathematics than he was with his other courses. His I.Q., as rated by the Henmon-Nelson Test, was 93. Barbara's I.Q. was 129 and her parents thought that she was doing extremely well in all her courses at school.

The next two subjects were Greg (Birth Date: Oct. 1, 1963) and Caroline (Birth Date: July 27, 1964), with whom we worked during the latter part of November and December, 1977. Greg and Caroline were both in the same grade 8 mathematics class at Chambly County High School in St. Lambert. Permission to work with these two students was obtained, by means of their teacher, from the Mathematics Department Head, the School Principal, and their parents.

Though they were in the top stream mathematics class, their teacher felt that they were perhaps two of her weakest students. Thus she considered them to be of average ability. The same Henmon-Nelson Test of Mental Ability was administered to Greg and Caroline whose I.Q.'s were thereby rated at 106 and 113 respectively.

The last two subjects were Piero (Birth Date: Dec. 21, 1964) and Patricia (Birth Date: July 21, 1965), with whom we worked during January and February, 1978. Piero and Patricia were both in grade 7 at LaSalle Catholic Comprehensive High School in LaSalle. As with our previous two subjects, we had approached the mathematics teacher first to ask if she wouldn't mind our working with a couple of her students. She then requested formal permission from the School Principal and the students' parents.

Piero and Patricia were both in the average stream grade 7 mathematics class. However, Piero was judged by his teacher to be quite bright, whereas, Patricia was thought to be fairly weak. As with our other four subjects, we administered the same I.Q. test to Piero and Patricia who were rated at 123 and 96 respectively.

Thus our six subjects are representative not only of different abilities but also of fairly diverse mathematical backgrounds, coming from four different high schools, in widely separated parts of metropolitan Montreal.

Following is a table summarizing the above information on our subjects:

NAME	Michel	Patricia	Greg	Caroline	Piero	Barbara
AGE AT 1st INTERVIEW	13-4	12-6	14-1	13-4	13-1	12-8
GRADE	8	7	8	8	7	7
TEACHER OR PARENT RATING	weak	weak	average	average	strong	strong
I.Q.	93	96	106	113	123	129

We were primarily interested in having the teachers' (parents') ratings in order to ensure a broad sampling of ability. Even though the I.Q. test was given only as an added measure and was not to be considered as being any more valid or indicative of ability than teacher rating, it is interesting to note how closely the I.Q.'s of our six subjects related to the teacher or parent ratings.

Experimental Procedure

In our teaching-learning experiment with our six subjects, we used the individual interview. The first four subjects (Barbara, Michel, Greg, Caroline) had one interview session per week, whereas the last two subjects (Piero, Patricia) had two interview sessions per week.

The total number of sessions per subject varied from five to eight, depending on his/her mathematical strength. It must be stated here that only the first two sessions which deal with the construction of meaning for the concept of equation and the justification of the rules to be used in the solution of equations will be described and analyzed in this thesis. The remainder of the interview sessions are part of another study, a pilot project on solution processes, and will not be discussed herein.

The interview sessions with Barbara and Michel were held in the home of the interviewer during the weekends. The interviews with the other four subjects were held in their respective schools either during their mathematics period or during part of their lunch break. Each session lasted from 25 to 45 minutes.

Each interview was audio-taped on a cassette-recorder following which the tape was transcribed in its entirety and then analyzed. The analysis after each interview session allowed us to examine in depth the thinking of the subject, to get a better overall view of his learning and to see difficulties or subtleties which may not have been so obvious during the actual interview session. Having a complete transcription of the protocols also gave us the opportunity to compare the various reactions of each of our subjects on a particular topic. In addition, it permitted us to discover certain areas of thinking which merited further exploration and gave us the chance to make those changes which could improve the next presentation with another subject.

As a matter of fact we made a few significant changes in our line of questioning as a result of the first sessions with Barbara and Michel. These changes will be brought out in the analysis of the data. Another variation that we tried during the course of the experiment was the assignment of homework with our last two subjects, Patricia and Piero. We wished to have another source of information which could further indicate the way our subjects were thinking and learning.

Following are the questions which we asked our subjects during each interview session. The rationale for the structure of this teaching-learning scheme has already been described in Chapter III. It must be mentioned that every interview session necessitated some slight deviation from the prepared questions in order to allow for individual differences and to give the interviewer the flexibility to explore further any interesting or unforeseen events.

Prepared Questions

(a) Session 1

Pretest: Have you seen equations before?
Can you explain what an equation is?
Are your familiar with this sign (=)?
Can you tell me what the equal sign means to you?
Can you give me some examples?

Extending the Notion of the Equal Sign:

If I look at your examples, on one side I see an operation, and on the other side I see the result. (Discuss the meaning of the word "operation").
Can you use the equal sign with one operation on each side?
Can you give me some examples?
Can you give an example where you have a different operation on each side?
Can you give an example where you have more than one operation on each side?
Have you learned the use of brackets?
Can you give me an example where you use brackets?

Giving a Name to the Equalities Above:

Let's give a name to all of these equalities you've written. We're going to call them ARITHMETIC IDENTITIES.
Are you familiar with the word identical?
Tell me what it means when two things are identical.
Could you explain to me why it makes sense to call these equalities arithmetic identities?

Introducing the Concept of Equation:

Let's take one of your arithmetic identities.

Let's hide one of the numbers of this arithmetic identity with our thumb.
When I hide a number of an arithmetic identity, I then have an EQUATION.
Let's use a box now for the hidden number instead of my thumb. (Repeat previous example with a box □).
Can you take an arithmetic identity and make it into an equation?
Let's use something else now in our equations instead of boxes. Let's use a small letter of the alphabet. This is the way equations are normally written.
Can you do it?

Giving a Name to the Letter:

The letter hides a number. This letter we will now call an UNKNOWN.
Can you explain to me why it's called an unknown?
What is the unknown hiding?
What would we have if we uncovered the number hidden by the unknown?

Examination of the Student's Interpretations:

I would like you to build for me five equations. While you're doing them, I would like you to explain to me how you are doing them. (In this relatively unstructured sequence, we would encourage variety and at the same time watch for any unnecessary restrictions on the concept of equation).
Does the unknown always have to be on the left side? (or right side, or at the end or beginning).
Does the unknown always have to be the letter "n"? (or any other letter).
Could you make more than one equation from this arithmetic identity?
Could you hide more than one number (i.e., different numbers)?
Could you hide the same number twice?
What would happen if you hid the "5" on the left side and also the "5" on the right side? (or any other number).
In your own words can you explain to me what an equation is?

Homework: (Only for Piero and Patricia)

Write down 5 different arithmetic identities -- some with many operations on both sides -- and then make an equation from each arithmetic identity.

(b) Session 2

Review: (Each session would begin with a review, both oral and written, in order to verify the understanding acquired by the subject in previous session(s). This would also be the time to examine any assigned homework).

Can you explain to me what an arithmetic identity is?
Can you give me an example?
Can you explain to me what is meant by an equation?
Can you give me an example?
Can you explain what an unknown is?

Inducing the Rules to be Used in the Solution of Equations:

The simplest arithmetic identity is one with one operation on one side and the result on the other. Can you give me an example?
We're now going to use your example to build new arithmetic identities. What happens if I add 2 to the left side?
Is it still an arithmetic identity?
Using only addition, how can I make it an arithmetic identity again?
What happens if I add 7 to the right side?
Using only addition, how can I make it an arithmetic identity again? (Repeat the last two questions with the numbers 13 and 20).
If we can only use addition in building new arithmetic identities, is there any rule we have to follow?
Will this rule be true for all arithmetic identities?
Can you give me an example?
What happens if we multiply the left side by 2? (Keep in mind a possible problem with bracketing and order of operations).
Is it still an arithmetic identity?
Using only multiplication, how can we make it an arithmetic identity again? (Repeat the last three questions using the numbers 7 and 13).
If we can only use multiplication in building new arithmetic identities, is there any rule we have to follow? (Repeat the above sequence of questions for the operations of subtraction and division).
We now have a rule for addition, a rule for subtraction, one for multiplication, and one for division. Could you make one rule that will cover all of these operations?
Will it be true for all arithmetic identities?
Can you give me an example?

Homework: (Only for Piero and Patricia)

Write down five different arithmetic identities. Beside each one, build a new arithmetic identity using the rule we discovered today.

CHAPTER V

ANALYSIS OF INTERVIEWS

In this chapter, we will be examining the protocols of the individual interviews with our six subjects within the framework of the teaching-learning scheme set out in Chapter III. We will attempt to describe the thinking of the learner while he is in the process of acquiring the concepts involved. In addition, we will point out any changes in our own way of thinking or in our manner of presentation which may have occurred as the result of one of the interview sessions.

In the ensuing excerpts, the coding system for identifying the individuals involved is as follows: I: Interviewer, B: Barbara, Pi: Piero, M: Michel, G: Greg, Pa: Patricia, C: Caroline. In addition, three dots . . . indicate a pause, whereas three dashes - - - indicate that a certain part of the original protocol has been omitted because of lack of significance to the discussion at hand.

Pretest

The first part of Session 1 was devoted to trying to determine the levels of cognition of our subjects. We wished to discover what were the students' existing ideas on equations and the equal sign.

I: Have you seen equations before?
C: Yeh.
G: Yes.
Pa: Yeh, we've been doing them in class.
Pi: Yes, we're starting it with our teacher.

I: Can you explain what an equation is?
C: It's when . . . well if two things . . . like 1 + 2 = 2 + 1.
G: Well . . . you mean how it works? - - -
An equation is a problem, sort of. - - - Like something divided by something, or addition, or subtraction, or something times something else. You have to find the answer.
Pa: Um . . . I don't know how to say it, but I can give examples - - - like, n divided by 7 times 3 minus 4 equals, let's say, 245.
Pi: It's the rule you go by. You write down numbers; you add the signs there and then you work it out.
B: All the multiplications, additions, subtractions, and divisions?
M: No.

Both Greg and Caroline thought that "equations" meant arithmetic equalities. Greg indicated that it was something where "you have to find the answer." According to their teacher, they had not yet been taught equations in their algebra course.

On the other hand, Piero and Patricia had just begun in school to work with and solve equations of the type, $ax + b = c$, where the result "c," was always on the right hand side. They had been taught to solve these by a process of undoing from right to left, all in one step, that is, $x = \frac{c - b}{a}$.

Though we hadn't asked Barbara or Michel if they had seen equations before, when they were asked what an equation was, neither of them knew. They indicated that they had not yet learned algebraic equations nor had they attached the name "equation" to arithmetic equalities. Thus it seems that only Piero and Patricia had had some prior exposure to algebraic equations.

We next asked our subjects about the meaning of the equal sign, which they verbalized as follows:

I: What does the equal sign mean to you?
M: The same thing.
G: It means what the answer is.

C: It is, like, after the equal sign is when . . . it's hard to say - - - after the equal sign is the answer.
Pa: A total of numbers is going to come after that.
Pi: You have to have two or more numbers before them and then they work in some way with a sign, multiplication, or addition and then whatever the sum or product is, the equal sign comes before it. Meaning that whatever the two numbers mean with the sign, it equals so and so.

I: Could you give me an example where you use the equal sign?
B: 6 x 3 = 18, 9 + 17 = 26.
M: 4 + 3 = 7.
G: At the end of an equation - - - before your answer - - - 5 + 3 = 8 - - - 5 x 3 = 15.
C: In multiplication, subtracting - - - 2 x 3 = 6 - - - (2 x 3 - 1) = (3 x 1 + 2).
Pa: 2 + 5 = 7 - - - 4 + 3 = 7.
Pi: 6 + 2 = 8, 8 - 2 = 6, 8 · 2 = 16, 16 ÷ 2 = 8.

One cannot help but notice the difficulty that Caroline had with verbalizing an answer to the question of what the equal sign meant to her. Patricia, also, in her answer to "Can you explain what an equation is?," said "I don't know how to say it, but I can give examples." This problem can be partially explained in terms of the preference of students to express themselves by giving examples. They show that they know something by their examples, even if they can't express a definition verbally. It is somewhat akin to Laborde's distinction between dynamic and static forms of a definition. As already mentioned, she states that junior high school children have a sensitivity for the dynamic form of a definition which involves an operational explanation of the event with the subevents leading up to it described in the order in which they occur and usually by means of a specific example.

From the examples given in the excerpts above, it seems that our subjects are more comfortable with the operations of addition and multiplication. Another interesting observation is that when

Patricia was asked for an example of an equation, she gave one involving an unknown; whereas, when asked immediately afterward for an example showing the use of the equal sign, she gave "2 + 5 = 7." It is possible that the equal sign is still anchored very strongly in her arithmetic and that the links between the equal sign and equations are still somewhat tenuous.

However, the prime observation to be made in this section involves the notion that "after the equal sign is the answer." Five of our six subjects expressed this idea in one form or another (Caroline was the only one who, in her second example [of the use of the equal sign], mentioned an equality which didn't have the answer on the right side). Their thinking in this regard does not appear to have changed from what it was at elementary school where, as has already been pointed out by Ginsburg, they looked at equalities in an operational way. As a matter of fact, the following excerpt indicates the difficulty that some students have with considering an equality which doesn't have the answer on the right side:

I: Do you know what this means? (= sign).
M: No.
\- - -
I: How about - - - 4 + 3 = 6 + 1?
M: That (the sign) is "equal to." - - - We don't do that kind of thing (in school). We just put 4 + 3 = 7.

Michel's reluctance to accept 4 + 3 = 6 + 1 points out the need of expanding the students' notion of the equal sign to include arithmetic equalities containing several operations on both left and right sides simultaneously. For if we don't do this expansion first, that is, if the student brings with him into the study of algebraic equations the "result on the right side" interpretation of the equal sign, then algebraic equations containing multiple operations on both sides may be meaningless to him. For example, "5 x 2 = 10" will lead to "5 x h = 10" which will be meaningful, as will "x + 363 = 542" and "3x + 5 = 26," because the result of the operation(s) is still clearly visible on the right side. But what happens when the student is presented with an equation such as "3x + 5 = 2x + 12"? How can we expect such an expression to be meaningful when the student is still under the misconception that the right side is for the result only?

Not only is the presence of this multiple operation on the right side foreign to him, but also seeing it for the first time within the context of an algebraic equation will add to the cognitive strain. Therefore, we feel that extending the notion of the equal sign within the framework of arithmetic equalities prior to the introduction of equations will greatly aid in the construction of meaning for a broad class of algebraic equations, including those of the form, $ax \pm b = cx \pm d$.

Extending the Notion of the Equal Sign

(a) One Operation on Each Side

The first phase in extending the notion of the equal sign involved the expansion of the concept of equality to include arithmetic equalities with one operation on each side.

I: Now, if I look at your example(s), I see one operation here on this side, and on the right side, I see the answer or the result. Can you use the equal sign with an operation on both sides?
Pi: 6 + 2 = 2 + 6.
Pa: 3 + 5 = 5 + 3.
C: 5 x 4 = 4 x 5.
G: What do you mean?

It took a fair amount of effort for Greg to realize what we were asking for, even after he knew what an operation was:

I: Which operation have you used in that equality (5 x 3 = 15)?
G: Multiplication.
I: And how many operations do you have written here?
G: One . . . two . . . 5 x 3.
I: You have two numbers, but one operation. Do you have an operation on this side (right)?
G: No.
I: Is it possible for you to write an equality with an operation on each side?
G: You mean like 5 x 3 = 15 and 15 = 10 + 5?
I: Would you write that down?
G: 5 x 3 = 15 = 10 + 5.
I: You have two equal signs.
G: Yes.
I: Is it possible to write the equal sign just once? An operation on one side, an operation on the other, with just one equal sign.
G: You mean two operations with one equal sign?
I: Uh, hm.
G: 5 x 3 = 10 + 5.

It is interesting to note that three of the four subjects, when asked for an equality with an operation on each side, responded with an example involving commutativity. Our fourth subject, Greg, suggested initially "5 x 3 = 15 and 15 = 10 + 5," which led to "5 x 3 = 15 = 10 + 5" and then finally to "5 x 3 = 10 + 5." His first suggestion, "5 x 3 = 15 and 15 = 10 + 5," shows how difficult it was for him to get away from having the result on one side. Even his writing, "5 x 3 = 15 = 10 + 5" indicates that he was still thinking that the result had to be shown somewhere. After his last effort, "5 x 3 = 10 + 5," we asked him if his equality was true, to which he responded: "Yes, 'cuz the answer is the same." Barbara and Michel had not been asked the question concerning an equality with an operation on each side. They had been <u>shown</u> one, whereupon Michel said that "they don't do that kind of thing (in school)."

We then asked some of our subjects who had responded to the above question ("Can you use the equal sign with an operation on both sides?") with the same operation on each side the following question: "Can you give me an example where you have one operation on one side and a different operation on the other side?" That even a very simple question such as this can be misunderstood is shown by Patricia's response:

I: Can you give me an example where you have one operation on one side and a different operation on the other side?
Pa: 6 + 3 = 3 x 6.
I: Is that true?
Pa: No, that's 9 (left side) and that's 18 (right side).
I: And what does the equal sign mean?
Pa: The equal sign means that this number, no, this problem here is supposed to be same as that one.
I: Uh, huh. So, if that's not true, can you fix it up so that it is true?
Pa: . . .
I: You've used the same numbers. You've used a 6 here, a 3 here, and a 6 here and a 3 here (pointing to the respective numbers). Do you have to use the same numbers?
Pa: No.
I: Well, let's leave the "6 + 3."
Pa: O.K.
I: The "6 + 3" has a value of ?
Pa: 9.
I: Now you have 9 on the left side. You want to have something that works out to be
Pa: 9.
I: On the right side.
Pa: 6 + 3 = 3 x 3.
I: Good!

Patricia seems to have thought at first that both sides had to have the same numbers. Although she quickly realized that 6 + 3 = 3 x 6 was not true, she seemed temporarily blocked and was unable to think of a suitable response. It was only when she was asked if one had to use the same numbers that she was able to construct an equality with a different operation on each side.

When we asked Piero the same question, "Can you give me an equality with a different operation on each side?", he showed none of the difficulty experienced by Patricia, although it had occurred to him also that we may have wanted an answer involving the same numbers:

I: Can you give me an equality with a different operation on each side?
Pi: Yes. 5 + 5 = 5 · 2. Unless you want the same numbers - I can do it with the same numbers.
I: O.K. Show me an example with the same numbers.
Pi: 0 + 0 = 0 · 0.
I: Very good!

That some students think we may want the same numbers used on both sides when we ask the question, "Can you give me an example where you have one operation on one side and a different operation on the other side?", can perhaps be explained in the following way. They have seen examples involving the commutative property of addition and multiplication in elementary school. This is obvious from their answers to the above question. However, these examples involving commutativity always have the same numbers on both sides. Thus, many of our students have only seen equalities with the result or the same numbers on the right side. Consequently, some of them have to be guided in breaking their pattern of thinking, as was necessary with Patricia, in order for them to consider equalities with different numbers on each side.

(b) Multiple Operations on Each Side

The next phase in extending the notion of the equal sign involved the expansion of the concept of equality to include arithmetic equalities with more than one operation on each side.

I: Can you give me an example (of an equality) where you have more than one operation on each side?
M: 3 x 4 ÷ 2 = 2 + 4.
B: 7 + 2 + 3 = 5 + 3 + 3 + 1.

Pi: 2 • 1 + 1 = 1 • 2 + 1.
C: 4 x 3 + 1 - 3 = 3 x 2 + 4.
Pa: And that add to the same thing?
G: 3 + 5 + 4 = . . . Do you want both sides to be equal?

Of our six subjects, only Patricia and Greg seemed to be still a little unsure of what thev were doing. Following are excerpts from the interviews with Patricia and Greq which show that their problem was of a temporary nature:

I: Can you give me an example (of an equality) where you have more than one operation on each side?
Pa: And that add to the same thing?
I: Sure, because it is still the equal sign in the middle.
Pa: O.K. 2 + 2 + 2 = 2 x 2.
I: You have 2 plus 2 plus 2 on the left side which, you know, comes out to be
Pa: 6.
I: So the right side must come out to be 6 too.
Pa: 2 + 2 + 2 = 2 x 3.
I: Could you do two operations on both sides?
Pa: 1 + 3 + 5 = 2 x 2 x 2.
I: One plus three, which is?
Pa: Oh no! . . . one minute . . . 0 + 3 + 5 = 2 x 2 x 2.

And with Greg:

I: Can you give me an example (of an equality) where you have more than one operation on each side?
G: 3 + 5 + 4 = . . . Do you want both sides to be equal?
I: Well, you have written the equal sign there and what does the equal sign mean?
G: Both sides equal.
I: O.K.
G: 3 + 5 + 4 = 12 ÷ 4 + 9.
I: What do we have on the left side there? - - -
G: 12.
I: And what do we have over here? - - -
G: 12. <u>It's O.K.</u>

In the equalities suggested by our subjects up to this point, all, save one [Caroline's second example showing how one can use the equal sign: (2 x 3 - 1) = (3 x 1 + 2)] were devoid of brackets. Moreover, our six subjects had all been taught bracketing and the conventional order of operations in class. As we proceeded into more complicated equalities, it became obvious that the topic of bracketing could not be ignored. The following excerpts indicate what occurred when we asked some of our subjects about bracketing:

I: Could you give me an example where you have two different operations on each side?
Pa: 2 + 3 x 5 = 3 x 5 + 2.
I: What is the value of the left side?
Pa: The left side - 18 . . . no, 17, wait a minute . . . yeh, 17.
I: What are you doing first?
Pa: I'm adding. No, I'm multiplying. No, . . .
I: Have you learned the use of brackets?
Pa: Yeh, I have. (2 + 3) x 5 = 3 x 5 + 2.

She rushed right in with a pair of brackets without any thought as to whether they were necessary or where to put them.

I: So that's 2 + 3, which is?
Pa: 5.
I: And 5 times 5?
Pa: 25.
I: So, what do we have over here? (right side).
Pa: . . . that's wrong - - - (2 + 3) x 5 = 3 x 5 + 2 + 8.
I: So if I see "2 + 3 x 5," why do you put brackets?
Pa: . . . to show which one you do first.

It's not at all sure that "2 + 3" was the operation that Patricia had initially thought of doing first. In view of her second response above ("The left side - 18 . . . no, 17, wait a minute . . . yeh, 17") it seems more likely that she had wanted to multiply 3 by 5 first and then add 2. She became a little confused only when we asked her which operation she was doing first, and then definitely switched her thinking when we asked her if she had learned the use of brackets.

A bracketing sequence with Piero:

I: How about different operations and different numbers?
Pi: 2 x 2 + 4 = 8 - 6 + 6.
I: Have you learned the use of brackets?
Pi: Yes.
I: Can you give me an example where you use brackets?
Pi: 2 + (1 x 3) - 2 = (1 x 3) - 2 + 2.

Piero seemed to have the notion that one brackets the operation of multiplication. He may have been confusing the use of bracketing with the rule that multiplication takes precedence over addition. Some other unusual ideas on bracketing were shown by Caroline. Though she was the only one who had used brackets in one of her examples prior to this phase of the questioning [i.e., (2 x 3 - 1) = (3 x 1 + 2)], her use of them on that occasion had been merely to set off one side from another. So we asked her specifically about bracketing during this "multiple-operation" section of our interview:

I: How about an example where you have more than one operation on each side?
C: 4 x 3 + 1 - 3 ≐ 3 x 2 + 4.
I: Have you learned the use of brackets yet?
C: Yes.
I: What if you wanted to add 3 + 1 first? How would you show that you wanted to add 3 + 1 first?
C: (3 + 1)4.
I: O.K. Now would you make an equality? Continue.
C: With this? [i.e., with (3 + 1)4].
I: Yes.
C: Using the same amount of steps?
I: Any combination you want. It doesn't matter.
C: (3 + 1)4 = (2 + 2)4.

Caroline seemed to think that whatever operation one wanted to do first should not only be bracketed but should also appear first. These various notions of Patricia, Piero, and Caroline on the use of brackets give some indication of the inadequate grasp that students can have of this notation. Bracketing and the order of operations was a problem which reoccurred often in later sequences with our students. A complete discussion of the bracketing difficulties which occurred throughout all of our interviews will be discussed later in this chapter.

When all of our subjects seemed comfortable with equalities containing not only one operation on each side but many operations on each side, we felt that they were ready to be given a name for this new concept. As has already been mentioned, Ginsburg and Beberman have pointed out the importance of acquiring a concept before giving it a name. The next section of this chapter will deal with our attempt to give a suitable name to this newly-expanded class of arithmetic equalities.

Finding a Suitable Name for Arithmetic Equalities

It seemed useful at this point to give a name to these strings of arithmetic operations joined by an equal sign. Even though we did not want to introduce any "unnecessary" new vocabulary, we felt that since these arithmetic equalities were so fundamental to the building of the concept of equation they should be given a name in order to help in identifying the concept. We decided to call them

ARITHMETIC IDENTITIES

thus leaving the name "equation" for those equalities having an unknown. It was thought that this name would reflect both the arithmetic nature of the equalities, and also the identical value borne by both sides.

However, this "naming" phase went through different stages. After trying out the name "arithmetic identity" with Michel and Barbara (our first two subjects), we went through a phase of experimenting to see if some other name might not be more suitable. This section of Chapter V outlines the path travelled in finding a name for our newly-expanded class of arithmetic equalities.

We begin by describing this "naming" phase as it occurred during Session 1 with Michel and Barbara:

I: Can you give me some examples of when you use the equal sign?
M: 4 + 3 = 7 - - - 4 + 3 = 8 - 1.
I: You know what we're going to call these. I'll write it out for you: ARITHMETIC IDENTITIES. Which means that what you have on one side
M: Equals the same.
- - -
I: Would you make me another arithmetic identity?
M: 2 + 1 = 10 - 7.
I: Alright. Can you think of one using the multiplication sign?
M: 6 x 7 = 42 . . . Can you use, like 42 ÷ 1? 6 x 7 = 42 ÷ 1.
I: Excellent! Sure. Now if you were trying to convince somebody that that is an arithmetic identity, how would you explain to them that that is an arithmetic identity?
M: They're both equal to 42.
I: Right. The left side is 42 and the right side is 42.
M: We never learned that (in school).
I: No?
M: I never heard of this (arithmetic identity).

We continued working with more examples of arithmetic identities, gradually increasing the number of operations on each side. (The sequence of questioning for Barbara and Michel had been slightly different than that for our other four subjects. They had been introduced to the term "arithmetic identity" immediately after seeing equalities with one operation on each side) Michel was able to build arithmetic identities, but seemed uncomfortable when using the terminology:

I: Is it still an arithmetic identity?
M: No.
I: The arithmetic identity means what?
M: Equals
- - -
I: 4 + 3 = 10 + 2 - 5. This is called . . .?
M: Arith . . . metic . . . (difficulty saying the word).
I: Arithmetic . . .
M: Identity.
- - -
I: And what is this [3 + (1 x 4) = 10 ÷ 2 + 2] called?
M: Arithmetic identity. It's hard to say.

That Michel was having no difficulty with the concept of arithmetic identity seemed clear. However, he was having trouble with saying the words. This was not surprising since Michel seemed, in general, to have difficulty expressing himself verbally. On the other hand, Barbara, an extremely bright girl who was capable both verbally and mathematically, had had no difficulty using the term arithmetic identity. However, it made us wonder if perhaps, for the benefit of the non-verbal child, we should not try to find some other easier name than "arithmetic identity." Thus, in light of Michel's difficulties, we decided to see if the name "equality" wouldn't serve just as well as "arithmetic identity." During Session 1 with Greg and Caroline, our next two subjects, we did not use the term "arithmetic identity," but instead, "equality":

I: Can you give me another example where you have one operation on the left side and one operation on the right side?
G: 3 + 5 = 2 x 4.
I: And why is that true?
G: You get the same answer in the equation.
I: Perhaps we should not use the word "equation" just now. We'll save the word "equation" for something which is a little bit different. Let's just call that an "equality."
G: O.K.
- - -
I: All of these are equalities [(5 + 2)3 = 2 x 10 + 1, 3 + 5 + 4 = 12 + 4 + 9]. What do you notice is true about all these equalities?
G: They're equal on both sides.
I: Does it matter how many operations you have for an equality?
G: No.

And with Caroline:

I: Let's look at what you have here: (2 x 3 - 1) = (3 x 1 + 2). Why is this an <u>equality</u>? (First time interviewer used the word).
Because the numbers in each group have the same value.
Alright, these are all equalities [5 x 4 = 4 x 5, 4 x 3 + 1 - 3 = 3 x 2 + 4, (3 + 1)4 = (2 + 2)4].

While it had seemed that with Greg the use of the term "equality" instead of "arithmetic identity" had been both functional and practical, we began to have some doubts during the session with Caroline. She seemed to be having some difficulty distinguishing between the words "equality" and "equation" and when she should use each (as evidenced by the following excerpt from Session 1 -- after she had learned about equations: I: What is this called "4b = 2 x 8?" C: An equality). It was decided to look for further evidence of this problem during the upcoming second session with both Greg and Caroline before coming to any final decision regarding the use or abandonment of the word "equality" instead of "arithmetic identity."

We began Session 2 with a review of Session 1:

I: Can you explain what is meant by an equation? (This had been taught during a later part of Session 1. As will be described in the next section of Chapter V, an equation was defined for Greg and Caroline as an equality with a hidden number).
G: It's got something to solve, you've got an equal sign, and you've got to solve it.
I: Can you give me an example of an equation?
G: 5 x 3 = 15.
I: Can you explain what is meant by an unknown?
G: The unknown part in this equation is 15.
I: Can you give me another?
G: An equation?
I: Yes.
G: 15 ÷ 5 = 3.
I: And where is the unknown?
G: What was unknown was the 3.
I: What do you mean: was unknown?
G: Now it's solved.
I: Can you show it to me before it's solved?
G: 15 ÷ 5 =
I: Just like that, with nothing written down on the right?
G: It's unknown.

And at the beginning of Session 2 with Caroline:

I: Can you explain what is meant by an equation?

C: An equation is when two different groups have the same value, like 2 + 3 and 4 + 1. They both equal 5.
I: Would you write down an example for me?
C: 4 + 6 = 8 + 2.
I: Can you explain to me what is meant by an unknown?
C: An unknown is a letter which takes the place of a number.
I: When do you see an unknown?
C: . . .
I: When do you have an unknown?
C: In an equation.
I: Now when I asked you for an equation, you put down 4 + 6 = 8 + 2.
C: Yeh.
I: So, is this (4 + 6 = 8 + 2) an equation?
C: No, it's . . .
I: Why is it not an equation?
C: Well, because there isn't an unknown.

It was, by this time, quite clear that Caroline could not remember what name to give to strings like 4 + 6 = 8 + 2 even though she had been quite aware of the different terminologies by the end of the first session. [I: Is there any difference between 5 + 4 = 7 + 2 and 5 + 4 = x + 2? C: Well, this one (5 + 4 = x + 2) is the equation and this one (5 + 4 = 7 + 2) is the equality]. Greg, who had been quite comfortable with both "equality" and "equation" during the first session, seemed unable to remember the distinguishing characteristics of each during the beginning of Session 2. We were, at this point, realizing that perhaps the term "equality," even though it was easier to say than "arithmetic identity," might be causing difficulties of another type. It seemed that the word "equality" was too similar to the word "equation." The concepts defined by these two terms were being confused because their names were not distinctively different enough. On the other hand, Michel who had had so much trouble saying "arithmetic identity" during his first session did not show such evidence of confusion between the two concepts during his second session:

I: Can you explain to me what an arithmetic identity is?
M: Something equal to both sides.
I: Could you give me an example?
M: 4 + 3 = 11 - 4.
I: Can you explain what is meant by an equation?
M: Is that something with n - with a letter?
I: Uh, hmm. Can you give me an example?
M: 4 + n = 11 - 4.
I: An equation is made from what?
M: Arithmetic identity.
I: What's the difference between the two of them?
M: There's a letter in an equation.

And during Session 2 with Barbara:

I: Can you explain what is meant by an equation?
B: It was an arithmetic identity which has had a number hidden.
I: Can you give me an example?
B: Of an equation?
I: Yes, please.
B: 2 x 9 x 6 = 2 x (3 x a) x 6.
I: Alright, can you give me an example of an arithmetic identity?
B: 3 x 6 = 3 x 2 x 3.

The outcome of the first two sessions with Michel and Barbara served to confirm what we had suspected during the second session with Greg and Caroline, that is, that "equality" is too much like "equation," whereas "arithmetic identity" is very different from "equation." This distinction between the two names was helping the student to clearly differentiate between both of the concepts involved, and also helping him to retain them. Thus we decided that even though the term "arithmetic identity" might be a mouthful for some students, it was so specific a term that it truly helped to anchor in their minds the concept of equality of strings of arithmetic operations.

And so, during Session 2, Caroline and Greg were introduced to the term "arithmetic identity."

With Caroline:

I; So if this (4 + 6 = 8 + 2) isn't an equation because it doesn't have an unknown in it, let's call it by a special name - an "arithmetic identity." Do you know what the word identical means?
C: Yeh, when two things look the same, or are the same value.
I: So why do you think that 4 + 6 = 8 + 2 is called an arithmetic identity?
C: Because both 4 plus 6 and 8 plus 2 are equal to 10. Both sides equal 10. They're identical.
I: Does an arithmetic identity have an unknown?
C: No.
I: Does an equation have an unknown?
C: Yes.

With Greg:

I: Last week when we were working on things like this (5 x 3 = 15, 15 ÷ 5 = 3), we called them equalities. But I think it would help if we gave them a special name. We're going to call them "arithmetic identities." Do you know what the word identical means?
G: Yes.
I: What?
G: When something is the same.
I: So why do you think these are called arithmetic identities?
G: Because 5 x 3 equals the same as 15.
I: These are very simple arithmetic identities. We just have one operation on one side and the result or the answer on the other side. Would you give me an example of an arithmetic identity that is not quite so simple? One that has two operations.
G: 3 x 5 + 3 = 18.
I: Alright, can you give me another arithmetic identity where you have an operation on the left side of the equal sign and an operation on the right side?
G: 3 + 6 x 2 = 9 + 6.
I: So, what are these called?
G: Arithmetic identities.

What was required at this point was to verify during the next session with Greg and Caroline (Session 3) whether or not they had become aware of the clear-cut distinction between an "arithmetic identity" and an "equation," which distinction had been lacking for the most part between an "equality" and an "equation."

At the beginning of Session 3 with Greg:

I: Can you explain to me what is an arithmetic identity?
G: It's, uhm, where each side is equal to one another.
I: O.K. Give me an example.
G: 5 x 3 + 2 = 7 + 10.
I: That's fine. Can you explain what is meant by an equation?
G: It's when something is unknown.
I: Alright, give me an example.
G: 5 + n = 10.

And at the beginning of Session 3 with Caroline:

I: Can you explain to me what an arithmetic identity is?
C: It's when both sides have the same value, like 2 times 3 plus 5 is the same as 6 plus 5. Both sides equal 11.
I: O.K. Can you explain what is meant by an equation?
C: An equation is, uhm, like when, eh, there is 2 times 3 and that equals 6. 6 is on the other side of the equal sign. Like . . .
I: Show me.
C: Uhm. 4 + 5 + 3 = 12.
I: And that's called an equation to you, is it? Actually, that's not quite right. This (4 + 5 + 3 = 12) is called an arithmetic identity.
C: Oh, right! O.K. An equation is when there's an unknown.
I: O.K. Would you give me an example then of an equation?
C: 4 + a = 8.

The above excerpt from Session 3 with Greg served to reinforce our decision to keep the term "arithmetic identity." However, our work with Caroline, though it indicated that she seemed to know what an arithmetic identity was, pointed out something else, that is, how very difficult it can be to discard old definitions and to relearn new ones. On beginning Session 1, Caroline had told us that she thought an equation was something like 1 + 2 = 2 + 1. After going through the learning experience of Session 1 and being taught that an equation had an unknown as one of its components, she returned the following week for Session 2, only to give as an example of an equation, 4 + 6 = 8 + 2. This occurred again at the beginning of Session 3 when she gave as an example of an equation, 4 + 5 + 3 = 12. However, Caroline was not alone in giving evidence of this problem. Greg had also begun

Session 1 with the notion that "equation" was another name for an arithmetic statement and had kept this idea even after being taught in Session 1 that equations had a letter.

At the beginning of Session 1:

I: Have you ever seen equations before?
G: Yes.
I: Could you explain what an equation is?
G: An equation is a problem, sort of.
I: Perhaps you can give me an example of what you mean.
G: Like something divided by something, or addition, or subtraction, or something times something else. You have to find the answer.
I: What does the equal sign mean to you?
G: It means what the answer is.
I: Where do you use the equal sign?
G: At the end of an equation.
I: Could you be more specific?
G: Before your answer.
I: Perhaps using some numbers.
G: 5 + 3 = 8.

And at the beginning of Session 2:

I: Can you explain what is meant by an equation?
G: It's got something to solve, you've got an equal sign, and you've got to solve it.
I: Can you give me an example?
G: Of an equation?
I: Yes.
G: 5 x 3 = 15.
I: Can you explain what is meant by an unknown?
G: The unknown part of this equation is 15.

As can be seen from the above excerpt from the beginning of Session 2, it became necessary to review with Greg what constituted an equation. It is not sure how much of his confusion can be attributed to his old notion of "equation" and how much, to our failure to use with him (and Caroline) a more distinctive term than "equality" to describe the equal strings of arithmetic operations. In any case, we introduced him to the term "arithmetic identity" during the second session (as with Caroline) and as has already been seen in a previous excerpt from Session 3, he had no further difficulty in distinguishing between arithmetic identities and equations.

Our last two subjects, Piero and Patricia, were also introduced to the term "arithmetic identity," but without going through any experimental period. They were given the "name" during Session 1 after they had acquired the concept involved. Following are some excerpts dealing with this aspect:

I: Alright, now we're going to give a name to all of these things (e.g., 2 + 3 = 4 + 1, etc.) you've written down: ARITHMETIC IDENTITY. They're all examples of arithmetic identities. Are you familiar with the word identical?
Pa: (She shook her head "no").
I: Have you heard of identical twins?
Pa: . . .
I: No? Alright, the word identical means two things are the same. So, why do you think it's a good idea for these to be called arithmetic identities?
Pa: Because some of the numbers aren't (???) the same?
I: The numbers aren't the same, but what is the same?
Pa: The total.
I: On? . . . The total on?
Pa: Both sides.
I: Is the same. So that's why they're called identities. Because the word identical means the same, that is, when two things are the same. So this side has the same?
Pa: As that side.
I: The same value. It doesn't have to have the same appearance, but it has the same value. So they're called arithmetic identities.

And during Session 1 with Piero:

I: Alright, let's give a name to all of these equalities (2 x 2 + 4 = 8 - 6 + 6, etc.) that you've written down. We're going to call them ARITHMETIC IDENTITIES. I'll write it out for you. Are you familiar with the word "identical?"
Pi: Yes.
I: What does it mean to you?
Pi: The same.
I: When two or more things are the same. So why does it make sense to call these equalities identities?
Pi: Because they're the same. Both sides.

It seemed that Piero had understood what was meant by the term "arithmetic identity." However, this might not have been the case, in view of what occurred later during the same session (after he had been taught what an equation was):

I: I want you to build for me 5 equations.
Pi: a + 2 = 4 + 2.
I: From which arithmetic identity did you make that equation?
Pi: I just made it up. (He continued to write): b - 8 = 8 - 8.
I: What are you thinking of while you're doing these?
Pi: I'm mostly thinking of equaling both of them, having them identical.
I: Is it necessary?
Pi: Yes. Or you don't need this. (As he points to the equal sign in b - 8 = 8 - 8).
I: But is it necessary to have "-8" here (on left side) and "-8" there (on right side)?
Pi: Yes.
I: What if you had this arithmetic identity, "6 · 2 = 8 + 4"?
Pi: You could say that, but it's not identical as much as "b - 8 = 8 - 8," because they don't have the same operation.
I: So you wouldn't call this (6 · 2 = 8 + 4) an arithmetic identity?
Pi: Not really.
I: Then I'll have to make something clear. To have an arithmetic identity, both sides have to have the identical value. They don't have to look the same. You don't have to have something like "10 - 8 = 10 - 8." That is an arithmetic identity, but so is this: 6 · 2 = 8 + 4. As long as both sides have the same value, it's an arithmetic identity. Alright, would you make me an arithmetic identity?
Pi: 12 · 8 = 90 + 6.
I: O.K. Now make an equation from that.
Pi: c · 8 = 90 + 6.

When Piero had been constructing multi-operation arithmetic equalities (prior to their being named "arithmetic identities"), he had exhibited great variety in his examples. However, as soon as the equalities were pegged with the name "arithmetic identity," he somehow assumed that this name referred only to the $a \pm b = a \pm b$ type. Since equations were formed from arithmetic identities, all his equations were of the $x \pm b = a \pm b$ type. This restriction on equations could possibly have been prevented if we had asked for several more examples of arithmetic identities immediately after exposing him to the term "arithmetic identity." We would thus have seen earlier, before going into equations, whether or not he was placing any limitations on his concept of arithmetic identity.

The next section of this chapter will deal with the acquisition of the concept of equation. However, the vocabulary used will not be the same across our six subjects, for, as has been described herein, Greg and Caroline were using the term "equality" rather than "arithmetic identity" when they were introduced to equations.

The Concept of Equation

(a) Introducing the Concept

Having extended the notion of the equal sign and given the name "arithmetic identities" to this extended class of arithmetic equalities (Greg and Caroline were at this point still using the name "equalities"), we were ready to introduce the concept of equation. We wished to introduce it by a three-step representational process.

For the first step, we would cover up with a finger one number of an arithmetic identity. Thus the notion of an equation as

AN ARITHMETIC IDENTITY WITH A HIDDEN NUMBER

would first be presented in a concrete manner. Again, as with "arithmetic identities," the name "equation" would not be given to the student at the outset, but only after he had seen part of an arithmetic identity being covered up.

For the second step, the covering-up of a number would be done by a box. This hiding of the number with a box would be an intermediate step in the gradual process of developing meaning for the new mathematical form.

The third step in the construction of meaning for the concept of equation would involve the hiding of a number in the arithmetic identity by means of a letter. Thus, it would only be after the student had acquired the concept of equation intuitively that he would be led to a more formal understanding (understanding of form) involving the use of letters. That is, by the first two steps we would express the mathematical idea involved in the concept of equation without resorting to unnecessary formalism. Only after meaning had been acquired by means of these preliminary steps would the symbolic representation of equation be presented.

Following is a sequence of excerpts dealing with the introduction of equations.

With Patricia:

I: Alright now, let's take one of your arithmetic identities. Let's take this one: 0 + 3 + 5 = 2 x 2 x 2.
I'm going to hide one of those numbers. I'll just hide it with my finger for now. Let's say I hide the 5. (Hiding the number 5): When I hide a number of an arithmetic identity, then it's called an EQUATION. An equation has something hidden. Now it's very difficult to go around always hiding numbers with your finger, so we might see it hidden with a box: 0 + 3 + □ = 2 x 2 x 2.
Pa: Or probably an "n."
I: Or an "n." You've seen it with an "n," have you?
0 + 3 + n = 2 x 2 x 2.
This is the most common way of writing an equation. We usually use a letter of the alphabet. Does it have to be "n"?
Pa: No, it can be "y." It can be "a" - any letter.
I: And we usually write it with the small letters of the alphabet and not the capital letters.

Patricia had already been introduced to formal equations in class before these sessions began, thus explaining her interjection, "Or probably an 'n'." She knew also that there was a choice of letter.

An important aspect of this introduction to the concept of equation was the choice of arithmetic identity used to illustrate the transformation into an equation. We selected one of the student's arithmetic identities, preferably one with at least one operation on each side of the equal sign. Any time we wanted to demonstrate something, we used one of their examples. And if a suitable example of theirs was lacking, we asked them to try to construct one. For if the students were to create meaning for the concept of equation, they had to be actively involved. This included constructing their own arithmetic identities and their own equations. It also provided us with the opportunity to study the types of examples they were inventing.

With Piero:

I: Let's take one of your arithmetic identities. Let's take this one: 5 + 5 = 5 · 2. Let's hide the number 5, perhaps the first one. (Hiding the first 5 with finger): When I hide a number of an arithmetic identity, I then have an EQUATION. So if I hide that (the first 5), we have: SOMETHING + 5 = 5 · 2. It's called an equation when a number is hidden. Now we're not always going to use a thumb or finger to hide a number. We could use a box. □ + 5 = 5 · 2. Something + 5 = 5 · 2. That (□ + 5 = 5 · 2) is called an equation. Can you take an arithmetic identity and make it into an equation?

Pi: 6 + 2 = 2 + 6
□ + 2 = 2 + 6
6 + □ = 2 + 6
6 + 2 = □ + 6
6 + 2 = 2 + □

I: Very good! Alright, let's use something new in our equations instead of boxes. Let's use a small letter of the alphabet. This is the way equations are usually written. So instead of □ + 5 = 5 · 2 we could use a + 5 = 5 · 2 or b + 5 = 5 · 2. Any letter of the alphabet, usually small letters and not capital letters, will do. Would you take an arithmetic identity and make it into an equation using a letter of the alphabet.

Pi: n + 2 = 6 + 2.

Piero, who, like Patricia, had already seen in class formal equations of the form ax ± b = c, surprised us nonetheless. His rapid run-off of four equations with the box in all the possible locations showed that he had not placed any unnecessary restrictions on the position of the hidden number, despite the fact that the equations he had worked with in class had always contained a letter, "n," and that the letter had invariably been placed first in the equation.

On the other hand, Greg, who had not done any prior work with equations in class, was not at all sure if the hidden number could be on either side:

I: - - - Now, we're not always going to be using a finger or a thumb to hide the number. Sometimes we use a box to hide a number. Suppose we take this last equality of yours (5 x 2 = 5 + 2 + 3). Let's rewrite it using a box to hide any number.
G: On which side?
I: It doesn't matter.
G: 5 x [ ] = 5 + 2 + 3.

We then went on with Greg to Step 3 of the hiding process, that is, hiding by using a letter of the alphabet. Greg gave as his example of an equation where a letter hides a number: 5 x n = 5 + 2 + 3. He also seemed to be fond of the letter "n," as Patricia and Piero were.

With Michel, as with the others, we first hid a number of an arithmetic identity with a finger and then made a move to go on to the more practical representations:

I: - - - Now we're not always going to be using our fingers or thumbs to hide a number.
M: You're going to cross it out?
I: No, we're going to use a box for the time being. [ ] + 3 = 10 + 2 - 5.
M: Oh, I've done that.
I: You have done this?
M: And then a 4 goes there.
I: Right. But we're not going to be figuring out just yet what goes into the box. Alright, would you write down another arithmetic identity.
- - -
M: 3 + (1 x 4) = 10 ÷ 2 + 2.
I: Make now an equation for me - using the box.
M: [ ] + (1 x 4) = 10 ÷ 2 + 2.
I: Alright now, let's go one step further. We won't always use boxes - - - We're going to use small letters of the alphabet instead of

the box. Rather than having □ + 3 = 10 + 2 - 5, you'll see something like x + 3 = 10 + 2 - 5.

M: That's algebra. (Sounding a little glum).

I: Right. We're going to use small letters of the alphabet. Now, it doesn't have to be x. It could be a + 3 = 10 + 2 - 5.

When Michel saw the boxes, Step 2 of the hiding process, he became quite pleased. Here was something familiar. He had seen boxes as placeholders during his years in elementary school and apparently had felt quite comfortable with them. Thus he was reassured by seeing something familiar which he knew he could handle. However, as soon as we began hiding a number with a letter, he seemed to become a little disappointed. He became aware that we were beginning algebra ("That's algebra.") and his comment reflected the fear that many children have about it.

(b) Giving a Name to the Letter

After having introduced the concept of equation by hiding a number of an arithmetic identity, we decided to give a name to the letter which was used. We selected the name "unknown" rather than "variable," since "unknown" corresponds more closely to the idea of a hidden number. In addition, as has already been mentioned, "unknown" involves a lower level of abstraction than "variable" (Wagner, Kuchemann).

Following are the responses of our six subjects to the introduction of the name "unknown":

T: We're going to give a name to this letter. Any letter that hides a number is going to be called an UNKNOWN. Why do you think we're calling that letter an unknown?

C: Because you don't know what the letter is, what the number is, what number it represents.

Pi: Cause it's unknown. Nobody knows what it is.
Pa: Til you have to find the number that's missing there.
G: Because, it hasn't been solved?
B: Because you don't know it. And you have to find out.
M: Because there's no number.

As can be seen, our six students could readily find justification for the letter being called an unknown. From these same responses, we also noticed that three of our subjects (Greg, Barbara, and Patricia) referred to the notion of solving equations. Even though we hadn't spoken about solving equations or even the need to solve equations, it seemed that some of our students felt that this was, in some way, an essential part of working with equations. As a matter of fact, it was quite natural for them, even at the construction-of-equations stage, to look at an equation and mentally try to slot in the required number. Michel showed evidence of this as soon as he began to construct equations:

I: We're going to try a box for the time being.
$\square + 3 = 10 + 2 - 5$.
M: Oh, I've done that.
I: You have done this?
M: And then a 4 goes there.

His spontaneous reaction seemed to illustrate that the operational process of hiding a number carries inherently the reverse process of uncovering. Another indication of the intuitive response of our subjects with respect to the uncovering component was shown when we asked Greg and Caroline the following question:

With Caroline:

I: Let's look at these two examples ($2 \times a = 10$; $3 \times a = 24$). If I replace the "a" by the correct number, what will I have back again?
C: The equality (arithmetic identity).

And with Greg:

I: Why do you think we're calling that letter an unknown?
G: Because it hasn't been solved.
I: And what if we put 6 in place of that "n" ($3 \times 5 \div 5 = 6 \times 3 \div n$), what would we be back at?
G: The equality (arithmetic identity).

Their answers indicated not only that they had the notion of uncovering (i.e., solving), but also that this uncovering would bring back the arithmetic identity. This awareness seemed to occur quite naturally. They had acquired the notion of solution as a result of the way they had constructed meaning for equations. Thus our students realized that not only could they go from the arithmetic identity to the equation, but also from the equation back to the arithmetic identity (if they knew the hidden number).

(c) Examination of the Student's Interpretations

Having introduced the concept of equation and the term "unknown," we next wished to examine the student's interpretations of the concept. This would be done within the context of having the student build several equations and at the same time asking him to explain how he was building them. This would give us an opportunity to see how the learner was thinking about the new material.

Discussion of this phase will center on four topics: i) the various approaches used by our subjects in building their own equations, ii) the misconceptions that students may develop in acquiring the concept of equation, iii) the extension of the concept by some of our subjects to include equations with more than one letter, iv) the explanation in their own words of the meaning of equation.

i) Various Approaches Used by Subjects in Building Equations

Greg's Method:

I: Can you give me three or four examples of equations and tell me what you're thinking of while you're making your equations?
G: Both sides have to be equal.
I: O.K. Let's try one. Think out loud. Tell me what you're thinking while you're writing these down.
G: Just trying to make one side equal, and the other side, put a letter to represent a number. 2 x 5 ÷ 2 = 2 + n.
I: Alright, what were you thinking of while you were making that equation?
G: Well, I got the answer to this side (left).
I: Which was what?
G: 5, and I just put something plus something would equal 5, and I put 2 and n for the number.
I: O.K. That's fine. Give me another equation now.
G: 6 ÷ 3 x 6 = 6 x 4 - n.
I: Alright now, what were you thinking of in that equation?
G: . . .
I: What did you make up first in your head? Before you wrote this equation?
G: 6 ÷ 3 is 2, times 6 is 12. 6 x 4 is 24, minus something.
I: Now, in these two examples, you have three numbers on the left side, your unknown at the end, and you used "n" in both cases. Does that mean that the unknown always has to be at the end?
G: No.
I: Does it always have to be the letter "n"?
G: No, just used a lot.
I: Give me another example.
G: 3 x n + 3 = 3 x 3.
I: What were you thinking here?
G: 3 times n, which would be 2, is 6, plus 3 is 9. It's the same as 3 times 3.

Greg's first two efforts consisted in writing the left side of an equation using numbers only, calculating its value, putting an equal sign, and then numbers with an unknown on the right. He was not beginning by writing out a complete arithmetic identity and then rewriting it with a number hidden. He would start with the left side of some arithmetic identity, and then, thinking of some equivalent form for the same value, he would use an unknown for the last number of that equivalent expression on the right side. It was only after he was asked if the unknown had to be at the end, that he came up with $3 \times n + 3 = 3 \times 3$. Here he had thought of 3×2 and covered up the 2, then added 3 to come up with a value of 9 on the left side. Thus he placed 3×3, an equivalent of 9, on the right side.

Greg was not the only one to build equations without first writing down a complete arithmetic identity. Piero did likewise, however, his equations were all of a trivial nature.

Piero's Method:

I: I want you to build for me five equations.
Pi: a + 2 = 4 + 2.
I: From which arithmetic identity did you make that equation?
Pi: I just made it up.
b - 8 = 8 - 8.
I: What are you thinking of while you're doing these?
Pi: I'm mostly thinking of equaling both of them, having them identical.
I: Is it necessary?
Pi: Yes, or you don't need this. (As he points to the equal sign in b - 8 = 8 - 8).
I: But is it necessary to have -8 here (on left side) and -8 there (on right side)?
Pi: Yes.
I: What if you had this arithmetic identity, 6 · 2 = 8 + 4?
Pi: You could say that, but it's not identical as much as b - 8 = 8 - 8, because they don't have the same operation.
I: So you wouldn't call this (6 · 2 = 8 + 4) an arithmetic identity?
Pi: Not really.
I: Then I'll have to make something clear. To have an arithmetic identity, both sides have to have the identical <u>value</u>. They don't have to <u>look</u> the same. You don't have to have something like 10 - 8 = 10 - 8. That is an arithmetic identity, but so is this: 6 · 2 = 8 + 4? As long as both sides have the same value, it's an arithmetic identity. Alright, would you make me an arithmetic identity?
Pi: 12 · 8 = 90 + 6.
I: O.K. Now make an equation from that.
Pi: c · 8 = 90 + 6.
I: Is it necessary for the small letter of the alphabet to come first?
Pi: No.

Piero's equations (a + 2 = 4 + 2 and b - 8 = 8 - 8) were built on the erroneous notion that arithmetic identities, from which equations were constructed, were all of the form $a \pm b = a \pm b$. Thus all his equations were of the $x \pm b = a \pm b$ form. As has already been mentioned in the previous section, this restriction on equations could possibly have been prevented if we had asked him for several more examples of arithmetic identities immediately after exposing him to the term "arithmetic identity." We would thus have seen, before going into equations, whether or not he was placing any limitations on his concept of arithmetic identity and could have corrected it sooner.

Another interesting feature of Piero's equation-building showed itself during Session 2. He had been asked at the end of Session 1 to do a little assignment -- to create five arithmetic identities and from them, five equations. This is what he prepared:

Pi:

ARITHMETIC IDENTITIES	EQUATIONS
5 + 2 = 2 + 5	□ + 2 = 2 + 5
10 - 6 = 8 - 4	10 - □ = 8 - 4
2 x 2 = 2 + 2	2 x 2 = □ + 2
10 ÷ 5 = 2 ÷ 1	10 ÷ 5 = 2 ÷ □
3 x 3 x 3 = 9 x 3	3 x 3 x □ = 9 x 3

I: Now I noticed you used boxes for your equations.
Pi: Yes.
I: Is there any particular reason?
Pi: No. I just thought, I like remembering old times when we used boxes.
I: We will be using small letters of the alphabet now though.
Pi: O.K.

Piero liked using boxes to hide his numbers. Even though we had passed through this phase in Session 1 and had gone on to using letters, he went back to the box stage when doing his assignment. It might be remembered that, during Step 2 of the covering-up process, Michel also liked very much the idea of using boxes. It was familiar ground; it was a reminder of "old times," according to Piero.

Caroline's Method:

Caroline used yet a different approach when she was constructing her equations.

I: Now what I'd like you to do is make up five equations and as you're making them up, I want you to explain to me what you're thinking while you're making up these equations.
C: 2 x a = 10. What times 2 equals 10?
3 x a = 24. What times 3 gives you 24?

Before continuing with Caroline, it is interesting to note how she read the two equations above. Rather than "2 times what equals 10?", she said "What times 2 equals 10?" It is possible that her knowledge of the commutative property (as already shown) gave her the freedom to re-express verbally an equation in whatever order she was most comfortable with. However, since she was not consistent in altering the order whenever the unknown appeared as the operator (as will be seen in the examples below), we cannot come to any conclusion.

I: Now in both of your examples you have just one operation on the left side and one number on the right. You have used the letter "a" both times. Would you give me in your next example of an equation, something where you don't have just one number on the right side?
C: 4 x b = 2 x 8 - 4 times what will give you 2 x 8?
I: Now you've got your unknown on the left side three times. Does it always have to be on the left side?
C: No.
I: Show me one where you have your unknown on the right side.
C: 3 x 7 = 7 x a. Three times seven is equal to seven times something.
I: What equality (Arithmetic Identity) were you thinking of when you made up that equation?
C: 7 x 3.
I: And what did you do when you wrote it down?
C: I reversed it and then I substitute the number by a letter.
I: O.K. Let's try one more equation where you have two operations on each side.

C: 6 x 5 + 1 = a x 5 + 1.
I: O.K. There you've got pretty well the same thing on both sides. How about one where it doesn't look quite so much the same on both sides?
C: 6 x 4 + 2 = 8 x □ + 2.
I: Now I notice that you put a box down for your unknown.
C: Oh! 6 x 4 + 2 = 8 x a + 2.

Caroline, in a similar fashion to Greg and Piero, was not building her equations by starting off with a written arithmetic identity and then hiding any number. The arithmetic identity stage was being done mentally. Her equation-building began in a very simple way: 2 x a = 10, 3 x a = 24, i.e., a single operation on one side (multiplication) and the result on the other. When asked for an example with more than one number on the right side, she gave 4 x b = 2 x 8. Again the unknown was on the left side, in contrast to Greg, whose first examples all had the unknown on the right. It was only when asked to have the unknown on the right side that she came up with 3 x 7 = 7 x a.

Putting the unknown on the right seemed to be a little difficult for Caroline, for not only was she avoiding it in her first examples, but also when asked for such an example she chose one of the simplest types -- an illustration of the commutative property. She invented an operation for her left side, "reversed" it for the right side and substituted one of the numbers on the right by a letter. This method helped her to avoid any mental effort that might have been required in building some more-complicated arithmetic identity. It eliminated the need to keep track of any numbers or even to compute the value of the left side. Then, when urged to give an equation with two operations on each side, she produced something simpler still than 3 x 7 = 7 x a.

She wrote the same thing on both sides and merely hid one number: 6 x 5 + 1 = a x 5 + 1. And when asked for an example where the two sides were not quite so similar, she structured both sides the same way (6 x 4 + 2 = 8 x □ + 2), though with 6 x 4 being replaced by 8 x □ . We noticed also the extent to which she favored the operation of multiplication in her examples.

Barbara showed a similar tendency to favor the operation of multiplication.

Barbara's Method:

I: Can you give me an example of an equation?
B: 2 x 9 x 6 = 2 x (3 x a) x 6.
I: Can you give me an example of an arithmetic identity?
B: 3 x 6 = 3 x 2 x 3.
I: I noticed you used multiplication again. Is there any particular reason?
B: It's easier.

Barbara used the operation of multiplication often because she found it easier. It was perhaps for this reason that multiplication was popular with several of our other subjects. It may also have been the case that some students were using only multiplication in their equations in order to avoid, either consciously or unconsciously, those ambiguous examples with multiplication following addition, where bracketing would be required. Examples of this nature had arisen during the previous work on building arithmetic identities, and, as has already been pointed out, caused problems for some of our subjects.

Something else which we noticed about Barbara's equation-building was the way in which she broke down a number from the left side into two factors, one of which she covered by a letter, on the right side [2 x 9 x 6 = 2 x (3 x a) x 6]. She followed the same approach of splitting a number into its factors when constructing arithmetic identities (3 x 6 = 3 x 2 x 3).

This subsection has dealt with the approaches of four of our subjects to equation-building. The thinking of Michel and Patricia (the two remaining subjects) will be examined in another context later in this section. It is quite interesting to note that each of our four subjects, when asked to build several equations, gave examples which they constructed mentally without first writing down an arithmetic identity and subsequently hiding some number in it. They followed the same approach several times in later sessions. However, in these later sessions, if they had been asked first to give an example of an arithmetic identity and then an example of an equation, they invariably used the arithmetic identity which was there, hiding one of its numbers. But if they had not been asked just prior for an arithmetic identity, most of them would construct an equation from the mental picture of an arithmetic identity. The resulting equations, both as seen in later sessions and in the examples shown above, were thus fairly limited in their complexity.

Greg's method of construction originally consisted in writing the left side with numbers only, calculating its value, then building the right side containing an unknown. Piero's equation-building, all of the $x \pm b = a \pm b$ type, had been founded on the erroneous notion that arithmetic identities were of the form $a \pm b = a \pm b$. Caroline's examples tended to have both left and right sides looking alike ($6 \times 5 + 1 = a \times 5 + 1$), presumably for the sake of convenience. Barbara in a similar fashion constructed her equations in such a way that one of the numbers on the left side was broken into two factors on the right side, one of which would be hidden by a letter.

The next topic in this section on student's interpretations of equations addresses itself to some of the possible misconceptions students can have regarding the unknown.

ii) Looking Out For Misconceptions

Two quite common misconceptions that students can have with respect to the unknown in an equation concern the question of whether or not the hidden number is the same ~~if~~ the letter of the equation is changed (Wagner), and the notion of whether or not the unknown must always be on one particular side. While our six students were building their own equations, we encouraged them to give a variety of examples and also asked certain questions in order to see if they had started with or developed either of these, or any other, misconceptions.

To check for the presence of the first misconception mentioned above, we asked our students the following questions:

With Patricia:

T: - - - This is the most common way of writing an equation; we usually use a letter of the alphabet. (As opposed to using a finger or a box). Does it have to be "n"?
Pa: No, it can be "y." It can be "a" - any letter.

With Caroline:

I: - - - Would you like to try to write this equation (3 x □ = 9 x 3) using a small letter of the alphabet instead of a box?
C: Sure: 3 x a = 9 x 3.
I: Could you do the same equation, but with a different letter of the alphabet?
C: Sure: 3 x c = 9 x 3.

With Greg:

I: Would you write it (5 x n = 5 + 2 + 3) again, using a different

letter of the alphabet?
G: 5 x y = 5 + 2 + 3.

and with Michel:

I: Could you use any other letter besides "n" here? (4 + n = 11 - 4).
M: Yes.
I: No matter what letter you use, is it still going to be hiding the same number?
M: Yes.

In view of Wagner's "conservation of equation" our students were asked to write the same equation twice using different letters to hide the same number. As seen above, they realized that the hidden number was the same, no matter which letter was used. The second misconception we checked for involved the question of whether or not the unknown of the equation had to be on one side or the other or at the beginning or end.

With Patricia:

I: Can you take an arithmetic identity and make it into an equation for me?
Pa: Any of them?
I: Uh, huh . . . Which one would you like to take?
Pa: 6 + 3 = 3 x 3.
I: Alright, make it into an equation.
Pa: 6 + y = 3 x 3.
I: Does the letter have to be on the left side?
Pa: Yes.
I: No, it doesn't have to be. Let's take the arithmetic identity: 2 + 2 + 2 = 2 x 3. Let's say I want to hide the number 3. You can write 2 + 2 + 2 = 2 x . . .
Pa: "a."
I: Or "c" or "d." So this is an equation: 2 + 2 + 2 = 2 x a.
Pa: And that's an equation: 6 + y = 3 x 3.

It was not surprising that Patricia had thought that the letter had to be on the left side, since in her brief exposure to formal equations in class, the only equations she had seen were those with a letter on the left side. Nevertheless, Piero who was in the same class as Patricia had no misconceptions about which side the letter could be on:

I: Alright, would you make me an arithmetic identity?
Pi: 12 · 8 = 90 + 6.
I: O.K. Now make an equation from that.
Pi: c · 8 = 90 + 6.
I: Is it necessary for the small letter of the alphabet to come first?
Pi: No . . . 9 · 9 = n.

Our other four subjects (Michel, Barbara, Greg, Caroline) were also free of this misconception as evidenced by the following two excerpts which are representative of the four subjects.

With Greg:

I: In these two examples (2 x 5 ÷ 2 = 2 + n, 6 ÷ 3 x 6 = 6 x 4 - n), your unknown is at the end. Does the unknown always have to be at the end?
G: No.
I: Give me an example where you change your style a bit.
G: 3 x n + 3 = 3 x 3.

And with Caroline:

I: Now in your last three examples (4 x b = 2 x 8, 3 x a = 24, 3 x c = 9 x 3), the letter is on the left side. Does it always have to be on the left side?
C: No.
I: Show me an example where you have it on the right side.
C: 3 x 7 = 7 x a.

Thus, with the exception of Patricia (who seemed to have started with the misconception regarding the location of the unknown), all of our subjects had been able to avoid developing the two common misconceptions described herein. The next subsection will deal with the extension of the concept of equation (by some of our subjects) to include equations with more than one letter.

iii) Extending the Concept to Include Equations With More Than One Letter

In the course of building equations, the question of hiding more than one number was discussed with three of our subjects.

With Patricia:

I: Would you build for me five equations and tell me how you're doing them?
Pa: 3 + n + 2 = . . . Can I put two letters on the two sides? No, eh! it's going to be difficult to find the answer.
I: Before I answer your question, why don't you start with the arithmetic identity first. Then underneath it put an equation.
Pa: (3 + 5) x 2 = 4 x 2 + 8.
(n + 5) x 2 = 4 x 2 + 8.
I: Now you asked if you could put a letter on
Pa: Both sides.
I: You can, on one condition. What number is "n" hiding here?
Pa: 3.
I: You can put an "n" on the other side, if "n" is hiding the number 3 on the other side too.
Pa: I see, yah.
I: You can't have "n" hiding the number 3 on this side, and the same "n" over here hiding a different number. You'd have to put a different letter, if you want to hide a different number. - - - But if you see two letters that are the same in an equation, they are both hiding the same number. Alright would you give me another example?
Pa: 2 + 2 = 2 x 2.
2 + n = n x 2.

It was not sure if Patricia was referring to an equation with two different letters (Diophantine) or simply an equation with the same letter occurring twice, when she asked her question "Can I put two letters on the two sides? No, eh! it's going to be difficult to find the answer." In any case, we pointed out the convention that the same letter could be used in an equation more than once, as long as it was used to hide the same number, otherwise one had to use two different letters.

A slightly different situation arose with Michel when he came face to face with a quadratic equation:

I: Now give me an arithmetic identity and then an equation using a small letter of the alphabet.
M: 3 + (1 x 4) = 10 ÷ 2 + 2.
b + (1 x 4) = 10 ÷ 2 + 2.
- - -
I: Instead of hiding the 3, could you hide something else?
M: Yes. 3 + (1 x 4) = 10 ÷ 2 + 2.
3 + (1 x 4) = 10 ÷ 2 + c.
- - -
I: What if you used "c" to hide the other 2 also?
M: 3 + (1 x 4) = 10 ÷ c + c. But now it's going to be harder.
I: Yes.
M: But you could have 10 ÷ 5 + 5 (slightly disturbed).

We were surprised by Michel's quick thinking of a second replacement for "c." It brought out a very nice example of a quadratic equation. We felt that since the students were involved in the building process, there didn't seem to be any need to restrict them to constructing first degree equations.

However, Michel wasn't very happy with the realization that, in, 3 + (1 x 4) = 10 ÷ c + c, the "c" could be replaced not only by 2, which was the number he was hiding in the arithmetic identity, but also by 5. It seemed that for the student, at this stage, it was one thing to have an equation with two letters (the same) where both were hiding one and only one number, but it was something quite different to have an equation with two letters (the same) where the replacement for the letters could be something other than the number that was hidden in the original arithmetic identity! There seemed to be a concern not only over the fact that there was more than one "right answer," but also over the difficulty that one would have in finding out these answers. As Michel expressed it: "But now it's going to be harder." Patricia had also remarked: "It's going to be difficult to find the answer." Michel's concern over the fact that there was more than one "right answer" was reflected by our next subject, Piero.

Piero too showed this same reluctance to consider an equation where the unknown could have more than one replacement:

I: You're hiding here the number 5 (□ + 2 = 2 + 5 -- reviewing his homework after Session 1). Here there's also a 5 (on right side). So you could hide both of them at the same time. You could have □ + 2 = 2 + □ .

Pi: But then it could be any number in the . . . you could make it; 1000000 + 2 = 2 + 1000000.

I: Yes.

Pi: So, you're always trying to find usually one number.

I: Yes.

Pi: So, you only box one number and leave the rest.

I: If it's an arithmetic identity like 5 + 2 = 2 + 5, where you have the commutative property shown, then it's true that by hiding 5 in both cases (n + 2 = 2 + n), "n" could be any one of an infinite number of choices. That is because of the special kind of example you have chosen . - - - But "n" is hiding the <u>same</u> number on both sides of this equation (n + 2 = 2 + n).

Piero's immediate solution to this situation in which the unknown (occurring twice) could be replaced by "any number" was to "only box one number." Thus one would avoid this "problem."

In extending the concept to include equations with more than one letter, we note that in the case of Patricia this extension occurred spontaneously. However, both Patricia and Michel realized that the presence of two letters would make the equation "harder," which is further evidence of the notion of solution being inherent in our construction. It is also quite remarkable that Michel "saw" the second solution of the quadratic, but then expressed the same fears as Piero. Both felt uncomfortable with equations having more than one solution. Piero's answer was to "box only one number" and thus avoid the "problem."

iv) Student's Explanation of the Meaning of Equation

Had our students, who represented a fairly wide range of abilities, been able to construct meaning for the concept of equation? How were we to judge this, save by the student's ability to explain what an equation was and by his capacity to construct equations by himself?

I: Can you explain in your own words what an equation is?
Pa: An equation is . . . an equation is when a letter is hiding a number.
Pi: It's a bunch of numbers with an operation, at least two numbers, an equal sign, and an unknown number. Or, I should say, an operation, a number, an equal sign, and an unknown.
C: An equation is when there's an unknown involved in the numbers.
G: There's an unknown number.
M: Something hidden.
B: It was an arithmetic identity which has had one number taken out and replaced by a letter.

As indicated above, all our students were able to verbalize what they meant by an equation, their definitions being expressed in their own words rather than merely repeating our own. In addition, we have seen in the previous pages how they were able to construct their own equations.

The questions they asked and the answers they gave us are further evidence of their mental constructs. Their questions showed that they were attempting to determine the boundaries of this concept, as signified by: "Can I put two letters on the two sides?", "Do you have to hide both of them?", "Can we have more than one letter in this?", "Does it always have to be an arithmetic identity?"

In this section we have introduced the concept of equation using three different representations and given a name to the letter. In the subsequent examination of the students' interpretations, we have seen their tendency to shortcut the construction process by building equations directly without first writing down an arithmetic identity. Some common misconceptions have been avoided. One of our subjects spontaneously extended the concept to "hiding more than one number," whereas others felt uncomfortable towards equations with more than one solution. All were able to explain in their own words what they meant by an equation. These results seem to indicate that this construction was accessible to all our students, since it could be readily assimilated into their existing cognition.

The next section of this chapter will show how arithmetic identities can be used to induce the rules involved in solving equations.

Operating on Arithmetic Identities

Having guided our students in acquiring an explicit meaning for the concept of equation, we then prepared the groundwork for the eventual justification of the algebraic operations used in the solution of equations. Even though solution processes are beyond the scope of this study, we will show how arithmetic identities can be used to induce the rules ("what you do to one side, do to the other side also") involved in solving equations.

To justify these algebraic operations, many teachers introduce the idea of a scale. This may be a very good way to justify these processes when limited to the simple operations of addition and subtraction of natural numbers; however, the scale does not lend itself readily to addition and subtraction of arbitrary rational numbers. Nor is it very helpful for the more complex operations of multiplication and division, for it is unlikely that high school students still think of these in terms of repeated addition and subtraction.

The physical limitations involved with the scale can be avoided by the use of arithmetic identities for these arithmetic representations of the concept of equilibrium are not subject to physical restrictions. In addition, any operation performed on an arithmetic identity is immediately verifiable by the student. Furthermore, since our students now define equations as arithmetic identities with a hidden number, the operations performed on arithmetic identities can be transferred to operations on equations.

This is not meant to imply that the student must keep on "doing the same thing to both sides" when solving equations and not take shortcuts, such as "transposing terms." In fact, he may even come to see by himself that it has the same effect. However, unless a fair amount of time is spent on solving by "doing the same thing to both sides" prior to "transposing terms," the student may easily overlook the reasons behind his transpositions.

Following is an analysis of the two sessions with Michel and Barbara (our first two subjects) which helped us evolve a line of questioning that could induce from the students the desired conclusions. As will be seen from the excerpts, certain changes were required in our approach in view of Barbara's and Michel's unforeseen reactions. The second part of this section incorporates these changes with our four remaining subjects.

(a) Initial Approach

With Michel:

I: We're going to operate on arithmetic identities. Do you know what I mean when I say "operate"?

M: No.
I: This is an operation sign (+); this is an operation sign (x); so are division and subtraction. There are four operations that we are going to be using. Suppose we take the arithmetic identity: 10 + 7 = 4(4) + 1. What happens if I add 2 to the left side? (10 + 7 + 2 = 4(4) + 1).
M: They're not going to be the same after.
I: What do I have on the left side, now that I've added 2?
M: 19. And here is 17 (on right side).
I: Is there anything that we can do?
M: Make this greater (i.e., change the "equal" sign to a "greater than" sign). 10 + 7 + 2 > 4(4) + 1.
I: Alright, but without using "greater than." Let's stay away from the "greater than" and "less than" symbols. We will only work with the equal sign. Is there anything that we can do to make that an arithmetic identity?
M: Add 2: 10 + 7 + 2 = 4(4) + 1 + 2.

As can be seen above, it was necessary to explain what was meant by the word "operation." Perhaps this might have been avoided by rephrasing the question to: "From your arithmetic identity, we're going to build another arithmetic identity." Michel's first reaction to the addition of "2" on the left was simply to change the equality symbol to an inequality. Our suggestion to remain with the equal sign brought the response, "Add 2" (on the right). This led us to believe that he would be coming up with the "rule" quite quickly; however, this was not the case:

I: O.K. Let's take another arithmetic identity: 4 + 2 = 9 - 3. Multiply the right side by 2.
M: . . . . 4 + 2 = 9 - 3 x 2.
- - (His notation on the right side will be discussed later).
I: Is it an arithmetic identity now?
M: No.
I: What can we do to make it into an arithmetic identity again?
M: Add 6? 4 + 2 + 6 = (9 - 3)2.

The notion of arithmetic identity was certainly clear, but it was obvious that our questioning was not leading to the rule, "Do the same thing to both sides." Any operation that yielded an arithmetic identity was considered acceptable by Michel. As is obvious, all of his suggestions thus far had been mathematically correct. That they did not correspond to the expected responses may be attributed to his previous work on arithmetic identities where a variety of operations had been encouraged and also to the open and general nature of our questions.

I: Another way?
M: Times 2.
I: Right. Is it an arithmetic identity now?
M: Yeh.
I: Let's take 4 + 2 = 9 - 3 again and divide the left side by 2.
M: 4 + 2 ÷ 2 = 9 - 3.
I: What do we have on the left side when we divide by 2?
M: 3.
I: Is it still an arithmetic identity?
M: No. - - -
I: How can we get an arithmetic identity?
M: Minus 3. 4 + 2 ÷ 2 = 9 - 3 - 3.
I: Another way?
M: Divide by 2.

Here again we see Michel bringing his arithmetic identity back into equilibrium using different operations on both sides. Since his arithmetic identities contained fairly small numbers, it is possible that this made it easy for him to see alternate ways of balancing the arithmetic identities.

I: Alright, let's try this arithmetic identity again (4 + 2 = 9 - 3). Subtract 3 from the right hand side.
M: 4 + 2 = 9 - 3 - 3.
I: What can I do to the left side to make it have the same value?
M: Subtract 3. 4 + 2 - 3 = 9 - 3 - 3.
I: If you subtract 3, what do you have on both sides?
M: 3.
I: Alright, can you reach any conclusion?
M: . . .

Since he did not seem to reach any conclusion, we decided to review with him his previous examples to see if he might recognize a pattern:

I: We've been operating here on arithmetic identities. We started with an identity. In the first example [10 + 7 = 4(4) + 1], what did we do?
M: We added 2.
I: We added 2 to the left side. And what did we do to the right side?
M: We added 2.
I: Then we took 4 + 2 = 9 - 3 and we multiplied
M: By 2.
I: We multiplied the right side by 2. And what did we do to the left side?
M: Multiplied it by 2.

As can be seen above, Michel did not bother with all the alternatives and indicated that he was aware of the rule we were trying to induce. However, our questioning was not specific enough to enable Michel to focus on the desired conclusion. Consequently we altered our questioning:

I: If you do something to one side, how can you make it into an arithmetic identity again?
M: By doing the same thing on the other side.
I: Does that always work?
M: Yes.

Michel was able to verbalize his conclusion and generalize it to all arithmetic identities, despite the confusion caused by the vagueness of our questions throughout this phase. However, we were reluctant to change immediately our line of questioning on the basis of one experience. Nor did we wish to "feed the answer." Thus we repeated the same approach with Barbara who responded in essentially the same manner as Michel with those arithmetic identities which involved small numbers.

We tried with Barbara an example involving larger numbers:

I: Can you think of another one (arithmetic identity), a little more complicated than the last one?
B: 72 x 3 = (9 x 8) x 3.
I: Alright, let's divide the right side by 3.
B: $\frac{(9 \times 8) \times 3}{3}$ (she wrote)

We expected Barbara to suggest dividing the left side by 3, for she had been doing so in the previous examples. However, since she did not, we tried to bring to her attention the result of dividing the right side by 3:

I: What do we have on the right side if we divide by 3?
B: $\frac{9 \times 8 \times 3}{3} = 72$
I: Why did you put "=72"?
B: Well, that's the answer.
I: You want to have the equal sign in front of "9 x 8 x 3" as it was before. - - - Divide by 3 - - - Keep the value of the right side in your head.
B: $72 \times 3 = \frac{9(8)(3)}{3}$
I: What is it we have on the left side?
B: 216.
I: And on the right side we have just 72. - - - And what are you going to do to the left side so that you have an arithmetic identity?
B: . . .

Quite obviously, all we achieved with this line of questioning was to confuse her. Perhaps we should have asked, "What can we do to the left side to make it an arithmetic identity?" rather than "What do we have on the right side ~~when we~~ divide by 3?" We proceeded to review in order to clarify the situation:

I: This is what we have so far: $72 \times 3 = \frac{9(8)(3)}{3}$
B: . . .
Divide it by 3: $\frac{72 \times 3}{3} = \frac{9(8)(3)}{3}$
I: What is the value of the left side now?
B: 72.
I: And the right side?
B: 72.

One more example involving a subtraction on both sides was sufficient to bring her to the desired conclusion:

I: So can you come to any conclusion when you operate on an arithmetic identity?
B: If your first arithmetic identity has the same value, and you do another one, and you add or subtract or multiply or divide a number, it'll be the same on the other side.

Although our two subjects reached the desired conclusions, it is obvious that our questions were not sufficiently specific to avoid confusion. We wanted the students to become aware of and be able to verbalize the rule, "Do the same thing to both sides." However, it seemed that Michel and Barbara had been temporarily sidetracked by having the choice of at least two operations that would bring the arithmetic identity back into equilibrium. Thus, we decided to try restricting the line of questioning.

We would take one of their arithmetic identities and after adding some small number to one side, we would ask them, "Using only addition, is there anything that you can do to make it an arithmetic identity again?" We would not ask them to compute the value of each side unless they were unable to answer the previous question. This procedure would be repeated four times in all with progressively larger numbers, following which we would ask the student if there seemed to be any rule that one could follow when building new arithmetic identities, using addition. If any student began to add on to both sides spontaneously, we would ask for the "rule" without going through any further examples. We would then apply the same process, but this time subtracting a number from one side and asking, "Using only subtraction, is there anything that you can do to make it an arithmetic identity again?" After using the same method for multiplication and division, we would ask them if they could make up one rule which would cover the four separate rules.

By using these directed questions, we felt that our subjects would be able to induce the rule, "Whatever you do to one side, do to the other side also." Thus we proceeded to try the above line of questioning on our remaining four subjects: Piero, Caroline, Greg, and Patricia.

(b) Revised Questioning

i) Addition

With Piero:

I: We're going to use this example of yours (3 · 3 = 9) to build a new arithmetic identity. What happens if I add 7 to the right hand side?
Pi: It doesn't come out to an arithmetic identity.
I: So, if we put down 3 · 3 = 9 and I add 7 to the right side, using addition is there any way I can make it into an arithmetic identity?
Pi: 3 · 3 + 7 = 9 + 7.
I: What if I add 13 to the left side? Is it an arithmetic identity?
Pi: No.
I: Using addition, how can I make it an arithmetic identity?
Pi: 3 · 3 + 13 = 9 + 13.
I: If I add 20 to the right side, how can I make it an arithmetic identity?
Pi: 3 · 3 + 10 + 10 = 9 + 20.
I: Does there seem to be any rule that you have to follow when you're building new arithmetic identities?
Pi: Yes.
I: Using addition, what does the rule seem to be?
Pi: The number that you added on the right side, you can equal it by any two numbers or three numbers, or you could put the same number on the left side.
I: Do you suppose that would be the case for all arithmetic identities?
Pi: Using addition?
I: Yes (using addition).
Pi: Yes.

Piero indicated immediately an awareness of the way to bring an arithmetic identity back into equilibrium by means of doing the same operation on both sides. He did not seem to be calculating any right or left side values, but rather "matched" both sides. He even added his own special variation to the "rule", when he suggested that the number added on the other side could be under another form and which he illustrated in the example above when he added "10 + 10" on the left side to balance the "20" on the other side. However, not all of our other subjects were able to grasp the essence of the above as easily as Piero, as will be seen in the next excerpt.

With Caroline:

I: We're going to use your example (2 x 5 = 10) to build new arithmetic identities. What happens if I add 2 to the left side?
C: Here? (to the 2 x 5).
I: Yes.
C: You'd have 12.
I: Is it an arithmetic identity now?
C: No.
I: Using only addition, is there anything that you can do to make it an arithmetic identity again?
C: Uh? . . .

Caroline did not seem to grasp what it was we wanted. But since her immediate reaction to our adding "2" on the left side was to give the new value of that side, we tried having her compare the values of both sides of the arithmetic identity to see if this would help her to answer our question.

I: We added 2 here (on the left side). What have we got on the left side?
C: 12.
I: What have we got on the right side?
C: 10.
I: Is it an arithmetic identity?
C: No.
I: What can we do to this (2 x 5 + 2 = 10) to make it an arithmetic identity again -- using addition?
C: Can we change it on the right side?
I: Right!
C: 2 x 5 + 2 = 10 + 2.
I: So you've added 2 to the right side, and it's now an arithmetic identity again, right?
C: (Nods, yes).

The breakthrough seemed to come when she asked, "Can we change it on the right side?". However, her difficulties weren't over, for when we continued with a different "add-on":

I: What happens if I add 7 to the right side?
C: 2 x 5 = 10 + 7. Then it's not an arithmetic identity because 2 x 5 is 10 and 10 + 7 is 17.
I: Alright, what can we do to that to make it an arithmetic identity, using only addition?
C: To the right side?
I: Whatever you think.
C: Using addition?
I: Yes.
C: 2 + 5 + 10 = 10 + 7.

We observed that in the process of re-establishing the arithmetic identity, Caroline changed the left side.

I: What did you do there?
C: 2 + 5 is 7, plus 10 is 17 and 10 plus 7 is 17.
I: What if we stick to this? (2 x 5 = 10 + 7).
C: Add 7.

We next proceeded to our third example of adding a number to one side of the arithmetic identity. We were wondering if, when we suggested adding to one side, Caroline would propose adding it on to both sides on her own, without our intervening questions. However, this did not occur:

I: Let's write 2 x 5 = 10 again. What happens if I add 13 to the left side? . . . Would you add 13 to the left side?
C: 2 x 5 + 13 = 10.
I: Is it an arithmetic identity?
C: No.
I: What can I do to that, (2 x 5 + 13 = 10), <u>using only addition</u>, to make that an arithmetic identity?
C: . . .

She did not seem to be aware of what had been done in the two previous examples.

I: What do we have on the left side? (2 x 5 + 13 = 10).
C: 23.
I: What do we have on the right side?
C: 10.
I: Is there anything that we can do, using addition, without rewriting this, to make it an arithmetic identity?
C: Yes, add 13. 2 x 5 + 13 = 10 + 13.
I: Is it an arithmetic identity?
C: Yah!

Caroline suddenly seemed to grasp what she was doing.

I: Again, let's use 2 x 5 = 10. What happens if I add 20 to the right side?
C: It's not an arithmetic identity: 2 x 5 = 10 + 20.
I: What do we have on the right side?
C: 30.
I: What do we have on the left side?
C: 10.
I: Using only addition, how can I make this statement into an arithmetic identity?
C: Add 20.

Her answers had begun to come much faster and with more certainty. So we then decided to ask for a "rule":

I: If we can only use addition in building new arithmetic identities, is there any rule that we must follow?
C: Yes. Both sides have to be identical; they have to have the same value.

This, however, was only a definition of an arithmetic identity. She was saying nothing about what one added to the arithmetic identity. So we then asked her to review what had been done in the last four examples. This seemed to be sufficient to elicit from her:

I: Is there any rule then that you think must be followed when you start with one arithmetic identity and build a new one from it?
C: Yeh. What you do to one side, you have to do to the other side.

Caroline had come up with the rule, but it had required a much greater effort than it had with Piero. We then proceeded to ask:

I: Will that (rule) be true for all arithmetic identities?
C: Uhm . . . no.
I: Can you give me an example?
C: 2 x 3 + 4 = 2 x 4 + 2.

She was trying to show that here was an arithmetic identity where 4 had been added on one side and 2 on the other. After we had reminded her that one must start off with an arithmetic identity before building on it, she proposed as another counterexample: 2 x 3 + 4 + (2 x 10) = 2 x 4 + 2 + (4 x 5). She said she was <u>not</u>

adding the same thing to both sides, yet it was still an arithmetic identity. She had thought that what was added to both sides had to be identical in appearance. We can contrast this with Piero who, as has been seen, spontaneously added to the other side a different form of the same number.

With Greg:

I: We're going to use this example (3 + 3 = 6) to build a new arithmetic identity. What happens if I add 2 to the left side?
G: It would become 8.
- - -
I: Is it an arithmetic identity now?
G: No, it's not.
I: Using addition, is there any way that we can take this and make it into an arithmetic identity?
G: Just addition?
I: Just addition.
G: Well, we can add 2 on the other side.

As can be seen above, we did not have to ask Greg to calculate the value of each side in order for him to answer the question, "... Is there any way that we can ... make it into an arithmetic identity?", as we did with Caroline. As a matter of fact, by the time we reached our fourth example, Greg was adding on to both sides simultaneously:

I: What happens if I add 20 to the right side?
G: To make it an equal arithmetic identity, we have to add 20 to the other side.
I: Is there any rule that you can give me that we must follow, if we start with an arithmetic identity and we want to build a new one, just using addition?
G: If you do something to the left side, you've got to do it to the right, the same thing.
I: Is that going to be true for all arithmetic identities?
G: Just addition?
I: Yes.
G: Yes.
I: Will you show me that the rule works for a more complicated arithmetic identity too?
- - -

G: 3 x 5 ÷ 3 = 3 + 2.
I: What did we do here (for 3 + 3 = 6)?
G: We kept on adding.
I: O.K.
G: Now multiply?
I: No, add. We're only building now with addition.
G: 2 + 3 x 5 ÷ 3 = 3 + 2 + 2.

This last minor confusion may have been avoided if we had originally begun with an arithmetic identity which did not involve the operation of addition.

With Patricia:

I: We're going to use your example (12 x 11 = 132) to build a new arithmetic identity. What happens if I add 2 to the left side?
Pa: A 2 over here?
I: Yes.
Pa: Oh! 14!
I: Well, keep the "12 x 11" as it is and put "+ 2."
Pa: O.K. 12 x 11 + 2 = 132.

Had we written down the adding-on of 2 for Patricia, she might have been able to avoid the above difficulty.

I: Is it still an arithmetic identity?
Pa: No.
- - -
I: Using addition, show me what has to be done to make it an arithmetic identity again.
Pa: With this problem here? (12 x 11 + 2 = 132).
I: Yes.
Pa: . . .
I: On the left side, you have how much?
Pa: 134.
I: And on the right side?
Pa: 132 . . . I have to make this part here (right side) equal to that one?
I: Yes, using addition.
Pa: 12 x 11 + 2 = 132 + 2.

Patricia seemed to have an idea of what was required when she asked, "I have to make this part here equal to that one?" We went through a similar sequence of questions and answers for the second and third add-ons during which time she always calculated the new value of one side before adding the same number on the other side. By the fourth example, she added on to both sides simultaneously, as Greg also had done.

I: What if I add 20 to the right side?
Pa: Add 20 (to the left side): (12 x 11) + 20 = 132 + 20.
I: Can you tell me if there's any rule that has to be followed?
Pa: Yes, you have to follow the rule that both sides have to be equal to the same amount.
I: And, how do you make both sides equal to the same amount?
Pa: . . . Let's say you have only 132 here and over here you have 152. You have to add a number to make it equal to the same side, to make it equal to the right side.
I: - - - Now do you suppose that's true for any arithmetic identity? - - -
Pa: Yes.

Our four subjects were able to verbalize the "addition rule," some of them requiring more effort than others. Our new line of questioning avoided the previous confusion and made us aware of some minor improvements, such as the use of examples not involving addition to induce the "addition rule" and the need to write down what was being added to one side. However, as will be seen below, it was the ease with which three of our four subjects were able to induce equivalent rules for the other operations that indicated the effect of our revised line of questioning.

ii) Remaining Operations

We planned to repeat the previous work with the other operations (subtraction, multiplication, division) and expected to use four examples in each case. Following are some excerpts from our session with Caroline:

I: Let's start with another arithmetic identity.
C: Any one?
I: Yes.

C: 4 x 6 = 12 x 2.
I: What happens if we subtract 4 from the left side?
C: We have to subtract 4 from the right side for it to be an arithmetic identity. 4 x 6 - 4 = 12 x 2 - 4.

Immediately Caroline knew what had to be done to bring the arithmetic identity back into equilibrium. Not only did she verbalize it, but she wrote down "- 4" on both sides immediately without any directive from us. She did no calculations. She had transferred the addition rule to subtraction. It was obviously not going to take four examples to induce the subtraction rule.

I: Do you think that that is true for all arithmetic identities?
C: Yes.
I: What is the rule then, using subtraction?
C: Whatever you subtract on one side, you have to subtract on the other side.

We decided to move immediately into examples involving building by multiplication and division. As with subtraction, it took only one example for her to come up with the rule for multiplication and the rule for division.

Just as Caroline had been operating simultaneously on both sides of the arithmetic identity for the remaining three operations, so did Greg and Piero. However, this was not the case with Patricia. Even though she had added on simultaneously to both sides of her fourth addition example, she did not transfer this immediately to the remaining operations as our other subjects had done:

I: What happens if I subtract 2 from the right-hand side? (10 - 7 = 3).
Pa: --- 10 - 7 = 3 - 2.
I: Now is that an arithmetic identity?
Pa: No.
I: Why not?
Pa: Because that's equal to 3 (left side) and that's equal to 1 (right side).

I: Alright, what can I do, using subtraction, to make it an arithmetic identity?
Pa: - - - Minus 2.
- - -
I: Take 10 - 7 = 3 and subtract 1 from the left side.
Pa: (10 - 7) - 1 = 3.
I: Is that an arithmetic identity?
Pa: No.
I: What do we have to do to it so that it is an arithmetic identity?
Pa: (10 - 7) - 1 = 3 - 1.

The above sequence involving another subtraction example was repeated with Patricia. Though she was not subtracting simultaneously on both sides, she was able to verbalize a "subtraction" rule. During her first two multiplication examples which followed, it was still necessary to ask "What do we have to do to the other side, using multiplication, to make it an arithmetic identity?" On her third multiplication example:

I: Let's multiply this side by 5.
Pa: 4 x 3 x 5 = 12 x 5.

She had done it simultaneously on both sides. It carried over to the operation of division:

I: Let's take this side and divide by 9.
Pa: (9 x 8) ÷ 9 = 72 ÷ 9.

Three of our four subjects were able to induce the rule for the three remaining operations on the basis of one example per operation. Patricia, one of our weaker students, needed a few more examples for subtraction and multiplication, but became as efficient as the others for the last operation, division.

iii) Generalization

After inducing individual rules for the four basic operations, our subjects were asked if they could give one rule which would be a generalization of the four separate rules.

I: Now we have a rule for addition, a rule for subtraction, a rule for multiplication, and a rule for division. Let's make one rule to cover everything.
C: Whatever you do, whether you add, subtract, or multiply on the left, you have to do the same operation on the right.

I: Today we were building new arithmetic identities using addition, using subtraction, using multiplication, and using division. Is there any one rule that you can give me for everything?
Pa: They all have to be . . . like when one side is, let's say, 23; the other side has to be 23. They have to be the same amount on both sides.

I: Is there one rule we can make that will be good for addition, subtraction, multiplication, and division?
G: Whatever you do to the left side, you do to the right side.

I: So when starting with any arithmetic identities, if you want to build new ones, what seems to be the rule?
Pi: You make another operation. Add another operation to either side. It must be the same as the other.

Thus they were all able to conclude that:

WHATEVER YOU DO TO ONE SIDE, YOU MUST DO TO THE OTHER SIDE ALSO

In summary, our initial line of questioning involved four examples, each with a different operation, after which we expected Barbara and Michel to generalize and conclude "what you do to one side, you do to the other." Although we succeeded, our questions were too general and created some confusion. Furthermore, since our scheme was too condensed, it forced our two subjects to induce a global awareness of the rule without constructing the intermediary concepts. Thus our revised sequence of questions with the four remaining subjects was an attempt to overcome these shortcomings.

We considered as intermediary concepts the four separate rules, one for each operation, which were elicited using operation-specific examples. In each case we decided that the subject could be asked for the rule when he was performing spontaneously the identical operation on both sides. Only after the student had induced the four individual rules, was he required to generalize.

In working with the examples leading to the addition rule, we restricted our question to, "Using only addition, is there anything that you can do to make it an arithmetic identity again?", thus avoiding the previous confusion that occurred with our first two subjects. With all four subjects, four examples or less were sufficient to elicit the desired rule. Thus we avoided the purely instrumental approach of giving the rule and having the students merely verify it by means of several examples.

For the remaining operations we used the same restricted line of questioning. Three of our four subjects could see the analogy with the previous work on addition, as evidenced by the facility with which they induced the remaining rules. In fact, one example was sufficient for each operation, since they were operating spontaneously on both sides of their arithmetic identity. The fourth subject was able to induce these rules but required a few more examples with subtraction and multiplication before operating spontaneously on both sides with division.

It was but an easy step for our four students to reach the next level of abstraction, that of considering the individual rules as examples of a more global one. This was achieved on the basis of the students' intuitive notions of arithmetic and their ability to think inductively.

In this last section, we have prepared the ground for "doing the same thing to both sides" of an equation by operating on arithmetic identities. In the next section, we shall be discussing a problem which arose intermittently during several phases of the experiment and especially during this last section. The problem concerns the difficulty which our students had with bracketing and the conventional hierarchy of operations.

Difficulties With Bracketing and the Conventional Order of Operations

(a) Introduction

Every one of our six subjects had been taught in class, at some time prior to working with us, the use of bracketing and the conventional order of operations. At first glance, a reading of the excerpts below would indicate that some students had forgotten the rules, others had either mislearned them originally or were remembering them incorrectly. However, we discovered that the root of the problem lies much deeper.

The standard textbook presentation of this topic shows the need for some convention by introducing a question believed to be ambiguous, such as, "What is the value of 2 + 5 x 4?" The student is asked, "Is it 28? Is it 22?" He is then informed that "mathematicians have agreed on the following steps to simplify such expressions,"[1] according to the following order:

1. perform the bracketed operations;
2. perform the multiplications and divisions in order from left to right;

[1] Dolciani and Wooton, Modern Algebra, Book 1, p. 41.

3. finally, do the additions and subtractions in order from left to right.

The student is then asked to evaluate strings of operations, such as, 5 + (4 - 1) + 2 x 3. Though he may very well be able to do these exercises at the time, we have evidence to suggest that these conventional rules for the order of operations run counter to the student's way of thinking. Thus it should come as no surprise that he seems to forget or discard these "rules without reasons."

We discovered during the teaching experiment, while our subjects were constructing and operating on arithmetic identities, that they were thinking operationally. For example, our students would not see 2 + 3 as the same operation as 3 + 2. Though the result is the same, in the first case one starts with 2 and adds on 3, which is not the same as beginning with 3 and adding on 2. Thus the written sequencing of their operations in a certain order reflects the order in which they are thinking about these operations. Furthermore, their evaluation of such a string of operations (i.e., 4 + 3 x 12 ÷ 4 + 2) begins with the first operation on the left and continues sequentially from left to right until the last operation is performed to yield a result of 23. This left-to-right tendency also reflects our cultural tradition in writing and reading.

In constructing their arithmetic identities (of which each side is a string of operations), our subjects were writing them down, operation by operation, as they were thinking of them, and were keeping a running total as they went along. They were not thinking of the conventions while constructing or evaluating their own strings. To then be asked to evaluate them according to the conventions conflicted with the more natural tendency of evaluating them in the order in which they were written, which was the order in which they had been conceived. In fact, the evaluation according to the conventions, at times, yielded a value different from what the student had intended. Whenever this occurred, we suggested bracketing certain operations in order not to violate the established conventions.

In the excerpts below, we shall examine the difficulties which our subjects experienced with bracketing and the order of operations, 1) while they were constructing arithmetic identities, and 2) while they were operating on arithmetic identities. The problems encountered while operating on arithmetic identities were of a slightly different nature from those met while constructing arithmetic identities.

(b) While Constructing Arithmetic Identities

This sectión is subdivided according to the following three topics: i) the tendency of our subjects to evaluate their arithmetic identities in a left-to-right order, ii) the effect of this tendency on bracketing, iii) other individual notions on the use of brackets.

i) The Tendency to Evaluate Arithmetic Identities in a Left-to-Right Order

While our subjects were constructing arithmetic identities, we often asked them to give the value of one side or the other. It soon became obvious that all of them were evaluating their arithmetic identities in a left-to-right sequence. This practice, which manifested itself repeatedly, is illustrated in the following two excerpts which are indicative of the thinking of our six subjects.

With Greg:

G: 5 + 2 x 3 =
I: What is the value of that left side?
G: 21.
I: So you're adding the 5 and 2 first, are you?
G: Yes.

With Michel:

I: Would you write down an arithmetic identity?
M: 5 + 10 ÷ 3 = 4 x 2 - 3.
I: What is being done first on this side (left side)? Is it the 5 + 10?
M: Yes.

When asked to evaluate a side of their arithmetic identity, all responded with a value arrived at by calculating from left to right. In fact, they were writing down their arithmetic identities, one operation at a time, as they were thinking of them, keeping track of the total as they went along. Though they had all worked in the past with bracketed expressions, even at elementary school, none of them used brackets in constructing their arithmetic identities. Only one student, Greg, remembered the conventional order of operations, but this was after he had first done a left-to-right calculation:

I: So you're adding the 5 and 2 first, are you? (5 + 2 x 3 = . . .)
G: Yes . . . Oh! you have to do the multiplication first.
5 + 2 x 3 = 1 x 10 + 1.

When we noticed that none of our students were following the conventional order of operations (except Greg), we suggested the use of brackets to overcome this problem.

ii) Some Effects of the Left-to-Right Tendency on Bracketing

Because the tendency to evaluate an arithmetic identity from left to right was so predominant, our suggestion to Michel to bracket the operation which appeared first seemed an unnecessary one to him. Michel didn't see the need of bracketing the first operation on the left of each side since these were the ones he was calculating first anyway. Though he acquiesced to our suggestion, as seen in the first excerpt below, he did not use brackets in any successive arithmetic identities except when asked for them, as seen in the second excerpt below:

I: What is being done first on this side (left side)?
(5 + 10 ÷ 3 = 4 x 2 - 3): Is it the 5 + 10?
M: Yes.
I: So, do you want to put that in brackets?
M: (5 + 10) ÷ 3 = (4 x 2) - 3.

And during another session:

I: Could you give me an example where you don't have just one number, the result, on this side?
M: 3 + 23 x 2 = 26 x 2.
I: What is the value of the left side?
M: 52.
- - -
I: Since 3 + 23 has to be added first, you put it in brackets.
M: (3 + 23) x 2 = 26 x 2.

Though Michel didn't see that brackets made any difference in the two examples above, he seemed more inclined to see their usefulness in the next example:

I: Would you write down another arithmetic identity?
M: 3 + 1 x 4 = 10 ÷ 2 + 2.
I: What is the left side?
M: . . . 7 . . . (but seemed puzzled).
I: What do you want to do first?
M: 1 x 4.
I: Since you want to do that first, let's put it in brackets.
M: 3 + (1 x 4) = 10 ÷ 2 + 2.

It's obvious why Michel seemed puzzled. The tendency to evaluate his written arithmetic identity in a left-to-right order was very strong. What had happened here was that Michel had thought of an arithmetic identity with a left side of "3 + 4". However, since we had been encouraging examples with multiple operations, he had written down

3 + 1 x 4 = ... Thus when he was asked to evaluate the left side of 3 + 1 x 4 = 10 ÷ 2 + 2, there was a conflict between the intended value of "7" and the value of "16" arrived at by his left-to-right method. Therefore, bracketing for this kind of example, where one wished to keep intact some operation other than the first one, seemed to make sense to Michel.

The influence of the left-to-right tendency on bracketing was seen also with Caroline. We did not ask her about bracketing an operation which appeared at the extreme left of either side of the arithmetic identity. However, when questioned about the bracketing of some operation other than the first one, Caroline not only inserted brackets around it but also rearranged the arithmetic identity so that the bracketed operation appeared first:

C: 4 x 3 + 1 - 3 = 3 x 2 + 4.
I: Have you learned the use of brackets yet?
C: Yes.
I: What if you wanted to add 3 + 1 first?
C: (3 + 1)4 = (2 + 2)4.

Caroline's approach implied that whatever operation was to be done first should be placed first at the extreme left of each side. We also noticed that Caroline changed not only the right side, but also dropped the "-3" from the left side of the arithmetic identity when doing her "rearranging."

Patricia was another student who seemed to think that whatever was bracketed was not only to be done first but also to be placed first. The following sequence indicates what took place prior to our first question about bracketing:

I: Could you give me an example where you have two operations on each side, but they're not both the same?

- - -

Pa: 2 + 3 x 5 = 3 x 5 + 2.

We observed that this particular example of Patricia's was different from several of her previous ones in that it involved commutativity. To continue:

I: What is the value of the left side?
Pa: 17.
I: What are you doing first?
Pa: I'm adding. No, I'm multiplying. No, . . .

A conflict was noticed here (very similar to Michel's with the arithmetic identity, 3 + 1 x 4 = 10 ÷ 2 + 2) between what was intended, "17", and the tendency to evaluate the left side from left to right. Patricia was becoming confused. To continue:

I: Have you learned the use of brackets?
Pa: Yeh, I have. (2 + 3) x 5 = 3 x 5 + 2.

She immediately put brackets around "2 + 3", the first operation, without any thinking about the effect on her arithmetic identity. We had thought that she might bracket "3 x 5" to maintain her arithmetic identity.

I: So that's 2 + 3 which is?
Pa: 5.
I: And 5 times 5?
Pa: 25.
I: O.K. So, what do we have over here (right side)?
Pa: . . . That's wrong.

- - -

I: Why do you put brackets?
Pa: To show which one you do first.

Thus even though she had wanted to do "3 x 5" first, the tendency to evaluate the written arithmetic identity from left to right and the notion that it is the first operation which is bracketed caused her to bracket "2 + 3" rather than "3 x 5."

We find in most elementary algebra textbooks the rule: "Anything that is bracketed must be done first." Caroline's and Patricia's examples are clear indications of how this can be misinterpreted. In their examples they show that the bracketed operation must not only be done first but also appear first. Other children, such as Michel, do not make this mistake. In fact, they do not see the need of bracketing until they construct an arithmetic identity which when evaluated (from left to right) conflicts with their mental construct.

We have seen the effect of the left-to-right tendency in the evaluation of an arithmetic identity and its subsequent influence on bracketing. We now examine some other ideas which our subjects had on bracketing.

iii) Other Bracketing Ideas

In this section we will observe some unconventional uses of bracketing by three of our subjects, Caroline, Barbara, and Piero.

In addition to the uses already cited, Caroline sometimes employed brackets to set off one side of an arithmetic identity (or equation) from another:

I: Let's try one more equation where you have two operations on each side, with an unknown hiding one of the numbers.
C: With brackets?
I: If you'd like.
C: (6 x 5 + 1) = (a x 5 + 1).

In another different type of example, Caroline used brackets to indicate alternate ways of expressing the same number, 2 x 3 + 4 + (2 x 10) = 2 x 4 + 2 + (4 x 5). Here she was not implying that these bracketed operations should be done first, otherwise they would appear first (as has been shown before). She was rather drawing attention to two different "replacements" of the same number, "20," in her arithmetic identity.

Barbara used brackets in a somewhat similar sense. She constructed several of her arithmetic identities by taking a number from the left side and expressing it as a bracketed product on the right side:

I: Make an arithmetic identity now, please.
B: 72 x 3 = (9 x 8) x 3.

Piero constructed his arithmetic identities without brackets and evaluated them from left to right, just as our other subjects had done. However, when asked about the use of brackets, he remembered some "old" rules:

I: How about an example with different operations and different numbers?
Pi: 2 x 2 + 4 = 8 - 6 + 6.
I: Have you learned the use of brackets?
Pi: Yes.
I: Can you give me an example where you use brackets?
Pi: 2 + (1 x 3) - 2 = (1 x 3) - 2 + 2.

And in another session:

I: So if we divide the right side by 2?
Pi: 6 x 4 ÷ 2 = 24 ÷ 2 . . . Shouldn't we put brackets around, (6 x 4) ÷ 2 = 24 ÷ 2?
I: Why?
Pi: Because then you might get mixed up in . . . well, in this case, it doesn't really matter because the multiplication is before division. But in cases like "7 + (3 x 3)" you should always use the brackets.
I: To show which operation you want to be done first?
Pi: Yes.

In learning the rule in school that multiplications and divisions are done before additions and subtractions, Piero had somehow picked up that multiplications take precedence over divisions. Furthermore, he had also mysteriously learned to associate bracketing with multiplication. For Piero, not only did multiplication take precedence, but it was also bracketed. In those examples where a multiplication occurred first, one was free to bracket or not. Piero was still evaluating his arithmetic identities from left to right, but was ensuring that any multiplication was kept intact by the use of brackets.

Thus we can see from the examples of these three students that they will construct a variety of rules which may or may not be erroneous.

(c) While Operating on Arithmetic Identities

The previous section described the bracketing difficulties encountered by our subjects while they were constructing arithmetic identities. This present section will describe the bracketing difficulties encountered in a different context. Here the students were operating on their already-constructed arithmetic identities. Most of our subjects merely tacked on the new operation at the right end of each side, without using brackets. In doing so, they were consistent with their own intuitive conventions of calculating from left to right.

However, two of our subjects, Patricia and Michel, in doing this "tacking on," experienced two different types of difficulties. Patricia's problem occurred as soon as she was confronted with the insertion of a second operation. In a previous example, 12 x 11 = 132, when asked to add 2 to the left side, she had responded with "Oh, 14!", having added 2 to 12. The same confusion is evident in the following excerpt:

I: What happens if we subtract 2 from the right side (10 - 7 = 3)?
Pa: - - - 10 - 7 = 3 - 2.

I: Now is that an arithmetic identity?
Pa: No.
- - -
I: Alright, what can I do, using subtraction, to make it an arithmetic identity?
Pa: - - - Minus 2.
I: You'd like to subtract 2?
Pa: Yeh, but you can't take 2 away from 10.

Patricia was focusing on the first number appearing on the left rather than the expression on the left. Therefore, we suggested that she put brackets around 10 - 7 to show that it was the original operation on the left side. Although in the conventional sense the use of brackets is unnecessary in this example, nevertheless it helped her. In fact, she continued on her own to insert brackets in subsequent examples and did not experience again the difficulty described above.

With Michel, we were the prime source of his confusion. We made a pedagogical mistake which brought to the foreground the possible effects of violating the student's natural thinking:

I: Let's take 4 + 2 = 9 - 3. Multiply the right side by 2.
M: 4 + 2 = 9 - 3 x 2.
I: You can put the 9 - 3 in brackets. Then it's clear which operation is done first. 4 + 2 = (9 - 3)2.

As has already been pointed out, Michel's left-to-right convention would not require bracketing in this type of example. To continue:

I: Is it an arithmetic identity now?
M: No.
I: What can we do to make it an arithmetic identity again?
- - -
M: Times 2. Do I put it in front?

This question may have been prompted very simply by the lack of space between "4 + 2" and the equal sign. However, since Michel was using brackets in this example, there didn't seem to be any need to write the multiplier after the brackets. To continue:

I: Yes, you could. It doesn't matter if you put it in front or in back. Let's put it in front then as you suggested.
M: 2(4 + 2) = (9 - 3)2.

But this was not a good idea, as the excerpt below will indicate. We should have maintained a left-to-right sequencing of operations, corresponding to Michel's natural tendencies. The blunder becomes obvious in an example where we were not interjecting with the use of brackets:

I: Take this arithmetic identity (4 x 12 = 48 - 0) and build a new one from it.
M: Plus 2? 2 + 4 x 12 = 48 - 0.
I: Is that still an arithmetic identity?
M: No. Put 2 on that side. 2 + 4 x 12 = 48 - 0 + 2.

Michel had added on to the left side by placing the "add on" at the front (following the same pattern as was set in the previous excerpt). Although he didn't notice the resulting discrepancy in this example, he did in the next one:

I: On the line below 4 ÷ 1 = 10 - 6, show me how you build a new arithmetic identity by doing some operation on 4 ÷ 1 = 10 - 6.
M: 4 x 4 ÷ 1 = 4 x 10 - 6. It won't go.

Michel had multiplied both sides by 4 and had placed the 4 at the beginning of each side. However, a quick evaluation (according to his left-to-right convention) made it clear that "It won't go." We then reminded him about bracketing the original operation on each side. Though this suggestion enabled him to balance his arithmetic identity, it was not getting at the root of Michel's difficulties. Tacking on at the beginning not only went against his method of evaluating, but also created for him notational problems:

I: I would like you to build a new arithmetic identity from this one, (5 + 10) ÷ 3 = (4 x 2) - 3.
M: + 7(5 + 10) ÷ 3 = + 7(4 x 2) - 3.

Note how Michel has added 7 on both sides by preceding the numeral with the operation symbol. This is possibly the best example one can find to illustrate the operational nature of his thinking.

I: Now what does the + 7 in front of the bracket mean to you?
M: Plus 7.
I: Oh, you're adding 7, are you?
M: After.
I: Are you adding 7 and then dividing the whole thing by 3, or do you want to do all of this [(5 + 10) ÷ 3] first, and then add the 7?
M: Add the 7 after.
I: O.K.
M: Do I do it after?

We corrected this problem by directing Michel to add on 7 at the end of each side and also suggested a second set of brackets to show that the original expressions were to be evaluated before performing the new operation, [(5 + 10) ÷ 3] + 7 = [(4 + 2) - 3] + 7. A week after this session, Michel was operating on his arithmetic identities by placing the new operation to be performed at the right end of each side. Furthermore, he was continuing to construct and operate on arithmetic identities without bracketing, and evaluating them according to his left-to-right convention.

(d) Summary and Conclusions

We have shown in this section the difficulties which our subjects experienced with bracketing and the conventional order of operations. This problem, uncovered within the context of our teaching experiment, proved to be much greater than expected. Though we proposed bracketing certain operations in some of their arithmetic identities to avoid contravening the conventional order of operations, our students (except

Greg) never seemed to be aware of any conflict and had to be shown every time such a conflict occurred.

Because our subjects were working with their own strings of arithmetic operations, while focusing on the construction of arithmetic identities, we were able to assess their natural tendency of evaluating strings in the order in which they were written, which was exactly the same as the order in which they had been conceived. As a result, the conventional order of operations (which they had all learned at some prior time in school) seemed quite arbitrary and unnatural, and therefore was easily forgotten (as indicated by most of our subjects) or remembered incorrectly (as indicated by Piero).

Since our students wrote down their arithmetic identities, operation by operation, as they were thinking of them, this had an effect on the way they operated on them. From our work with all our subjects, we found that they have a natural tendency to perform additional operations in the order in which they are conceived. The problems caused by not following this natural order were especially evident in the excerpts with Michel [e.g., + 7(5 + 10) ÷ 3 = + 7(4 x 2) - 3].

The use of bracketing, when proposed to some of our students, indicated problems associated with their interpretation of the rule that "whatever is bracketed should be done first." For Caroline and Patricia, bracketed operations in arithmetic identities were not only to be done first, but also to appear first. By way of contrast, other students, such as Michel, saw no need at all for bracketing the first operation. This was the operation they would be performing first in any case, since it reflected the beginning of the sequence of their mentally-constructed operations. Such students could see a need for bracketing only when some operation other than the first happened to be a written replacement for their mental construct of a single number, as seen in the case of Michel's "3 + (1 x 4) = 10 ÷ 2 + 2."

These findings explain why standard textbook presentations on bracketing and the order of operations have little effect on many students whose left-to-right tendency prevents them from seeing any ambiguity in the evaluation of 2 + 5 x 4. Our work suggests a more compelling approach to the topic of bracketing. By using an arithmetic identity (such as 2 x 5 = 10) rather than a string of operations (such as 2 + 5 x 4), one can effectively establish an awareness of the need for bracketing. Taking 2 x 5 = 10 and subsequently replacing 5 by 4 + 1 yields 2 x 4 + 1 = 10. Since the left side must still have a value of 10, the child, whether he evaluates by his own left-to-right method or by the standard ordering conventions, will come to see that the only way to maintain this arithmetic identity is by bracketing, hence 2 x (4 + 1) = 10. It is only by creating in the student's mind a need for such a notation that he will accept it and thereby make use of it.

CHAPTER VI

SUMMARY AND CONCLUSIONS

Since this study has addressed itself not only to the development of a teaching scheme for equations, but also to an examination of the way in which students think about and understand the concepts involved during the actual teaching process, it would seem appropriate to divide this concluding chapter into three parts: a summary of the teaching outline and the rationale behind it, experimental conclusions (based on the individual interviews), and general conclusions.

The summary covers the first four chapters: statement of the problem, theoretical framework, a teaching-learning scheme for equations, methodology.

The experimental conclusions which are based on the analysis of the protocols of the individual interviews as presented in the fifth chapter include: pretest, extending the notion of the equal sign, the concept of equation, operating on arithmetic identities, difficulties with bracketing and the conventional order of operations.

The general conclusions deal with the success of the experiment, the intuitive and operational aspects of the constructions, the operational nature of students' thinking, pedagogical implications, and suggestions for further research.

Summary

(a) Statement of the Problem

Recent studies by Wagner[1] and Firth[2] have pointed out the widespread difficulties experienced by high school students with algebra. Wagner has shown the existence of problems in understanding the concept of equation. Firth has found that many students have very little grasp of algebraic symbols and find them difficult to use. However, their problems with algebra are not merely a reflection of their intellectual abilities, as pointed out by Skemp,[3] but also a reflection of the way they are taught.

A very common approach to the teaching of first-degree equations in one unknown introduces the terms "variable," "open sentence," "truth set," etc., to define equations and then proceeds to have the students solve these equations. Such an excessive use of terminology may prove to be counterproductive and leave the student with the problem of having to create meaning for equations through the manipulations involved in solving them. However, the meaningless manipulation of meaningless symbols may not yield any understanding at all.

Another popular approach uses word-problems to introduce equations. However, the work of Clement et al[4] implies that, though this may create

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[1]Wagner, "Conservation of Equation, Conservation of Function, and Their Relationship to Formal Operational Thinking."

[2]Firth, "A Study of Rule Dependence in Algebra."

[3]Skemp, The Psychology of Learning Mathematics, p. 15.

[4]Clement et al, "Translating Between Symbol Systems: Isolating a Common Difficulty in Solving Algebra Word Problems."

relevance, the meaning of an equation can be obscured by the cognitive problems particular to the process of translating word-problems into equations.

A third presentation involves the teaching of first-degree equations within the framework of functions. However, Wagner has shown that the concept of function involves a higher level of abstraction than the concept of equation[1] and may thereby create unnecessary obstacles.

Thus we have sought an alternate approach to the teaching of first-degree equations in one unknown which would allow the student both to construct meaning for the concept of equation and also to lay the groundwork for the eventual justification of the algebraic operations used in the solution of equations.

(b) Theoretical Framework

The presentations currently in use confront the student with a new mathematical form for which he has not, as yet, developed any meaning. Thus in the context of Piaget's theory of equilibration,[2] this amounts to a problem of accommodation. In developing a new approach, we have tried to transform this into a problem of assimilation. As pointed out by Steffe, this can be achieved whenever the new, more complex concept to be learned is based on a simpler one existing in the learner's cognition.[3]

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[1]Wagner, communication to N. Herscovics.

[2]Flavell, The Developmental Psychology of Jean Piaget, p. 50.

[3]Steffe, "Constructivist Models for Children's Learning in Arithmetic," p. 5.

In linking the new material with the existing knowledge, one can either transform the new concept to reach the student's cognition or transform the student's cognition to reach the new concept.[1] This latter approach is particularly important in developing meaning for a new mathematical form, such as algebra. Such a construction, by necessity, avoids a formal presentation and must use an intuitive one (applying Bruner's enactive and iconic modes), in order to achieve "continuity of content,"[2] before introducing the new form. These intuitive presentations can then gradually be transformed into more formal ones to yield corresponding modes of understanding.[3]

In developing a new approach to equations, we have used to a large extent the teaching-learning model, "Didactic Reversal,"[4] which incorporates the above theoretical considerations. By applying this model, which stresses reversibility, it is possible to construct meaning not only for the concept of equation but also for the concept of the solution of an equation.

(c) A Teaching-Learning Scheme For Equations

Based on the theoretical considerations summarized above, we developed a teaching outline which aimed at 1) constructing meaning for the concept of equation, and 2) laying the groundwork for justification

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[1] Herscovics, "A Learning Model for Some Algebraic Concepts."

[2] Byers, "Essays in Mathematics Education."

[3] Byers and Herscovics, "Understanding School Mathematics," pp. 24-27.

[4] Herscovics, "A Learning Model for Some Algebraic Concepts."

of the algebraic operations used in the solution of equations.

In order to have the student construct meaning for the new mathematical form, using as a starting point his already-acquired arithmetic knowledge, it was first necessary to expand his notion of the equal sign. This was done by having the student construct arithmetic identities with several operations on each side. Thus he no longer saw the equal sign as being a link between an operation on the left side and the result on the right, but rather as a sign of the equilibrium between two sides "having the same value."

The next step, going from arithmetic identity to equation, included both a change in content and a change in form. In order to lead the student gradually to the acquisition of meaning for the new form of an equation, Bruner's three modes of representation were used. A number in one of the student's arithmetic identities was hidden first by a finger (enactive), then by a box (iconic), and finally by a letter (symbolic). Thus an equation was defined as an arithmetic identity with a hidden number. In following these three stages the student was able to acquire an intuitive understanding of the concept and then gradually transform this to a formal understanding.

Arithmetic identities were also used to establish the justification of the rule ("Whatever you do to one side, do to the other side.") which would be used in the solution of equations (solution processes, however, remain beyond the scope of this study). By performing an operation on one side of an arithmetic identity, the student was able to see the need for performing an operation on the other side to restore the balance. When restricted to the use of the same operation, he was led to doing the same thing on both sides.

(d) Methodology

The methodology chosen for this study was a version of the Soviet "teaching experiment," a method which, according to Menchinskaya, "is directed at disclosing the very process of learning, as it takes place under the influence of pedagogy."[1] The "dynamic" nature of this methodology allowed the researcher, by means of individual interviews, to examine the way that the learner was thinking about and understanding specific concepts during the actual teaching process.

The number of subjects for this study was six, three from grade 7 and three from grade 8, who represented a wide range in ability. The individual interviews, which were 20-45 minutes in length, were all audio-taped and transcribed in their entirety for the purpose of analysis.

Experimental Conclusions

This segment of the chapter will present the thinking patterns which we have been able to discover through the section-by-section analysis of the interviews.

(a) Pretest

The object of the pretest was to determine the student's existing ideas on equations and the equal sign.

For five of our six students, their notion of the equal sign was

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[1] Menchinskaya, "The Psychology of Mastering Concepts: Fundamental Problems and Methods of Research," p. 89.

restricted to a single operation on the left side and the result on the right. Their thinking in this regard seemed to be similar to that found at the elementary level where, according to Ginsburg,[1] children look at equalities in such an operational way. This pointed out the need for expanding the student's notion of the equal sign to include arithmetic equalities containing several operations on both left and right sides simultaneously.

Some of our students had difficulty in verbalizing what they meant by "equation" and the equal sign. They didn't know how to say "it," but could give examples. This confirmed Laborde's findings that junior high school students have a tendency to define "dynamically",[2] that is, they give an operational explanation of the event with the subevents leading up to it described in the order in which they occur, and usually by means of specific examples.

(b) Extending the Notion of the Equal Sign

The notion of the equal sign was extended to include multiple operations on both sides. This expanded class of equalities was given the name "arithmetic identity," a term which reflected both the arithmetic nature of these equalities and the identical value borne by both sides.

When asked for an equality with an operation on each side, three of the four subjects questioned (the other two subjects, Barbara and

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[1]Ginsburg, Children's Arithmetic, p. 90.

[2]Laborde, "Relations Arithmétiques -- Aspect Statique -- Aspect Dynamique," p. 41.

Michel, had been shown such arithmetic identities) responded with an example involving commutativity. When asked for an equality with a different operation on each side, two students thought initially that they had to use the same numbers on both sides. Our six students were able to go on to construct arithmetic identities with multiple operations on each side, the "value" of each side being the criterion for equality.

In these constructions all our subjects wrote down their arithmetic identities, operation by operation, as they were thinking of them, keeping track of the total as they went along. Thus they viewed their arithmetic identities sequentially (a sequence of operations which yielded a certain value) rather than globally (each side as "another name for" the same number).

(c) The Concept of Equation

By means of a three-step representational process, arithmetic identities were transformed into equations. A number in an arithmetic identity was hidden first by a finger, then by a box, then by a letter. In this way an equation was defined as an arithmetic identity with a hidden number.

Our six students could readily find justification for the letter being called an "unknown." From their responses, we also noticed that three subjects (one of whom had seen equations in class) referred to the notion of solving equations. Even though we hadn't spoken about solving equations or even the need to solve equations, it seemed that some of our students felt that this was, in some way, an essential part of working with equations. As a matter of fact, it was quite natural for them, even at the construction-of-equations stage, to look at an equation and mentally try to slot in the required number. Their answers indicated not only that they had the notion of uncovering (i.e., solving), but also that this uncovering would bring back the arithmetic identity. This awareness seemed to occur quite naturally. They had acquired the notion of solution as a result of the way they had constructed meaning for equations. Thus our students realized that not only could they go from the arithmetic identity to the equation, but also from the equation back to the arithmetic identity (if they knew the hidden number).

It is quite interesting to note that, when asked to build equations, five of our six subjects did not follow the sequence of our construction, i.e., starting with a written arithmetic identity and then hiding a number, but instead wrote equations immediately (the sixth student was instructed to write the arithmetic identity first). This seemed to indicate that the concept of equation had acquired sufficient meaning to allow our students to skip the intermediary step.

In looking for common misconceptions in their interpretations we noticed that all our students were able to "conserve equation,"[1] and all (except one who had been taught equations in school) avoided restricting the unknown to any one side.

Our results indicated that this method of constructing meaning for equations was accessible to all of our students who represented a

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[1]Wagner, "Conservation of Equation, Conservation of Function, and their Relationship to Formal Operational Thinking."

wide range of abilities. They not only could construct their own equations but also were able to express in their own words what was meant by an equation. In addition, we were unable to detect any significant differences with respect to the time required to acquire the concept of equation or in the sophistication of their examples that could be directly related to their varying abilities.

(d) Operating on Arithmetic Identities.

We did not use the scale to induce the general rule ("Do the same thing to both sides") because of its limitation to addition and subtraction of natural numbers. Arithmetic identities were used instead, for these mathematical representations of equilibrium were not restricted by the physical limitations of the scale. Furthermore, any operation performed on an arithmetic identity was immediately verifiable by the student. Thus, since our students now defined equations as arithmetic identities with a hidden number, the operations performed on arithmetic identities could be eventually transferred to operations on equations.

In trying to induce the rule with the first two subjects, our questioning was found to be too vague, leading them to perform different operations on each side. Since we wanted our students to perform the same operation on each side, we restricted our line of questioning with the four remaining subjects in the following way: after adding a number to one side of an arithmetic identity, we asked them, "Using only addition, is there anything that you can do to make it an arithmetic identity again?"

Our four remaining subjects were able to induce the addition rule (Add the same thing to both sides), some of them requiring more examples than others. However, it was the ease with which three of them were able to induce equivalent rules for the other operations that indicated the effect of our revised line of questioning. One of our weaker students needed a few more examples for subtraction and multiplication, but became as efficient as the others for the last operation, division.

It was but an easy step for our four students to reach the next level of abstraction, that of considering the individual rules as examples of the general rule.

(e) Difficulties With Bracketing and the Conventional Order of Operations

Although every one of our six subjects had been taught in class the use of bracketing and the conventional order of operations, they experienced difficulties in using them while constructing arithmetic identities and also while operating on arithmetic identities.

In constructing their arithmetic identities, our students were writing them down (without brackets), operation by operation, as they were thinking of them, and were keeping a running total as they went along. To then be asked to evaluate them according to the conventions conflicted with the more natural tendency of evaluating them in the order in which they were written, which was the order in which they had been conceived. The evaluation according to the conventions at times yielded a value different from what the student had intended. Whenever this occurred, we suggested bracketing certain operations in order not to violate the established conventions.

The tendency to evaluate an arithmetic identity from left to right was so predominant that it gave rise to different and unexpected responses to the use of bracketing. Michel was one student who didn't see the need of bracketing the first operation on the left of each side since these were the ones he was calculating first anyway. Two other students, Caroline and Patricia, seemed to think that whatever was bracketed was not only to be done first but was also to be placed first.

When operating on their already-constructed arithmetic identities, most of our subjects merely "tacked on" the new operation at the right end of each side, without using brackets. In doing so, they were consistent with their own intuitive conventions of calculating from left to right. The response of one of our students provided striking evidence of the sequential nature of his thinking and of the problems which can arise as a result of not following this natural order. Michel, who was told that he could add-on to either the left or right end of each side, wanted to add 7 to both sides of $(5 + 10) \div 3 = (4 \times 2) - 3$ by writing $+7(5 + 10) \div 3 = +7(4 \times 2) - 3$.

The difficulties experienced with bracketing have led us to suggest alternate approaches to the teaching of this topic, which shall be discussed under "Pedagogical Implications."

General Conclusions

(a) Success of Experiment

This experiment was designed to investigate whether or not the constructions involved were accessible to students representing a wide range of ability. From the written and verbal evidence of our six subjects, we could conclude that all of them were able to construct meaning for each of the main concepts: extending the meaning of the equal sign; constructing equations; operating on arithmetic identities.

In the first two constructions there did not seem to be any significant differences among the students' responses which could be directly related to their varying abilities. In the third construction, operating on arithmetic identities, Patricia required a few more examples than the others in order to induce the rules. However, we could not draw any conclusion relating this to ability, because our other weaker student, Michel, was subjected to a different line of questioning.

In assessing the reasons why all our students were able to build meaning for these concepts, we have identified the two main factors to be the intuitive and the operational aspects of the constructions. Though these two aspects are fundamentally interrelated, we will view them separately for purposes of discussion.

(b) The Intuitive Nature of the Constructions

In describing the intuitive nature of the constructions, we shall use the word "intuitive" as a general designation for the following: the sudden insight, the spontaneous response, the inductive leap, and the global perception of a problem[1] -- all of these as evidenced in the student.

To expand the meaning of the equal sign from the sense of "an operation on the left and the result on the right", we asked four subjects for an example with an operation on each side (the other two

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[1] Byers and Herscovics, "Understanding School Mathematics," p. 24.

subjects, Barbara and Michel, had been shown such an example). Three gave examples involving commutativity. When asked subsequently for an example with a different operation on each side, they responded spontaneously with an example. That this can be interpreted as being an intuitive jump at the conceptual level can be illustrated by contrasting the above students with one who did not make the same intuitive jump, and who had to go through an intermediate step before making the transition from a result to a different operation on the right side. When Greg was asked for an operation on each side of the equal sign, he wrote 5 x 3 = 15 = 10 + 5, and had to be guided into dropping the middle step. We feel that the transformation which occurs in the meaning of the equal sign from an operation and its result to a different operation on each side is an intuitive construction for it does not seem to be taught in school, as indicated by Michel's, "We don't do that kind of thing (in school)." Expanding the concept further to include multiple operations is merely a quantitative extension.

In constructing meaning for the concept of equation, by hiding a number of an arithmetic identity first with a finger and then with a box, Bruner's first two modes of representation (enactive, iconic) were used to present the concept at a very concrete level. Thus the student acquired an intuitive notion of equation (i.e., a global perception of what was meant by an equation) before being confronted with the symbolic form.

In operating on arithmetic identities, the intuitive nature of the construction occurred on various levels. When one student began adding spontaneously on to both sides of the arithmetic identity without ever having done any verification, he showed an intuitive understanding of the equilibrium of the arithmetic identity. When on the basis of a few examples our students were able to induce the "addition rule," this gave evidence of intuitive understanding. Furthermore, the ease with which they came up with the rules for the other three operations and the general rule ("Do the same thing to both sides") was additional evidence of their ability to think inductively -- which at this level can be considered to be an example of intuitive thinking. Thus the three constructions -- extending the meaning of the equal sign, constructing equations, operating on arithmetic identities -- were accessible to the student on an intuitive level.

(c) The Operational Nature of the Constructions

In describing the operational nature of these constructions, as a factor contributing to the accessibility of the concepts involved, we shall use the word "operational", first, in the sense of the learner transforming, i.e., "operating on", his existing knowledge in order to construct meaning for the new concepts, and second, in the sense of the learner performing, i.e., doing, operations, be they mathematical or non-mathematical, in the process of constructing meaning.

The transformation of the learner's existing knowledge was evidenced in all three constructions. From the results of the pretest, it was clear that the student was accustomed to using the equal sign to express an operation on the left side and the result on the right. By asking him if he could use different operations on each side, we guided him in expanding his notion of the equal sign. This new meaning was extended further to include multiple operations on each side. The student thus acquired the notion of an arithmetic identity which ~~in turn~~ was used to construct the concept of equation. This step-by-step process insured that there was no break in the continuity of content, thus avoiding gaps which could prevent assimilation from occurring. The student was also able to use his newly-acquired knowledge of arithmetic identities to derive the general rule that would be used in solving equations. At every stage the learner was operating on (transforming) his existing knowledge in order to extend it further.

The second sense of the word "operational," that of performing operations, is self-evident in the case of extending the meaning of the equal sign and in the case of operating on arithmetic identities since both involved the use of arithmetic operations. In the construction of meaning for the concept of equation, the act of hiding a number was operational in the non-mathematical sense. Thus, during these three constructions the learner was actively involved not only in the physical sense of "doing," but also in the sense of transforming his own cognition, step by step.

(d) Operational Nature of Students' Thinking

We often observed on different occasions with our subjects evidence of a certain way of thinking which could be called "operational." In describing this mode of thinking, we shall use the word "operational" in three senses: first, in the sense of Laborde, that is, where pupils describe a mathematical idea or expression "dynamically" (how to?) rather than "statically" (what is?);[1] second, in the sense of Ginsburg,

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[1] Laborde, "Relations Arithmétiques -- Aspect Statique -- Aspect Dynamique," pp. 41-50.

that is, where children interpret mathematical symbols such as equality in terms of an arithmetic operation;[1] and third, in the sense of Kieran, that is, where students view strings of arithmetic operations sequentially.[2]

When asked in the pretest to give a definition of what they thought an equation was or of the use of the equal sign, some of our subjects had difficulty verbalizing such an explanation and offered to give examples instead. This bears out Laborde's findings that children aged 11-13 years have a tendency to define "dynamically," that is, to give an operational explanation of the event with the subevents leading up to it described in the order in which they occur, and usually by means of specific examples.

When asked to give an example involving the equal sign, our subjects gave an equality with an operation on the left and the result on the right. What Ginsburg found with elementary school children (i.e., that children view the equality symbol in terms of an operation and the result) seems to hold also for junior high school children.

When constructing their arithmetic identities, our students wrote them down operation by operation as they were thinking of them, keeping a running total as they went along. Whenever they were asked to evaluate one side or the other, they did so, from left to right, which was the same order as that in which the operations had been written. As

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[1] Ginsburg, Children's Arithmetic, p. 90.

[2] Carolyn Kieran, "Children's Operational Thinking Within the Context of Bracketing and the Order of Operations," Proceedings of Third International Conference for the Psychology of Mathematics Education, Warwick, England, July 1979.

has already been pointed out, this had an effect on the way they viewed bracketing and the conventional order of operations.

When operating on their already-constructed arithmetic identities, five out of six students tacked the new operation on at the right end of each side, in keeping with their own notion of operating sequentially from left to right.

We have wondered if the operational aspect (i.e., "performing operations") of our method of construction encouraged our students to think sequentially (operation by operation). However, on the basis of their tendency to overlook the need for bracketing and the need for a convention regarding the order of operations, we must conclude that this is their natural way of thinking. In fact, we would tend to believe that it was because of the operational (and intuitive) nature of the constructions that the concepts were readily assimilated by our students.

(e) Pedagogical Implications

In reviewing the various introductions to algebra (see Chapter I), we have seen that most presentations were formal and made excessive use of difficult terminology. We think that the teaching outline developed in this study provides the teacher with an alternative approach which is both intuitive and operational, and thus is accessible to students of various ability. Classroom use does not require any major adaptation as indicated by our experience with a remedial algebra class for adults.

By explicitly building algebra on the student's existing arithmetic knowledge, the relationship between these two mathematical subjects is self-evident to the learner. The algebraic concepts become more concrete and thus can be more readily assimilated by students. By giving meaning to algebraic forms such as equations, the student need not manipulate meaningless symbols. The rules derived from operating on arithmetic identities lay the ground for the eventual justification of algebraic operations. It can also be stressed that these constructions do not take any more classroom time than the more standard presentations.

The standard introduction to algebra presents equations as expressions which may or may not have a solution. This approach is assuredly more general than ours, but overlooks the cognitive problems of the learner confronted with an equation that has no solution. The distinction between classes of equations which have or do not have a solution may be easier if one has first established meaning for solvable equations. The construction of equations based on arithmetic identities provides the student with equations that have a solution. Later, after these have become familiar, the student can be introduced to equations which don't have a solution, e.g., $2x + 3 = 2x + 4$. Since there is no possible arithmetic identity from which such an "equation" could have been formed, it has no solution. Thus the notion of equation is extended to include any algebraic expression containing a letter.

Equations with infinitely many solutions can also arise if a student constructs an equation from an arithmetic identity such as $2 + 4 = 4 + 2$ and wishes to hide the same number twice, e.g., $2 + a = a + 2$. This context could present a good opportunity for discussion of the axioms of our number systems. Many students may never have understood the symbolism used to express these axioms and, in this new frame of reference, these postulates could become more meaningful.

Another important pedagogical implication to arise from this study concerns the sequential operational thinking of students and the use of brackets. As has already been pointed out, our students, in following their natural left-to-right tendency for writing and evaluating strings of arithmetic operations, neither saw any need for the conventional order of operations nor for the necessity of bracketing certain operations to avoid contravening these conventions. However, our work suggests one approach to the topic of bracketing which will help create in the student's mind a need for such a notation.

By using an arithmetic identity, such as 2 x 5 = 10, rather than a string of operations, such as 2 + 5 x 4 (as are found in standard textbook presentations of the topic), one can effectively establish an awareness of the need for bracketing. Taking 2 x 5 = 10 and subsequently replacing 5 by 4 + 1 yields 2 x 4 + 1 = 10. Since the left side must still have a value of 10, the student, whether he evaluates by his own left-to-right method or by the standard ordering conventions, will come to see that the only way to maintain this arithmetic identity is by bracketing, hence 2 x (4 + 1) = 10. In this context the use of brackets does not seem superfluous, and, in fact, establishes the need for this notation.

Because this study was carried out on an individual basis, it allowed us to identify modes of thinking which would have been inaccessible in a group study. There were many patterns observed which can be expected to be found in a normal classroom situation. Thus we believe that the findings of this study can assist a teacher in becoming more aware of students' thinking in this area, and as a result help her/him in communicating these concepts.

(f) Suggestions For Further Research

The version of the Russian "teaching experiment" which we used was, we feel, a valuable methodology for examining the thinking and understanding of the learner while he is in the process of forming new concepts. Our version differed from that being used by Steffe[1] who teaches small groups of four to eight children and then afterwards interviews on an individual basis, and from that being used by Rachlin[2] who does not teach himself but who interviews some students individually after they have been taught in a classroom situation by their teacher. We tend to believe that our interpretation of this methodology has the advantage of providing us with spontaneous responses, and furthermore avoids the influence of group interactions on the student's thinking.

One of the major problems we experienced was striking the right balance between our teaching role and our observing role. Too often we were so intent on teaching that we missed important clues expressed by the student. This would suggest that the presence of a second researcher during each interview would help in picking up comments missed by the interviewer. Of course, this may create other problems such as the discontinuities caused by the observer's interventions, the distraction caused by his/her presence, and the fact that the learner may feel less

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[1]Leslie P. Steffe, "Analysis and Critique of the Teaching Experiment as Exemplified by a Particular Investigation," Paper presented at NCTM Convention, Boston, April 1979.

[2]Sid Rachlin, personal communication, April 19, 1979.

comfortable in the presence of two adults rather than one. However, we think that these disadvantages would be outweighed by the benefits derived. In addition, we found that the difficulties involved in analyzing individual-interview protocols for common thinking patterns are so great that we recommend that this kind of research be undertaken by at least two researchers working together.

Another advantage of this methodology was the flexibility it gave us to alter our original teaching scheme when we saw that it was not producing the predicted or desired effect. We have tried to include in this dissertation all of these changes in our approach, for Easley has suggested, "It is regrettable that few writers of case or clinical studies keep a tally of the kind of changes in perspective the researcher is forced to make by the events of the clinical studies themselves."[1]

Our research suggests many different areas for further investigation. It would be interesting to study the effect of this method of constructing meaning for equations on the development of solution strategies for equations. Do children who have constructed meaning for equations and who have laid the groundwork for doing the same thing to both sides of an equation approach the solution of equations any differently from those children who have not constructed meaning for equations by means of our scheme?

We have observed the very strong tendency of students to view strings of arithmetic operations sequentially. How does this sequential thinking effect the processes they use to solve equations?

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[1]Easley, "On Clinical Studies in Mathematics Education," p. 4.

Clement et al have pointed out the difficulties involved in translating word problems into equations.[1] Will the prior construction of meaning for equations eliminate some of these problems?

We think that these are interesting research problems with pedagogical implications. In fact, any answers to these questions will provide teachers with the means of helping students overcome the major mathematical obstacle they face in high school, namely algebra.

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[1]Clement et al, "Translating Between Symbol Systems: Isolating a Common Difficulty in Solving Algebra Word Problems."

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