Fundamenta Informaticae 42(1): 29-60 (2000)
IOS Press

Constructing the Least Models for Positive Modal Logic Programs

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Abstract. We give algorithms to construct the least L-model for a given positive modal logic program P, where L can be one of the modal logics KD, T, KDB, B, $KD4$, $S4$, $KD5$, $KD45$, and $S5$. If $L \in \{KD5, KD45, S5\}$, or $L \in \{KD, T, KDB, B\}$ and the modal depth of P is finitely bounded, then the least L-model of P can be constructed in PTIME and coded in polynomial space. We also show that if P has no *flat* models then it has the least models in KB, $K5$, $K45$, and $KB5$. As a consequence, the problem of checking the satisfiability of a set of modal Horn formulae with finitely bounded modal depth in KD, T, KB, KDB, or B is decidable in PTIME. The known result that the problem of checking the satisfiability of a set of Horn formulae in $K5$, $KD5$, $K45$, $KD45$, $KB5$, or $S5$ is decidable in PTIME is also studied in this work via a different method.

1. Introduction

It is well-known that for any positive classical logic program P there exists the least model of P. Moreover, the least model can be constructed in PTIME, and its size is bounded by a polynomial in the size of P. How can we extend these results for modal logics? First of all, we must define what is a (positive) modal logic program. There is a correspondence between Horn clauses and positive logic programs. The definition of modal Horn clauses can be found in [3]. We use the translation method used in [10, 6] to translate this definition to a simpler form, which formulates the so called *positive modal logic programs.*

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In [9] Ladner showed that the complexity of provability in the modal logics K, T, and $S4$ is PSPACE-complete, and in $S5$ is co-NP-complete. We can raise a question: What fragments of modal logics are computable in polynomial time? Some authors have studied the case [3, 2, 5]. In [3] Fariñas del Cerro and Penttonen showed that the problem of checking the satisfiability of a set of modal Horn clauses in $S5$ is decidable in PTIME. In [2] Chen and Lin showed that the similar problem for a normal modal logic L which is a normal extension of $K5$, write $K5 \leq L$, is also decidable in PTIME. Chen and Lin also proved that for a normal modal logic L such that $K \leq L \leq S4$ or $K \leq L \leq B$, the problem is PSPACE-complete. In [5] Halpern showed that if we restrict to finite modal depth, and restrict the language to having only finitely many primitive propositions, then the complexity of the problem of checking the satisfiability of a set of modal formulae is reduced to linear time for the modal logics K, T, and $S4$. Although the latter restriction is very strong, the idea is interesting. We will show that the combination of restriction to Horn formulae and restriction to having finitely bounded modal depth gives polynomial time complexity for the logics KD, T, KB, KDB, and B.

In this paper, we give algorithms to construct the least L-model for a given positive modal logic program P, where L can be one of the modal logics KD, T, KDB, B, $KD4$, $S4$, $KD5$, $KD45$, and $S5$. The above listed modal logics are, as a characterization, serial, and their frame restrictions, excluding D, are Horn formulae. For the modal logics KB, $K5$, $K45$, and $KB5$, which are *almost serial* (see Definition 6.3), we show that if P has no *flat* models (see Definition 6.1) then P has the least models in such logics.

The idea of how to construct the least model for a positive modal logic program P in a modal logic L is the following: We construct a L-model graph realizing P (the method of constructing model graphs can be found in [12, 4]). To guarantee the constructed model graph to be the smallest, each new world is connected to an empty world at the time of its creation.

The algorithms of constructing the least models for positive modal logic programs are useful from the point of view of modal deductive databases. A modal deductive database consists of facts and positive modal logic rules, which can be universally quantified. Deductive databases can be treated as logic programs. Having the least model of a database, answers for a query to that database can be effectively computed (see [11] for details). The algorithms present a bottom-up method for answering queries to modal deductive databases.

2. Preliminaries

2.1. Syntax and Semantics of Propositional Modal Logics

The sentences of modal logics are built from primitive propositions p_1, p_2, ..., classical connectives $\wedge$, $\vee$, $\neg$, $\rightarrow$, nonclassical unary modal connectives $\Box$, $\Diamond$, and parentheses), (.

A modal formula, hereafter simply called a *formula*, is any finite sequence of these symbols obtained by applying the following rules: any proposition p_i is a formula, and if ϕ and ψ are formulae then so are $\neg\phi$, $\phi \wedge \psi$, $\phi \vee \psi$, $\phi \rightarrow \psi$, $\Box\phi$, and $\Diamond\phi$.

The symbols $\neg$, $\wedge$, $\vee$ and $\rightarrow$, respectively, stand for logical negation, logical conjunction, logical disjunction and logical (material) implication. The symbols $\Box$ and $\Diamond$ can take various meanings but traditionally stand for "necessity" and "possibility". To enable us to omit parentheses, we adopt the convention that the connectives $\neg$, $\Box$, $\Diamond$ are of equal binding strength but bind stronger than $\wedge$, which binds stronger than $\vee$, which binds stronger than $\rightarrow$.

We use letters p and q to denote primitive propositions, and Greek letters ϕ, ψ, ζ to denote formulae. We denote the set of primitive propositions by $\mathcal{P}$, and the set of formulae by $\mathcal{F}$.

Definition 2.1. (Kripke Frames) A *Kripke frame* is a triple $\langle W, \tau, R\rangle$, where W is a nonempty set of possible worlds, $\tau \in W$ is the actual world, and R is a binary relation on W, called the accessibility relation. If $R(w, u)$ holds then we say that the world u is accessible from the world w, or that u is reachable from w.

If R is a binary relation then by R^* we denote its transitive closure. A frame $\langle W, \tau, R\rangle$ is said to be *connected* if every world from W is directly or indirectly reachable from τ (i.e. $\forall x \in W\ x = \tau \vee R^*(\tau, x)$).

Definition 2.2. (Kripke Models) A *Kripke model* is a tuple $\langle W, \tau, R, h\rangle$, where $\langle W, \tau, R\rangle$ is a Kripke frame, $h : W \rightarrow P(\mathcal{P})$, and $h(w)$ is the set of primitive propositions which are "true" at the world w.

Definition 2.3. (Model Graphs) A *model graph* is a tuple $\langle W, \tau, R, H\rangle$, where $\langle W, \tau, R\rangle$ is a Kripke frame, $H : W \rightarrow P(\mathcal{F})$, and $H(w)$ is the set of formulae which should be "true" at the world w.

We sometimes treat model graphs as models with H being restricted to the set of primitive propositions.

Definition 2.4. (Satisfiability) Given some Kripke model $M = \langle W, \tau, R, h\rangle$, some world $w \in W$, the satisfaction relation $\vDash$ is defined as follows:

$M, w \vDash p$	iff	$p \in h(w)$;
$M, w \vDash \neg\phi$	iff	$M, w \nvDash \phi$;
$M, w \vDash \phi \wedge \psi$	iff	$M, w \vDash \phi$ and $M, w \vDash \psi$;
$M, w \vDash \phi \vee \psi$	iff	$M, w \vDash \phi$ or $M, w \vDash \psi$;
$M, w \vDash \phi \rightarrow \psi$	iff	$M, w \nvDash \phi$ or $M, w \vDash \psi$;
$M, w \vDash \Box\phi$	iff	for all $v \in W$ such that $R(w, v)$, $M, v \vDash \phi$;
$M, w \vDash \Diamond\phi$	iff	there exists some $v \in W$ such that $R(w, v)$ and $M, v \vDash \phi$.

We say that M *satisfies* ϕ *at* w iff $M, w \vDash \phi$. We say that M *satisfies* ϕ, or ϕ *is satisfied in* M, and write $M \vDash \phi$, iff $M, \tau \vDash \phi$.

Definition 2.5. (Size of a Kripke Model) We define the *size* of a Kripke model $M = \langle W, \tau, R, h\rangle$ (resp. a model graph $M = \langle W, \tau, R, H\rangle$) to be the sum of the number of its worlds, the size of its accessibility relation, and the total number of primitive propositions (resp. formulae) from its worlds, i.e. $|W| + |R| + \Sigma_{w\in W}|h(w)|$ (resp. $|W| + |R| + \Sigma_{w\in W}|H(w)|$).

Definition 2.6. (Length and Modal Depth of a Formula) We define the *length* of a formula ϕ to be the number of connectives and primitive propositions in ϕ. We define the *modal depth* of a formula, denoted by *mdepth*, as follows:

$$\begin{aligned}&mdepth(p) = 0;\;\; mdepth(\neg\phi) = mdepth(\phi);\\ &mdepth(\phi \wedge \psi) = mdepth(\phi \vee \psi) =\\ &mdepth(\phi \to \psi) = max(mdepth(\phi), mdepth(\psi));\\ &mdepth(\Box\phi) = mdepth(\Diamond\phi) = mdepth(\phi) + 1.\end{aligned}$$

Lemma 2.1. *Given a model M and a formula ϕ, the problem of checking whether $M \vDash \phi$ is solvable in polynomial time.*

Proof:
Let the function $f_M(\phi)$ denote the cost of computing the set of all worlds u such that M satisfies ϕ at u. We estimate f_M by induction on the construction of ϕ. Let n be the size of M, and m be the length of ϕ. We have:

$$\begin{aligned} f_M(p) &\leq O(n)\\ f_M(\neg\phi) &\leq f_M(\phi) + O(n)\\ f_M(\phi \wedge \psi) &\leq f_M(\phi) + f_M(\psi) + O(n)\\ f_M(\phi \vee \psi) &\leq f_M(\phi) + f_M(\psi) + O(n)\\ f_M(\phi \to \psi) &\leq f_M(\phi) + f_M(\psi) + O(n)\\ f_M(\Box\phi) &\leq f_M(\phi) + O(n)\\ f_M(\Diamond\phi) &\leq f_M(\phi) + O(n)\end{aligned}$$

We conclude that $f_M(\phi) = O(n.m)$. □

Definition 2.7. (Depth of a World) For a Kripke model $M = \langle W, \tau, R, h\rangle$ (or a model graph $M = \langle W, \tau, R, H\rangle$), for $w \in W$, $w \neq \tau$, we define the *depth* of w to be the smallest number k such that $R^k(\tau, w)$. The depth of τ is assumed to be 0. The depth of a world that is not directly or indirectly reachable from τ is undefined. We denote the depth of x by $depth(x)$.

Definition 2.8. (Real Diameter of a Kripke Model) For a Kripke model $M = \langle W, \tau, R, h\rangle$ (or a model graph $M = \langle W, \tau, R, H\rangle$), we define the *real diameter* of M to be the maximal depth of some non-empty world in M (i.e. a world w with $h(w)$ (resp. $H(w)$) not empty). If there is no upper bound on depths, the real diameter is infinite.

Definition 2.9. (Restricted Models and Model Graphs) Let M be a Kripke model (resp. a model graph). For $k \geq 0$, we define $M|_k$ to be the model (resp. model graph) obtained from M by restricting it to the worlds with depth not greater then k.

Lemma 2.2. *Let M be a Kripke model. A formula ϕ with modal depth m is satisfied in M iff it is satisfied in $M|_m$.*

This lemma can be easily proved by induction on the construction of ϕ.

Axiom	Schemata	First-Order Formula
D	$\Box\Phi \to \Diamond\Phi$	$\forall w\ \exists u\ R(w,u)$
T	$\Box\Phi \to \Phi$	$\forall w\ R(w,w)$
B	$\Phi \to \Box\Diamond\Phi$	$\forall w,u\ R(w,u) \to R(u,w)$
4	$\Box\Phi \to \Box\Box\Phi$	$\forall w,u,v\ R(w,u) \land R(u,v) \to R(w,v)$
5	$\Diamond\Phi \to \Box\Diamond\Phi$	$\forall w,u,v\ R(w,u) \land R(w,v) \to R(u,v)$

Table 1. Axioms and corresponding conditions on R

2.2. Modal Logic Correspondences

The simplest normal modal logic (called K) is axiomatized by the standard axioms for the classical propositional logic, the *modus ponens* inference rule, the K-*axiom* schema $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$, plus the additional *necessitation rule*

$$\frac{\vdash \phi}{\vdash \Box\phi}$$

It can be shown that a modal logic formula is provable in this axiomatization iff it is satisfied in every Kripke model (i.e. without any special R-properties) [8]. It is known that certain axiom schemata added to this axiomatization correspond to certain properties of the accessibility relation (see also [7, 1]).

Many of such correspondences are definable as formulae of first-order logic where the binary predicate $R(x,y)$ represents the accessibility relation, as shown in Table 1. Different modal logics are distinguished by their respective additional axiom schemata. Some of the most popular modal logics together with their axiom schemata are listed in Table 2. We refer to the properties of the accessibility relation of a modal logic L as the L-*frame axioms* or L-*frame restrictions*.

Definition 2.10. (L-satisfiability) We call a model M a L-*model* if the accessibility relation of M satisfies all L-frame restrictions. We say that ϕ is L-*satisfiable* if there exists a L-model of ϕ, i.e. a L-model satisfying ϕ. A formula ϕ is a *tautology* in a logic L if ϕ is satisfied in every L-model. We write $\phi \vDash_L \psi$ to denote that ψ is satisfied in every L-model of ϕ.

We call modal logics that are characterized by classes of Kripke models *normal modal logics*. Given two normal modal logics L and L', we say that L' is a *normal extension* of L, and write $L \leq L'$, if all L-frame restrictions are also L'-frame restrictions.

Let L be one of the modal logics listed in Table 2. For a binary relation R, we use $Ext_L(R)$ to denote the least extension of R that satisfies all L-frame axioms, excluding the axiom D. It is clear that this operator is well defined for such logics.

2.3. Modal Horn Formulae and Positive Modal Logic Programs

In this section, we give the definition of positive propositional modal logic programs. At a first sight, the definition seems a bit restrictive, however, we will show that for any set X of so called

Logic	Axioms	Frame Restriction
K	K	no restriction
KD	KD	serial
T	KT	reflexive
KB	KB	symmetric
KDB	KDB	serial and symmetric
B	KTB	reflexive and symmetric
$K4$	K4	transitive
$KD4$	KD4	serial and transitive
$S4$	KT4	reflexive and transitive
$K5$	K5	euclidean
$KD5$	KD5	serial and euclidean
$K45$	K45	transitive and euclidean
$KD45$	KD45	serial, transitive and euclidean
$KB5$	KB5	symmetric and euclidean
$S5$	KT5	reflexive and euclidean

Table 2. Modal logics and frame restriction

Horn formulae (see Definition 2.12) in any normal modal logic L there exists a positive program P and a query Q (i.e. a positive formula) such that X is L-satisfiable iff $P \not\models_L Q$.

We call formulae of the form p or $\neg p$ *classical literals*, and use letters a, b, c to denote them. We call formulae of the form a, $\Box a$, or $\Diamond a$ and their negations *atoms*, and use letters A, B, C to denote them. A *simple clause* is an atom or a disjunction of atoms. If ϕ is a simple clause then we call $\Box^s \phi$ a *clause*. The length of a clause is the length of the formula it stands for.

A formula is in the *negative normal form* if it does not contain the connective $\rightarrow$, and each of its negations occurs immediately before a primitive proposition. Given a formula, we can translate it to the equivalent negative normal form, by applying the following rules:

$$\begin{aligned}
\phi \rightarrow \psi &\equiv \neg\phi \vee \psi \\
\neg\neg\phi &\equiv \phi \\
\neg(\phi \wedge \psi) &\equiv \neg\phi \vee \neg\psi \\
\neg(\phi \vee \psi) &\equiv \neg\phi \wedge \neg\psi \\
\neg\Box\phi &\equiv \Diamond\neg\phi \\
\neg\Diamond\phi &\equiv \Box\neg\phi
\end{aligned}$$

Definition 2.11. (Negative and Positive Formulae) A formula is called *negative* if in its negative normal form every primitive proposition is prefixed by negation. A formula is called *non-negative* if it is not negative, and *positive* if its negation is a negative formula.

Definition 2.12. (Modal Horn Formulae and Clauses) A formula ϕ is called a *Horn formula* iff one of the following conditions holds:

- ϕ is a primitive proposition
- ϕ is a negative formula
- $\phi = \Box\psi$, or $\phi = \Diamond\psi$, or $\phi = \psi \wedge \zeta$, where ψ and ζ are Horn formulae
- ϕ is a disjunction of a negative formula and a Horn formula.

We call a clause a *Horn clause* if it is a Horn formula.

In [3] Fariñas del Cerro and Penttonen have given a definition of Horn clauses. It is different from our definition of Horn clauses and Horn formulae. Each of our Horn clauses is also a Horn clause in the meaning by Fariñas del Cerro and Penttonen, and each of the latter is a Horn formula, but not vice versa. However, these definitions are equivalent in the sense that every Horn formula ϕ can be translated to a set X of Horn clauses such that for any normal modal logic L, ϕ is L-satisfiable iff X is L-satisfiable.

Definition 2.13. (Positive Modal Logic Programs) A *positive propositional modal logic program* is a finite set of rules of the following form:

$$\Box^s(B_1 \wedge \ldots \wedge B_k \rightarrow A)$$

where $s \geq 0$, $k \geq 0$, and $A, B_1, \ldots, B_k$ are atoms of one of the forms p, $\Box p$, $\Diamond p$, where p is a primitive proposition. We often write program rules in the form:

$$\Box^s(A \leftarrow B_1, \ldots, B_k).$$

We write $[\phi_1, \ldots, \phi_k]$ to denote the disjunction $\phi_1 \vee \ldots \vee \phi_k$, and write $\phi_1; \phi_2; \ldots; \phi_k$ to denote the set $\{\phi_1, \phi_2, \ldots, \phi_k\}$. If X and Y are sets of formulae then we write $X; Y$ to denote the sum of them. A set of formulae is sometimes considered as the conjunction of its formulae, in particular when we are talking about length, modal depth, or satisfiability.

Definition 2.14. (Equisatisfiability) We call two sets of formulae X and Y *equisatisfiable* in a logic L iff (X is L-satisfiable iff Y is L-satisfiable).

Lemma 2.3. (cf. Mints [10]) *In the following let p and q be new primitive propositions (i.e. p and q occur only at the indicated positions). Then the following pairs of sets of formulae are equisatisfiable in any normal modal logic:*

$$X; \Box^s[\psi, \zeta \vee \xi] \quad \textit{and} \quad X; \Box^s[\psi, \zeta, \xi] \tag{1}$$

$$X; \Box^s[\psi, \zeta \wedge \xi] \quad \textit{and} \quad X; \Box^s[\psi, \neg p]; \Box^s[p, \zeta]; \Box^s[p, \xi] \tag{2}$$

$$X; \Box^s[\psi, \zeta \wedge \xi] \quad \textit{and} \quad X; \Box^s[\psi, q]; \Box^s[\neg q, \zeta]; \Box^s[\neg q, \xi] \tag{3}$$

$$X; \Box^s[\phi, \Box\psi] \quad \textit{and} \quad X; \Box^s[\phi, \Box p]; \Box^{s+1}[\neg p, \psi] \tag{4}$$

$$X; \Box^s[\phi, \Box\psi] \quad \textit{and} \quad X; \Box^s[\phi, \Box\neg q]; \Box^{s+1}[q, \psi] \tag{5}$$

$$X; \Box^s[\phi, \Diamond\psi] \quad \textit{and} \quad X; \Box^s[\phi, \Diamond p]; \Box^{s+1}[\neg p, \psi] \tag{6}$$

$$X; \Box^s[\phi, \Diamond\psi] \quad \textit{and} \quad X; \Box^s[\phi, \Diamond\neg q]; \Box^{s+1}[q, \psi] \tag{7}$$

Proof:

$\rightarrow$) Fix one of the pairs. Suppose that the LHS set is satisfied in a model $M = \langle W, \tau, R, h\rangle$. Let $M' = \langle W, \tau, R, h'\rangle$ with $h'(u)(x) \equiv h(u)(x)$ for $x \neq p$ and $x \neq q$, $p \in h'(u)$ iff $M, u \vDash \psi$, and $q \in h'(u)$ iff $M, u \vDash \neg\psi$, where p and q are new primitive propositions. It is easily seen that the RHS set is satisfied in M'.

$\leftarrow$) Fix one of the pairs. We show that the RHS set of formulae implies the LHS set in any normal modal logic. This claim is clear for the pairs (1), (2), and (3). The pairs (5) and (7) are dual to the pairs (4) and (6), respectively. The assertion about the pair (4) holds because the following formulae are tautologies in any normal modal logic:

$\Box(p \wedge (\neg p \vee \psi)) \rightarrow \Box\psi$;

$(\phi \vee \Box p) \wedge \Box(\neg p \vee \psi) \rightarrow (\phi \vee \Box\psi)$.

The assertion about the pair (6) holds since the following formulae are tautologies in any normal modal logic:

$\Diamond p \wedge \Box(\neg p \vee \psi) \rightarrow \Diamond(p \wedge (\neg p \vee \psi))$;

$\Diamond p \wedge \Box(\neg p \vee \psi) \rightarrow \Diamond\psi$;

$(\phi \vee \Diamond p) \wedge \Box(\neg p \vee \psi) \rightarrow (\phi \vee \Diamond\psi)$. □

Proposition 2.1. *For any set X of Horn formulae there exists a set Y of Horn clauses such that:*

- *X and Y are equisatisfiable in any normal modal logic.*
- *The modal depth of Y is equal to the modal depth of X, and the length of Y is of quadratic order of the length of X.*

Moreover, if we divide Y into two groups P and Q such that P contains only non-negative clauses and Q contains only negative clauses, then P can be treated as a positive program, and X is L-satisfiable iff $P \nvDash_L \neg Q$, where L is any normal modal logic. The translation from X to Y is solvable in polynomial time.

Proof:

Let n be the length of X. We first translate X to the negative normal form, to X'. This task is done in $O(n)$ steps (assume that formulae are stored as trees), and the length of X' is of order $O(n)$.

We refer to the pairs of equisatisfiable sets of formulae given in Lemma 2.3 as translation rules (with left to right direction of application). We then apply these translation rules to X'. The rules (4), (5), (6), and (7) are applied only when ψ is not a classical literal. In situations when both of the rules (2) and (3), or both (4) and (5), or both (6) and (7), are applicable, the appropriate one must be chosen in order to guarantee that the resulting set contains only Horn formulae. We apply the rules until no more changes can be made to the set. Let Y be the resulting set. Observe that there are no more than $O(n)$ times we can apply the rules to X'. Each application takes $O(n)$ steps and increases the length of the set by $O(n)$. Therefore the process of translating X to Y terminates in $O(n^2)$ steps.

It is easily seen that Y satisfies the two mentioned assertions. Let Y be divided into two groups P and Q such that P contains only non-negative clauses and Q contains only negative clauses. Y is L-satisfiable iff $\neg(P \wedge Q)$ is not a tautology in L, and iff $P \not\models_L \neg Q$. □

3. Ordering Kripke Models

In this section, we define an order between Kripke models. This order has the property that if a model M is less than or equal to N, then for any positive formula ϕ, if ϕ is satisfied in M then ϕ is also satisfied in N. Using the order we then define the least L-model of a positive modal logic program.

Definition 3.1. (Order between Kripke Models) Let $M = \langle W, \tau, R, h\rangle$ and $N = \langle W', \tau', R', h'\rangle$ be Kripke models. We say that M is less than or equal to N wrt. a binary relation $r \subseteq W \times W'$, and write $M \leq N$ wrt. r, if the following conditions hold:

1. $r(\tau, \tau')$
2. $\forall x, x' \;\; r(x, x') \rightarrow (h(x) \subseteq h'(x'))$
3. $\forall x, x', y \;\; R(x, y) \wedge r(x, x') \rightarrow \exists y' \; R'(x', y') \wedge r(y, y')$
4. $\forall x, x', y' \;\; R'(x', y') \wedge r(x, x') \rightarrow \exists y \; R(x, y) \wedge r(y, y')$.

We say that a model M is less than or equal to N, and write $M \leq N$, if $M \leq N$ wrt. some r.

Lemma 3.1. *If $M \leq N$ wrt. r, then for every positive formula ϕ and for every u and u' such that $r(u, u')$, if $M, u \models \phi$ then $N, u' \models \phi$. In particular if $M \leq N$, then for every positive formula ϕ, if $M \models \phi$ then $N \models \phi$.*

Proof:
Let $M \leq N$ wrt. r. We prove the lemma by induction on the construction of ϕ. Suppose that $r(u, u')$ and $M, u \models \phi$ hold.

Case $\phi = p$: We have $p \in h(u)$. By the condition 2, we have $p \in h'(u')$, hence $N, u' \models p$.

The case $\phi = \psi \wedge \zeta$ or $\phi = \psi \vee \zeta$ is trivial.

Case $\phi = \Box\psi$: By the condition 4, we have $\forall v' \;\; R'(u', v') \rightarrow \exists v \; r(v, v') \wedge R(u, v) \wedge (M, v \models \psi)$. By the inductive assumption, we have $\forall v' \;\; R'(u', v') \rightarrow (N, v' \models \psi)$, hence $N, u' \models \Box\psi$.

Case $\phi = \Diamond\psi$: Since $M, u \models \Diamond\psi$, it follows that $\exists v \;\; R(u, v) \wedge (M, v \models \psi)$. By the condition 3, we have $\exists v, v' \;\; R'(u', v') \wedge r(v, v') \wedge (M, v \models \psi)$. By the inductive assumption, it yields that $\exists v' \;\; R'(u', v') \wedge (N, v' \models \psi)$. Therefore $N, u' \models \Diamond\psi$. □

Definition 3.2. (The Least L-Model of a Positive Program) Let P be a positive program in a normal modal logic L. We say that M is the least L-model of P if M is a L-model of P and M is less than or equal to every L-model of P.

Observe that if P is a positive program in a normal modal logic L, and M is the least L-model of P, then for any positive formula ϕ, $M \models \phi$ iff $P \models_L \phi$.

Proposition 3.1. *The relation $\leq$ between Kripke models is a pre-order.*

Proof:
Let $M = \langle W, \tau, R, h\rangle$, $M' = \langle W', \tau', R', h'\rangle$, and $M'' = \langle W', \tau', R', h'\rangle$ be Kripke models. We have $M \leq M$ wrt. the identity relation. Let $M \leq M'$ wrt. r_1, and $M' \leq M''$ wrt. r_2. Let $r_3 = r_1 * r_2$. We show that $M \leq M''$ wrt. r_3.

It is obvious that $r_3(\tau, \tau'')$ and $\forall u, u''\ r_3(u, u'') \rightarrow (h(u) \subseteq h''(u''))$.

For the condition 3 : Suppose that $R(u, v) \wedge r_3(u, u'')$ holds. By the definition of r_3, there exists u' such that $R(u, v) \wedge r_1(u, u') \wedge r_2(u', u'')$. By Definition 3.1, the following formulae hold:

$$\exists v'\ R'(u', v') \wedge r_1(v, v') \wedge r_2(u', u'');$$
$$\exists v''\ R''(u'', v'') \wedge r_2(v', v'') \wedge r_1(v, v');$$
$$\exists v''\ R''(u'', v'') \wedge r_3(v, v'').$$

For the condition 4 : Suppose that $R''(u'', v'') \wedge r_3(u, u'')$ holds. By the definition of r_3 there exists u' such that $R''(u'', v'') \wedge r_1(u, u') \wedge r_2(u', u'')$. By Definition 3.1, the following formulae hold:

$$\exists v'\ R'(u', v') \wedge r_2(v', v'') \wedge r_1(u, u');$$
$$\exists v\ R(u, v) \wedge r_1(v, v') \wedge r_2(v', v'');$$
$$\exists v\ R(u, v) \wedge r_3(v, v'').$$

□

4. Results of this Work

Before presenting algorithms of constructing the least models for positive modal logic programs, we list the main results of this paper:

- For any positive modal logic program P of size n in a modal logic $L \in \{KD, T, KDB, B, KD4, S4, KD5, KD45, S5\}$, there exists the least L-model M of P. If $L \in \{KD5, KD45, S5\}$, or $L \in \{KD, T, KDB, B\}$ and the modal depth of P is finitely bounded, then the least L-model of P can be constructed in PTIME and coded in polynomial space.
- Any positive modal logic program P in $L \in \{KB, K5, K45, KB5\}$ can be characterized by one or two models, in the sense that a positive formula follows from P iff it is satisfied in these models. If $L \in \{K5, K45, KB5\}$, or L is KB and the modal depth of P is finitely bounded, then the models can be constructed in PTIME and coded in polynomial space.
- The problem of checking the satisfiability of a set of Horn formulae with finitely bounded modal depth in KD, T, KB, KDB, or B is decidable in PTIME.
- A new proof for the known result [3, 2] that the problem of checking the satisfiability of a set of Horn formulae in $K5$, $KD5$, $K45$, $KD45$, $KB5$, or $S5$ is decidable in PTIME.

5. Constructing the Least Models for Positive Programs in KD, T, KDB, B, $KD5$, $KD45$, and $S5$

In this section we give an algorithm to construct the least L-model for a given positive modal logic program P, where L can be one of the modal logics KD, T, KDB, B, $KD5$, $KD45$, and $S5$. The complexity of the algorithm is analyzed, and some corollaries are presented.

To illustrate the idea of the algorithm, let us consider an example. Let $P = \{\Diamond p, \Diamond q\}$ be a positive program. $M = \langle \{\tau, u, v\}, \tau, R, h \rangle$, with $R = \{(\tau, u), (\tau, v), (u, u), (v, v)\}$, $h(\tau) = \{\}$, $h(u) = \{p\}$, and $h(v) = \{q\}$, is a KD-model of P. But M is not the least KD-model of P. Why not? The reason is that $M \vDash \Box(p \vee q)$ but $\Diamond p \wedge \Diamond q \nvDash_{KD} \Box(p \vee q)$. How can we modify M to make it the least KD-model of P? First, note that if we connect τ to a new empty world, then M no longer satisfies $\Box(p \vee q)$. But it still satisfies, for example, $\Diamond\Box p$, which does not follow from $\Diamond p \wedge \Diamond q$ in KD. Now connect u to a new empty world, and note that M no longer satisfies $\Diamond\Box p$. Generalizing these observations, one can hope that connecting each newly created world to an empty world is the key to construct the least models for positive modal logic programs.

The least L-model for P will be constructed by building a L-model graph for P. Formulae from the contents of worlds of the model graph are treated as requirements to be realized. We will realize formulae of the form $(B \leftarrow C_1, \ldots, C_k)$ or $\Box\psi$ in a usual way (see Algorithm 5.1). To realize a formula $\Diamond\psi$ at a world u we connect u to a new world containing ψ. To guarantee the model graph to be the smallest, we connect each newly created world to an infinite chain of empty worlds. For the logics $KD5$, $KD45$, and $S5$, these infinite chains can be replaced by two special worlds. From time to time, we extend the accessibility relation in the way to satisfy the frame restrictions.

In the algorithm given below, as a data structure we have a model graph $M = \langle W, \tau, R, H \rangle$. We sometimes refer to M as the *model* $\langle W, \tau, R, h \rangle$, with $h(x) = \{p : p$ is a primitive proposition belonging to $H(x)\}$. We write $M, u \vDash \phi$ to mean that the formula ϕ is satisfied at the world u in the *model* M; and write $M \leq N$ to denote that the *model* M is less than or equal to N.

We will use a procedure called $CreateEmptyTail_L(x_0)$, which is defined as follows:

- If L is one of the logics KD, T, KDB, and B : add an infinite chain of new empty worlds $(x_1, x_2, \ldots)$ to W and set $R = R \bigcup \{(x_i, x_{i+1}) \mid i \geq 0\}$. (Note that the chain can be coded as a finite chain, which can be dynamically expanded when necessary).
- If L is one of the logics $KD5$, $KD45$, and $S5$, and $x_0 = \tau$: add two new empty worlds ρ and ϖ to W, and add the edges (τ, ρ) and (ρ, ϖ) to R.

Algorithm 5.1.
Input: A positive modal logic program P
in a modal logic $L \in \{KD, T, KDB, B, KD5, KD45, S5\}$
Output: The least L-model $M = \langle W, \tau, R, h \rangle$ of P.
Steps:

1. Set $W = \{\tau\}$, $H(\tau) = P$, $R = \emptyset$.
   $CreateEmptyTail_L(\tau)$.
   Set $R = Ext_L(R)$.
2. For every world $u \in W$ with $H(u)$ not empty, and for every formula $\phi \in H(u)$:
   (a) Case $\phi = (B \leftarrow C_1, \ldots, C_k)$, where $k \geq 1$, and $M, u \vDash C_i$ for all $1 \leq i \leq k$: Set $H(u) = H(u) \bigcup \{B\}$.
   (b) Case $\phi = \Box\psi$: For every world $v \in W$ satisfying $R(u, v)$, set $H(v) = H(v) \bigcup \{\psi\}$.
   (c) Case $\phi = \Diamond\psi$, and $\neg(\exists x\ R(u, x) \wedge \psi \in H(x))$:

   Let v be a new world.
   Set $W = W \bigcup \{v\}$, $H(v) = \psi$, $R = R \bigcup \{(u, v)\}$.
   $CreateEmptyTail_L(v)$.
   Set $R = Ext_L(R)$.
3. While some change occurred, repeat step 2.

We give below two auxiliary lemmas.

Lemma 5.1. *At the end of each numerated step of the above algorithm, $\langle W, \tau, R\rangle$ is a connected L-frame.*

The proof of this lemma is straightforward. When $KD5 \leq L$, the frame of the model graph satisfies the condition $\forall x, y \in W\ (x \neq \tau \wedge y \neq \tau) \to R(x, y)$. Moreover, if $L = S5$ then the frame satisfies $\forall x, y \in W\ R(x, y)$; if $L = KD45$ then the frame satisfies $\forall x \in W\ x \neq \tau \to R(\tau, x)$.

Lemma 5.2. *Consider step 2c. Let R denote itself at the beginning of the step, and let R_2 denote R at the end of the step. If $KD5 \leq L$ then the following assertion holds:*

$$\forall x\ x \neq v \wedge (R_2(v, x) \vee (x \neq \tau \wedge R_2(x, v))) \to R^*(u, x).$$

This lemma can be verified using Lemma 5.1.

We give below the main lemma of this section. It informally states that during the execution of Algorithm 5.1, the model graph M is always less than or equal to any L-model of P. What we really want to obtain is the first four assertions, however, to prove them the assertions 5 and 6 are also needed.

Lemma 5.3. *Let $N = \langle W', \tau', R', h'\rangle$ be an arbitrary L-model of P. It is an invariant of the above algorithm[1] that there exists a relation $r \subseteq W \times W'$ such that the following assertions hold:*

1. $r(\tau, \tau') \wedge (\forall x'\ r(\tau, x') \to x' = \tau')$
2. $\forall x, x'\ \forall\, \zeta \in H(x)\ \ r(x, x') \to N, x' \vDash \zeta$
3. $\forall x, y, x'\ R(x, y) \wedge r(x, x') \to \exists y'\ R'(x', y') \wedge r(y, y')$
4. $\forall x, x', y'\ R'(x', y') \wedge r(x, x') \to \exists y\ R(x, y) \wedge r(y, y')$

[1] i.e. the invariant holds at the end of each numerated step of the algorithm

When $L \in \{KD5, KD45, S5\}$*, this assertion has a stronger form:*
$\forall x, x', y'\ R'(x', y') \wedge r(x, x') \wedge x \neq \tau \rightarrow R(x, \varpi) \wedge r(\varpi, y')$
$\wedge\ \forall x', y'\ R'(x', y') \wedge r(\tau, x') \rightarrow R(\tau, \rho) \wedge r(\rho, y')$

5. $\forall x, y, y'\ R(x, y) \wedge r(y, y') \rightarrow \exists x'\ R'(x', y') \wedge r(x, x')$
6. *If* $KD5 \leq L$ *then*
$\forall x, x', x''\ r(x, x') \wedge r(x, x'') \rightarrow x' = x'' \vee R'(x', x'')$.

Proof:
First, we prove the lemma for step 1.

Case $L \in \{KD, T, KDB, B\}$:

Let $(v_1, v_2, \ldots)$ be the infinite chain of worlds created by the call $CreateEmptyTail_L(\tau)$. Denote $v_0 = \tau$.

Let $r = \{(\tau, \tau')\} \bigcup \{(v_i, v_i') \mid R'^i(\tau', v_i')\ for\ i \geq 1\}$.

It is obvious that the assertions 1 and 2 hold.

3) Suppose that $R(x, y) \wedge r(x, x')$ holds. We show that $\exists y'\ R'(x', y') \wedge r(y, y')$. There exists $i \geq 0$ such that ($i \geq 1$, $KDB \leq L$, $x = v_i$, and $y = v_{i-1}$) or ($T \leq L$ and $x = y = v_i$) or ($x = v_i$ and $y = v_{i+1}$). If $x = v_i$, there exist v_0', v_1', …, v_{i+1}' such that $v_0' = \tau'$, $v_i' = x'$, and $\forall j\ 0 \leq j \leq i\ R'(v_j', v_{j+1}')$. It is easy to check that if $y = v_j$ for some $i-1 \leq j \leq i+1$, then by choosing $y' = v_j'$ we have $R'(x', y') \wedge r(y, y')$.

4) Suppose that $R'(x', y') \wedge r(x, x')$ holds. There exists $i \geq 0$ such that $x = v_i$. It is easy to check that for $y = v_{i+1}$ we have $R(x, y) \wedge r(y, y')$.

5) The proof of the assertion is similar to the proof of the assertion 3.

Case $L \in \{KD5, KD45, S5\}$: Let
$r = \{(\tau, \tau')\} \bigcup \{(\rho, \rho') \mid R'(\tau', \rho')\} \bigcup \{(\varpi, \varpi') \mid \exists \rho'\ R'(\tau', \rho') \wedge R'(\rho', \varpi')\}$.

It is easy to check that the assertions 1, 2, 4, 6 hold.

3) Suppose that $R(x, y) \wedge r(x, x')$ holds. We show that $\exists y'\ R'(x', y') \wedge r(y, y')$.

Case $y = \tau$: We have $L = S5$. For $y' = \tau'$, we have $R'(x', y') \wedge r(y, y')$.

Case $y = \rho$: If $x = \rho$, we have $R'(\tau', x')$, hence $R'(x', x')$; for $y' = x'$, we have $R'(x', y') \wedge r(y, y')$. If $x = \tau$ or $x = \varpi$, then for y' such that $R'(\tau', y')$, it is easy to check that $R'(x', y') \wedge r(y, y')$ holds.

Case $y = \varpi$: Let y' be a world such that $R'(x', y')$. It is easy to check that $r(y, y')$ holds.

5) Suppose that $R(x, y) \wedge r(y, y')$ holds. We show that $\exists x'\ R'(x', y') \wedge r(x, x')$.

Case $x = \tau$: Let $x' = \tau'$. We have $r(x, x')$. If $y = \tau$, we have $L = S5$, hence $R'(x', y')$. If $y = \rho$, we have $R'(\tau', y')$, hence $R'(x', y')$. If $y = \varpi$, we have $KD45 \leq L$, hence from $r(y, y')$ we derive $R'(\tau', y')$, and $R'(x', y')$.

Case $x = \rho$ and $y = \varpi$: Since $r(y, y')$, there exists x' such that $R'(\tau', x')$ and $R'(x', y')$, which implies $R'(x', y') \wedge r(x, x')$.

Case $x = \varpi$ and $y = \rho$: By the assertion 3, there exist ρ' and ϖ' such that $R'(\tau', \rho') \wedge R'(\rho', \varpi') \wedge r(\varpi, \varpi')$. Since $r(y, y')$ and $y = \rho$, we have $R'(\tau', y')$. It is easily seen that for $x' = \varpi'$ we have $R'(x', y') \wedge r(x, x')$.

Case $x = y \neq \tau$: Let $x' = y'$. We have $r(x, x')$. It is easily seen that $R'(x', y')$.

We now prove the lemma for step 2. Fix one of steps 2a, 2b, 2c, and suppose that before executing that step, M with r satisfies all the assertions 1 - 6. Let $M_2 = \langle W_2, \tau, R_2, H_2 \rangle$ be the model graph obtained as a result of executing the step. We construct a relation r_2 such that M_2 with r_2 satisfies all the mentioned assertions.

In the remainder of this proof, the phrase "by the assumption about the assertion i", where $1 \leq i \leq 6$, is understood as "because the assertion i is assumed to hold for M and r". When we are proving the assertion i for M_2 and r_2, the phrase "by assumption" in most of cases stands for "by the assumption about the assertion i".

For step 2a: Let $r_2 = r$. We only need to prove the assertion 2 for $x = u$ and $\zeta = B$. Suppose that $r(u, x')$ holds. We show that $N, x' \vDash \zeta$. Assumptions 1 - 4 guarantee that $M \leq N$, hence for every $1 \leq i \leq k$ we have $N, x' \vDash C_i$. On the other hand, it follows from $(B \leftarrow C_1, \ldots, C_k) \in H(u)$ and $r(u, x')$ that $N, x' \vDash (B \leftarrow C_1, \ldots, C_k)$. Therefore $N, x' \vDash B$, and $N, x' \vDash \zeta$.

For step 2b: Let $r_2 = r$. We only need to prove the assertion 2 for $x = v$ and $\zeta = \psi$. Suppose that $r(v, x')$ holds. We show that $N, x' \vDash \zeta$. By the assumption 5, we have $\exists u'\ R'(u', x') \wedge r(u, u')$. Then by the assumption 2, we derive $\exists u'\ R'(u', x') \wedge (N, u' \vDash \phi)$, which implies $N, x' \vDash \zeta$.

For step 2c:

Case $L \in \{KD, T, KDB, B\}$:

Let v be the world mentioned in the algorithm; we denote it also by v_0. Let $(v_1, v_2, \ldots)$ be the chain of worlds created by $CreateEmptyTail_L(v_0)$. Let

$$r_2 = r \bigcup \ \{(v_i, v_i') \mid i \geq 0 \text{ and } \exists u', v_0' \in W'\ r(u, u') \wedge R'(u', v_0') \wedge N, v_0' \vDash \psi \wedge R'^i(v_0', v_i')\}.$$

The assertion 1 obviously holds.

2) It suffices to verify the assertion for $x = v$ and $\zeta = \psi$; but this case is also trivial, by the definition of r_2.

3) Suppose that $R_2(x, y) \wedge r_2(x, x')$ holds. We show that $\exists y'\ R'(x', y') \wedge r_2(y, y')$. If ($x \neq u$ and $\forall i \geq 0\ x \neq v_i$) or ($x = u$ and $y \neq v_0$), we have $R(x, y) \wedge r(x, x')$, and then by assumption, there exists y' such that $R'(x', y') \wedge r(y, y')$, which implies $R'(x', y') \wedge r_2(y, y')$.

- Case $x = u$ and $y = v_0$: We have $r(u, x')$ since $r_2(x, x')$ and $x = u$. Consequently, $N, x' \vDash \Diamond\psi$, and hence there exists y' such that $R'(x', y')$ and $N, y' \vDash \psi$. For such y' we have $R'(x', y') \wedge r_2(y, y')$.
- Case $x = v_i$, $i \geq 0$: Denote $v_{-1} = u$. Since $R_2(x, y)$, we have $y = v_{i-1}$ or $y = v_i$ or $y = v_{i+1}$. Since $r_2(x, x')$, there exist u' and v_0' such that $r(u, u') \wedge R'(u', v_0') \wedge N, v_0' \vDash \psi \wedge R'^i(v_0', x')$. Let v_1', …, v_{i+1}' be worlds such that $v_i' = x'$ and $R'(v_j', v_{j+1}')$, for $0 \leq j \leq i$. Denote

$v_{-1}' = u'$. We have $r_2(v_j, v_j')$, for $-1 \leq j \leq i+1$. It is easily seen that for $y = v_j$, $i-1 \leq j \leq i+1$, by choosing $y' = v_j'$, we have $R'(x', y') \wedge r_2(y, y')$.

4) Suppose that $R'(x', y') \wedge r_2(x, x')$ holds. We show that $\exists y\ R_2(x, y) \wedge r_2(y, y')$. If $\forall i \geq 0\ x \neq v_i$, we have $r(x, x')$, and by assumption, $\exists y\ R(x, y) \wedge r(y, y')$, which implies $\exists y\ R_2(x, y) \wedge r_2(y, y')$. If $x = v_i$ for some $i \geq 0$, then by choosing $y = v_{i+1}$, we have $R_2(x, y) \wedge r_2(y, y')$.

5) Suppose that $R_2(x, y) \wedge r_2(y, y')$ holds. We show that $\exists x'\ R'(x', y') \wedge r_2(x, x')$.

Case $L = KDB$ or $L = B$: We have $R_2(y, x) \wedge r_2(y, y')$, and it follows from the assertion 3 that there exists x' such that $R'(y', x') \wedge r_2(x, x')$. Since R' is symmetric, we also have $R'(x', y')$.

Case $L = KD$ or $L = T$: If ($x \neq u$ and $\forall i \geq 0\ x \neq v_i$) or ($x = u$ and $y \neq v_0$), we have $R(x, y) \wedge r(y, y')$, and then by assumption, there exists x' such that $R'(x', y') \wedge r(x, x')$, which implies $R'(x', y') \wedge r_2(x, x')$. Denote $v_{-1} = u$. Now assume that $x = v_i$ for some $i \geq -1$, and that if $x = v_{-1}$ then $y = v_0$. Since $R_2(x, y)$, we have $y = v_j$ for $j = i$ or $j = i+1$. Since $r_2(y, y')$, there exist u' and v_0' such that $r(u, u') \wedge R'(u', v_0') \wedge N, v_0' \vDash \psi \wedge R'^j(v_0', y')$. Let $v_1', \ldots, v_j'$ be worlds such that $v_j' = y'$ and $R'(v_{k-1}', v_k')$, for $1 \leq k \leq j$. Denote $v_{-1}' = u'$. We have $r_2(v_k, v_k')$, for $-1 \leq k \leq j$. It is easily seen that for $x = v_i$, by choosing $x' = v_i'$, we have $R'(x', y') \wedge r_2(x, x')$.

Case $L \in \{KD5, KD45, S5\}$:

Let $r_2 = r \bigcup \{(v, v') \mid \exists u'\ r(u, u') \wedge R'(u', v') \wedge N, v' \vDash \psi\}$.

The assertion 1 holds by assumption.

2) It suffices to verify the assertion for $x = v$ and $\zeta = \psi$; but this case is also trivial, by the definition of r_2.

3) Suppose that $R_2(x, y) \wedge r_2(x, x')$ holds. We show that $\exists y'\ R'(x', y') \wedge r_2(y, y')$.

- Case $x \neq v$ and $y \neq v$: The assertion holds by assumption.
- Case $x = v$ and $y = v$: It is easy to check that for $y' = x'$ we have $R'(x', y') \wedge r_2(y, y')$.
- Case $x = v$ and $y \neq v$: There exists u' such that $r(u, u') \wedge R'(u', x') \wedge (N, x' \vDash \psi)$. Since $R_2(x, y)$, by Lemma 5.2, we have $R^*(u, y)$. Applying the assumption 3 one or more times for $R^*(u, y) \wedge r(u, u')$, we conclude that there exists y' such that $R'^*(u', y') \wedge r(y, y')$. Since $R'(u', x')$ and $KD5 \leq L$ and $r \subseteq r_2$, it follows that $R'(x', y') \wedge r_2(y, y')$.
- Case $y = v$ and $x \neq v$: By Lemma 5.1 and the assumption 3, there exists u' such that $((R'^*(\tau', u') \wedge u \neq \tau) \vee (u' = \tau' \wedge u = \tau)) \wedge r(u, u')$. Hence, by the assumption 2, we have $N, u' \vDash \phi$. Hence there exists y' such that $R'(u', y')$ and $N, y' \vDash \psi$. Thus we have $r_2(y, y')$. We now show that $R'(x', y')$ holds.

  Case $x = \tau$: By the assumption 1, we have $x' = \tau'$. If $u = \tau$, we have $u' = \tau' = x'$, and $R'(x', y')$. Suppose that $u \neq \tau$. In this case, from $R_2(x, y)$, $x = \tau$, and $y = v$, we claim that $KD45 \leq L$. It follows that $R'(\tau', y')$, hence $R'(x', y')$.

  Case $x \neq \tau$: Since $R_2(x, y)$, by Lemma 5.2, we have $R^*(u, x)$. Applying the assumption 3 one or more times for $R^*(u, x) \wedge r(u, u')$, we conclude that there exists x'' such that $R'^*(u', x'') \wedge r(x, x'')$, which implies $R'(x'', y')$, since $R'(u', y')$ and $KD5 \leq L$. Since $r(x, x') \wedge r(x, x'')$, by the assumption 6, we have $x' = x'' \vee R'(x'', x')$. This together with $R'(x'', y')$ implies $R'(x', y')$.

4) It is sufficient to show that for any x, x', y' belonging to W, the following implication holds: $R'(x', y') \wedge r_2(x, x') \wedge x \neq \tau \rightarrow r_2(\varpi, y')$. Suppose that the antecedent of this implication holds. The case $x \neq v$ is trivial, so assume that $x = v$. Since $r_2(x, x')$, there exists u' such that $r(u, u') \wedge R'(u', x') \wedge N, x' \vDash \psi$. If $u = \tau$, by assumption, we derive $r(\rho, x')$, and then $r(\varpi, y')$. If $u \neq \tau$, we have $R'(u', x') \wedge r(u, u') \wedge u \neq \tau$, which by assumption implies $r(\varpi, x')$, and then $r(\varpi, y')$. Hence $r_2(\varpi, y')$ holds.

5) Suppose that $R_2(x, y) \wedge r_2(y, y')$ holds. We show that $\exists x'\ R'(x', y') \wedge r_2(x, x')$.

- Case $x \neq v$ and $y \neq v$: The assertion holds by assumption.
- Case $x = v$ and $y = v$: It is easy to check that for $x' = y'$ we have $R'(x', y') \wedge r_2(x, x')$.
- Case $x = v$ and $y \neq v$: We have $r(y, y')$. It is easily seen that there exists u' such that $r(u, u')$. By the assumption 2, we have $N, u' \vDash \phi$, hence there exists v' such that $R'(u', v')$ and $N, v' \vDash \psi$. This implies $r_2(v, v')$. Since $R_2(x, y)$, by Lemma 5.2, we have $R^*(u, y)$. One or more times applying the assumption 3 for $R^*(u, y) \wedge r(u, u')$, we conclude that there exists y'' such that $R'^*(u', y'') \wedge r(y, y'')$. Consequently, $R'(v', y'')$ and $R'(y'', v')$ hold, since $R'(u', v')$ and $KD5 \leq L$. Since $r(y, y') \wedge r(y, y'')$, by the assumption 6, we have $y' = y'' \vee R'(y'', y')$. As a consequence, we have $R'(v', y')$, since $R'(v', y'')$, $R'(y'', v')$, and $KD5 \leq L$. Therefore, for $x' = v'$, we have $R'(x', y') \wedge r_2(x, x')$.
- Case $x \neq v$ and $y = v$: Since $r_2(y, y')$, there exists u' such that $r(u, u') \wedge R'(u', y') \wedge N, y' \vDash \psi$.

  Case $x = \tau$: Let $x' = \tau'$. We have $r_2(x, x')$, and need to prove that $R'(\tau', y')$ holds. If $u = \tau$, we have $u' = \tau'$, and $R'(\tau', y')$. Suppose that $u \neq \tau$. From $R_2(x, y)$, $x = \tau$, and $y = v$, we claim that $KD45 \leq L$. By Lemma 5.1, we have $R^*(\tau, u)$, and hence $R(\tau, u)$. Applying the assumption 3 for $R(\tau, u) \wedge r(\tau, \tau')$, we derive that there exists u'' such that $R'(\tau', u'') \wedge r(u, u'')$. From the assumption 6, we derive that $u' = u'' \vee R'(u', u'')$. This together with $R'(\tau', u'')$, $R'(u', y')$, and $KD45 \leq L$ implies $R'(\tau', y')$.

  Case $x \neq \tau$: Since $R_2(x, y)$, by Lemma 5.2, we have $R^*(u, x)$. Once or more times applying the assumption 3 for $R^*(u, x) \wedge r(u, u')$, we derive that there exists x' such that $R'^*(u', x') \wedge r(x, x')$. Thus $R'(x', y') \wedge r_2(x, x')$, since $R'(u', y')$ and $KD5 \leq L$ and $r \subseteq r_2$.

6) Suppose that $r_2(x, x') \wedge r_2(x, x'')$ holds. We show that $x' = x'' \vee R'(x', x'')$. If $x \neq v$, the assertion holds by assumption. Suppose that $x = v$. There exist u' and u'' such that $(r(u, u') \wedge R'(u', x') \wedge N, x' \vDash \psi)$ and $(r(u, u'') \wedge R'(u'', x'') \wedge N, x'' \vDash \psi)$. Since $r(u, u') \wedge r(u, u'')$, by assumption, we have $u' = u'' \vee R'(u', u'')$. If $u' = u''$, we derive $R'(x', x'')$ from $R'(u', x')$ and $R'(u'', x'')$. If $R'(u', u'')$, we have $R'(u'', x')$, since $R'(u', x')$ and $KD5 \leq L$; hence $R'(x', x'')$, since $R'(u'', x'')$ and $KD5 \leq L$. □

We now estimate the complexity of the algorithm and the size of the resulting model.

Lemma 5.4. *Let n and k be the size (i.e. the length) and the modal depth of a program P. Let M be the model constructed by Algorithm 5.1 for P. If L is one of the logics KD, T, KDB, and B, then the algorithm terminates in $O(n^{2k+2})$ steps, the real diameter of M is less than or* *equal to k, M can be coded using $O(n^{k+1})$-space, and for any m, the model $M|_m$ has no more than $m.n^k$ worlds. If L is one of the logics $KD5$, $KD45$, and $S5$, then the algorithm terminates in $O(n^4)$ steps, and the size of M is of order $O(n^2)$.*

Proof:
Case $L \in \{KD, T, KDB, B\}$: It is easily seen that, during the execution of the algorithm, the real diameter of M is always less than or equal to k. The number of worlds in $M|_k$ is bounded by n^k, the number of edges in $M|_k$ are bounded by $3n^k$, and the number of formulae at each world is bounded by n. Hence the size of $M|_k$ is of order $O(n^{k+1})$. It is easily seen that checking whether the model graph can further be extended is solvable in $O(n^{k+1})$ steps. Because each time step 2 is executed, it either increases the size of $M|_k$ by at least 1 or certifies that no more changes can be made to M, we conclude that the algorithm terminates in $O(n^{2k+2})$ steps. For each world $w \in W$, there is at most one infinite chain of empty worlds starting from w, hence the resulting model M can be coded using $O(n^{k+1})$-space, and for any $m \geq 1$, the model $M|_m$ has no more than $m.n^k$ worlds.

Case $L \in \{KD5, KD45, S5\}$: It is easily seen that, during the execution of the algorithm, there are no more than $n + 3$ worlds. The size of each world is bounded by n. Hence the size of M is of order $O(n^2)$. Checking whether step 2 can make some change to M can be solved in $O(n^2)$ steps. Each time step 2 is executed, it either increases the size M by at least 1 or certifies that no more changes can be made to the model graph. Hence the algorithm terminates in $O(n^4)$ steps. □

Here is the main theorem of this section:

Theorem 5.2. *For any positive modal logic program P of size n, in a modal logic $L \in \{KD, T, KDB, B, KD5, KD45, S5\}$, there exists the least L-model M of P. If $L \in \{KD5, KD45, S5\}$, then M can be constructed in $O(n^4)$ steps and its size is of order $O(n^4)$. If $L \in \{KD, T, KDB, B\}$ and the modal depth of P is finitely bounded, then M can be constructed in PTIME, and for any $m \geq 1$, the size of the model $M|_m$ is bounded by a polynomial in n and m.*

Proof:
Let M be the model constructed by Algorithm 5.1 for P in L. It is easy to prove by induction (on the construction of ϕ) that for any world u, and any formula $\phi \in H(u)$, $M, u \models \phi$. Hence $M \models P$. Lemma 5.1 asserts that M is a L-model. These together with Lemma 5.3 assert that M is the least L-model of P. The estimation of the complexity of the algorithm and of the size of M follows from Lemma 5.4. □

Corollary 5.1. *The problem of checking the satisfiability of a set of Horn formulae with finitely bounded modal depth in KD, T, KDB, or B is decidable in PTIME.*

Corollary 5.2. (Fariñas del Cerro and Penttonen [3], Chen and Lin [2]) *The problem of checking the satisfiability of a set of Horn formulae in $KD5$, $KD45$, or $S5$ is decidable in PTIME.*

These corollaries immediately follow from Proposition 2.1, Theorem 5.2, and Lemmas 2.2 and 2.1.

Although the model M constructed by Algorithm 5.1 for a program P in a logic $L \in \{KD, T, KDB, B\}$ is infinite, by Lemmas 5.4 and 2.2, M can be treated as a finite model. For L being KD or T, we will show that M can actually be transformed to an equivalent finite model.

Lemma 5.5. *Let $M = \langle W, \tau, R, h\rangle$ be a connected KD-model (i.e. $\langle W, \tau, R\rangle$ is a connected KD-frame). Suppose that W can be divided into two groups W_1 and W_2 such that W_1 contains τ, W_2 is a non-empty set of empty worlds, and there is no edges connecting W_2 to W_1. Let w_2 be a new world, and let*

$$\begin{aligned}
&W' = W_1 \bigcup \{w_2\};\\
&R' = R - \{(x, y) \mid y \in W_2\}\\
&\qquad\quad \bigcup \{(x, w_2) \mid x \in W_1 \wedge \exists y \in W_2\ R(x, y)\} \bigcup \{(w_2, w_2)\};\\
&h'(x) = h(x)\ for\ x \neq w_2, \quad h'(w_2) = \emptyset;\\
&M' = \langle W', \tau, R', h'\rangle;\\
&r = \{(x, x) \mid x \in W_1\} \bigcup \{(y, w_2) \mid y \in W_2\};\\
&s = \{(x, x) \mid x \in W_1\} \bigcup \{(w_2, y) \mid y \in W_2\}.
\end{aligned}$$

Then $M' \leq M$ wrt. s, and $M \leq M'$ wrt. r. Moreover, for any formula ϕ, $M \vDash \phi$ iff $M' \vDash \phi$.

The proof of the first assertion is straightforward. The second assertion can be easily proved by induction on the construction of ϕ.

Let M be the model constructed by Algorithm 5.1 for a program P in a logic $L \in \{KD, T\}$. By the above lemma, if we replace all infinite chains of empty worlds in M by a new empty world w_2 connected to itself, we can obtain the least L-model of P. Thus, we have:

Corollary 5.3. *For any positive program P with a finitely bounded modal depth, the least KD-model and the least T-model of P can be constructed in PTIME, and their sizes are bounded by a polynomial in the size of P.*

6. Characterizations of Positive Programs in KB, $K5$, $K45$, and $KB5$

In this section, we show that for any positive program P in a modal logic $L \in \{KB$, $K5$, $K45$, $KB5\}$ we can construct one or two L-models such that a positive formula ϕ follows from P iff ϕ is satisfied in these models.

Definition 6.1. (Flat Models) We call a model $\langle W, \tau, R, h\rangle$ *flat* if $W = \{\tau\}$ and $R = \emptyset$.

Definition 6.2. (The Least Flat Model of a Positive Program) A model M is called the *least flat model* of a positive program P if it is a flat model of P and is less than or equal to any flat model of P.

Definition 6.3. (Almost Serial Modal Logics) Let L be a normal modal logic. We say that L is *almost serial* if every L-frame $\langle W, \tau, R\rangle$ satisfies the formula
$\exists x\, R(\tau, x) \rightarrow (\forall y\, R^*(\tau, y) \rightarrow \exists z\, R(y, z))$.

The logics KB, $K5$, $K45$, and $KB5$ are almost serial. Hence any non-flat connected frame in KB (resp. $K5$, $K45$, $KB5$) is a frame in KDB (resp. $KD5$, $KD45$, $S5$). We say that KDB, $KD5$, $KD45$, and $S5$ are the serial logics corresponding to KB, $K5$, $K45$, and $KB5$, respectively. For $L \in \{KB, K5, K45, KB5\}$, we use LD to denote the serial logic corresponding to L.

Theorem 6.1. *Let P be a positive program in a modal logic $L \in \{KB,\ K5,\ K45,\ KB5\}$. Let M be the least LD-model of P constructed by Algorithm 5.1, and let $N = \langle W', \tau', R', h'\rangle$ be a non-flat L-model of P. Then $M \leq N$.*

The proof of this theorem is straightforward.

Theorem 6.2. *The problem of checking whether a positive modal logic program P has flat models is decidable in PTIME. If P has flat models then it has the least flat model, which can be constructed in PTIME and has size bounded by a polynomial in the size of P.*

Proof:
Let P' be the program obtained from P by replacing every formula $\Box\phi$ in P by *true*, and every atom $\Diamond p$ by a new primitive proposition p'. The program P' is a positive program in the classical logic. Therefore it has the least model M, which can be constructed in PTIME and has size bounded by a polynomial in the size of P. If there is a primitive proposition p' satisfied in M then P has no flat models; for otherwise let N be a flat model of P, we would have $\Diamond p$ satisfied in N, which is impossible. If there is no primitive proposition p' satisfied in M then M is a flat model of P. If N is a flat model of P then it is easily seen that it is also a model of P', hence $M \leq N$. □

Corollary 6.1. *Let P be a positive program in a modal logic $L \in \{KB,\ K5,\ K45,\ KB5\}$. Let M be the least LD-model of P. If P has no flat models then M is the least L-model of P. If P has flat models then it has the least flat model M', and for every model N of P, if N is flat then $M' \leq N$, otherwise $M \leq N$.*

This corollary follows from Theorems 6.1 and 6.2.

Corollary 6.2. *The problem of checking the satisfiability of a set of Horn formulae with finitely bounded modal depth in KB is decidable in PTIME.*

Corollary 6.3. (Chen and Lin [2]) *The problem of checking the satisfiability of a set of Horn formulae in $K5$, $K45$, or $KB5$ is decidable in PTIME.*

These corollaries immediately follow from Proposition 2.1, Corollary 6.1, Theorems 5.2 and 6.2, and Lemmas 2.2 and 2.1.

7. Constructing the Least Models for Positive Programs in $KD4$ and $S4$

Algorithm 5.1 cannot be applied for the logics $KD4$ and $S4$, because it will not terminate, for example, when P contains the formula $\Box\Diamond p$. Before giving an algorithm of constructing the least models for positive programs in $KD4$ and $S4$, let us introduce some notations.

In Algorithm 7.1 given below, we use the following data structures:

- $M = \langle W, \tau, R^*, H\rangle$ - a model graph, with R^* being the transitive closure of R. We sometimes refer to M as a model in the same way as described in Section 5.
- $StatusW : W \to \{normal, minimal\}$;
- $StatusR : R \to \{permanent, temporal\}$;
- $Next : W \times \mathcal{F} \to W$, which is interpreted as follows: $Next(u, \Diamond\phi) = v$ means $R(u, v)$, $\phi \in H(v)$, and the world u *realizes* the requirement $\Diamond\phi$ by going to v.

We define the binary relation $Permanent$ on W as follows

$$Permanent(x, y) \equiv (R(x, y) \wedge StatusR(x, y) = permanent),$$

and use $Permanent^*$ to denote its transitive closure. We also need the following notations:

$$\begin{aligned} H^+(x) &= \{\phi \mid \exists y\ Permanent^*(y, x) \text{ and} \\ &\quad (\Box\phi \in H(y) \text{ or } (\phi \in H(y) \text{ and } \phi = \Box\psi \text{ for some } \psi))\}, \\ H^*(x) &= H^+(x) \bigcup \\ &\quad \{\phi \mid \Box\phi \in H(x) \text{ or } (\phi \in H(x) \text{ and } \phi = \Box\psi \text{ for some } \psi)\}, \end{aligned}$$

where $H^+(x)$ can be interpreted as the potentiality inherited from the predecessors of x, and $H^*(x)$ - the potentiality inherited from both x and its predecessors.

Consider Algorithm 7.1 given below. We build the least L-model for P by constructing a L-model graph for it. The relation $Permanent$, without self-referencing edges, forms a tree with root τ, which is a skeleton of the model graph.

Consider step 2c. If we create a new world to realize the formula $\Diamond p$, then it should contain p and $H^*(u)$. We first try to find a world v on the path from τ to u (in the tree generated by $Permanent$) such that it contains $\Diamond p$, and $H^*(v) = H^*(u)$. If such v exists and the formula $\Diamond p$ has been realized in v by a connection from v to w, then we just connect u to w; but this edge is not *permanent*, because if a new formula $\Box\zeta$ is later added to u while ζ or $\Box\zeta$ is not present in w then we will delete the edge. If such v exists but the formula $\Diamond p$ has not been realized in v, then we do nothing; this means that the formula $\Diamond p$ will be realized in v before in u. If such v does not exist, then we realize the formula $\Diamond p$ in u by creating a new world, denoted by v, assigning p and $H^*(u)$ to it, and connecting u to v, and v to a new *minimal* world. $Minimal$ worlds are elements that make M the least L-model of P. They have the same role as the infinite chains of empty worlds used in Algorithm 5.1. In step 2(b)i, we do not add ψ to u because of the nature of *minimal* worlds.

In Algorithm 7.1, we use a procedure called $CreateNextMinNode(x)$, which is defined as:

Let y be a new world.
Set $W = W \bigcup \{y\}$, $StatusW(y) = minimal$, $H(y) = H^*(x)$,
$R = R \bigcup \{(x, y), (y, y)\}$, $StatusR(x, y) = permanent$.
If $L = S4$, set $StatusR(y, y) = permanent$;
otherwise, set $StatusR(y, y) = temporal$.

Algorithm 7.1.
Input: A positive modal logic program P in a modal logic $L \in \{KD4, S4\}$
Output: The least L-model $M = \langle W, \tau, R^*, h\rangle$ of P.
Steps:

1. Set $W = \{\tau\}$, $StatusW(\tau) = normal$, $H(\tau) = P$.
   If $L = S4$, set $R = \{(\tau, \tau)\}$ and $StatusR(\tau, \tau) = permanent$;
   otherwise, set $R = \emptyset$.
   $CreateNextMinNode(\tau)$.
2. For every world $u \in W$, and for every formula $\phi \in H(u)$,
   (a) Case $\phi = B \leftarrow C_1, \ldots, C_k$ for some $k \geq 1$, and $M, u \vDash C_i$ for all $1 \leq i \leq k$:
   Set $H(u) = H(u) \bigcup \{B\}$.
   (b) Case $\phi = \Box\psi$ for some ψ :
   i. If $StatusW(u) = minimal$, $StatusR(u, u) = temporal$, and $\psi \notin H(u)$, then
   Set $R = R - \{(u, u)\}$, $StatusW(u) = normal$,
   $CreateNextMinNode(u)$.
   ii. Otherwise,
   For every v such that $Permanent^*(u, v)$,
   Set $H(v) = H(v) \bigcup \{\phi, \psi\}$,
   For every v such that $R(u, v)$, $StatusR(u, v) = temporal$,
   and $\{\phi, \psi\} \nsubseteq H(v)$,
   Set $R = R - \{(u, v)\}$,
   For every $\Diamond p$ such that $Next(u, \Diamond p) = v$,
   Set $Next(u, \Diamond p)$ undefined.
   (c) Case $\phi = \Diamond p$ for some primitive proposition p, and $Next(u, \Diamond p)$ is undefined:
   i. If there exist v and w such that:
   $$Permanent^*(v, u) \wedge H^*(v) = H^*(u) \wedge \phi \in H(v) \wedge$$
   $$Permanent(v, w) \wedge Next(v, \Diamond p) = w,$$
   Set $R = R \bigcup \{(u, w)\}$, $StatusR(u, w) = temporal$, $Next(u, \Diamond p) = w$.
   ii. Otherwise, if $\neg(\exists x \;\; x \neq u \wedge Permanent^*(x, u) \wedge H^*(x) = H^*(u) \wedge \phi \in H(x))$,
   Let v be a new world, and set $Next(u, \Diamond p) = v$.
   Set $W = W \bigcup \{v\}$, $StatusW(v) = normal$, $H(v) = H^*(u) \bigcup \{p\}$,
   $R = R \bigcup \{(u, v)\}$, $StatusR(u, v) = permanent$.

If $L = S4$, set $R = R \bigcup \{(v, v)\}$, $StatusR(v, v) = permanent$.
$CreateNextMinNode(v)$.

3. While some change occurred, repeat step 2.

We give below an auxiliary lemma. Its proof is straightforward.

Lemma 7.1. *The following assertions hold at the end of each numerated step of the above algorithm:*

1. $L = KD4 \rightarrow$
   $\forall x \ \ (StatusW(x) = normal \rightarrow \neg R(x, x)) \wedge$
   $(StatusW(x) = minimal \rightarrow StatusR(x, x) = temporal)$
2. $L = S4 \rightarrow \forall x \ Permanent(x, x)$
3. $\forall x \ Permanent^*(x, x) \rightarrow Permanent(x, x)$
4. *For every formula* $\Diamond q$,
   $\forall x, y \ Next(x, \Diamond q) = y \rightarrow R(x, y) \wedge q \in H(y) \wedge (Permanent(x, y) \vee$
   $(\exists z \ Permanent^*(z, x) \wedge H^*(z) = H^*(x) \wedge \Diamond q \in H(z)$
   $\wedge \ Permanent(z, y) \wedge Next(z, \Diamond q) = y))$
5. $(L = KD4 \rightarrow \forall x \ \exists y \ R(x, y)) \wedge (L = S4 \rightarrow \forall x \ R(x, x))$.

Lemma 7.2. *If Algorithm 7.1 terminates, then the resulting model M is a L-model of P.*

Proof:
We prove that for every world u, and a formula $\phi \in H(u)$, $M, u \vDash \phi$. It is easily seen that the assertion holds for the case when ϕ is of the form $\Box\psi$ or $(B \leftarrow C_1, \ldots C_k)$, with $k \geq 1$. Suppose that for some u and $\Diamond q \in H(u)$, $Next(u, \Diamond q)$ is undefined. Without loss of generality, we can assume that for every world x such that $x \neq u$ and $Permanent^*(x, u)$, $Next(x, \Diamond p)$ is defined for every $\Diamond p \in H(x)$. Since no more changes can be made to the model graph, there exists v such that $v \neq u \wedge Permanent^*(v, u) \wedge H^*(v) = H^*(u) \wedge \Diamond q \in H(v)$. Thus, we have $Next(v, \Diamond q)$ defined. Let $Next(v, \Diamond q) = w$. By Lemma 7.1:4, we have:

$R(v, w) \wedge q \in H(w) \wedge (Permanent(v, w) \vee$
$(\exists z \ Permanent^*(z, v) \wedge H^*(z) = H^*(v) \wedge \Diamond q \in H(z) \wedge Permanent(z, w) \wedge Next(z, \Diamond q) = w))$.

If $Permanent(v, w)$ holds, M can be extended by step 2(c)i, and we have a contradiction. If there is z such that $Permanent^*(z, v) \wedge (H^*(z) = H^*(v)) \wedge \Diamond q \in H(z) \wedge Permanent(z, w) \wedge Next(z, \Diamond q) = w$, then we have $Permanent^*(z, u) \wedge H^*(z) = H^*(u) \wedge \Diamond q \in H(z) \wedge Permanent(z, w) \wedge Next(z, \Diamond q) = w$, hence step 2(c)i can be applied to extend M, which is a contradiction. Therefore for any world $u \in W$ and $\Diamond p \in H(u)$, $Next(u, \Diamond p)$ is defined, which by Lemma 7.1:4 implies that $N, u \vDash \Diamond p$. The consequence is that $M, \tau \vDash P$, since $P \subseteq H(\tau)$. Lemma 7.1:5 asserts that R^* satisfies all L-frame axioms. □

We give below the main lemma of this section. What we really want to claim by this lemma is that the model M constructed by Algorithm 7.1 is less than or equal to any L-model of P. One would expect only the first four assertions of the lemma, in a simpler form as follows:

1. $r(\tau, \tau')$
2. $\forall x, x' \; \forall \zeta \in H(x) \;\; r(x, x') \rightarrow N, x' \vDash \zeta$
3. $\forall x, y, x' \; R^*(x, y) \wedge r(x, x') \rightarrow \exists y' \; R'(x', y') \wedge r(y, y')$
4. $\forall x, x', y' \; R'(x', y') \wedge r(x, x') \rightarrow \exists y \; R^*(x, y) \wedge r(y, y')$.

We make the assertions stronger and add additional ones in order to prove themselves.

Lemma 7.3. *Let $N = \langle W', \tau', R', h' \rangle$ be an arbitrary L-model of P. It is an invariant of the algorithm*[2] *that there exists a relation $r \subseteq W \times W'$ such that the following assertions hold:*

1. $r(\tau, \tau')$
2. $\forall x, x' \;\; \forall \zeta \in H(x) \bigcup H^+(x) \quad r(x, x') \rightarrow N, x' \vDash \zeta$
3. $\forall x, y, x' \; R^*(x, y) \wedge r(x, x') \rightarrow \exists y' \; R'^*(x', y') \wedge r(y, y')$
4. $\forall x, x', y' \; R'^*(x', y') \wedge r(x, x') \rightarrow \exists y \; (Permanent^*(x, y) \vee (y = x \wedge R(x, y))) \wedge r(y, y')$
5. $\forall x, y, y' \; x \neq y \wedge Permanent^*(x, y) \wedge r(y, y') \rightarrow \exists x' \; R'^*(x', y') \wedge r(x, x')$
6. *For every formula $\Diamond q$,*
   $\forall x, y, x', y' \; (Permanent(x, y) \wedge Next(x, \Diamond q) = y \wedge \; r(x, x') \wedge R'^*(x', y') \wedge N, y' \vDash q)$
   $\rightarrow r(y, y')$.

Note that $R'^* \equiv R'$ since R' is transitive. We write R'^* instead of R' for symmetry.

Proof:
For step 1:

Let u be the world created by $CreateNextMinNode(\tau)$.

Let $r = \{(\tau, \tau')\} \bigcup \{(u, x') \mid R'^*(\tau', x')\}$.

It is obvious that the assertion 1 holds. The assertion 6 holds because the function $Next$ is totally undefined at this step.

2) Suppose that $\zeta \in H(x) \bigcup H^+(x)$ and $r(x, x')$ holds. We show that $N, x' \vDash \zeta$.

- Case $x = \tau$: We have $x' = \tau'$ since $r(x, x')$.

  Case $L = KD4$: We have $\neg R(\tau, \tau)$, hence $H^+(\tau) = \emptyset$ and $\zeta \in H(\tau)$. Since $H(\tau) = P$ and $N \vDash P$, it follows that $N, x' \vDash \zeta$.

  Case $L = S4$: We have $H(\tau) \bigcup H^+(\tau) = P \bigcup \{\phi \mid \Box\phi \in P\}$. If $\zeta \in P$, we have $N, x' \vDash \zeta$ since $N \vDash P$. If $\Box\zeta \in P$, we have $N, x' \vDash \Box\zeta$ since $N \vDash P$; and $N, x' \vDash \zeta$ since $L = S4$.
- Case $x = u$: We have $R'^*(\tau', x')$. Since $\zeta \in H(u) \bigcup H^+(u)$, there are cases: $\Box\zeta \in H(\tau)$, or ($\zeta \in H(\tau)$ and $\zeta = \Box\psi$ for some ψ), or ($\Box\Box\zeta \in H(\tau)$ and $L = S4$).

  Case $\Box\zeta \in H(\tau)$: We have $N, \tau' \vDash \Box\zeta$, since $H(\tau) = P$ and $N, \tau' \vDash P$. It follows that $N, x' \vDash \zeta$, since $R'^*(\tau', x')$.

  Case $\zeta = \Box\psi$ and $\zeta \in H(\tau)$: We have $N, \tau' \vDash \Box\psi$, since $H(\tau) = P$ and $N, \tau' \vDash P$. Hence $N, \tau' \vDash \Box\Box\psi$, since N is a $K4$-model. It follows that $N, x' \vDash \zeta$, since $R'^*(\tau', x')$.

  Case $\Box\Box\zeta \in H(\tau)$ and $L = S4$: We have $N, \tau' \vDash \Box\Box\zeta$, since $H(\tau) = P$ and $N, \tau' \vDash P$. Hence $N, x' \vDash \Box\zeta$, since $R'^*(\tau', x')$. It follows that $N, x' \vDash \zeta$, since N is a $S4$-model.

[2] i.e. the invariant holds at the end of each numerated step of the algorithm

3) Suppose that $R^*(x,y) \wedge r(x,x')$ holds. We show that $\exists y'\ R'^*(x',y') \wedge r(y,y')$.

Case $x = y = \tau$: It follows that $L = S4$. We have $x' = \tau'$ since $r(x,x')$. For $y' = \tau'$, we have $R'^*(x',y') \wedge r(y,y')$.

Case $x = \tau$ and $y = u$: We have $x' = \tau'$ since $r(x,x')$. There exists y' such that $R'(\tau',y')$, since $KD \leq L$. For such y' we have $R'^*(x',y') \wedge r(y,y')$.

Case $x = u$ and $y = u$: We have $R'^*(\tau',x')$ since $r(x,x')$. There exists y' such that $R'(x',y')$, hence $R'^*(\tau',y')$. For such y' we have $R'^*(x',y') \wedge r(y,y')$.

4) Suppose that $R'^*(x',y') \wedge r(x,x')$ holds. We show that
$\exists y\ (Permanent^*(x,y) \vee (y = x \wedge R(x,y))) \wedge r(y,y')$.

Case $x = \tau$: It follows that $x' = \tau'$. For $y = u$, we have $Permanent^*(x,y)$ and $r(y,y')$.

Case $x = u$: We have $R'^*(\tau',x')$ since $r(x,x')$, hence $R'^*(\tau',y')$. For $y = u$, we have $y = x \wedge R(x,y) \wedge r(y,y')$.

5) Suppose that $x \neq y \wedge Permanent^*(x,y) \wedge r(y,y')$ holds. We show that $\exists x'\ R'^*(x',y') \wedge r(x,x')$. It is true that $x = \tau$ and $y = u$. We have $R'^*(\tau',y')$ since $r(y,y')$. For $x' = \tau'$, we have $R'^*(x',y') \wedge r(x,x')$.

We now prove the lemma for step 2. Fix one of steps 2a, 2(b)i, 2(b)ii, 2(c)i, 2(c)ii, and suppose that before executing the step, M with $StatusW$, $StatusR$, $Next$, and r satisfies all of the assertions 1 - 6. Let $M_2 = \langle W_2, \tau, R_2, H_2 \rangle$, $StatusW_2$, $StatusR_2$, and $Next_2$ be obtained as a result of executing the step. We construct a relation r_2 such that all of the above mentioned assertions hold for $M_2, StatusW_2, StatusR_2, Next_2, r_2$.

In the remainder of this proof, the phrase "by the assumption about the assertion i", where $1 \leq i \leq 6$, is understood as "because the assertion i is assumed to hold for M, $StatusW$, $StatusR$, $Next$, and r". When we are proving the assertion i for M_2, $StatusW_2$, $StatusR_2$, $Next_2$, and r_2, the phrase "by assumption" in most of cases stands for "by the assumption about the assertion i".

For step 2a:

Let $r_2 = r$. We only need to prove the assertion 2.

Suppose that $\zeta \in H_2(x) \bigcup {H_2}^+(x)$ and $r(x,x')$ holds. We show that $N, x' \vDash \zeta$. The assumptions 1 - 4 assert that $M \leq N$.

- Case $x = u$ and $\zeta = B$: We have $N, x' \vDash \phi$ and $N, x' \vDash C_i$ for all $1 \leq i \leq k$, since $r(x,x')$. Therefore $N, x' \vDash B$, and $N, x' \vDash \zeta$.
- Case $x \neq u$ or $\zeta \neq B$: It follows that if $\zeta \in H_2(x)$ then $\zeta \in H(x)$. If B is not of the form $\Box\xi$, or $\neg Permanent^*(u,x)$, or ($B = \Box\xi$, $\zeta \neq \xi$, and $\zeta \neq B$), then $\zeta \in {H_2}^+(x)$ implies $\zeta \in H^+(x)$, hence $\zeta \in H(x) \bigcup H^+(x)$, and finally $N, x' \vDash \zeta$. Suppose that $B = \Box\xi$ for some ξ, $Permanent^*(u,x)$, and ($\zeta = \xi$ or $\zeta = B$).

  Case $x \neq u$: By the assumption about the assertion 5, there exists u' such that $R'^*(u',x') \wedge r(u,u')$. Thus $N, u' \vDash \phi$ and $\forall i\ 1 \leq i \leq k\ \ N, u' \vDash C_i$. It follows that $N, u' \vDash \Box\xi$, hence $N, x' \vDash \xi$ and $N, x' \vDash \Box\xi$, since $R'^*(u',x')$ and $K4 \leq L$. Hence $N, x' \vDash \zeta$.

Case $x = u$: Since $Permanent^*(u, x)$ holds, it follows from Lemma 7.1, items 3 and 1 that $L = S4$. Similarly as for the case $x = u$ and $\zeta = B$, we derive $N, x' \vDash B$. The consequence is that $N, x' \vDash \zeta$.

For step 2(b)i:

By Lemma 7.1:2, we have $L = KD4$ since $StatusR(u, u) = temporal$.

Let v be the world created by $CreateNextMinNode(u)$.

Let $r_2 = r \bigcup \{(v, v') \mid \exists u'\ r(u, u') \wedge R'^*(u', v')\}$.

1) The assertion holds by assumption.

2) Suppose that $\zeta \in H_2(x) \bigcup {H_2}^+(x)$ and $r_2(x, x')$ holds. We show that $N, x' \vDash \zeta$.

- Case $x \neq v$: We have $H_2(x) \bigcup {H_2}^+(x) = H(x) \bigcup H^+(x)$ and $r(x, x')$. Hence, by assumption, $N, x' \vDash \zeta$.
- Case $x = v$: Since $r_2(x, x')$, there exists u' such that $r(u, u') \wedge R'^*(u', x')$. We have ${H_2}^+(v) = H_2(v) = H^*(u)$. Therefore $\Box\zeta \in H(u) \bigcup H^+(u)$ or ($\zeta \in H(u) \bigcup H^+(u)$ and $\zeta = \Box\xi$ for some ξ).

  Case $\Box\zeta \in H(u) \bigcup H^+(u)$: By assumption, we have $N, u' \vDash \Box\zeta$. Hence $N, x' \vDash \zeta$, since $R'^*(u', x')$.

  Case $\zeta \in H(u) \bigcup H^+(u)$ and $\zeta = \Box\xi$: By assumption, we have $N, u' \vDash \Box\xi$. Since $K4 \leq L$, we have $N, u' \vDash \Box\Box\xi$. Hence $N, x' \vDash \zeta$, since $R'^*(u', x')$.

3) Suppose that $R_2^*(x, y) \wedge r_2(x, x')$ holds. We show that $\exists y'\ R'^*(x', y') \wedge r_2(y, y')$.

- Case $x \neq v$ and $y \neq v$: We have $R^*(x, y) \wedge r(x, x')$. By assumption and that $r \subseteq r_2$, we derive $\exists y'\ R'^*(x', y') \wedge r_2(y, y')$.
- Case $x = v$: We have $y = v$ and $\exists u'\ r(u, u') \wedge R'^*(u', x')$. Let y' be a world such that $R'(x', y')$. Thus $\exists u'\ r(u, u') \wedge R'^*(u', y')$, and $r_2(v, y')$. Therefore $\exists y'\ R'^*(x', y') \wedge r_2(y, y')$.
- Case $y = v$ and $x \neq v$: Since ${R_2}^*(x, y)$, we have $x = u$ or $R^*(x, u)$.

  Case $x = u$: We have $r(u, x')$ since $r_2(x, x')$. Let y' be a world such that $R'(x', y')$. We have $r(u, x') \wedge R'(x', y')$, hence $r_2(v, y')$. Therefore $\exists y'\ R'^*(x', y') \wedge r_2(y, y')$.

  Case $R^*(x, u)$: We have $r(x, x')$ since $r_2(x, x')$ and $x \neq v$. Hence, by assumption, there exists u' such that $R'^*(x', u') \wedge r(u, u')$. Since $KD \leq L$, there exists y' such that $R'(u', y')$. Thus we have $r(u, u') \wedge R'^*(u', y')$, hence $r_2(v, y')$, and finally $R'^*(x', y') \wedge r_2(y, y')$.

4) Suppose that $R'^*(x', y') \wedge r_2(x, x')$ holds. We show that
$\exists y\ (Permanent_2{}^*(x, y) \vee (y = x \wedge R_2(x, y))) \wedge r_2(y, y')$.

Case $x \neq v$ and $x \neq u$: We have $r(x, x')$ since $r_2(x, x')$. The assertion holds by assumption.

Case $x = v$: We have $\exists u'\ r(u, u') \wedge R'^*(u', x')$ since $r_2(x, x')$. Hence $\exists u'\ r(u, u') \wedge R'^*(u', y')$, since $R'^*(x', y')$. It follows that $r_2(v, y')$, and by choosing $y = x$ we have $R_2(x, y) \wedge r_2(y, y')$.

Case $x = u$: We have $R'^*(x', y') \wedge r(u, x')$. Hence $r_2(v, y')$, and by choosing $y = v$ we have $Permanent_2{}^*(x, y) \wedge r_2(y, y')$.

5) Suppose that $x \neq y \wedge Permanent_2{}^*(x, y) \wedge r_2(y, y')$ holds.
We show that $\exists x'\ R'^*(x', y') \wedge r_2(x, x')$. We have $x \neq v$.

- Case $y \neq v$: We have $Permanent^*(x,y) \wedge r(y,y')$. By assumption, it follows that $\exists x'\ R'^*(x',y') \wedge r(x,x')$. Hence $\exists x'\ R'^*(x',y') \wedge r_2(x,x')$, since $r \subseteq r_2$.
- Case $y = v$: We have $x = u$ or $Permanent^*(x,u)$, since $Permanent_2{}^*(x,y)$. Since $r_2(y,y')$, there exists u' such that $r(u,u') \wedge R'^*(u',y')$.

  Case $x = u$: For $x' = u'$, we have $R'^*(x',y') \wedge r(x,x')$. Thus $\exists x'\ R'^*(x',y') \wedge r_2(x,x')$.

  Case $x \neq u$: We have $x \neq u \wedge Permanent^*(x,u) \wedge r(u,u')$, hence, by assumption, $\exists x'\ R'^*(x',u') \wedge r(x,x')$. Thus $\exists x'\ \ R'^*(x',y') \wedge r(x,x')$, which implies $\exists x'\ R'^*(x',y') \wedge r_2(x,x')$.

6) Suppose that $Permanent_2(x,y) \wedge Next_2(x,\Diamond q) = y \wedge r_2(x,x') \wedge R'^*(x',y') \wedge (N,y' \vDash q)$ holds. We show that $r_2(y,y')$. Since $Next_2(x,\Diamond q) = y$, we have $y \neq v$ and $x \neq v$. It follows that $Permanent(x,y) \wedge Next(x,\Diamond q) = y \wedge r(x,x')$. By assumption, we derive $r(y,y')$, and $r_2(y,y')$.

For step 2(b)ii:

Since $StatusW(u) = normal$ or $StatusR(u,u) = permanent$ or $\psi \in H(u)$, by Lemma 7.1:1, we have $\neg R(u,u)$ or $L = S4$ or $\psi \in H(u)$.

Let $r_2 = r$. It is easily seen that $Permanent_2{}^* = Permanent^*$. We also have $R_2{}^* \subseteq R^*$, and $Next$ is an extension of $Next_2$, i.e. for any $\Diamond p$, $\forall x\ \forall y\ Next_2(x,\Diamond p) = y \rightarrow Next(x,\Diamond p) = y$. Therefore, by assumption, the assertions 1, 3, 5, and 6 hold. It is easily seen that $\forall x\ R(x,x) \rightarrow R_2(x,x)$, hence the assertion 4 also holds.

2) Suppose that $\zeta \in H_2(x) \bigcup H_2{}^+(x)$ and $r(x,x')$ holds. We show that $N,x' \vDash \zeta$.

- Case $\neg Permanent^*(u,x)$: We have $\zeta \in H(x) \bigcup H^+(x)$, and then by assumption, $N,x' \vDash \zeta$.
- Case $Permanent^*(u,x)$:
  We have $H_2(x) \bigcup H_2{}^+(x) \subseteq H(x) \bigcup H^+(x) \bigcup \{\xi \mid \psi = \Box\xi\}$.
  If $\zeta \in H(x) \bigcup H^+(x)$, then by assumption, we have $N,x' \vDash \zeta$. Suppose that $\zeta \notin H(x) \bigcup H^+(x)$ and $\psi = \Box\zeta$. It follows that $\psi \notin H(u)$.

  - Case $x = u$: By Lemma 7.1:3, we have $Permanent(u,u)$, and $L = S4$. By assumption, we have $N,x' \vDash \phi$ since $r(x,x')$. Hence $N,x' \vDash \zeta$.
  - Case $x \neq u$: There are two cases: $L = S4$, or $\exists y\ y \neq u \wedge y \neq x \wedge Permanent^*(u,y) \wedge Permanent^*(y,x)$; (for otherwise we would have $\zeta \in H(x) \bigcup H^+(x)$).

    Case $L = S4$: By the assumption about the assertion 5, we have $\exists u'\ R'^*(u',x') \wedge r(u,u')$. By assumption, this implies $\exists u'\ R'^*(u',x') \wedge N,u' \vDash \phi$. Thus $N,x' \vDash \zeta$, since $L = S4$.

    The second case : By the assumption about the assertion 5, there exists y' such that $R'^*(y',x') \wedge r(y,y')$, and u' such that $R'^*(u',y') \wedge r(u,u')$. By assumption, we have $N,u' \vDash \phi$, hence $N,x' \vDash \zeta$.

For step 2(c)i:

Let $r_2 = r$. It is easily seen that $Permanent_2{}^* = Permanent^*$. Therefore the assertions 1, 2, 4, 5, and 6 hold by assumption.

3) Suppose that ${R_2}^*(x, y) \wedge r(x, x')$ holds. We show that $\exists y' \; R'^*(x', y') \wedge r(y, y')$. If $R^*(x, y)$, then by assumption, we have $\exists y' \; R'^*(x', y') \wedge r(y, y')$. Suppose that $R^*(x, y)$ does not hold. Since ${R_2}^*(x, y)$, we have ($x = u$ or $R^*(x, u)$) and ($y = w$ or $R^*(w, y)$). If $x = u$, let $u' = x'$, we have $r(u, u')$. If $R^*(x, u)$, by assumption, there exists u' such that $R'^*(x', u') \wedge r(u, u')$. We have $v \neq u$ because $Next(u, \phi)$ is undefined while $Next(v, \phi)$ is defined. By the assumption about the assertion 5, there exists v' such that $R'^*(v', u') \wedge r(v, v')$. We have $N, u' \vDash \phi$ since $r(u, u')$ and $\phi \in H(u)$. Therefore there exists w' such that $R'(u', w')$ and $N, w' \vDash p$. We have $Permanent(v, w) \wedge Next(v, \Diamond p) = w \wedge r(v, v') \wedge R'^*(v', w') \wedge N, w' \vDash p$, hence, by the assumption about the assertion 6, $r(w, w')$ holds.

If $y = w$, we have $\exists y' = w' \; R'^*(x', y') \wedge r(y, y')$. If $y \neq w$, we have $R^*(w, y)$, hence, by assumption, $\exists y' \; R'^*(w', y') \wedge r(y, y')$, and finally $\exists y' \; R'^*(x', y') \wedge r(y, y')$.

For step 2(c)ii:

Let v be the world mentioned in the step, and let w be the world created by $CreateNextMinNode(v)$. Let

$$\begin{aligned} r_2 = r \; & \bigcup \{(v, v') \mid \exists u' \; r(u, u') \wedge R'^*(u', v') \wedge N, v' \vDash p\} \\ & \bigcup \{(w, w') \mid \exists u', v' \; r(u, u') \wedge R'^*(u', v') \wedge N, v' \vDash p \wedge R'^*(v', w')\} \end{aligned}$$

1) The assertion holds by assumption.

2) Suppose that $\zeta \in H_2(x) \bigcup {H_2}^+(x)$ and $r_2(x, x')$ holds. We show that $N, x' \vDash \zeta$.

- Case $x \neq v$ and $x \neq w$: We have $H_2(x) \bigcup {H_2}^+(x) = H(x) \bigcup H^+(x)$ and $r(x, x')$. Therefore, by assumption, $N, x' \vDash \zeta$.
- Case $x = v$: Since $r_2(x, x')$, there exists u' such that $r(u, u') \wedge R'^*(u', x') \wedge N, x' \vDash p$. Because $\zeta \in H_2(x) \bigcup {H_2}^+(x)$ and $H_2(v) = H^*(u) \bigcup \{p\}$, there are the following cases:
  - Case $\zeta \in H(u) \bigcup H^+(u)$ and $\zeta = \Box\xi$ for some ξ : By assumption, we have $N, u' \vDash \Box\xi$. It follows that $N, u' \vDash \Box\Box\xi$, since $K4 \leq L$. Hence $N, x' \vDash \zeta$, since $R'^*(u', x')$.
  - Case $\Box\zeta \in H(u) \bigcup H^+(u)$: By assumption, we have $N, u' \vDash \Box\zeta$. Hence $N, x' \vDash \zeta$, since $R'^*(u', x')$.
  - Case $\Box\Box\zeta \in H(u) \bigcup H^+(u)$ and $L = S4$: By assumption, we have $N, u' \vDash \Box\Box\zeta$. Hence $N, u' \vDash \Box\zeta$, since $L = S4$. Consequently, $N, x' \vDash \zeta$, since $R'^*(u', x')$.
  - Case $\zeta = p$: We have $N, x' \vDash \zeta$ since $N, x' \vDash p$.
- Case $x = w$: Since $r_2(x, x')$, there exist u' and v' such that $r(u, u') \wedge R'^*(u', v') \wedge N, v' \vDash p \wedge R'^*(v', x')$. Since $\zeta \in H_2(x) \bigcup {H_2}^+(x)$, there are the following cases:
  - Cases ($\zeta \in H(u) \bigcup H^+(u)$ and $\zeta = \Box\xi$ for some ξ) or ($\Box\zeta \in H(u) \bigcup H^+(u)$): Reasoning as for the case $x = v$, we obtain $N, x' \vDash \zeta$.
  - Case $\Box\Box\zeta \in H(u) \bigcup H^+(u)$: By assumption, we have $N, u' \vDash \Box\Box\zeta$. Hence $N, x' \vDash \zeta$, since $R'^*(u', v')$ and $R'^*(v', x')$.

– Case $\Box^3\zeta \in H(u) \bigcup H^+(u)$ and $L = S4$: By assumption, we have $N, u' \vDash \Box^3\zeta$. Hence $N, u' \vDash \Box\zeta$, since $L = S4$. Consequently, $N, x' \vDash \zeta$, since $R'^*(u', x')$.

3) Suppose that $R_2{}^*(x, y) \wedge r_2(x, x')$ holds. We show that $\exists y'\ R'^*(x', y') \wedge r_2(y, y')$.

- Case $x \neq v$ and $x \neq w$: It follows that $r(x, x')$.

  Case $y \neq v$ and $y \neq w$: It follows that $R^*(x, y)$. By assumption, we have $\exists y'\ \ R'^*(x', y') \wedge r(y, y')$, which implies $\exists y'\ R'^*(x', y') \wedge r_2(y, y')$.

  Case $y = v$ or $y = w$: Since $R_2{}^*(x, y)$, we have $x = u$ or $R^*(x, u)$. If $x = u$, then let $u' = x'$, and we have $r(u, u')$. If $R^*(x, u)$, then by assumption, there exists u' such that $R'^*(x', u') \wedge r(u, u')$. Thus, in both of the cases, $N, u' \vDash \phi$. Therefore there exists v' such that $R'(u', v')$ and $N, v' \vDash p$. Let w' be a world such that $R'(v', w')$. We have $r_2(v, v')$ and $r_2(w, w')$. If $y = v$ then choose $y' = v'$, if $y = w$ then choose $y' = w'$. Thus $R'^*(x', y') \wedge r_2(y, y')$.
- Case $x = v$: We have $y = v$ or $y = w$. Since $r_2(x, x')$, we have $\exists u'\ r(u, u') \wedge R'^*(u', x') \wedge N, x' \vDash p$.

  Case $y = v$: Since $R_2{}^*(x, y)$, we have $L = S4$. Therefore, for $y' = x'$, we have $R'^*(x', y') \wedge r_2(y, y')$.

  Case $y = w$: Let y' be a world such that $R'(x', y')$. It is easily seen that $R'^*(x', y') \wedge r_2(y, y')$ holds.
- Case $x = w$: We have $y = w$, and $\exists u', v'\ r(u, u') \wedge R'^*(u', v') \wedge (N, v' \vDash p) \wedge R'^*(v', x')$. Let y' be a world such that $R'(x', y')$. It is easily seen that $R'^*(x', y') \wedge r_2(y, y')$ holds.

4) Suppose that $R'^*(x', y') \wedge r_2(x, x')$ holds. We show that
$\exists y\ (Permanent_2{}^*(x, y) \vee (y = x \wedge R_2(x, y))) \wedge r_2(y, y')$.

Case $x \neq v$ and $x \neq w$: The assertion holds by assumption.

Case $x = v$: Since $r_2(x, x')$, we have $\exists u'\ r(u, u') \wedge R'^*(u', x') \wedge (N, x' \vDash p)$. Hence $r_2(w, y')$, since $R'^*(x', y')$. Therefore $\exists y = w\ Permanent_2{}^*(x, y) \wedge r_2(y, y')$.

Case $x = w$: Since $r_2(x, x')$, we have $\exists u', v'\ r(u, u') \wedge R'^*(u', v') \wedge (N, v' \vDash p) \wedge R'^*(v', x')$. Hence $r_2(w, y')$, since $R'^*(x', y')$. By choosing $y = w$ we have $y = x \wedge R_2(x, y) \wedge r_2(y, y')$.

5) Suppose that $x \neq y \wedge Permanent_2{}^*(x, y) \wedge r_2(y, y')$ holds. We show that $\exists x'\ R'^*(x', y') \wedge r_2(x, x')$.

Case $y \neq v$ and $y \neq w$: The assertion holds by assumption.

Case $y = v$: We have $x = u$ or $Permanent^*(x, u)$. Since $r_2(y, y')$, there exists u' such that $r(u, u') \wedge R'^*(u', y') \wedge N, y' \vDash p$. If $x = u$, for $x' = u'$, we have $R'^*(x', y') \wedge r_2(x, x')$. If $x \neq u$, we have $Permanent^*(x, u)$, hence, by assumption, $\exists x'\ R'^*(x', u') \wedge r(x, x')$, and finally $\exists x'\ R'^*(x', y') \wedge r_2(x, x')$.

Case $y = w$: We have $x = v$ or $x = u$ or $Permanent^*(x, u)$. Since $r_2(y, y')$, there exist u' and v' such that $r(u, u') \wedge R'^*(u', v') \wedge (N, v' \vDash p) \wedge R'^*(v', y')$. If $x = v$ then choose $x' = v'$, if $x = u$ then choose $x' = u'$; we will have $R'^*(x', y') \wedge r_2(x, x')$. Now suppose that $x \neq v$ and

$x \neq u$. We have $Permanent^*(x,u)$. Hence, by assumption, $\exists x' \; R'^*(x',u') \wedge r(x,x')$, and finally $\exists x' \; R'^*(x',y') \wedge r_2(x,x')$.

6) Suppose that $Permanent_2(x,y) \wedge Next_2(x,\Diamond q) = y \wedge r_2(x,x') \wedge R'^*(x',y') \wedge (N,y' \vDash q)$ holds. We show that $r_2(y,y')$ holds. Since $Next_2(x,\Diamond q)$ is defined, we have $x \neq v$ and $x \neq w$. It follows that $r(x,x')$.

Case $x \neq u$ or ($x = u$ and $y \neq v$) : We have $Next(x,\Diamond q) = Next_2(x,\Diamond q) = y$, and $Permanent(x,y)$. Hence, by assumption, $r(y,y')$, and then $r_2(y,y')$.

Case $x = u$ and $y = v$: We have $q = p$, hence $r_2(y,y')$. □

Corollary 7.1. *Let M be the model obtained as a result of executing Algorithm 7.1 for a positive program P in a modal logic L. Then M is less than or equal to every L-model of P.*

We now estimate the complexity of the given algorithm. In the remainder of this section we assume that at step 2 of Algorithm 7.1, every world x such that $Permanent^*(x,u)$ has been previously processed.

Lemma 7.4. *Let P be a positive modal logic program. Let k be the modal depth of P, n be the number of rules in P of the form $\Box^s(\Box A \leftarrow B_1,\ldots,B_p)$, where $s \geq 1$, $p \geq 1$, and let m be the number of rules in P of the form $\Box^s(\Diamond A \leftarrow B_1,\ldots,B_p)$, where $s \geq 1$, $p \geq 1$. Then during the execution of Algorithm 7.1 for P, the depth of every world in W is less than $k+(n+1).(2m+1)$.*

Proof:
When a new world x is created, define $f(x)$, $f_1(x)$, $f_2(x)$, and $H_0(x)$ as follows:

$$
\begin{aligned}
H_0(x) &= \{\Box\phi \mid \exists y\,(y = x \vee Permanent^*(y,x)) \wedge \Box\phi \in H(y)\};\\
f_1(x) &= \#\{y \in W \mid \; (y = x \vee Permanent^*(y,x)) \wedge H_0(y) = H_0(x);\\
&\qquad\qquad\quad \text{and } y \text{ was created at step 2(c)ii }\}\\
f_2(x) &= \#\{y \in W \mid \; (y = x \vee Permanent^*(y,x))\\
&\qquad\qquad\quad \text{and } y \text{ was created at step 2(b)i }\};\\
f(x) &= |H_0(x)|.2m + f_1(x) + f_2(x).
\end{aligned}
$$

Note that $depth(x) = i$ iff there are i different worlds $w_1,\ldots,w_i$ such that: $Permanent(\tau,w_1) \wedge Permanent(w_1,w_2) \wedge \ldots \wedge Permanent(w_{i-1},w_i) \wedge w_i = x$.

For every $x \in W$ such that x is created at step 2(c)ii, at the moment of creating x we have $\forall y \; Permanent^*(y,x) \wedge H_0(y) = H_0(x) \rightarrow H^*(y) = H^*(x)$. We claim that $\forall x \; depth(x) \geq k \rightarrow f_1(x) \leq 2m$. As a consequence, we have $\forall x,y \;\; depth(x) \geq k \wedge Permanent(x,y) \rightarrow f(x) < f(y)$.

We now show that $\forall x,y \; depth(x) \geq k \wedge Permanent^*(x,y) \rightarrow (f_2(y) - f_2(x) \leq n)$. Suppose that $depth(u) \geq k$ and it is the time we are creating a new world v at step 2(b)i by $CreateNextMinNode(u)$. Since $depth(u) \geq k \wedge \Box\psi \in H(u) \wedge \psi \notin H(u)$, we claim that ψ must be a primitive proposition p such that $\Box p$ has been added to $H(u)$ at step 2a. Moreover, by the assumption made immediately above this lemma, we have $\forall x \;\; Permanent^*(x,u) \rightarrow \Box p \notin H(x)$. We conclude that $\forall x,y \; depth(x) \geq k \wedge Permanent^*(x,y) \rightarrow (f_2(y) - f_2(x) \leq n)$.

Let $t > k$, and let $w_1,\ldots,w_t$ be different worlds such that
$Permanent(\tau,w_1) \wedge Permanent(w_1,w_2) \wedge \ldots \wedge Permanent(w_{t-1},w_t)$.

We have $|H_0(w_t) - H_0(w_k)| \leq n$, hence
$f(w_t) - f(w_k) \leq (n+1).2m + f_2(w_t) - f_2(w_k) < (n+1).(2m+1)$.
Since $\forall x, y\ depth(x) \geq k \wedge Permanent(x,y) \rightarrow f(x) < f(y)$, we derive that $t - k \leq f(w_t) - f(w_k) < (n+1).(2m+1)$, and finally $t < k + (n+1).(2m+1)$. This completes our proof. □

Lemma 7.5. *Let P be a positive program with size n. If during the execution of Algorithm 7.1 for P, the depth of every world in W is less than t, then the algorithm terminates in $O(n^{3t+2})$ steps.*

Proof:
The number of worlds in M is always bounded by n^t. For any action of adding a formula $\Box\phi$ to the model graph, we need no more than $O(n^{t+1})$ units of time to update the model graph with respect to the action. Therefore the total time of execution of step 2b is bounded by $O(n^{2t+2})$.

We pay $O(n^{2t+1})$ units of time to extend the model graph at step 2a or to certify that no more changes can be made by step 2a.

We pay $O(t.n^{t+1})$ units of time to extend the model graph at step 2c or to satisfy all formulae of the form $\Diamond q$ at every worlds.

There are no more than $O(n^{t+1})$ times the model graph is extended by steps 2a and 2c, therefore the algorithm terminates in $O(n^{3t+2})$ steps. □

Here is the main result of this section:

Theorem 7.2. *For any positive program P in a modal logic $L \in \{KD4, S4\}$, there exists the least L-model of P. Moreover, if the modal depth of P and the number of rules of the form $\Box^s(A \leftarrow B_1, \ldots, B_p)$, where $s \geq 1$, $p \geq 1$, and A is of the form $\Box p$ or $\Diamond p$, are finitely bounded, then the least L-model can be constructed in PTIME and its size is bounded by a polynomial in the size of P.*

Note that the modal depth of any program in $S4$ can be assumed to be less than or equal to 2, since in $S4$ we have $\Box\Box\phi \equiv \Box\phi$. The theorem immediately follows from Lemmas 7.4, 7.5, 7.2, and Corollary 7.1.

One can observe that the time complexity of Algorithm 7.1 is bounded by a polynomial in the size of the resulting model and the size of the input program. If the least $S4$-model constructed by Algorithm 7.1 for any positive program with finitely bounded modal depth always has polynomial size, then the problem of checking the satisfiability of a set of Horn formulae with finitely bounded modal depth in $S4$ would be decidable in PTIME. But this problem is PSPACE-complete, because the problem of checking the satisfiability of a set of Horn formulae in $S4$ is PSPACE-complete, and in $S4$ we have $\Box\Box\phi \equiv \Box\phi$. That is why the least model constructed by Algorithm 7.1 can be large even when the input program has finitely bounded modal depth.

8. A Problem with Positive Programs in K and $K4$

In this section, we show that there are positive programs that cannot be *characterized* (see Definition 8.1) in the logic K by a finitely bounded number of models. We also give a positive program that cannot be characterized in $K4$ by a finite set of models. In particular, there is a positive program that has neither the least K-model nor the least $K4$-model.

Definition 8.1. (Characterizing Positive Programs by Models) We say that a positive program P is *characterized* in a normal modal logic L by a set $\mathcal{M}$ of models if $\mathcal{M}$ contains only L-models of P, and for any positive formula ϕ, $P \vDash_L \phi$ iff ϕ is satisfied in every model of $\mathcal{M}$.

Definition 8.2. (Short Diameter of a Kripke Model) For a Kripke model $M = \langle W, \tau, R, h\rangle$, we define the *short diameter* of M to be the smallest number k such that there exists a world $w \in W$ such that $R^k(\tau, w)$ and there are no worlds reachable from w. If such k does not exist then we define the short diameter of M to be infinite.

We show that the program $P = \{\Box p, \Box^2 p, \ldots, \Box^n p\}$, $n \geq 1$, cannot be characterized in K by less than $n+1$ models. Suppose oppositely that it can. Without loss of generality we can assume that P is characterized in K by exactly n models, denoted by $M_i = \langle W_i, \tau_i, R_i, h_i\rangle$, for $1 \leq i \leq n$. Let $k \geq 0$ be the smallest number such that k is not the short diameter of any model M_i, for $1 \leq i \leq n$. It is clear that $k \leq n$. Let $\phi = \Box q \vee \Diamond\Box q \vee \ldots \vee \Diamond^{k-1}\Box q \vee \Diamond^{k+1} p$. It is easy to check that $M_i \vDash \phi$ for all $1 \leq i \leq n$, hence $P \vDash_K \phi$. Consider the model $M = \langle\{\tau, 1, 2, \ldots, k\}, \tau, R, h\rangle$ with $R = \{(\tau, 1), (1, 2), \ldots, (k-1, k)\}$ and $h(w) = \{p\}$ for any $w \in W$. It is easily seen that $M \vDash P$ but $M \nvDash \phi$, hence $P \nvDash_K \phi$, which is a contradiction.

We now show that the program $\{\Box p\}$ cannot be characterized in $K4$ by a finite set of models. Suppose oppositely that $\{\Box p\}$ is characterized in $K4$ by n models, denoted by $M_i = \langle W_i, \tau_i, R_i, h_i\rangle$, for $1 \leq i \leq n$. If in each of these models, there exists an infinite path starting from the actual world, then $M_i \vDash \Diamond p$ for all $1 \leq i \leq n$, and hence $\Box p \vDash_{K4} \Diamond p$, which is a contradiction. So, we can assume that there are some models without any infinite path starting from the actual world. Denote these models by $M_{i_1}, \ldots, M_{i_k}$, where $k \geq 1$ and $1 \leq i_1 < \ldots < i_k \leq n$. Let $m > 0$ be a number such that there is no h, $1 \leq h \leq k$, and no world $w \in W_{i_h}$ such that ${R_{i_h}}^m(\tau_{i_h}, w)$. Let $\phi = \Box^m \Diamond q \vee \Diamond^{m+1} p$. It is easily seen that $M_i \vDash \phi$ for all $1 \leq i \leq n$. It follows that $\Box p \vDash_{K4} \phi$, which is a contradiction (to see this just consider the model $M = \langle\{\tau, 1, 2, \ldots, m\}, \tau, R, h\rangle$ with $R = Ext_{K4}(\{(\tau, 1), (1, 2), \ldots, (m-1, m)\})$ and $h(w) = \{p\}$ for any $w \in W$). Therefore the program $\{\Box p\}$ cannot be characterized in $K4$ by a finite set of models.

As a consequence, the program $\{\Box p\}$ has neither the least K-model nor the least $K4$-model.

9. Conclusions

We have shown that for any positive modal logic program P in a modal logic $L \in \{KD, T, KDB, B, KD4, S4, KD5, KD45, S5\}$ there exists the least L-model of P. If P has no flat models then P also has the least models in KB, $K5$, $K45$, and $KB5$. Algorithms of constructing the least models have been given and analyzed.

We have also shown that the problem of checking the satisfiability of a set of Horn formulae with finitely bounded modal depth in KD, T, KB, KDB, or B is decidable in PTIME. The known result [3, 2] that the problem of checking the satisfiability of a set of Horn formulae in $K5$, $KD5$, $K45$, $KD45$, $KB5$, or $S5$ is decidable in PTIME has been also studied in this work via a different method.

Our results for the modal logics KD, T, KB, KDB, B, $K5$, $KD5$, $K45$, $KD45$, $KB5$, and $S5$ can be useful from the point of view of modal deductive databases. In [11] the results of this work are extended for positive modal logic programs with universally quantified rules; and a modal query language, called *MDatalog*, is defined and studied.

Acknowledgements

I would like to thank prof. Andrzej Szałas for careful reading the draft of this paper and giving many helpful suggestions, the anonymous reviewers for many helpful comments, and my friend, Hai Hung Bui, for improving the English of this paper.

References

[1] B.F. Chellas. *Modal Logic: An Introduction.* Cambridge Univ. Press, 1980.

[2] C.C. Chen and I.P. Lin. The computational complexity of the satisfiability of modal Horn clauses for modal propositional logics. *Theoretical Computer Science*, 129:95–121, 1994.

[3] L. Fariñas del Cerro and M. Penttonen. A note on the complexity of the satisfiability of modal Horn clauses. *Logic Programming*, 4:1–10, 1987.

[4] R. Goré. Tableau methods for modal and temporal logics. In D'Agostino, Gabbay, Hänle, and Posegga, editors, *Handbook of Tableau Methods*, pages 297–396. Kluwer Academic Publishers, 1999.

[5] J.Y. Halpern. The effect of bounding the number of primitive propositions and the depth of nesting on the complexity of modal logic. *Artificial Intelligence*, 75, 1995.

[6] J. Hudelmaier. Improved decision procedures for the modal logics K, T, S4. In *H. Kleine Büning, editor, Proceedings of CSL'95, LNCS 1092*, pages 320–334. Springer, 1996.

[7] G.E. Hughes and M.J. Cresswell. *An Introduction to Modal Logic.* Methuen, 1968.

[8] S.A. Kripke. Semantical analysis of modal logic, i. normal modal propositional calculus. *Z. Math. Logic Grundlag*, 9:67–96, 1963.

[9] R. Ladner. The computational complexity of provability in systems of modal propositional logic. *SIAM Journal of Computing*, 6:467–480, 1977.

[10] G.E. Mints. Gentzen-type systems and resolution rules. In *P.Martin-Löf, G. Mints (eds.): COLOG-88, LNCS 417*, pages 198–231. Springer, 1988.

[11] L.A. Nguyen. *Results on Modal Reasoning with Applications to Modal Deductive Databases.* PhD thesis, Warsaw University. Forthcoming.

[12] W. Rautenberg. Modal tableau calculi and interpolation. *Journal of Philosophical Logic*, 12:403–423, 1983.