Constructing the symplectic Evans matrix using maximally-analytic individual vectors

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Abstract

For linear systems with a multi-symplectic structure, arising from the linearization of Hamiltonian PDEs about a solitary wave, the Evans function can be characterized as the determinant of a matrix, and each entry of this matrix is a restricted symplectic form. This variant of the Evans function is useful for a geometric analysis of the linear stability problem. But, in general this matrix of two-forms may have branch points at isolated points, shrinking the natural region of analyticity. In this paper, a new construction of the symplectic Evans matrix is presented which is based on individual vectors but is analytic at the branch points – indeed maximally analytic. In fact this result has greater generality than just the symplectic case: it solves the following open problem in the literature: can the Evans function be constructed in a maximally analytic way when individual vectors are used? Although the non-symplectic case will be discussed in passing, the paper will concentrate on the symplectic case, where there are geometric reasons for evaluating the Evans function on individual vectors. This result simplifies and generalizes the multi-symplectic framework for the stability analysis of solitary waves, and some of the implications are discussed.

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1 Introduction

The motivation for this paper is linear systems of the form

$$\mathbf{u}_x = \mathbf{A}(x,\lambda)\mathbf{u}\,, \quad \mathbf{u}\in\mathbb{C}^n\,, \quad x\in\mathbb{R}\,, \quad \lambda\in\Lambda\subset\mathbb{C}\,, \tag{1.1}$$

when $\mathbf{A}(x,\lambda)$ has the special form

$$\mathbf{A}(x,\lambda) = \mathbf{J}^{-1}(\,\mathbf{B}(x) - \lambda\,\mathbf{M}\,)\,, \tag{1.2}$$

with the real matrices

$$\mathbf{J}^T = -\mathbf{J}\,, \quad \mathbf{M}^T = -\mathbf{M}\,, \quad \mathbf{B}(x)^T = \mathbf{B}(x) \quad \text{and} \quad \det(\mathbf{J}) \neq 0\,. \tag{1.3}$$

A precise list of hypotheses will be given in §2. Here, we will give an informal overview of the main results of the paper for this class of systems.

Systems such as (1.1) arise in the linearization about solitary wave solutions of nonlinear Hamiltonian PDEs, in which case $\mathbf{B}(x)$ is asymptotic to a constant real symmetric matrix as $x\to\pm\infty$ (Bridges & Derks [4, 6]). The spectrum of the matrix

$$\mathbf{A}^\infty(\lambda) = \mathbf{J}^{-1}(\,\mathbf{B}^\infty - \lambda\,\mathbf{M}\,) \quad \text{where} \quad \mathbf{B}^\infty = \lim_{x\to\pm\infty}\mathbf{B}(x)\,, \tag{1.4}$$

which is associated with the *system at infinity*, $\mathbf{u}_x = \mathbf{A}^\infty(\lambda)\mathbf{u}$, is a function of λ and we will assume that it is *weakly hyperbolic*: for each $\lambda\in\Lambda$, where Λ is an open simply-connected subset of the complex λ-plane, there are k eigenvalues with strictly negative real part and $n-k$ eigenvalues with non-negative real part. (This hypothesis is not the most general, and can be relaxed in various directions, see discussion in §2.) Denote the $k-$dimensional subspace associated with the strictly negative eigenvalues by $\mathbb{S}^+(\lambda)$, and the $(n-k)-$dimensional subspace associated with the non-negative eigenvalues by $\mathbb{S}^-(\lambda)$. Since the two sets of eigenvalues are disjoint, the subspaces $\mathbb{S}^\pm(\lambda)$ are analytic for all $\lambda\in\Lambda$ (cf. Kato [17], p. 67).

On the other hand, individual vectors – eigenvectors of $\mathbf{A}^\infty(\lambda)$, for example – in either $\mathbb{S}^+(\lambda)$ or $\mathbb{S}^-(\lambda)$ will not in general be analytic for all $\lambda\in\Lambda$. An individual vector $\xi(\lambda)$ in a subspace $\mathbb{S}(\lambda)$ will be called *maximally analytic* if it is analytic on the same set Λ as $\mathbb{S}(\lambda)$. Note that maximal is used here *relative* to the fixed set Λ. Typically, an individual vector will have branch points in Λ, restricting its region of analyticity. A maximally analytic basis for an analytic subspace, which is defined as the image of an analytic projection, can be constructed using Kato's Theorem ([17], p. 99). However, in the context that arises in this paper, Kato's Theorem will not be immediately applicable and therefore a new approach to finding maximally analytic vectors will be introduced.

For systems of the type (1.1) – whether symplectic or otherwise – with the above splitting of the system at infinity, there is a standard asymptotic theory for linear systems of ODEs [9, 11], from which it follows that the system (1.1) has a k-dimensional subspace of solutions which decays exponentially as $x\to+\infty$ and an $(n-k)$-dimensional subspace of solutions which grows at most algebraically as $x\to-\infty$. Each of these subspaces can be represented by characterizing forms: a k-form $\mathbf{U}^+(x,\lambda)$ and an $(n-k)$-form $\mathbf{U}^-(x,\lambda)$. These forms represent the $x-$dependent extensions of the characterizing forms for $\mathbb{S}^\pm(\lambda)$ to all $x\in\mathbb{R}$. An important feature of the forms is that their natural construction is maximally analytic.

The *Evans function*, $D(\lambda)$, is a complex analytic function which measures the intersection of the above k and $(n-k)$ dimensional subspaces (Alexander, Gardner & Jones [2]),

$$D(\lambda) = \mathrm{e}^{-\int_0^x \tau(s,\lambda)\,\mathrm{d}s}\,\mathbf{U}^-(x,\lambda)\wedge\mathbf{U}^+(x,\lambda) \quad \text{where} \quad \tau(x,\lambda) = \mathrm{Trace}(\mathbf{A}(x,\lambda))\,. \tag{1.5}$$

With suitable hypotheses, the Evans function has the property that roots in the complex λ-plane are related to eigenvalues of the spectral problem associated with the linearization about solitary waves. The Evans function is analytic for all $\lambda \in \Lambda$ where Λ determined by the region of analyticity of $\mathbb{S}^{\pm}(\lambda)$.

The construction and analyticity of the Evans function is independent of the symplectic or other geometric property of (1.1). However, when $\mathbf{A}(x,\lambda)$ has the multi-symplectic decomposition (1.2), it is possible to obtain more refined information about the geometric structure of the Evans function. The Evans function can be transformed in a way that leads to a representation in terms of the *symplectic Evans matrix* which is expressed in terms of individual vector-valued solutions of (1.1) (cf. Bridges & Derks [6]). The symplectic Evans matrix is

$$\mathbf{\Omega}(\lambda) = \begin{bmatrix} \Omega(\mathbf{w}_1,\mathbf{u}_1) & \cdots & \Omega(\mathbf{w}_1,\mathbf{u}_k) \\ \vdots & \ddots & \vdots \\ \Omega(\mathbf{w}_k,\mathbf{u}_1) & \cdots & \Omega(\mathbf{w}_k,\mathbf{u}_k) \end{bmatrix}, \tag{1.6}$$

where each entry is a restricted two form based on the symplectic operator $\mathbf{J}$,

$$\Omega(\mathbf{a},\mathbf{b}) = \langle \mathbf{J}\overline{\mathbf{a}}, \mathbf{b}\rangle\,, \quad \text{for any} \quad \mathbf{a},\mathbf{b}\in\mathbb{C}^n\,, \tag{1.7}$$

and $\langle\cdot,\cdot\rangle$ is a standard Hermitian inner product on $\mathbb{C}^n$, with conjugation on the first element. Each $\mathbf{u}_i(x,\lambda)$ is a solution of (1.1) and each $\mathbf{w}_j(x,\lambda)$ is a solution of an adjoint equation associated with (1.1) (details are given in §5). On subsets of the λ-plane where *both (1.6) and the right-hand side of (1.5) are defined*, it is proved in [6] that

$$D(\lambda) = \det[\mathbf{\Omega}(\lambda)]\,\mathcal{V}\,, \tag{1.8}$$

where $\mathcal{V}$ is a volume form on $\mathbb{C}^n$.

For systems of the form (1.1) without any symplectic structure, an *Evans matrix* can still be formed by using solutions of the adjoint equation,

$$\mathbf{E}(\lambda) = \begin{bmatrix} \langle\mathbf{z}_1,\mathbf{u}_1\rangle & \cdots & \langle\mathbf{z}_1,\mathbf{u}_k\rangle \\ \vdots & \ddots & \vdots \\ \langle\mathbf{z}_k,\mathbf{u}_1\rangle & \cdots & \langle\mathbf{z}_k,\mathbf{u}_k\rangle \end{bmatrix}, \tag{1.9}$$

where $\langle\cdot,\cdot\rangle$ is a standard Hermitian inner product on $\mathbb{C}^n$, with conjugation on the first element. Each $\mathbf{u}_i(x,\lambda)$ is a solution of (1.1) and each $\mathbf{z}_j(x,\lambda)$ is a solution of an adjoint equation associated with (1.1) (cf. Swinton [21], Bridges & Derks [5], Benzoni-Gavage, Serre & Zumbrun [3]). On subsets of the λ-plane where *both (1.9) and the right-hand side of (1.5) are defined*,

$$D(\lambda) = c(\lambda)\det[\mathbf{E}(\lambda)]\,\mathcal{V}\,, \tag{1.10}$$

where $\mathcal{V}$ is a volume form on $\mathbb{C}^n$ and $c(\lambda)\in\mathbb{C}$ is analytic and non-vanishing.

The advantage of (1.8) over (1.5) is that it can encode geometrical information implicit in (1.2), associated with the multi-symplectic structure. This geometric information has been used to prove abstract results about the Evans function for a wide range of Hamiltonian PDEs [6]. However, the disadvantage of the Evans matrix representation is that the individual vector-valued solutions of (1.1) may have branch points in the complex λ-plane, associated with multiple eigenvalues of $\mathbf{A}^\infty(\lambda)$, and therefore the right-hand side of (1.8) or (1.10) may be analytic only on a subset of Λ. Indeed, it has been claimed in the literature that it is not possible to construct individual vectors which are globally analytic (cf. page 802 of GARDNER & ZUMBRUN [12]).

One of the main results of this paper is to prove that the determinant of the symplectic Evans matrix (or more generally, the determinant of the Evans matrix) can in fact be extended in such a way that it is analytic on the same subset of the complex λ-plane as (1.5). The idea will be to construct the entries in the symplectic Evans matrix (or the Evans matrix) using maximally-analytic individual vectors.

The backbone of the argument is the solution of the following problem, which as far as we are aware, is an open problem: given a decomposable k-form $\mathbf{U}(\lambda) \in \bigwedge^k(\mathbb{C}^n)$ which is analytic for all $\lambda \in \Lambda$, find a basis $\mathbf{u}_1(\lambda), \ldots, \mathbf{u}_k(\lambda)$ which is analytic and such that $\mathbf{U}(\lambda) = \mathbf{u}_1(\lambda) \wedge \cdots \wedge \mathbf{u}_k(\lambda)$. This can always be done *locally* in regions where $\mathbf{A}^\infty(\lambda)$ has semisimple eigenvalues; see GARDNER & ZUMBRUN [12], page 820-821 for a local construction of this type. Here, the aim is to solve this problem *globally*.

The natural inclination towards solving this problem is to construct an analytic projection whose image is the space required, and then Kato's Theorem [17] can be applied. However, a proof along these lines is not obvious. We solve this problem by approaching it a different way. A linear operator on the exterior algebra is constructed whose kernel is equal to the required space. Then results from complex function theory, and a theorem of GOHBERG & RODMAN [14] – on the analyticity of the kernel of an analytic operator – is applied to conclude.

This argument will be developed in the setting of systems of the form (1.1) with a multi-symplectic decomposition, because there are geometric reasons for evaluating the Evans function on individual vectors. The result is however more general, and shows that an Evans function can be constructed for (1.1) based on individual vectors which has the same region of analyticity as (1.5). In other words, the Evans function (1.5) has the equivalent representation

$$D(\lambda) = \mathrm{e}^{-\int_0^x \tau(s,\lambda)\,\mathrm{d}s}\, \mathbf{q}_1(x,\lambda) \wedge \cdots \wedge \mathbf{q}_{n-k}(x,\lambda) \wedge \mathbf{v}_1(x,\lambda) \wedge \cdots \wedge \mathbf{v}_k(x,\lambda)\,, \tag{1.11}$$

where each individual vector $\mathbf{q}_i(x,\lambda), \mathbf{v}_j(x,\lambda),\ i = 1, \ldots, n-k,\ j = 1, \ldots, k$ is maximally analytic.

Another important observation is that if the systems at plus and minus infinity are conjugate, i.e., the matrices $\mathbf{A}^{+\infty}(\lambda)$ and $\mathbf{A}^{-\infty}(\lambda)$ are similar, then there exists a natural normalisation for the k-form $\mathbf{U}^+(x,\lambda)$ and the $(n-k)$-form $\mathbf{U}^-(x,\lambda)$, and this normalization induces a normalization of the individual vector solutions. An important consequence is that the Evans function $D(\lambda)$ and the determinant of the Evans matrix are identical and unique in any subset of the complex plane where they are both defined. With the chosen normalisation, all definitions of the Evans function lead to the same unique function (without any freedom in scaling). A curious by-product of this result is that the Evans function is analytic even if non-analytic individual solutions are used in the construction! This last observation has useful consequences for the numerical calculation of the Evans function [8].

In Section 2, the basic properties of the system (1.1) are recorded and the details of the construction of the Evans function (1.5) and of an equivalent definition by using the adjoint system are given. Section 3 contains the main new result on the existence of an analytic basis for an analytic decomposable k-form.

In Section 4, the main result on analytic extension of a matrix-based definition of the Evans function to all of Λ is presented and it is shown that $D(\lambda)$ is the analytic extension of (1.10), or (1.6) in the symplectic case. In § 5 we apply the new construction to the symplectic Evans matrix and show how it simplifies the proof of instability for the solitary wave solutions of the Boussinesq model. The paper has three appendices. The first Appendix sketches how an explicit proof of the analytic basis extraction in §3 would proceed. The second appendix contains a new large$-\lambda$ result for the Evans function, which is used in the proof of Proposition 5.1, which is recorded in Appendix C.

2 The Evans function: theme and variations

The starting point is the linear system (1.1) with the hypotheses

H1. The set $\Lambda \subset \mathbb{C}$ is an open simply-connected subset of the complex λ-plane.

H2. The matrix $\mathbf{A}(x, \lambda)$ is a continuously differentiable function of x.

H3. The matrix $\mathbf{A}(x, \lambda)$ is asymptotically constant, with $\mathbf{A}^{\infty}(\lambda) = \lim_{x\to\pm\infty} \mathbf{A}(x, \lambda)$ and the approach is exponential: $\lim_{x\to\pm\infty} \mathrm{e}^{\sigma|x|}\|\mathbf{A}(x, \lambda) - \mathbf{A}^{\infty}(\lambda)\| = 0$ for some $\sigma > 0$, uniform for $\lambda \in \Lambda$.

H4. For all $\lambda \in \Lambda$, the matrix $\mathbf{A}^{\infty}(\lambda)$ has k eigenvalues with strictly negative real part, and $(n-k)$ with non-negative real part, with $1 < k < n-1$.

Note that the multi-symplectic structure is not required in these hypotheses. It will arise *a posteriori* in a geometric analysis of the symplectic Evans matrix in § 5.

The fourth hypothesis can be relaxed in a number of directions. First, k can equal 1 or $n-1$ but these cases are trivial in the sense that $\mathbf{S}^{+}(\lambda)$ (or $\mathbf{S}^{-}(\lambda)$) is one dimensional and so the subspace is spanned by an individual vector which is naturally maximal. With the Gap Lemma, the hypothesis that the spectrum of $\mathbf{A}^{\infty}(\lambda)$ split into nonoverlapping sets can be weakened (cf. Gardner & Zumbrun [12], Kapitula & Sandstede [16]). The Gap Lemma provides a sufficient condition for enlarging the basic set Λ on which (1.5) is analytic. On the other hand, the size of Λ is independent of the question addressed in this paper: if the basic set Λ is larger, it becomes the fixed set, and the question here would remain the same: to construct individual vectors with the requirement that they are analytic on all of Λ.

The theory can easily be extended to the case where the limit matrices, $\mathbf{A}^{\pm\infty}(\lambda)$, are different at plus and minus infinity (as long as they are conjugate [6]), but adding this is straightforward and so to avoid complicating notation we will restrict to the case where they are equal.

In the remainder of this section, the background of the construction of the Evans function needed for the later analysis is recorded. There will be several formulations of the Evans function, and to calibrate and compare them it will be useful to fix the volume form. The space $\mathbb{C}^n$ will be taken to be a Hermitian inner product space with standard Hermitian inner product $\langle\cdot,\cdot\rangle$ with complex conjugation on the first element. Given any λ independent orthonormal basis for $\mathbb{C}^n$, say $\mathbf{e}_1, \cdots, \mathbf{e}_n$, we will assume henceforth that the volume form is fixed, say

$$\mathcal{V} = \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n \,. \tag{2.1}$$

Associated with (1.1) are the induced systems

$$\mathbf{U}_x^+ = \mathbf{A}^{(k)}(x,\lambda)\mathbf{U}^+\,, \quad \mathbf{U}^+ \in \bigwedge\nolimits^k(\mathbb{C}^n)\,, \tag{2.2}$$

and

$$\mathbf{U}_x^- = \mathbf{A}^{(n-k)}(x,\lambda)\mathbf{U}^-\,, \quad \mathbf{U}^- \in \bigwedge\nolimits^{(n-k)}(\mathbb{C}^n)\,, \tag{2.3}$$

where for any decomposable $\mathbf{U} = \mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_k \in \bigwedge^k(\mathbb{C}^n)$,

$$\mathbf{A}^{(k)}\mathbf{U} = \sum_{j=1}^{k} \mathbf{u}_1 \wedge \cdots \wedge \mathbf{A}\mathbf{u}_j \wedge \cdots \wedge \mathbf{u}_k\,, \tag{2.4}$$

with extension to any $\mathbf{U} \in \bigwedge^k(\mathbb{C}^n)$ by linearity.

The mapping $\mathbf{A}(x,\lambda) \mapsto \mathbf{A}^{(k)}(x,\lambda)$ is clearly linear, in the sense that the entries of $\mathbf{A}^{(k)}(x,\lambda)$ are linear functions of the entries of $\mathbf{A}(x,\lambda)$. Therefore, with the above hypotheses, the induced systems will have the property that

$$\lim_{x\to\pm\infty} \mathbf{A}^{(k)}(x,\lambda) = \mathbf{A}_\infty^{(k)}(\lambda)\,, \quad \lim_{x\to\pm\infty} \mathbf{A}^{(n-k)}(x,\lambda) = \mathbf{A}_\infty^{(n-k)}(\lambda)\,,$$

and the matrices $\mathbf{A}^{(k)}(x,\lambda)$ and $\mathbf{A}^{(n-k)}(x,\lambda)$ will inherit the differentiability properties of $\mathbf{A}(x,\lambda)$. The set of eigenvalues of $\mathbf{A}_\infty^{(k)}(\lambda)$ consists of the k-fold sums of the eigenvalues of $\mathbf{A}(x,\lambda)$ [18]. Therefore, under the above hypotheses, the eigenvalue of $\mathbf{A}_\infty^{(k)}(\lambda)$ with largest negative real part is simple, and the eigenvalue of $\mathbf{A}_\infty^{(n-k)}(\lambda)$ of largest positive real part is simple. Denote these eigenvalues by $\alpha_+(\lambda)$ (satisfying $\mathrm{Re}(\alpha_+(\lambda)) < 0$) and $\alpha_-(\lambda)$ (satisfying $\mathrm{Re}(\alpha_-(\lambda)) > 0$) with their respective eigenvectors

$$\zeta_+(\lambda) \in \bigwedge\nolimits^k(\mathbb{C}^n) \quad \text{and} \quad \zeta_-(\lambda) \in \bigwedge\nolimits^{(n-k)}(\mathbb{C}^n)\,,$$

satisfying

$$\mathbf{A}_\infty^{(k)}(\lambda)\zeta_+(\lambda) = \alpha_+(\lambda)\zeta_+(\lambda) \quad \text{and} \quad \mathbf{A}_\infty^{(n-k)}(\lambda)\zeta_-(\lambda) = \alpha_-(\lambda)\zeta_-(\lambda)\,. \tag{2.5}$$

The eigenvectors can be chosen in a natural way to be maximally analytic – that is, analytic on the same set Λ as $\mathbb{S}^\pm(\lambda)$ – and also chosen so that $\zeta_+(\lambda) \in \bigwedge^k(\mathbb{R}^n)$ and $\zeta_-(\lambda) \in \bigwedge^{(n-k)}(\mathbb{R}^n)$ when $\lambda \in \mathbb{R}$ (cf. [20], pages 54-55, [12], Lemma 2.7). The two eigenvectors $\zeta_\pm(\lambda)$ will be normalized by

$$\zeta_-(\lambda) \wedge \zeta_+(\lambda) = \mathcal{V} \quad \forall\ \lambda \in \Lambda\,. \tag{2.6}$$

This normalization may appear odd, but it is in fact the standard normalization for a simple eigenvalue. This relationship becomes evident using Hodge duality. The Hodge star operator, which maps elements in $\bigwedge^{(n-k)}(\mathbb{C}^n)$ to $\bigwedge^k(\mathbb{C}^n)$, exists on any oriented Hermitian inner product space [22], and is defined by

$$\mathbf{V} \wedge \mathbf{U} = [\![\bigstar\mathbf{V}, \mathbf{U}]\!]_k\, \mathcal{V} \quad \text{for any} \quad \mathbf{U} \in \bigwedge\nolimits^k(\mathbb{C}^n),\ \mathbf{V} \in \bigwedge\nolimits^{(n-k)}(\mathbb{C}^n)\,, \tag{2.7}$$

(cf. Wells [22], page 155), where $[\![\cdot,\cdot]\!]_k$ is the induced inner product on $\bigwedge^k(\mathbb{C}^n)$: for any decomposable elements,

$$\mathbf{U} = \mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_k \in \bigwedge\nolimits^k(\mathbb{C}^n) \quad \text{and} \quad \mathbf{V} = \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_k \in \bigwedge\nolimits^k(\mathbb{C}^n)\,,$$

the inner product of $\mathbf{U}$ and $\mathbf{V}$ is defined by

$$[\![\mathbf{U},\mathbf{V}]\!]_k = \det \begin{bmatrix} \langle \mathbf{u}_1,\mathbf{v}_1\rangle & \cdots & \langle \mathbf{u}_1,\mathbf{v}_k\rangle \\ \vdots & \ddots & \vdots \\ \langle \mathbf{u}_k,\mathbf{v}_1\rangle & \cdots & \langle \mathbf{u}_k,\mathbf{v}_k\rangle \end{bmatrix}. \tag{2.8}$$

The definition extends to any pair of elements of $\bigwedge^k(\mathbb{C}^n)$ (i.e. not necessarily decomposable) by linearity, and satisfies all the conditions of an inner product [18].

Lemma 2.1 *Let $(\mathbf{A}_\infty^{(k)}(\lambda))^*$ be the adjoint of $\mathbf{A}_\infty^{(k)}(\lambda)$ with respect to the inner product $[\![\cdot,\cdot]\!]_k$. For any $\lambda\in\Lambda$, the k-form $\bigstar\zeta_-$ is an eigenvector of $(\mathbf{A}_\infty^{(k)}(\lambda))^*$ with eigenvalue $\overline{\alpha_+(\lambda)}$.*

Proof For any $\mathbf{U}\in\bigwedge^k(\mathbb{C}^n)$,

$$\begin{aligned} [\![(\mathbf{A}_\infty^{(k)})^*\bigstar\zeta_-\,,\,\mathbf{U}]\!]_k\,\mathcal{V} &= [\![\bigstar\zeta_-\,,\,\mathbf{A}_\infty^{(k)}\mathbf{U}]\!]_k\mathcal{V} = \zeta_-\wedge\mathbf{A}_\infty^{(k)}\mathbf{U} \\ &= \mathrm{Trace}(\mathbf{A}^\infty)\,\zeta_-\wedge\mathbf{U} - \mathbf{A}_\infty^{(n-k)}\zeta_-\wedge\mathbf{U} \\ &= (\alpha_+ + \alpha_-)\,\zeta_-\wedge\mathbf{U} - \alpha_-\,\zeta_-\wedge\mathbf{U} \\ &= \alpha_+\,\zeta_-\wedge\mathbf{U} = [\![\overline{\alpha_+}\,\bigstar\zeta_-\,,\,\mathbf{U}]\!]_k\,. \end{aligned}$$

In the second line, the following identity has been used

$$\mathbf{V}\wedge\mathbf{A}_\infty^{(k)}\mathbf{U} + \mathbf{A}_\infty^{(n-k)}\mathbf{V}\wedge\mathbf{U} = \mathrm{Trace}(\mathbf{A}^\infty)\mathbf{V}\wedge\mathbf{U}\,, \quad \mathbf{U}\in\bigwedge^k(\mathbb{C}^n)\,,\ \mathbf{V}\in\bigwedge^{(n-k)}(\mathbb{C}^n)\,.$$

For any $\lambda\in\Lambda$, $\alpha_+(\lambda)$ is simple; hence it follows that $\mathrm{Ker}[(\mathbf{A}_\infty^{(k)})^* - \overline{\alpha_+}\mathbf{I}] = \mathrm{span}\{\bigstar\zeta_-\}$. □

Therefore the natural normalization for the simple eigenvalue $\alpha_+(\lambda)$ is $[\![\bigstar\zeta_-(\lambda),\zeta_+(\lambda)]\!]_k = 1$, and so (2.6) is justified by

$$\zeta_-(\lambda)\wedge\zeta_+(\lambda) = [\![\bigstar\zeta_-(\lambda),\zeta_+(\lambda)]\!]_k\mathcal{V} = \mathcal{V}\,. \tag{2.9}$$

An important consequence of this normalization is that the Evans function (1.5) can be calibrated in a unique way. Indeed, by standard asymptotic theory (e.g. [9, 11]), there exists a $\mathbf{U}^+(x,\lambda)\in\bigwedge^k(\mathbb{C}^n)$ satisfying (2.2) and a $\mathbf{U}^-(x,\lambda)\in\bigwedge^{(n-k)}(\mathbb{C}^n)$ satisfying (2.3) with the properties that

$$\lim_{x\to+\infty} \mathrm{e}^{-\alpha_+(\lambda)x}\mathbf{U}^+(x,\lambda) = \zeta_+(\lambda) \quad \text{and} \quad \lim_{x\to-\infty} \mathrm{e}^{-\alpha_-(\lambda)x}\mathbf{U}^-(x,\lambda) = \zeta_-(\lambda)\,, \tag{2.10}$$

with the limits uniform on compact subsets of the λ-plane (cf. [2], Lemma 4.1, [20], Proposition 1.2).

The Evans function is the complex analytic function on $\bigwedge^n(\mathbb{C}^n)$,

$$D(\lambda) = \mathrm{e}^{-\tau(x,\lambda)}\mathbf{U}^-(x,\lambda)\wedge\mathbf{U}^+(x,\lambda)\,, \quad \text{with} \quad \tau(x,\lambda) = \int_0^x \mathrm{Trace}(\mathbf{A}(s,\lambda))\,\mathrm{d}s\,. \tag{2.11}$$

The function $D(\lambda)$ is analytic for all $\lambda\in\Lambda$ and independent of x [2].

Now, observe that the only freedom in the functions $\mathbf{U}^\pm(x,\lambda)$ is in the choice of (scaling of) one of the eigenvectors $\zeta_\pm(\lambda)$. If another eigenvector is chosen, say $\widetilde{\zeta}_+(\lambda) = C(\lambda)\zeta_+(\lambda)$, for some analytic nonvanishing function $C(\lambda)$, then the normalization (2.6) implies that $\widetilde{\zeta}_-(\lambda) = (C(\lambda))^{-1}\zeta_-(\lambda)$. Replacing $\zeta_\pm(\lambda)$ by $\widetilde{\zeta}_\pm(\lambda)$ in (2.10) generates new functions $\widetilde{\mathbf{U}}^\pm(x,\lambda)$, and the associated Evans function becomes

$$\begin{aligned}\widetilde{D}(\lambda) &= \mathrm{e}^{-\tau(x,\lambda)}\widetilde{\mathbf{U}}^-(x,\lambda)\wedge\widetilde{\mathbf{U}}^+(x,\lambda)\\ &= C(\lambda)(C(\lambda))^{-1}\,\mathrm{e}^{-\tau(x,\lambda)}\mathbf{U}^-(x,\lambda)\wedge\mathbf{U}^+(x,\lambda) = D(\lambda)\,.\end{aligned}$$

The Hodge star operator also leads to a dual expression for the Evans function. In [5], the following lemma is proved.

Lemma 2.2 *The k-form $\mathbf{W}(x,\lambda) = e^{-\overline{\tau(x,\lambda)}}(\bigstar\mathbf{V}(x,\lambda)) \in \bigwedge^k(\mathbb{C}^n)$, where $\mathbf{V}(x,\lambda)$ is a solution of (2.3), satisfies the adjoint differential equation*

$$\frac{d}{dx}\mathbf{W} = -(\mathbf{A}^{(k)}(x,\lambda))^*\mathbf{W}\,. \tag{2.12}$$

It follows that the k-form $\bigstar\mathcal{U}^-(x,\lambda) = e^{-\overline{\tau(x,\lambda)}}(\bigstar\mathbf{U}^-(x,\lambda)) \in \bigwedge^k(\mathbb{C}^n)$ solves the adjoint differential equation (2.12) and has the asymptotic behaviour

$$\lim_{x\to-\infty} e^{+\alpha_+(\lambda)x}\overline{\bigstar\mathcal{U}^-(x,\lambda)} = \overline{\bigstar\zeta_-(\lambda)} \in \bigwedge^k(\mathbb{C}^n)\,.$$

Hence the solution $\bigstar\mathcal{U}^-(x,\lambda)$ is a dual characterization of the subspace of solutions which decay exponentially as $x\to-\infty$. Note that $\overline{\bigstar\zeta_-(\lambda)}$ is analytic for all $\lambda\in\Lambda$, since $\bigstar$ includes conjugation which is then cancelled by the overall conjugation (cf. [5, 6]). This equation is to be contrasted with the asymptotics of $\mathbf{U}^-(x,\lambda)$ as $x\to-\infty$ in (2.10). Now we have the following dual expression for the Evans function

$$D(\lambda) = \mathrm{e}^{-\tau(x,\lambda)}\,\mathbf{U}^-(x,\lambda)\wedge\mathbf{U}^+(x,\lambda) = [\![\bigstar\mathcal{U}^-,\mathbf{U}^+]\!]_k\,\mathcal{V}\,. \tag{2.13}$$

The following observation is another convenient consequence of the normalisation.

Proposition 2.3 *The Evans function $D(\lambda)$ is real when $\lambda\in\Lambda\cap\mathbb{R}$.*

Proof. If $\lambda\in\mathbb{R}$, then the matrix $\mathbf{A}^\infty(\lambda)$ has only real coefficients. Hence $\mathbf{A}_\infty^{(k)}(\lambda)$ and $\mathbf{A}_\infty^{(n-k)}(\lambda)$ are real too. The eigenvalues $\alpha_\pm(\lambda)$ must be real, since they are simple and have the largest postive or negative real part. This implies that $\zeta_\pm(\lambda)$ can be chosen to be real and since the normalisation is real too, the Evans function is real (see also Lemma 2.7 and its proof in [12].) □

3 Analyticity and decomposability

In this section we consider the following general question. Given a decomposable k-form, depending analytically on a parameter λ for all $\lambda\in\Lambda$, with Λ an open simply-connected subset of $\mathbb{C}$, does there exist a basis for the k-dimensional space that $\mathbf{U}(\lambda)$ characterizes, which is also analytic for all $\lambda\in\Lambda$?

A decomposable k-form $\mathbf{U}\in\bigwedge^k(\mathbb{C}^n)$ is said to characterize a k-dimensional subspace of $\mathbb{C}^n$ if there exists linearly independent vectors $\mathbf{u}_1,\dots,\mathbf{u}_k\in\mathbb{C}^n$ with the property that

$$\mathbf{U} = \mathbf{u}_1\wedge\cdots\wedge\mathbf{u}_k\,.$$

Without the requirement of analyticity, the result is trivial and reduces to an exercise in linear algebra – see pages 95-98 of Crampin & Pirani [10]. The central question here is how to construct such a basis in an analytic way. We will give a positive answer to the above question by first constructing a linear operator whose kernel is the required vector space. The problem is then reduced to that of finding an analytic basis for the kernel of an analytic linear operator.

In order to construct an analytic basis for the kernel of a linear operator, we will need the following two results due to Gohberg & Rodman [14]. The form of the first result that we need is Lemma S6.2 on page 389 in Gohberg, Lancaster & Rodman [13].

Lemma 3.1 ([14, 13]) *Let* $\mathbf{u}_1(\lambda),\ldots,\mathbf{u}_k(\lambda)$ *be* n*-dimensional vector functions which are analytic for all* $\lambda\in\Lambda$*. Suppose that for some* $\lambda_0\in\Lambda$ *the vectors* $\mathbf{u}_1(\lambda_0),\ldots,\mathbf{u}_k(\lambda_0)$ *are linearly independent, and let*

$$\Lambda_0=\{\,\lambda\in\Lambda\ :\ \mathbf{u}_1(\lambda),\ldots,\mathbf{u}_k(\lambda)\ \text{are linearly}\ \textit{dependent}\,\}\,.$$

Then there exist n*-dimensional vector-valued functions* $\mathbf{v}_1(\lambda),\ldots,\mathbf{v}_k(\lambda)$ *with the properties:* $\mathbf{v}_1(\lambda),\ldots,\mathbf{v}_k(\lambda)$ *are analytic and linearly independent for all* $\lambda\in\Lambda$*, and*

$$\mathrm{span}\{\mathbf{v}_1(\lambda),\ldots,\mathbf{v}_k(\lambda)\}=\mathrm{span}\{\mathbf{u}_1(\lambda),\ldots,\mathbf{u}_k(\lambda)\}\quad \textit{for every}\ \lambda\in\Lambda\setminus\Lambda_0\,.$$

The proof of this Lemma can be found on pages 389-392 in [13]. The proof is inductive, and it systematically divides out each singularity using a variant of Weierstrass's Theorem. Moreover, if for some $\ell\le k$ the vector functions $\mathbf{u}_1(\lambda),\ldots,\mathbf{u}_\ell(\lambda)$ are linearly independent for all $\lambda\in\Lambda$ then $\mathbf{u}_i(\lambda)=\mathbf{v}_i(\lambda)$ for $i=1,\ldots,\ell$. Lemma 3.1 is the basic result needed to prove the result on analyticity of the image and kernel of a linear operator due to Gohberg & Rodman [14]. The form of this result needed here is a minor variation of Theorem S6.1 on page 388 in [13].

Theorem 3.2 ([14, 13]) *Let* $\mathbf{\Phi}(\lambda)$ *be a complex* $d\times n$ *matrix-valued function with* $d\ge n$ *which is analytic in a domain* $\Lambda\subset\mathbb{C}$*. Let*

$$r=n-\max_{\lambda\in\Lambda}\{\mathrm{rank}\,\mathbf{\Phi}(\lambda)\}\,.$$

Then there exist n*-dimensional vector-valued functions* $\mathbf{v}_1(\lambda),\ldots,\mathbf{v}_r(\lambda)$ *which are analytic for all* $\lambda\in\Lambda$ *with the properties:* $\mathbf{v}_1(\lambda),\ldots,\mathbf{v}_r(\lambda)$ *are linearly independent for every* $\lambda\in\Lambda$*,*

$$\mathrm{span}\{\mathbf{v}_1(\lambda),\ldots,\mathbf{v}_r(\lambda)\}=\mathrm{Ker}(\mathbf{\Phi}(\lambda))\,.$$

at every point $\lambda\in\Lambda$*, except for a set of isolated points which consists exactly of those* $\lambda_0\in\Lambda$ *at which* $\mathrm{rank}\,\mathbf{\Phi}(\lambda)<n-r$*. For such exceptional* $\lambda\in\Lambda$*, the following inclusion holds*

$$\mathrm{span}\{\mathbf{v}_1(\lambda),\ldots,\mathbf{v}_r(\lambda)\}\subset\mathrm{Ker}(\mathbf{\Phi}(\lambda))\,.$$

On pages 392-394 in [13] this Theorem is proved for the case $d=n$, but with elementary modification, the proof goes through when $d\ge n$.

This Theorem is now applied to the question stated in the first paragraph of this section. Let $\mathbf{U}(\lambda)$ be any analytic decomposable k-form. The idea will be to construct a linear operator whose kernel is equal to the required space. The appropriate operator is suggested by the following Theorem whose proof can be found on page 5 in Marcus [18].

Theorem 3.3 ([18]) *Let* $\mathbf{U} \in \bigwedge^k(\mathbb{C}^n)$ *. Then* $\mathbf{U}$ *is decomposable if and only if there exists a linearly independent set of vectors* $\mathbf{u}_1, \ldots, \mathbf{u}_k$ *in* $\mathbb{C}^n$ *such that* $\mathbf{U} \wedge \mathbf{u}_i = 0$ *for* $i = 1, \ldots, k$ *.*

For fixed $\mathbf{U} \in \bigwedge^k(\mathbb{C}^n)$, introduce the linear mapping

$$\varphi_k(\mathbf{U}) : \bigwedge^1(\mathbb{C}^n) \to \bigwedge^{k+1}(\mathbb{C}^n) \quad \text{defined by} \quad \varphi_k(\mathbf{U})\,\mathbf{v} = \mathbf{U} \wedge \mathbf{v}\,.$$

There are two immediate consequence of Theorem 3.3

Corollary 3.4 ([18, 15]) *A form* $\mathbf{U} \in \bigwedge^k(\mathbb{C}^n)$ *is decomposable if and only if the rank of the mapping* $\varphi_k(\mathbf{U})$ *is* $n - k$ *.*

Corollary 3.5 ([18, 15]) *Suppose* $\mathbf{U} \in \bigwedge^k(\mathbb{C}^n)$ *is decomposable. An element* $\mathbf{u}_i \in \mathbb{C}^n$ *is in the space characterized by* $\mathbf{U}$ *if and only if* $\mathbf{u}_i \in \mathrm{Ker}(\varphi_k(\mathbf{U}))$ *.*

Corollary 3.5 together with Theorem 3.2 are the main tools needed to solve the problem stated in the first paragraph of this section.

Theorem 3.6 *Let* $\mathbf{U}(\lambda) \in \bigwedge^k(\mathbb{C}^n)$ *be nonzero and decomposable, and suppose that it is analytic for each* $\lambda \in \Lambda$. *Then there exists vectors* $\mathbf{v}_j(\lambda) \in \mathbb{C}^n$ *,* $j = 1, \ldots, k$ *which are analytic and linearly independent for each* $\lambda \in \Lambda$ *such that* $\mathbf{U}(\lambda) = \mathbf{v}_1(\lambda) \wedge \cdots \wedge \mathbf{v}_k(\lambda)$ *.*

Proof. Choose any constant basis for $\bigwedge^1(\mathbb{C}^n)$, $\bigwedge^k(\mathbb{C}^n)$ and $\bigwedge^{k+1}(\mathbb{C}^n)$. Then the operator $\varphi_k(\mathbf{U}(\lambda))$ can be represented by a $d \times n$ matrix $\mathbf{\Phi}(\lambda)$ where d is the dimension of $\bigwedge^{k+1}(\mathbb{C}^n)$. The entries of the matrix $\mathbf{\Phi}(\lambda)$ are linear functions of the coefficients of $\mathbf{U}(\lambda)$ and hence analytic for each $\lambda \in \Lambda$. Now, by Corollary 3.4 and Corollary 3.5, the Kernel of $\mathbf{\Phi}(\lambda)$ has dimension k for all $\lambda \in \Lambda$, and so $\mathbf{\Phi}$ has constant rank $(n-k)$. Therefore, an analytic basis $\mathbf{v}_1(\lambda), \ldots, \mathbf{v}_k(\lambda)$ for the kernel exists by Theorem 3.2. The kernel is independent of the basis chosen, and so the construction is independent of the bases chosen for $\bigwedge^1(\mathbb{C}^n)$, $\bigwedge^k(\mathbb{C}^n)$ and $\bigwedge^{k+1}(\mathbb{C}^n)$: the basis of the kernel of $\mathbf{\Phi}(\lambda)$ is precisely a basis for the space characterized by $\mathbf{U}(\lambda)$. Therefore there exists a complex analytic nonzero function $b(\lambda)$ such that

$$\mathbf{U}(\lambda) = b(\lambda)\,\mathbf{v}_1(\lambda) \wedge \cdots \wedge \mathbf{v}_k(\lambda) \quad \text{for all} \quad \lambda \in \Lambda\,.$$

By scaling elements in the kernel, the function $b(\lambda)$ can be taken to be unity. □

In Appendix A, a sketch of how an *explicit* proof would proceed for the case $n = 4$ and $k = 2$ is given.

4 Maximally analytic construction of Evans matrices

In this section, the results of the previous two sections are combined to construct a new maximally analytic expression for the Evans function by using individual solution vectors. By using the dual expression for the Evans function (2.13) and the analytic decomposition of § 3 a maximally-analytic Evans matrix can be constructed whose determinant equals the Evans function. Finally, we show that the Evans function and matrix constructed with the maximally analytic solution vectors are analytic extensions of the Evans function and matrix constructed by using non-maximally analytic solution vectors. This observation has important practical consequences.

The strategy will be to start with the definition of the Evans function using elements from $\bigwedge^k(\mathbb{C}^n)$ and $\bigwedge^{(n-k)}(\mathbb{C}^n)$, which are naturally maximally analytic, and then to extract individual vectors from these forms.

Now, let $\mathbf{U}^+(x,\lambda)$ satisfy (2.2) with the asymptotic condition (2.10). Then

$$\mathbf{U}^+(0,\lambda) \in \bigwedge^k(\mathbb{C}^n)$$

is a decomposable k-form which is analytic for all $\lambda \in \Lambda$. By Theorem 3.6, there exist vectors $\theta_1(\lambda), \ldots, \theta_k(\lambda)$ which are analytic and linearly independent for all $\lambda \in \Lambda$, and satisfy

$$\mathbf{U}^+(0,\lambda) = \theta_1(\lambda) \wedge \ldots \wedge \theta_k(\lambda)\,. \tag{4.1}$$

The decomposed form $\mathbf{U}^+(0,\lambda)$ can be extended to all $x \in \mathbb{R}$.

Lemma 4.1 *There exist vectors* $\mathbf{v}_1(x,\lambda), \ldots, \mathbf{v}_k(x,\lambda)$ *for all* $x \in \mathbb{R}$ *satisfying*

$$\frac{d}{dx}\mathbf{v}_j = \mathbf{A}(x,\lambda)\mathbf{v}_j\,, \quad \mathbf{v}_j(0,\lambda) = \theta_j(\lambda)\,, \quad j = 1, \ldots, k\,, \tag{4.2}$$

which are analytic for all $\lambda \in \Lambda$ *such that* $\mathbf{U}^+(x,\lambda) = \mathbf{v}_1(x,\lambda) \wedge \cdots \wedge \mathbf{v}_k(x,\lambda)$.

Proof. By standard results in the theory of ODEs, each solution $\mathbf{v}_j(x,\lambda)$ of (4.2) exists and is analytic for all x. Furthermore, $\mathbf{U}(x,\lambda) = \mathbf{v}_1(x,\lambda) \wedge \cdots \wedge \mathbf{v}_k(x,\lambda)$ satisfies the ODE $\mathbf{U}_x = \mathbf{A}^{(k)}(x,\lambda)\mathbf{U}$ in $\bigwedge^k(\mathbb{C}^n)$ and has the initial condition $\mathbf{U}(0,\lambda) = \theta_1(\lambda) \wedge \ldots \wedge \theta_k(\lambda)$ (cf. Lemma 4.7). But also $\mathbf{U}^+(x,\lambda)$ satisfies this ODE and initial condition. Uniqueness of solutions of ODEs gives that $\mathbf{U}^+(x,\lambda) = \mathbf{v}_1(x,\lambda) \wedge \cdots \wedge \mathbf{v}_k(x,\lambda)$. □

Similarly, suppose $\mathbf{U}^-(x,\lambda)$ satisfies (2.3) and the asymptotic condition (2.10). Then $\mathbf{U}^-(0,\lambda) \in \bigwedge^{(n-k)}(\mathbb{C}^n)$ is a decomposable $(n-k)$-form which is analytic for all $\lambda \in \Lambda$. By Theorem 3.6, there exist vectors $\psi_1(\lambda), \ldots, \psi_{n-k}(\lambda)$ which are analytic and linearly independent for all $\lambda \in \Lambda$, and satisfy

$$\mathbf{U}^-(0,\lambda) = \psi_1(\lambda) \wedge \ldots \wedge \psi_{n-k}(\lambda)\,. \tag{4.3}$$

The decomposed form $\mathbf{U}^-(0,\lambda)$ can be extended to all $x \in \mathbb{R}$.

Lemma 4.2 *There exist vectors* $\mathbf{q}_1(x,\lambda), \ldots, \mathbf{q}_{n-k}(x,\lambda)$ *for all* $x \in \mathbb{R}$ *satisfying*

$$\frac{d}{dx}\mathbf{q}_j = \mathbf{A}(x,\lambda)\mathbf{q}_j\,, \quad \mathbf{q}_j(0,\lambda) = \psi_j(\lambda)\,, \quad j = 1, \ldots, n-k\,, \tag{4.4}$$

which are analytic for all $\lambda \in \Lambda$ *such that* $\mathbf{U}^-(x,\lambda) = \mathbf{q}_1(x,\lambda) \wedge \cdots \wedge \mathbf{q}_{n-k}(x,\lambda)$.

Proof. Same as proof of Lemma 4.1. □

Corollary 4.3 *The Evans function (2.11) can be represented in terms of individual vectors,*

$$\begin{aligned} D(\lambda) &= \mathrm{e}^{-\tau(x,\lambda)}\mathbf{U}^-(x,\lambda) \wedge \mathbf{U}^+(x,\lambda) \\ &= \mathrm{e}^{-\tau(x,\lambda)}\mathbf{q}_1(x,\lambda) \wedge \cdots \wedge \mathbf{q}_{n-k}(\lambda) \wedge \mathbf{v}_1(x,\lambda) \wedge \cdots \wedge \mathbf{v}_k(x,\lambda)\,. \end{aligned}$$

and it is analytic for all $\lambda \in \Lambda$.

This result gives an expression for the Evans function in terms of maximally analytic individual solutions. In a similar way, the dual expression for the Evans function (2.13) can be expressed in terms of maximally analytic individual solutions. This is shown by extracting k maximally-analytic individual vectors from the *complex conjugate of the* k-form $\bigstar\mathcal{U}^-(x,\lambda)$.

Evaluating $\overline{\bigstar\mathcal{U}^-(x,\lambda)} \in \bigwedge^k(\mathbb{C}^n)$ at $x=0$ gives a decomposable k-form (since Hodge star preserves decomposability [18, 5]) which is analytic for all $\lambda \in \Lambda$. By Theorem 3.6, there exist vectors $\chi_1(\lambda),\ldots,\chi_k(\lambda)$ which are analytic and linearly independent for all $\lambda\in\Lambda$, and satisfy

$$\overline{\bigstar\mathcal{U}^-(0,\lambda)} = \chi_1(\lambda)\wedge\ldots\wedge\chi_k(\lambda)\,. \tag{4.5}$$

Comparison with (4.3) leads to

$$\overline{\chi_1(\lambda)\wedge\ldots\wedge\chi_k(\lambda)} = \bigstar\mathbf{U}^-(0,\lambda) = \bigstar\left(\psi_1(\lambda)\wedge\ldots\wedge\psi_{n-k}(\lambda)\right) \in \textstyle\bigwedge^k(\mathbb{C}^n)\,. \tag{4.6}$$

Lemma 4.4 *There exist vectors $\mathbf{y}_1(x,\lambda),\ldots,\mathbf{y}_k(x,\lambda)$ for all $x\in\mathbb{R}$ satisfying*

$$\frac{d}{dx}\mathbf{y}_j = -\mathbf{A}(x,\lambda)^T\mathbf{y}_j\,,\quad \mathbf{y}_j(0,\lambda)=\chi_j(\lambda)\,,\quad j=1,\ldots,k\,, \tag{4.7}$$

which are analytic for all $\lambda\in\Lambda$ such that $\overline{\bigstar\mathcal{U}^-(x,\lambda)} = \mathbf{y}_1(x,\lambda)\wedge\ldots\wedge\mathbf{y}_k(x,\lambda)$.

Proof. Similar to proof of Lemma 4.1 with the additional observation that the adjoint in $\mathbb{C}^n$ induces an adjoint in $\bigwedge^k(\mathbb{C}^n)$. Indeed, it is elementary algebra to show that $(\mathbf{A}^{(k)})^* = (\mathbf{A}^*)^{(k)}$, where $(\mathbf{A}^{(k)})^*$ denotes the adjoint of $\mathbf{A}^{(k)}$ on $\bigwedge^k(\mathbb{C}^n)$ defined using the Hermitian inner product $[\![\cdot,\cdot]\!]_k$ and $\mathbf{A}^*$ is the adjoint of $\mathbf{A}$ on $\mathbb{C}^n$ using the Hermitian inner product $\langle\cdot,\cdot\rangle$. □

The results of Lemmas 4.1 and 4.4 can now be introduced into the dual expression for the Evans function in (2.13). First, write $D(\lambda) = E(\lambda)\,\mathcal{V}$, with $E(\lambda) = [\![\bigstar\mathcal{U}^-,\mathbf{U}^+]\!]_k$. Then, using the definition of the inner product on $\bigwedge^k(\mathbb{C}^n)$,

$$E(\lambda) = \det\begin{bmatrix} \langle\overline{\mathbf{y}_1(x,\lambda)},\mathbf{v}_1(x,\lambda)\rangle & \cdots & \langle\overline{\mathbf{y}_1(x,\lambda)},\mathbf{v}_k(x,\lambda)\rangle \\ \vdots & \ddots & \vdots \\ \langle\overline{\mathbf{y}_k(x,\lambda)},\mathbf{v}_1(x,\lambda)\rangle & \cdots & \langle\overline{\mathbf{y}_k(x,\lambda)},\mathbf{v}_k(x,\lambda)\rangle \end{bmatrix}. \tag{4.8}$$

The matrix in (4.8) will be called an Evans matrix.

With the constructions in this section, we have proved the following,

Theorem 4.5 *Let $\mathbf{y}_j(x,\lambda)$ as defined in Lemma 4.4, let $\mathbf{v}_i(x,\lambda)$ be the maximally analytic vectors given by Lemma 4.1, and let $\mathbf{q}_i(x,\lambda)$ be the maximally analytic vectors given by Lemma 4.2. Then for each $\lambda\in\Lambda$, the Evans function has the representations*

$$\begin{aligned} D(\lambda) &= \mathrm{e}^{-\tau(x,\lambda)}\,\mathbf{U}^-(x,\lambda)\wedge\mathbf{U}^+(x,\lambda) \\ &= \mathrm{e}^{-\tau(x,\lambda)}\,\mathbf{q}_1(x,\lambda)\wedge\cdots\wedge\mathbf{q}_{n-k}(x,\lambda)\wedge\mathbf{v}_1(x,\lambda)\wedge\cdots\wedge\mathbf{v}_k(x,\lambda) \\ &= [\![\bigstar\mathcal{U}^-,\mathbf{U}^+]\!]_k\,\mathcal{V} \\ &= \det\begin{bmatrix} \langle\overline{\mathbf{y}_1(x,\lambda)},\mathbf{v}_1(x,\lambda)\rangle & \cdots & \langle\overline{\mathbf{y}_1(x,\lambda)},\mathbf{v}_k(x,\lambda)\rangle \\ \vdots & \ddots & \vdots \\ \langle\overline{\mathbf{y}_k(x,\lambda)},\mathbf{v}_1(x,\lambda)\rangle & \cdots & \langle\overline{\mathbf{y}_k(x,\lambda)},\mathbf{v}_k(x,\lambda)\rangle \end{bmatrix}\mathcal{V}\,. \end{aligned}$$

with all four equal and maximally analytic.

This result extends the Evans matrices introduced by SWINTON [21] and BENZONI-GAVAGE, SERRE & ZUMBRUN [3] to maximally analytic Evans matrices.

4.1 The Evans function with non-maximally analytic individual solutions

The individual solutions constructed in Lemmas 4.1, 4.2, and 4.4 are obtained by extracting a basis from the k and $n-k$ forms at $x=0$. On the other hand, the usual way to construct an Evans matrix based on individual vectors is to use solutions which are asymptotic to eigenvectors of $\mathbf{A}^\infty(\lambda)$. The latter individual vectors are not in general analytic functions for all $\lambda \in \Lambda$. Let $\mu_1(\lambda),\dots,\mu_k(\lambda)$ be the k eigenvalues of $\mathbf{A}^\infty(\lambda)$ with strictly negative real part. Let $\Lambda_b \subset \Lambda$ be any simply-connected subset of Λ with the branch points and branch cuts associated with the eigenvalues excised.

For each $\lambda \in \Lambda_b$ the k eigenvalues are simple or semisimple and therefore analytic, and an analytic eigenvector, denoted by $\xi_j(\lambda)$, $j=1,\dots,k$, can be associated with each of the eigenvalues. As in §2, the standard form of Levinson's Theorem can be applied to show that there exists $\mathbf{u}_j(x,\lambda) \in \mathbb{C}^n$ satisfying (1.1) and

$$\lim_{x\to+\infty} \mathrm{e}^{-\mu_j(\lambda)x}\mathbf{u}_j(x,\lambda) = \xi_j(\lambda)\,, \quad j=1,\dots,k\,,$$

with the limits uniform on compact subsets of the λ-plane. The following result is an immediate consequence of the definition of the vectors $\mathbf{u}_j$.

Proposition 4.6 *For all $\lambda \in \Lambda_b$ and $x \in \mathbb{R}$, $\mathbf{U}^+(x,\lambda)\wedge\mathbf{u}_j(x,\lambda) = 0$, $j=1,\dots,k$.*

In other words, the subspace $\mathrm{span}\{\mathbf{u}_1(x,\lambda),\dots,\mathbf{u}_k(x,\lambda)\}$ is precisely the subspace characterized by $\mathbf{U}^+(x,\lambda)$, for each $\lambda\in\Lambda_b$.

The adjoint equation associated with (1.1) takes the form

$$\mathbf{u}^*_x = -\mathbf{A}(x,\lambda)^*\mathbf{u}^*\,, \quad \mathbf{u}^* \in \mathbb{C}^n\,. \tag{4.9}$$

The adjoint matrix $-[\mathbf{A}^\infty(\lambda)]^T$ has k eigenvalues with strictly positive real part for $\lambda\in\Lambda_b$, with analytic eigenvectors $\eta_j(\lambda)$, $j=1,\dots,k$, which are normalised, i.e., for all $i,j=1,\dots,k$

$$-[\mathbf{A}^\infty(\lambda)]^T\eta_j(\lambda) = -\mu_j(\lambda)\eta_j(\lambda) \quad \text{and} \quad \langle\overline{\eta_i(\lambda)},\xi_j(\lambda)\rangle = \begin{cases} 1, & \text{if } i=j \\ 0, & \text{otherwise}\,. \end{cases} \tag{4.10}$$

Invoking Levinson's Theorem for the x-asymptotic behaviour, there exist $\mathbf{z}_j(x,\lambda)\in\mathbb{C}^n$ satisfying the $\mathbf{u}^*$-system in (4.9) and

$$\lim_{x\to+\infty} \mathrm{e}^{+\mu_j(\lambda)x}\mathbf{z}_j(x,\lambda) = \overline{\eta_j(\lambda)}\,, \quad j=1,\dots,k\,,$$

with the limits uniform on compact subsets of the λ-plane.

An Evans matrix for $\lambda\in\Lambda_b$ is then

$$\mathbf{E}_b(\lambda) = \begin{bmatrix} \langle\mathbf{z}_1(x,\lambda),\mathbf{u}_1(x,\lambda)\rangle & \cdots & \langle\mathbf{z}_1(x,\lambda),\mathbf{u}_k(x,\lambda)\rangle \\ \vdots & \ddots & \vdots \\ \langle\mathbf{z}_k(x,\lambda),\mathbf{u}_1(x,\lambda)\rangle & \cdots & \langle\mathbf{z}_k(x,\lambda),\mathbf{u}_k(x,\lambda)\rangle \end{bmatrix}. \tag{4.11}$$

This matrix is independent of x, an analytic function of λ for all $\lambda\in\Lambda_b$. Note that the order in which the eigenvalues $\mu_1,\dots,\mu_k$ are chosen, does not affect the value of the determinant of the Evans matrix, since permutation of eigenvalues μ_i and μ_j leads to permutation of the i^{th} and j^{th} rows *and* columns, leaving the determinant invariant.

For $\Lambda\cap\Lambda_b$ the determinant of $\mathbf{E}_b(\lambda)$ is equal to the Evans function. To prove this we first need the following preliminary result.

Lemma 4.7 *Let* $\mathbf{u}_1(x,\lambda),\ldots,\mathbf{u}_k(x,\lambda)$ *be solutions of* $\mathbf{u}_x = \mathbf{A}(x,\lambda)\mathbf{u}$, *and let* $\mathbf{U}(x,\lambda) \in \bigwedge^k(\mathbb{C}^n)$ *be a solution of* $\mathbf{U}_x = \mathbf{A}^{(k)}(x,\lambda)\mathbf{U}$, *for* $x \in \mathbb{X} \subset \mathbb{R}$, *where* $\mathbf{A}^{(k)}(x,\lambda)$ *is the induced matrix on* $\bigwedge^k(\mathbb{C}^n)$ *associated with* $\mathbf{A}(x,\lambda)$. *Suppose that for some* $x_0 \in \mathbb{X}$,

$$\mathbf{u}_j(x_0,\lambda) = \xi_j(\lambda)\,, \quad j = 1,\ldots,k \quad \textit{and} \quad \mathbf{U}(x_0,\lambda) = \zeta(\lambda)\,.$$

If $\zeta(\lambda) \wedge \xi_j(\lambda) = 0$ *for each* j, *then for all* $x \in \mathbb{X}$, $\mathbf{U}(x,\lambda) \wedge \mathbf{u}_j(x,\lambda) = 0$ *for* $j = 1,\ldots,k$.

Proof. Differentiate $\mathbf{U}(x,\lambda) \wedge \mathbf{u}_j(x,\lambda)$ with respect to x,

$$\begin{aligned} \frac{d}{dx}\mathbf{U}(x,\lambda) \wedge \mathbf{u}_j(x,\lambda) &= \mathbf{A}^{(k)}\mathbf{U}(x,\lambda) \wedge \mathbf{u}_j(x,\lambda) + \mathbf{U}(x,\lambda) \wedge \mathbf{A}\mathbf{u}_j(x,\lambda) \\ &:= \mathbf{A}^{(k+1)}(x,\lambda)\mathbf{U}(x,\lambda) \wedge \mathbf{u}_j(x,\lambda)\,. \end{aligned}$$

For every j, the initial data is trivial. By uniqueness of solutions of ordinary differential equations, this linear system on $\bigwedge^{k+1}(\mathbb{C}^n)$, has only the trivial solution. □

Combining Proposition 4.6 with Lemma 4.1 proves the following result.

Corollary 4.8 $\operatorname{span}\{\mathbf{v}_1(x,\lambda),\ldots,\mathbf{v}_k(x,\lambda)\} = \operatorname{span}\{\mathbf{u}_1(x,\lambda),\ldots,\mathbf{u}_k(x,\lambda)\}$ $\forall\lambda \in \Lambda_b$.

Hence $\mathbf{v}_1(x,\lambda) \wedge \ldots \wedge \mathbf{v}_k(x,\lambda) = b(\lambda)\mathbf{u}_1(x,\lambda) \wedge \ldots \wedge \mathbf{u}_k(x,\lambda)$ for some $b(\lambda)$, which is analytic and non-zero for all $\lambda \in \Lambda_b$.

In a similar way, we have for the solutions of the adjoint system

Corollary 4.9 $\operatorname{span}\{\mathbf{y}_1(x,\lambda),\ldots,\mathbf{y}_k(x,\lambda)\} = \operatorname{span}\{\mathbf{z}_1(x,\lambda),\ldots,\mathbf{z}_k(x,\lambda)\}$ $\forall\lambda \in \Lambda_b$.

Because of the normalisation (2.6), it follows that $\mathbf{y}_1(x,\lambda) \wedge \ldots \wedge \mathbf{y}_k(x,\lambda) = (b(\lambda))^{-1}\mathbf{z}_1(x,\lambda) \wedge \ldots \wedge \mathbf{z}_k(x,\lambda)$ for $\lambda \in \Lambda_b$. So we can conclude that the Evans function is an analytic extension of the determinant of the Evans matrix $\mathbf{E}_b(\lambda)$, which is constructed using the eigenvectors.

Theorem 4.10 *For* $\lambda \in \Lambda_b$, *we have* $\det(\mathbf{E}_b(\lambda)) = D(\lambda)$.

5 Analyticity of the symplectic Evans matrix

In this section we consider the case when $\mathbf{A}(x,\lambda)$ has the special form (1.2)-(1.3). The asymptotic matrix is

$$\mathbf{A}^\infty(\lambda) = \mathbf{J}^{-1}(\mathbf{B}^\infty - \lambda\mathbf{M}) \quad \text{where} \quad \mathbf{B}^\infty = \lim_{x\to\pm\infty}\mathbf{B}(x)\,, \tag{5.1}$$

For applications, the following hypothesis is added to the list in § 2.

H5. The matrix $\mathbf{A}(x,\lambda)$ has the form (1.2)-(1.3) and the set Λ includes a neighbourhood of the positive real axis, with the origin in the closure of Λ.

When $\mathbf{A}(x,\lambda)$ has the multi-symplectic decomposition (1.2), it is possible to obtain more refined information about the geometric structure of the Evans function. The Evans matrix can be expressed in a way that explicitly shows the symplectic structure of the original PDE. With hypothesis H5, the adjoint equation (4.9) takes the form

$$\mathbf{u}_x^* = -\mathbf{A}(x,\lambda)^*\mathbf{u}^*\,, \quad \mathbf{u}^* \in \mathbb{C}^n\,, \quad -\mathbf{A}(x,\lambda)^* = (\mathbf{B}(x) + \overline{\lambda}\mathbf{M})\,\mathbf{J}^{-1}\,. \tag{5.2}$$

It is immediate from (5.2) that

$$\mathbf{z}_j(x,\lambda) = \mathbf{J}\,\overline{\mathbf{w}_j(x,\lambda)}\,, \tag{5.3}$$

with $\mathbf{u}_j(x,\lambda)$ and $\mathbf{w}_j(x,\lambda)$ satisfying

$$\mathbf{J}\,(\mathbf{u}_j)_x = [\mathbf{B}(x) - \lambda\,\mathbf{M}]\,\mathbf{u}_j \quad \text{and} \quad \mathbf{J}\,(\mathbf{w}_j)_x = [\mathbf{B}(x) + \lambda\,\mathbf{M}]\,\mathbf{w}_j\,, \quad j = 1,\ldots,k\,. \tag{5.4}$$

The Evans matrix $\mathbf{E}_b(\lambda)$ can be written as the *symplectic Evans matrix*, which is

$$\boldsymbol{\Omega}_b(\lambda) = \begin{bmatrix} \Omega(\mathbf{w}_1(x,\lambda),\mathbf{u}_1(x,\lambda)) & \cdots & \Omega(\mathbf{w}_1(x,\lambda),\mathbf{u}_k(x,\lambda)) \\ \vdots & \ddots & \vdots \\ \Omega(\mathbf{w}_k(x,\lambda),\mathbf{u}_1(x,\lambda)) & \cdots & \Omega(\mathbf{w}_k(x,\lambda),\mathbf{u}_k(x,\lambda)) \end{bmatrix}. \tag{5.5}$$

where each entry is a restricted two form based on the symplectic operator $\mathbf{J}$,

$$\Omega(\mathbf{a},\mathbf{b}) = \langle \mathbf{J}\bar{\mathbf{a}}, \mathbf{b}\rangle\,, \quad \text{for any} \quad \mathbf{a},\mathbf{b} \in \mathbb{C}^n\,. \tag{5.6}$$

For $\lambda \in \Lambda_b$, define $D_b(\lambda) = \det[\boldsymbol{\Omega}_b(\lambda)]$. Since $\mathbf{E}_b(\lambda) = \boldsymbol{\Omega}_b(\lambda)$, the following result is an immediate consequence of Theorem 4.10

Theorem 5.1 *For $\lambda \in \Lambda_b$, the Evans function satisfies $D(\lambda) = \det[\boldsymbol{\Omega}_b(\lambda)]\,\mathcal{V} = D_b(\lambda)\,\mathcal{V}$.*

The main consequence of this Theorem is that $D(\lambda)$ is an analytic extension of $D_b(\lambda)$ and therefore $D_b(\lambda)$ can be used in regions of the $\lambda-$plane where geometric information is required, for example near $\lambda = 0$, and $D(\lambda)$ can be used as an extension in regions where $D_b(\lambda)$ is not defined.

5.1 Example: the Boussinesq equation revisited

The Boussinesq equations form a class of model partial differential equations for dispersive shallow-water waves, nonlinear strings and condensed matter physics. In [6] a geometric instability condition based on the symplectic Evans matrix was applied to show instability for a class of solitary waves of the following Boussinesq model:

$$u_{tt} = \partial_{xx}(u - u^2 - u_{xx}), \quad x \in \mathbb{R}\,, \quad t > 0\,, \tag{5.1}$$

where $u(x,t)$ is a scalar-valued function. The two-parameter family of solitary waves is

$$u(x,t) = \tfrac{1}{2}(1-c^2) - 2\delta^2 + 6\delta^2 \mathrm{sech}^2(\delta(x-ct))\,, \tag{5.2}$$

where c is the wave speed,

$$\delta = \tfrac{1}{2}\sqrt{\sqrt{1+4a} - c^2} \quad \text{with the condition} \quad a > \tfrac{1}{4}(c^4 - 1)\,.$$

In [6] it is proved that this two parameter family of solitary waves is unstable whenever $a > -\frac{1}{4}$ and $c^2 < \frac{1}{4}\sqrt{1+4a}$. The system (5.1) can be formulated as a multi-symplectic Hamiltonian PDE. The proof of instability in [6] proceeds by constructing the symplectic Evans matrix, and proving that

$$D_b(\lambda) = \frac{\lambda^2}{192\delta^4\sqrt{1+4a}}\left(4c^2 - \sqrt{1+4a}\right) + \mathcal{O}(\lambda^3)\,, \quad \text{as } \lambda \to 0\,, \tag{5.3}$$

and that $D_b(\lambda) \to 1$ as $\lambda \to +\infty$ along the real axis. One then concludes by an intermediate value theorem (IVT) argument that there exists an unstable real positive eigenvalue when $D_b''(0) < 0$. In [6] the IVT argument is delicate because the symplectic Evans matrix $\mathbf{\Omega}_b(\lambda)$ has a branch point on the real axis.

With our result in Theorem 5.1 a simplified proof of the instability result in [6] can be given. Since $D(\lambda)$ is analytic on the full real axis and an analytic extension of $D_b(\lambda)\,\mathcal{V}$, the application of the IVT is elementary: $D_b(\lambda)$ is analyzed for λ near zero, and $D(\lambda)$ is analyzed for large $|\lambda|$. It remains only to study the asymptotic behaviour of $D(\lambda)$.

Proposition 5.1 *The Evans function for the Boussinesq model linearized about the two parameter family of solitary waves (5.2) has the asymptotic property,* $D(\lambda) \to 1$ *as* $\lambda \to +\infty$ *along the real* λ*-axis.*

A Proof of this result is given in Appendix C. For the Boussinesq model hypothesis H5 is satisfied. Therefore combining Proposition 5.1 with (5.3) proves that the solitary wave (5.2) is unstable when $D_b''(0) < 0$.

References

[1] L. Allen & T.J. Bridges. *Numerical exterior algebra and the compound matrix method*, Numerische Mathematik (in press, 2001).

[2] J. Alexander, R. Gardner & C.K.R.T. Jones. *A topological invariant arising in the stability analysis of traveling waves*, J. Reine Angew. Math. **410**, 167–212 (1990).

[3] S. Benzoni-Gavage, D. Serre & K. Zumbrun. *Alternate Evans functions and viscous shock waves*, SIAM J. Math. Anal. **32**, 929–962 (2001).

[4] T.J. Bridges & G. Derks. *Unstable eigenvalues and the linearization about solitary waves and fronts with symmetry*, Proc. R. Soc. Lond. A **455**, 2427–2469 (1999).

[5] T.J. Bridges & G. Derks. *Hodge duality and the Evans function*, Phys. Lett. A **251**, 363–372 (1999).

[6] T.J. Bridges & G. Derks. *The symplectic Evans matrix, and the linearization about solitary waves and fronts*, Arch. Rat. Mech. Anal. **156**, 1–87 (2001).

[7] T.J. Bridges & G. Derks. *Linear instability of solitary wave solutions of the Kawahara equation and its generalizations*, SIAM J. Math. Anal. (in press, 2002).

[8] T.J. Bridges, G. Derks & G. Gottwald. *Stability and instability of solitary waves of the fifth-order KdV equation: a numerical framework*, Preprint, University of Surrey (2001).

[9] W.A. Coppel. *Stability and Asymptotic Behavior of Differential Equations*, D.C. Heath and Co. Boston (1965).

[10] M. Crampin & F.A.E. Pirani. *Applicable Differential Geometry*, CUP: Cambridge (1986).

[11] M.S.P. Eastham. *The Asymptotic Solution of Linear Differential Systems: Applications of the Levinson Theorem*, LMS Monographs New Series, Oxford U Press (1989).

[12] R.A. Gardner & K. Zumbrun. *The gap lemma and geometric criteria for instability of viscous shock profiles.* Comm. Pure Appl. Math. **51** 797–855 (1998).

[13] I. Gohberg, P. Lancaster & L. Rodman. *Matrix Polynomials*, Academic Press: New York (1982).

[14] I. Gohberg & L. Rodman. *Analytic matrix functions with prescribed local data*, J. d'Analyse Mathématique **40**, 90–128 (1981).

[15] J. Harris. *Algebraic Geometry*, Springer-Verlag: Berlin (1992).

[16] T. Kapitula & B. Sandstede. *Stability of bright solitary-wave solutions to perturbed nonlinear Schrödinger equations*. PhysicaD **124** 58–103 (1998).

[17] T. Kato. *Perturbation Theory for Linear Operators*. Second Edition, Springer Verlag: Heidelberg (1984).

[18] M. Marcus. *Finite-dimensional Multilinear Algebra, Part II.* Marcel-Dekker: New York (1975).

[19] A.I. Markushevich. *Theory of Functions of a Complex Variable, Volumes I,II,III.* Chelsea Publishing Co: New York (1977).

[20] R.L. Pego & M.I. Weinstein. *Eigenvalues, and instabilities of solitary waves.* Phil. Trans. Roy. Soc. Lond. **340** 47–94 (1992).

[21] J. Swinton. *The stability of homoclinic pulses: a generalisation of Evans' method*. Phys. Lett. A **163** 57–62 (1992).

[22] R.O. Wells. *Differential Analysis on Complex Manifolds*. Springer-Verlag: Berlin (1980).

A An explicit calculation of a maximally analytic basis

In this Appendix, a sketch of an explicit proof of the existence of a maximally analytic basis – i.e. Theorem 3.6 – is given for the case $k = 2$ and $n = 4$.

Let $\mathbf{e}_1, \ldots, \mathbf{e}_4$ be an orthonormal basis for $\mathbb{C}^4$, and take a standard constant lexical basis for $\bigwedge^2(\mathbb{C}^4)$,

$$\begin{aligned} \omega_1 &= \mathbf{e}_1 \wedge \mathbf{e}_2\,, \quad \omega_2 = \mathbf{e}_1 \wedge \mathbf{e}_3\,, \quad \omega_3 = \mathbf{e}_1 \wedge \mathbf{e}_4\,, \\ \omega_4 &= \mathbf{e}_2 \wedge \mathbf{e}_3\,, \quad \omega_5 = \mathbf{e}_2 \wedge \mathbf{e}_4\,, \quad \omega_6 = \mathbf{e}_3 \wedge \mathbf{e}_4\,, \end{aligned}$$

and let $\mathbf{U}(\lambda)$ be an element of $\bigwedge^2(\mathbb{C}^4)$ which depends analytically on λ for all $\lambda \in \Lambda$ where Λ is an open simply-connected subset of the complex λ-plane. Then $\mathbf{U}(\lambda)$ can be expressed as

$$\mathbf{U}(\lambda) = \sum_{j=1}^{6} a_j(\lambda)\, \omega_j\,,$$

where each of the coefficients $a_j(\lambda)$ is an analytic function for all $\lambda \in \Lambda$. It is a standard result in algebraic geometry that a 2-form in $\bigwedge^2(\mathbb{C}^4)$ is decomposable if and only if $\mathbf{U}(\lambda)$ wedged with itself vanishes [15], and in terms of the above basis

$$\mathbf{U}(\lambda) \wedge \mathbf{U}(\lambda) = I(\lambda)\, \mathcal{V} \quad \text{where} \quad I(\lambda) = a_1(\lambda)a_6(\lambda) - a_2(\lambda)a_5(\lambda) + a_3(\lambda)a_4(\lambda)\,,$$

and $\mathcal{V} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4$ is the standard volume form. Hence $\mathbf{U}(\lambda)$ is decomposable if and only if $I(\lambda) = 0$.

Let $\mathbf{U}(\lambda)$ be a decomposable k-form (i.e., $I(\lambda) = 0$). A matrix representation for $\varphi_2(\mathbf{U}(\lambda))$, which we will denote by $\mathbf{\Phi}(\lambda) \in \mathbb{C}^{4\times 4}$, can be constructed by introducing a basis for $\bigwedge^1(\mathbb{C}^4)$ and for $\bigwedge^3(\mathbb{C}^4)$. For $\bigwedge^1(\mathbb{C}^4)$ take the standard basis, $\mathbf{e}_1, \ldots, \mathbf{e}_4$, and volume form (2.1). For $\bigwedge^3(\mathbb{C}^4)$ take the natural basis generated by Hodge duality,

$$\begin{aligned} \sigma_1 &= \bigstar\mathbf{e}_1 = -\mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4\,, \quad \sigma_2 = \bigstar\mathbf{e}_2 = \mathbf{e}_1 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4 \\ \sigma_3 &= \bigstar\mathbf{e}_3 = -\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_4\,, \quad \sigma_4 = \bigstar\mathbf{e}_4 = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3\,. \end{aligned}$$

In terms of this basis we find,

$$\begin{aligned} \varphi_2(\mathbf{U})\mathbf{e}_1 &= \mathbf{U}(\lambda) \wedge \mathbf{e}_1 = a_6(\lambda)\sigma_1 - a_5(\lambda)\sigma_3 + a_4(\lambda)\sigma_4 \\ \varphi_2(\mathbf{U})\mathbf{e}_2 &= \mathbf{U}(\lambda) \wedge \mathbf{e}_2 = -a_6(\lambda)\sigma_1 + a_3(\lambda)\sigma_3 - a_2(\lambda)\sigma_4 \\ \varphi_2(\mathbf{U})\mathbf{e}_3 &= \mathbf{U}(\lambda) \wedge \mathbf{e}_3 = a_5(\lambda)\sigma_1 - a_3(\lambda)\sigma_2 + a_1(\lambda)\sigma_4 \\ \varphi_2(\mathbf{U})\mathbf{e}_4 &= \mathbf{U}(\lambda) \wedge \mathbf{e}_4 = -a_4(\lambda)\sigma_1 + a_2(\lambda)\sigma_2 - a_1(\lambda)\sigma_3 \end{aligned}$$

and so, writing $\mathbf{\Phi}(\lambda)$ for this matrix representation,

$$\mathbf{\Phi}(\lambda) = \begin{bmatrix} 0 & -a_6(\lambda) & a_5(\lambda) & -a_4(\lambda) \\ a_6(\lambda) & 0 & -a_3(\lambda) & a_2(\lambda) \\ -a_5(\lambda) & a_3(\lambda) & 0 & -a_1(\lambda) \\ a_4(\lambda) & -a_2(\lambda) & a_1(\lambda) & 0 \end{bmatrix}. \tag{1.1}$$

An analytic basis for the kernel of $\mathbf{\Phi}(\lambda)$, when $\mathbf{U}(\lambda)$ is decomposable, provides the required basis. Although the matrix $\mathbf{\Phi}(\lambda)$ is skew symmetric when the entries are real, this is irrelevant in the present context because the entries are complex in general.

For general k, the matrix representation $\mathbf{\Phi}(\lambda)$ is rectangular. However, for the special case $k=2$ and $n=4$ it is square, and therefore it is useful to use the spectrum of $\mathbf{\Phi}(\lambda)$ to illuminate its properties.

The characteristic polynomial associated with $\mathbf{\Phi}(\lambda)$ in (1.1) is

$$\Delta(\mu,\lambda)=\det[\Phi(\lambda)-\mu\mathbf{I}]=\mu^4+\left(\sum_{j=1}^{6}a_j(\lambda)^2\right)\mu^2+I(\lambda)^2\,,$$

and since $\mathbf{U}(\lambda)$ is decomposable, $I(\lambda)=0$, this reduces to

$$\Delta(\mu,\lambda)=\mu^2\left(\mu^2+\sum_{j=1}^{6}a_j(\lambda)^2\right)\,.$$

Zero is a root of algebraic multiplicity at least two. It is also easy to check that the rank of $\mathbf{\Phi}(\lambda)$ is always two, when $\|\mathbf{U}(\lambda)\|\neq 0$. When the entries of $\mathbf{\Phi}(\lambda)$ are real, the algebraic multiplicity of the zero eigenvalue of $\mathbf{\Phi}(\lambda)$ is equal to two. However, when the entries of $\mathbf{\Phi}(\lambda)$ complex – the case of interest – the algebraic multiplicity of zero as an eigenvalue may be greater than 2.

It is tempting to try to characterize the kernel as the image of an analytic projection of the form

$$\mathbf{P}(\lambda)=\frac{1}{2\pi\mathrm{i}}\oint[\mu\mathbf{I}-\mathbf{\Phi}(\lambda)]\,\mathrm{d}\mu\,,$$

taking the contour to be any Jordan curve encircling the origin in the μ-plane which does not include any of the nonzero eigenvalues of $\mathbf{\Phi}(\lambda)$ inside or on the contour. However, this approach fails because the dimension of the image of $\mathbf{P}(\lambda)$ will be equal to the *algebraic* multiplicity of the zero root, which may exceed the dimension of the kernel in general.

However, in this case, the kernel of $\mathbf{\Phi}(\lambda)$ can be obtained by explicit calculation, and explicit application of Weierstrass' Theorem. The argument is straightforward (albeit lengthy) and so the details are omitted.

B Large λ behaviour of the Evans function

There are many results in the literature on the large λ behaviour of the Evans function (e.g. [2], [6], [12], [16], [20]). The result of this paper shows that any of the equivalent representations of the Evans function, which are valid for large λ, can be analyzed. In other words, particular forms of the Evans function may be better suited for large λ analysis.

Here we present a large λ result which is simple to verify. It is new, but it is really a reinterpretation of Proposition 1.17 of Pego & Weinstein [20]: indeed, the proof carries over almost verbatim. It generalizes their result for the case $k=1$ to $k>1$ by simply applying the result on $\bigwedge^k(\mathbb{C}^n)$, rather than to the system on $\mathbb{C}^n$. The result gives a sufficient condition for $D(\lambda)\to 1$ as $\lambda\to\infty$ along the real axis. It provides a neat proof in the case of the Boussinesq analysis, and may be useful for other examples (see [7] for an application of this result to the fifth-order KdV equation).

Lemma 2.1 (Pego-Weinstein Lemma) *Consider the system*

$$\mathbf{u}_x=\mathbf{A}(x,\lambda)\mathbf{u}\,,\quad \mathbf{u}\in\mathbb{C}^n\,,\quad \lambda\in\Lambda\,,$$

where $\mathbb{C}^n$ *is considered to be an oriented inner product space, and* Λ *is as defined in §2. Suppose that the spectrum of*

$$\mathbf{A}^{\infty}(\lambda) = \lim_{x\to\pm\infty} \mathbf{A}(x,\lambda)\,,$$

is such that, for all $\lambda \in \Lambda$, *the eigenvalue of* $\mathbf{A}_{\infty}^{(k)}(\lambda)$ *with largest negative real part is unique and simple. Denote this eigenvalue by* $\alpha(\lambda)$ *and its right eigenvector by* $\zeta_+(\lambda)$ *and its left eigenvector by* $\bigstar\zeta_-(\lambda)$ *with normalization* $\zeta_-(\lambda)\wedge\zeta_+(\lambda) = \mathcal{V}$ *(see §2 and §4 for definition).*

Let $\mathbf{U}(x,\lambda) \in \bigwedge^k(\mathbb{C}^n)$ *be the solution of the system*

$$\mathbf{U}_x = \mathbf{A}^{(k)}(x,\lambda)\mathbf{U} \quad \textit{satisfying} \quad \lim_{x\to+\infty} \mathrm{e}^{-\alpha(\lambda)x}\mathbf{U}(x,\lambda) = \zeta_+(\lambda) \in \textstyle\bigwedge^k(\mathbb{C}^n)\,.$$

Similarly, let $\mathbf{W}(x,\lambda) \in \bigwedge^k(\mathbb{C}^n)$ *be the solution of the system*

$$\mathbf{W}_x = -\mathbf{A}^{(k)}(x,\lambda)^T\mathbf{W} \quad \textit{satisfying} \quad \lim_{x\to-\infty} \mathrm{e}^{\alpha(\lambda)x}\mathbf{W}(x,\lambda) = \bigstar\zeta_-(\lambda) \in \textstyle\bigwedge^k(\mathbb{C}^n)\,.$$

Using $\mathbf{U}(x,\lambda)$ *and* $\mathbf{W}(x,\lambda)$, *the Evans function can be expressed in the form*

$$D(\lambda) = \mathbf{W}(0,\lambda)\cdot\mathbf{U}(0,\lambda) = \langle\overline{\mathbf{W}(0,\lambda)}, \mathbf{U}(0,\lambda)\rangle\,,$$

where $\langle\cdot,\cdot\rangle$ *is the standard inner product on* $\mathbb{C}^d$, *with* d *the dimension of* $\bigwedge^k(\mathbb{C}^n)$.

Now, suppose $\mathbf{A}_{\infty}^{(k)}(\lambda)$ *is diagonalisable for large* λ *and let* $\mathbf{V}(\lambda)$ *be the matrix of right eigenvectors such that first column corresponds to* $\zeta_+(\lambda)$. *If*

$$\int_{-\infty}^{\infty} \|\mathbf{V}(\lambda)^{-1}[\mathbf{A}^{(k)}(x,\lambda) - \mathbf{A}_{\infty}^{(k)}(\lambda)]\mathbf{V}(\lambda)\|\,\mathrm{d}x \;\le\; C, \quad \textit{independent of } \lambda$$

$$\int_{|x|\ge x_0} \|\mathbf{V}(\lambda)^{-1}[\mathbf{A}^{(k)}(x,\lambda) - \mathbf{A}_{\infty}^{(k)}(\lambda)]\mathbf{V}(\lambda)\|\,\mathrm{d}x \;\to\; 0, \quad \textit{as } x_0\to\infty, \textit{ uniformly in } \lambda$$

$$\int_{-\infty}^{\infty} \|\mathbf{V}(\lambda)^{-1}[\mathbf{A}^{(k)}(x,\lambda) - \mathbf{A}_{\infty}^{(k)}(\lambda)]\zeta(\lambda)\|\,\mathrm{d}x \;\to\; 0, \quad \textit{as } |\lambda|\to\infty.$$

Then

$$\mathbf{V}(\lambda)^{-1}\mathbf{U}(0,\lambda) = \mathbf{V}(\lambda)^{-1}\zeta_+(\lambda) + o(1), \quad \textit{for} \quad |\lambda|\to\infty$$

and $\mathbf{W}(0,\lambda)\mathbf{V}(\lambda)$ *is bounded with*

$$\mathbf{W}(0,\lambda)\mathbf{V}(\lambda)\mathbf{e}_1 = \mathbf{W}(0,\lambda)\zeta_+(\lambda) = 1 + o(1) \quad \textit{for} \quad |\lambda|\to\infty.$$

This implies that $D(\lambda)\to 1$ *as* $|\lambda|\to\infty$.

Proof. The proof of Proposition 1.17 in [20] carries over almost verbatim. □

C Proof of Proposition 5.1

The matrix $\mathbf{A}^{\infty}(\lambda)$ associated with the system at infinity for the multi-symplectic form of the Boussinesq model takes the following form [6]

$$\mathbf{A}^{\infty}(\lambda) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1-2u_0 & 0 & -\lambda & c \\ 0 & 0 & 0 & 1 \\ \lambda & -c & 0 & 0 \end{pmatrix},$$

where $u_0 = \frac{1}{2}[1 - \sqrt{1+4a}]$ is the asymptotic ($x \to \pm\infty$) value of the u-component of the solitary wave solution.

Hence the induced matrix on $\bigwedge^2(\mathbb{C}^4)$, relative to the standard basis,

$$\{\, \mathbf{e}_1 \wedge \mathbf{e}_2,\ \mathbf{e}_1 \wedge \mathbf{e}_3,\ \mathbf{e}_1 \wedge \mathbf{e}_4,\ \mathbf{e}_2 \wedge \mathbf{e}_3,\ \mathbf{e}_2 \wedge \mathbf{e}_4,\ \mathbf{e}_3 \wedge \mathbf{e}_4 \,\}$$

is

$$\mathbf{A}_\infty^{(2)}(\lambda) = \begin{pmatrix} 0 & -\lambda & c & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ -c & 0 & 0 & 0 & 1 & 0 \\ 0 & 1-2u_0 & 0 & 0 & 1 & -c \\ -\lambda & 0 & 1-2u_0 & 0 & 0 & -\lambda \\ 0 & -\lambda & 0 & c & 0 & 0 \end{pmatrix}.$$

For $|\lambda|$ large, the eigenvalues of $\mathbf{A}^\infty(\lambda)$ are

$$\mu = \mu_0\sqrt{\lambda} - \frac{c}{2}\mu_0^2 + \mathcal{O}(\tfrac{1}{\sqrt{\lambda}}), \quad \text{where} \quad \mu_0^4 = -1.$$

Note that at leading order, order λ, the term in the expansion vanishes. This is due to the fact that at order λ the matrix $\mathbf{A}^\infty(\lambda)$ is given by $\mathbf{J}_c^{-1}\mathbf{M}$ and this matrix has zero as an eigenvalue of algebraic multiplicity four. The six eigenvalues of the induced matrix are

$$\pm 2\sqrt{\lambda} + \mathcal{O}(\sqrt{\lambda}), \quad \pm \mathrm{i}2\sqrt{\lambda} + \mathcal{O}(\sqrt{\lambda}), \quad \text{and} \quad \pm\, \mathrm{i}c + \mathcal{O}(\sqrt{\lambda}).$$

An eigenvector of $\mathbf{A}_\infty(\lambda)$ associated with an eigenvalue μ takes the form

$$\xi = (\mu^2, \mu^3, (\lambda - \mu c), \mu(\lambda - \mu c)) \in \mathbb{C}^4\,.$$

¿From this expression, the induced eigenvectors on $\bigwedge^2(\mathbb{C}^4)$ can be constructed, and it can be easily shown, by explicit construction using the eigenvectors, that $\mathbf{A}_\infty^{(2)}(\lambda)$ is diagonalisable when $|\lambda|$ is large. The matrix of eigenvectors for $\mathbf{A}_\infty^{(2)}(\lambda)$ for $|\lambda|$ large is given by

$$\mathbf{V}(\lambda) = \frac{1}{\sqrt{\lambda}} \begin{pmatrix} 4 & -4 & 4 & 4 & 4i & -4 \\ \frac{2}{\sqrt{\lambda}} & 0 & \frac{2i}{\sqrt{\lambda}} & \frac{-2i}{\sqrt{\lambda}} & 0 & \frac{2}{\sqrt{\lambda}} \\ -2 & 2i & 2 & 2 & -2 & 2 \\ -2 & -2i & 2 & 2 & 2 & 2 \\ 4\sqrt{\lambda} & 0 & -4i\sqrt{\lambda} & 4i\sqrt{\lambda} & 0 & 4\sqrt{\lambda} \\ 1 & 1 & 1 & 1 & -i & -1 \end{pmatrix} + \mathcal{O}\left(\frac{1}{\lambda}\right) \mathbf{V}_1(\lambda)$$

where

$$\mathbf{V}_1(\lambda) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \frac{1}{\sqrt{\lambda}} & \frac{1}{\sqrt{\lambda}} & \frac{1}{\sqrt{\lambda}} & \frac{1}{\sqrt{\lambda}} & \frac{1}{\sqrt{\lambda}} & \frac{1}{\sqrt{\lambda}} \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

The induced residual matrix associated with the x-dependent part of $\mathbf{A}(x,\lambda)$ is

$$\mathbf{R}^{(2)}(x,\lambda) \stackrel{\text{def}}{=} [\mathbf{A}^{(2)}(x,\lambda) - \mathbf{A}_\infty^{(2)}(\lambda)] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & f(x) & 0 & 0 & 0 & 0 \\ 0 & 0 & f(x) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $f(x) = 6\delta^2 \text{sech}^2(\delta x)$. The inverse of $\mathbf{V}(\lambda)$ is

$$\mathbf{V}(\lambda)^{-1} = \frac{\sqrt{\lambda}}{32} \begin{pmatrix} 1 & 4\sqrt{\lambda} & -2 & -2 & \frac{2}{\sqrt{\lambda}} & 4 \\ \frac{-2}{\sqrt{\lambda}} & \mathcal{O}(\frac{1}{\lambda}) & \frac{-4i}{\sqrt{\lambda}} & \frac{4i}{\sqrt{\lambda}} & \mathcal{O}(\frac{1}{\lambda}) & \frac{8}{\sqrt{\lambda}} \\ 1 & -4i\sqrt{\lambda} & 2 & 2 & \frac{2i}{\sqrt{\lambda}} & 4 \\ 1 & 4i\sqrt{\lambda} & 2 & 2 & \frac{-2i}{\sqrt{\lambda}} & 4 \\ \frac{-2i}{\sqrt{\lambda}} & \mathcal{O}(\frac{1}{\lambda}) & \frac{-4}{\sqrt{\lambda}} & \frac{4}{\sqrt{\lambda}} & \mathcal{O}(\frac{1}{\lambda}) & \frac{8i}{\sqrt{\lambda}} \\ -1 & 4\sqrt{\lambda} & 2 & 2 & \frac{2}{\sqrt{\lambda}} & -4 \end{pmatrix} + h.o.t,$$

where the higher order terms are the next order in $\frac{1}{\sqrt{\lambda}}$ for each entry in the matrix.

Altogether this gives

$$\mathbf{V}(\lambda)^{-1}\mathbf{R}^{(2)}(x,\lambda)\mathbf{V}(\lambda) = \frac{f(x)}{8} \begin{pmatrix} 0 & i & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -i & 0 \\ 0 & 1 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & -1 & 0 \end{pmatrix} + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)$$

Thus

$$\|\mathbf{V}(\lambda)^{-1}\mathbf{R}^{(2)}(x,\lambda)\mathbf{V}(\lambda)\| = \frac{f(x)}{8} + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)$$

and

$$\begin{aligned} \|\mathbf{V}(\lambda)^{-1}\mathbf{R}^{(2)}(x,\lambda)\zeta\| &= \|\mathbf{V}(\lambda)^{-1}[\mathbf{A}^{(2)}(x,\lambda) - \mathbf{A}_\infty^{(2)}(\lambda)]\mathbf{V}(\lambda)\mathbf{e}_1\| \\ &= \mathcal{O}\left(\frac{f(x)}{\sqrt{\lambda}}\right) \end{aligned}$$

So all three integral conditions of the Pego-Weinstein Lemma in Appendix B are satisfied and we can conclude from that Lemma that the Evans function converges to unity for $\lambda \to \infty$ along the real axis.