Tôhoku Math. J.
42 (1990), 17–44

CONSTRUCTION AND STABILITY ANALYSIS OF TRANSITION LAYER SOLUTIONS IN REACTION-DIFFUSION SYSTEMS

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(Received August 8, 1988, revised July 24, 1989)

1. Introduction. In mathematical biology, reaction-diffusion equations have been of great interest as a model describing spatial pattern formations. One of the most powerful approaches to the existence of spatially inhomogeneous solutions is a singular perturbation method. In fact, this method enables us to construct solutions with sharp spatial transition layers [5], [6], [12], [13]. It is the purpose of this paper to present a method to construct solutions with internal transition layers in the context of singular perturbation problems. We also emphasize the stability analysis of the solutions so obtained as above. Our method is slightly different from those in [5], [6], [12], [13] in that existence and stability analysis are carried out simultaneously.

For d_i, $i=1, 2$, positive parameters, consider the following pair of reaction-diffusion equations

$$\text{(PDE)} \qquad \begin{cases} u_t = d_1 u_{xx} + f(u, v) \\ v_t = d_2 v_{xx} + g(u, v) \end{cases} \qquad x \in (0, 1)\,, \quad t > 0$$

under the homogeneous Neumann boundary conditions

$$\text{(BC)} \qquad u_x = 0 = v_x \qquad x = 0, 1\,, \quad t > 0\,.$$

The problem (PDE)+(BC) has been studied rather extensively for the case in which both diffusion coefficients d_1, d_2 are very large by, among others, Conway, Hoff and Smoller [3], Hale [9] and Hale and Rocha [10]. Roughly speaking, the asymptotic dynamics of (PDE)+(BC) is qualitatively the same as that of

$$\text{(ODE)} \qquad u_t = f(u, v)\,, \qquad v_t = g(u, v)$$

when min (d_1, d_2) is sufficiently large.

On the other hand, there has been a series of works by Nishiura, Mimura, et al. [6], [7], [8], [12], [13], [14], [15] on (PDE)+(BC) from a viewpoint of pattern formation when $d_1 > 0$ is very small with d_2 remaining large. These authors have established the existence of equilibrium solutions with interior transition layers [13] as well as their

This research was supported in part by National Science Foundation of the United States under NSF grant DMS-8507056.

stability when the nonlinearity (f, g) satisfies such conditions as stated below. The latter result seems to be the first satisfactory one on the stability of large amplitude equilibrium states of a system of nonlinear parabolic equations, and is based on the so called "SLEP-Method" due to Fujii and Nishiura [8].

In order to show the existence of interior transition layers, Mimura, et al. [13] used the earlier work of Fife [5] (see also Ito [12]) on Dirichlet boundary layers. They first split the interval [0, 1] into two subintervals $[0, x^*]$ and $[x^*, 1]$, construct, on each subinterval, a solution with a Dirichlet boundary layer at x^* so that the boundary values of each solution coincide at $x=x^*$, and then adjust x^* so that the resulting solution be of class C^2 on [0, 1]. The stablity analysis of the transition layer solution requires subtle estimates on small eigenvalues which go to zero when the first diffusion coefficient d_1 tends to zero. Fujii and Nishiura [8] succeeded in reducing the original eigenvalue problem to a second order differential equation involving Dirac's δ-functions and determine the behavior of the small (or, critical) eigenvalues.

In this paper, we will present an alternative approach to the construction and the stability analysis of transition layer solutions of the following problem:

$$(1.1)\qquad \begin{cases} u_t=\varepsilon^2 u_{xx}+f(u, v)\,, \\ v_t=\sigma^{-1} v_{xx}+g(u, v)\,, & x\in(0, 1)\,,\quad t>0\,, \\ u_x=0=v_x \quad \text{at} \quad x=0, 1\,, \end{cases}$$

in which ε and σ are positive constants and f and g satisfy the conditions (A.0)–(A.5) below (see Figure 1).

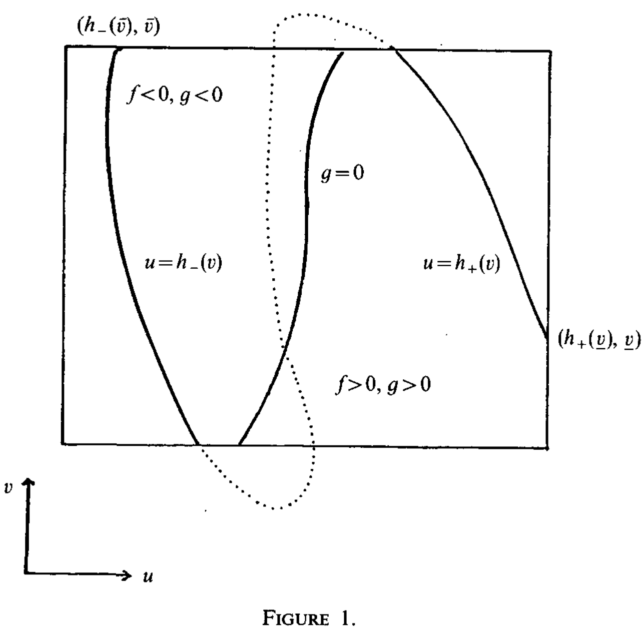

FIGURE 1.

(A.0) The functions f and g are C^3-functions defined on some open set $O \subset R^2$.
(A.1) The nullcline $\{(u, v) \in O; f(u, v) = 0\}$ of f contains at least two curves $C_i = \{(u, v) \in O;\ u = h_i(v),\ v \in I_i\}$ where $h_i(v)$ is a C^3-function defined on a closed interval I_i, $i = -, +$, and satisfies

$$h_-(v) < h_+(v) \qquad \text{for} \quad v \in I = [\underline{v}, \bar{v}] = I_- \cap I_+ \,.$$

(A.2) If $J(v)$ is defined by $J(v) = \int_{h_-(v)}^{h_+(v)} f(s, v) ds$, $v \in I$, then there exists v^*, $\underline{v} < v^* < \bar{v}$ such that $J(v^*) = 0$, $J'(v^*) \neq 0$ and

$$\int_{h_-(v^*)}^{u} f(s, v') ds < 0 \,, \qquad \text{for} \quad u \in (h_-(v^*),\ \ h_+(v^*)) \,.$$

(A.3) $f_u < 0$ on $C_-^* \cup C_+^*$ where $C_\pm^*$ are defined by

$$C_-^* = \{(u, v) \in C_-;\ v \leq v^*\} \,, \qquad C_+^* = \{(u, v) \in C_+;\ v \geq v^*\} \,.$$

(A.4) The following inequalities hold true: $g|_{C_-^*} < 0 < g|_{C_+^*}$, $[f_u g_v - f_v g_u]|_{C_-^* \cup C_+^*} > 0$.
(A.5) The following inequality holds true: $g_v|_{C_-^* \cup C_+^*} \leq 0$.

Under these conditions, the following is the main theorem to be proved in this paper.

THEOREM A. (i) *If the conditions* (A.1) *through* (A.4) *are satisfied, then there exist* $\varepsilon_* > 0$, $\sigma_* > 0$, *a* C^1-*function* $x^*(\sigma)$, *and a family of equilibrium solutions* $(u(x, \varepsilon, \sigma),$ $v(x, \varepsilon, \sigma))$ *of* (1.1) *for* $(\varepsilon, \sigma) \in (0, \varepsilon_*] \times (0, \sigma_*]$ *such that*

$$|u(x, \varepsilon, \sigma) - h_-(v(x, \varepsilon, \sigma))| \to 0 \text{ uniformly on every compact subinterval of } [0, x^*(\sigma))$$

and

$$|u(x, \varepsilon, \sigma) - h_+(v(x, \varepsilon, \sigma))| \to 0 \text{ uniformly on every compact subinterval of } (x^*(\sigma), 1]$$

as $\varepsilon \to 0$, *while* $v(x, \varepsilon, \sigma)$ *converges to a* C^1-*function* $\hat{V}(x, \sigma)$, *which is monotone increasing in* x (*see Theorem* 2.1), *as* $\varepsilon \to 0$ *in* $C^1[0, 1]$-*norm.*

Moreover, for each $\delta > 0$ *the set*

$$\{x \in [0, 1];\ |u(x, \varepsilon, \sigma) - h_-(v(x, \varepsilon, \sigma))| > \delta,\ |u(x, \varepsilon, \sigma) - h_+(v(x, \varepsilon, \sigma))| > \delta\}$$

is an open interval around a uniquely determined (*see Theorem* 2.1) *point* $x^*(\sigma) \in (0, 1)$ *with width of* $O(\varepsilon)$ as $\varepsilon \to 0$.

(ii) *If the condition* (A.5) *is satisfied in addition to* (A.0)–(A.4), *then there is a positive constant* ρ^* *such that the eigenvalue problem*:

$$\Gamma^{\varepsilon, \sigma} \begin{bmatrix} w \\ z \end{bmatrix} = \rho \begin{bmatrix} w \\ z \end{bmatrix}$$

has a unique, simple, real eigenvalue $\rho_1(\varepsilon, \sigma)$ *in the region* $\{\rho \in C; \operatorname{Re} \rho \geq -\rho^*\}$ *for* $(\varepsilon, \sigma) \in (0, \varepsilon_*] \times (0, \sigma_*]$, *where* $\Gamma^{\varepsilon, \sigma}$ *is the linearization, under the homogeneous Neumann boundary conditions, of the right hand side of* (1.1) *around the equilibrium solution*

$(u(x, \varepsilon, \sigma), v(x, \varepsilon, \sigma))$. *Moreover, the eigenvalue* $\rho_1(\varepsilon, \sigma)$ *is a continuous function of* (ε, σ) *and satisfies* $\rho_1(\varepsilon, \sigma) = \varepsilon\hat{\rho}_1(\varepsilon, \sigma)$ *and* $\lim_{\varepsilon \to 0} \hat{\rho}_1(\varepsilon, \sigma) = \gamma(\sigma)J'(v^*)$, *where* $\gamma(\sigma)$ *is a continuous function with* $\gamma(\sigma) > 0$, $\sigma \in (0, \sigma_*]$ *and* $\lim_{\sigma \to 0} \gamma(\sigma) > 0$ (*see Figure* 2).

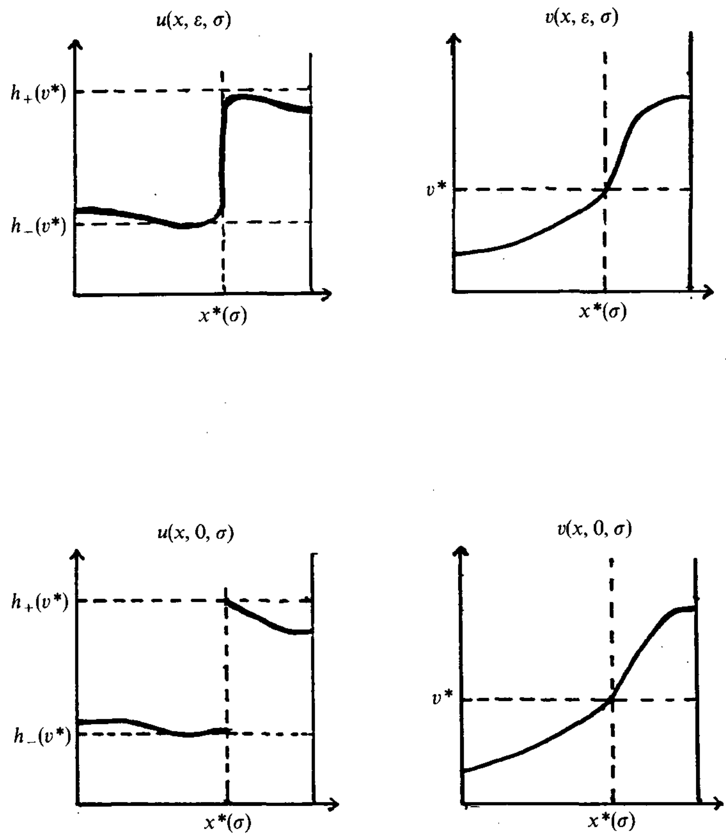

FIGURE 2.

COROLLARY. *If* $J'(v^*) < 0$ (*resp.* > 0) *in* (A.2) *then the equilibrium solution in Theorem* A *is stable* (*resp. unstable*) *relative to the parabolic equations* (1.1).

Our strategy for the proof of Theorem A is as follows. We start with a family of smooth (i.e. C^2) approximate solutions which is constructed in Section 2, and reduce the problem to finding a fixed point of an operator equation on appropriate function spaces. It should be noted that an idea in Ito [12] plays an important role to obtain a variational equation suitable for the subsequent analysis of the operator equation. In Section 3, we examine some spectral properties of the linear differential operator. The method of Liapunov-Schmidt applied to the linear differential operator singles out the unique small eigenvalue which goes to zero as ε tends to zero. It turns out that a scaled version of the small eigenvalue is a solution of a Singular Limit Eigenvalue Problem due to Fujii and Nishiura [8]. Theorem A is proved in Section 4. Section 5 is devoted to the stability analysis of multiple transition layer solutions.

The method employed in this paper seems to have several advantages. In terms of constructing approximate solutions, our approach is more natural than the matching method employed in [5], [13]. It enables us to make the accuracy of the approximation of internal transition layers as high as we wish under generic conditions. It also clarifies the obstruction of constructing approximations of higher accuracy (see [17] in this regard). Another advantage of smooth approximations is that we have the approximate solutions residing in the same space as the true solutions. This fact renders us a dynamic approach to the parabolic equations (1.1). The way in which we construct approximate solution also simplifies the stability analysis considerably. The method of Liapunov-Schmidt reveals in a natural way the equations which determine the small eigenvalues, as well as the order of magnitude of these eigenvalues.

Throughout this paper, the following function spaces are frequently referred to.

$H^2(0, 1)$: the usual Sobolev space on $[0, 1]$.

$H_N^2=\{u\in H^2(0, 1);\ u'(0)=0=u'(1)\}$

$C_N^2=\{u\in C^2[0, 1];\ u'(0)=0=u'(1)\}$

$\|u\|_2=\|u\|_0+\|u'\|_0+\|u''\|_0$: the usual Sobolev H^2-norm.

$|u|_2=|u|_0+|u'|_0+|u''|_0$: the usual C^2-norm.

$H_{N,\varepsilon}^2=H_N^2$ with the weighted norm $\|u\|_{2,\varepsilon}=\|u\|_0+\varepsilon\|u'\|_0+\varepsilon^2\|u''\|_0$.

$C_{N,\varepsilon}^2=C_N^2$ with the weighted norm $|u|_{2,\varepsilon}=|u|_0+\varepsilon|u'|_0+\varepsilon^2|u''|_0$.

Throughout this paper, prime "$'$" is used to indicate differentiation with respect to the spatial variable x as well as derivatives of a function of a single variable.

ACKNOWLEDGEMENT. The author is very grateful to Professor Jack K. Hale for his continuous encouragement and his inspiring conversation throughout the course of this work. Many thanks also go to Professor Y. Nishiura for stimulation through a series of publications as well as personal communication. Finally, but not least, the author would like to give utmost appreciation to the referee who carefully read the manuscript and pointed out many mistakes, typographical and otherwise.

2. Construction of approximate solutions. In this section, we construct a family of smooth approximate solutions of the problem:

$$(2.1)_{\varepsilon,\sigma}\qquad \begin{cases} \varepsilon^2 u''+f(u, v)=0\,, & x\in(0, 1)\\ \sigma^{-1}v''+g(u, v)=0\,, & x\in(0, 1)\\ u'=0=v'\,, \quad \text{at} & x=0, 1\,. \end{cases}$$

The accuracy of approximations is measured relative to the magnitude of the small parameter $\varepsilon > 0$.

2.1 Outer approximations. If one puts $\varepsilon = 0$ in (2.1), then there results the following reduced problem:

$$(2.1)_{0,\sigma} \qquad \begin{cases} f(u, v) = 0, & x \in (0, 1) \\ \sigma^{-1} v'' + g(u, v) = 0, & x \in (0, 1) \\ v' = 0, & \text{at } x = 0, 1. \end{cases}$$

The first equation $f(u, v) = 0$ has a continuum of solutions. Not all of them are of interest to us here, namely, not all of them can be extended to $\varepsilon > 0$ small. For each $v \in I$, the condition (A.1) gives $u = h_{\pm}(v)$ as a solution of $f(u, v) = 0$. For $v \in I$, they are stable equilibria of the kinetic equation $u_t = f(u, v)$. According to the condition (A.2), we choose

$$u = h_{-}(v) \quad \text{for} \quad v < v^*, \quad u = h_{+}(v) \quad \text{for} \quad v \geq v^*$$

as a solution of $f(u, v) = 0$. We would like to substitute this function into the second of $(2.1)_{0,\sigma}$ and to solve the resulting equation. For this purpose, define $G(v)$ for $v \in I$, by

$$G(v) = \begin{cases} G_{-}(v), & \text{if} \quad v < v^* \\ G_{+}(v), & \text{if} \quad v \geq v^* \end{cases}$$

where $G_{\pm}(v) = g(h_{\pm}(v), v)$ for $v \in I$. The problem $(2.1)_{0,\sigma}$ now reduces to

$$(\mathrm{RP})_\sigma \qquad \begin{cases} \sigma^{-1} v'' + G(v) = 0, & x \in [0, 1] - \{x^*(\sigma)\} \\ v^*(0) = 0 = v'(1), & v(x^*(\sigma)) = v^*. \end{cases}$$

By a solution V of $(\mathrm{RP})_\sigma$, we mean a function V belonging to $C^1[0, 1] \cup C^2([0, 1] - \{x^*(\sigma)\})$ and satisfying the relations in $(\mathrm{RP})_\sigma$. It should be noted that the determination of the "transition point" $x^*(\sigma)$ is a part of the problem. We have the following theorem available.

THEOREM 2.1. ([13, Theorem 1], [8, Lemma A.1]) *Under the conditions* (A.0), (A.1), *and* (A.4), *there exist a uniquely determined constant σ_0 and a C^1-function $x^*(\sigma)$ such that for $\sigma \in (0, \sigma_0]$ the problem $(\mathrm{RP})_\sigma$ has a unique solution $\hat{V}(x, \sigma)$, which is C^1 in $(x, \sigma) \in [0, 1] \times [0, \sigma_0]$. Moreover, $\hat{V}(\cdot, \sigma)$ is continuous in $\sigma \in [0, \sigma_0]$ with respect to $C^1[0, 1]$-topology and $x^*(0) = G_{+}(v^*)/[G_{+}(v^*) - G_{-}(v^*)]$.*

By using the functions $\hat{V}(x, \sigma)$ and $h_{\pm}$, our outer approximation is given by the pair $(\hat{U}(x, \sigma), \hat{V}(x, \sigma))$ where $\hat{U}(x, \sigma)$ is defined by

$$\hat{U}(x, \sigma) = \begin{cases} h_{-}(\hat{V}(x, \sigma)), & x \in [0, x^*(\sigma)) \\ h_{+}(\hat{V}(x, \sigma)), & x \in [x^*(\sigma), 1] \end{cases}$$

2.2 Inner approximation. The functions $\hat{U}(x, \sigma)$ and $\hat{V}''(x, \sigma)$ have a jump discontinuity at the transition point $x^*(\sigma)$. This can be smoothed out by adding inner approximations in a neighborhood of $x=x^*(\sigma)$.

For this purpose, we first review the procedure of obtaining inner approximations for the following scalar problem:

$$\text{(IP)} \qquad \begin{cases} \varepsilon^2 u'' + A(u, x) = 0, & x \in (0, 1) \\ u'(0) = 0 = u'(1) \end{cases}$$

where the function $A(u, x)$ satisfies:

(a.1) $A: R \times [0, 1] \to R$ is of C^2-class, and there are two functions $u = b_\pm(x)$ of C^2-class such that

$$b_-(x) < b_+(x), \qquad A(b_\pm(x), x) \equiv 0, \qquad x \in [0, 1]$$

(a.2) $A_u(b_\pm(x), x) \leq -3\beta^2 < 0$ $x \in [0, 1]$ for some constant $\beta > 0$.

(a.3) If we define $j(x) = \int_{b_-(x)}^{b_+(x)} A(s, x)ds$, then there exists a point $x^* \in (0, 1)$ such that $j(x^*) = 0, j'(x^*) \neq 0$, *and*

$$\int_{b_-(x)}^{u} A(s, x^*)ds < 0 \qquad \text{for} \quad u \in (b_-(x^*), b_+(x^*)).$$

Transition layer phenomena for (IP) under the conditions (a.1)–(a.3) are studied rather extensively by [1], [4], [11]. Here, we follow the method in [11].

Let us introduce the fast variable η around x^* by $x = x^* + \varepsilon\eta$ and the transformation $u = b(x)z + b_-(x)$ in (IP), where $b(x) = b_+(x) - b_-(x)$. The equation for z reads:

$$\text{(2.2)} \qquad b\ddot{z} + 2\varepsilon b'\dot{z} + \varepsilon^2(b''z + b''_-) + A(bz + b_-, x^* + \varepsilon\eta) = 0$$

where all the functions are evaluated at $x = x^* + \varepsilon\eta$, and the dot designates the differentiation with respect to the fast variable η, while the prime stands for differentiation with respect to x. Formally substituting the expression $z = z_0(\eta) + \varepsilon z_1(\eta) + o(\varepsilon)$ in (2.2) and equating the coefficient of each power of ε separately to zero, the relation (2.2) gives rise to the equations for z_0 and z_1:

$$\text{(2.3)} \qquad b(x^*)\ddot{z}_0 + A(b(x^*)z_0 + b_-(x^*), x^*) = 0, \qquad \eta \in R$$

$$\text{(2.4)} \qquad \begin{aligned} &b(x^*)\ddot{z}_1 + A_u(\#)b(x^*)z_1 + \{A_u(\#)[b'(x^*)z_0(\eta) + b'_-(x^*)] \\ &\quad + A_x(\#)\}\eta + b'(x^*)\eta\ddot{z}_0(\eta) + 2b'(x^*)\dot{z}_0(\eta) = 0 \end{aligned}$$

where $A_u(\#)$ and $A_x(\#)$ are evaluated at $(u, x) = (b(x^*)z_0(\eta) + b_-(x^*), x^*)$. For each constant $\gamma \in (0, 1)$, the equation (2.3) has a unique solution $z_0(\eta, \gamma)$ such that $z_0(0, \gamma) = \gamma$ and $\lim_{\eta \to \infty} z_0(\eta) = 1$, $\lim_{\eta \to -\infty} z_0(\eta) = 0$ exponentially. Moreover $\max\{|\dot{z}_0(\eta)|, |\ddot{z}_0(\eta)|\} = O(e^{-2\beta|\eta|})$, as $|\eta| \to \infty$. Once $z_0(\eta)$ is specified as above, the equation (2.4) takes the following form

$$b(x^*)\ddot{z}_1 + A_u(\#)b(x^*)z_1 + q(\eta, z_0(\eta)) = 0, \quad \text{with} \quad |q(\eta, z_0(\eta)| \leq \text{const.} e^{-2\beta|\eta|}, \quad \eta \in R.$$

By using theorems based on exponential dichotomies and Fredholm alternatives (see Chow and Hale [2]), it is easy to see that (2.4) has a solution bounded on R if and only if the condition (2.5) below is satisfied.

$$\int_{-\infty}^{\infty} q(\eta, z_0(\eta, \gamma))\dot{z}_0(\eta, \gamma)d\eta = 0 . \tag{2.5}$$

Following the procedure in [11], one can verify that there exists a unique $\gamma \in (0, 1)$ for which (2.5) is fulfilled. For such a choice of γ, the equation (2.4) has a unique family of bounded solutions (which decay exponentially as $|\eta| \to \infty$) $z_1(\eta) = c\dot{z}_0(\eta) + \bar{z}_1(\eta)$, where $\bar{z}_1(0) = \dot{\bar{z}}_1(0) = 0$, $c \in R$. We choose the coefficient c of $\dot{z}_0$ so that

$$\int_{-\infty}^{\infty} [A_{uu}(\#)(z_1(\eta))^2 + 2A_{ux}(\#)\eta z_1(\eta) + A_{xx}(\#)\eta^2]\dot{z}_0(\eta)d\eta = 0$$

is satisfied. This condition can be written as $I_2c^2 + I_1c + I_0 = 0$. It turns out that $I_2 = 0$, $I_1 = -j'(x^*) \neq 0$. Therefore, we can determine the coefficient c such that the condition above is satisfied. This choice of c is best possible in the sense that only for this choice of c can we determine the second order approximation. See [11] for more detail.

We shall apply the procedure described above to the problem:

$$(\mathrm{IP})_\sigma \qquad \begin{cases} \varepsilon^2 u'' + f(u, \hat{V}(x, \sigma)) = 0 , & x \in (0, 1) \\ u'(0) = 0 = u'(1) . \end{cases}$$

In order to do so, we simply set $A(u, x) = f(u, \hat{V}(x, \sigma))$, $b_\pm = h_\pm(\hat{V}(x, \sigma)$, and $x^* = x^*(\sigma)$. Although the functions $f(u, \hat{V}(x, \sigma))$ and $h_\pm(\hat{V}(x, \sigma)$ are not twice continuously differentiable in x at $x = x^*(\sigma)$, the procedure above still works, since the equations (2.3) and (2.4) involves at most the first x-derivatives of the functions $b(x)$, $b_\pm(x)$ and $A(u, x)$. It is easy to verify that the condition (a.3) is satisfied with $j(x) = J(\hat{V}(x, \sigma))$ (see the condition (A.2)).

Now let $Z(\eta, \varepsilon) = z_0(\eta) + \varepsilon z_1(\eta)$ be constructed from $(\mathrm{IP})_\sigma$ through the procedure described for (IP) above. Let $\zeta_0(x) \in C^\infty(R)$ be such that

$$\zeta_0(x) = 1 , \quad \text{for} \quad |x| \leq 1/4 , \quad \zeta_0(x) = 0 , \quad \text{for} \quad |x| \geq 1/2 , \quad 0 \leq \zeta_0(x) \leq 1$$

and $\zeta_+(x) = 1 - \zeta_0(x)$ for $x \geq 0$ and $\zeta_+(x) = 0$ for $x \leq 0$, $d = \min\{x^*(\sigma), 1 - x^*(\sigma);$ $\sigma \in [0, \sigma_0]\}$ and $H(v) = h_+(v) - h_-(v)$. Now our tentative approximation for the u-component is given by

$$\bar{U}(x, \varepsilon, \sigma) = H(\hat{V}(x, \sigma))[Z(\eta, \varepsilon)\zeta_0([x - x^*(\sigma)]/d) + \zeta_+([x - x^*(\sigma)]/d)] + h_-(\hat{V}(x, \sigma)) \tag{2.6}$$

where $\eta = [x - x^*(\sigma)]/\varepsilon$ is the stretched variable. The function $\bar{U}(x, \varepsilon, \sigma)$ is a C^1-function of x, but its second derivative has a jump discontinuity at $x = x^*(\sigma)$. In order to smooth this out, let us first observe the influence of substituting $(\bar{U}, \hat{V})$ in $(\mathrm{RP})_\sigma$ instead of $(\hat{U}, \hat{V})$, namely the difference

$$\sigma^{-1}\hat{V}'' + g(\bar{U}, \hat{V}) = g(\bar{U}, \hat{V}) - g(\hat{U}, \hat{V}) .$$

Although the difference goes to zero as fast as ε^2 uniformly outside a fixed neighborhood of $x=x^*(\sigma)$, it remains "large", i.e., of the order of one in the fixed neighborhood of the same point. This could be overcome by adding an "inner correction" to $\hat{V}$, which simultaneously balances the jump discontinuity of $\hat{V}''$ at $x=x^*(\sigma)$. In order to do so, let us put $(\bar{U}, \hat{V}+\varepsilon^2\sigma Y)$ into $(\mathrm{RP})_\sigma$ to obtain

$$\varepsilon^2 Y''+g(\bar{U}, \hat{V})-g(\hat{U}, \hat{V})+[g(\bar{U}, \hat{V}+\sigma\varepsilon^2 Y)-g(\bar{U}, \hat{V})]=0\,.$$

The difference $[g(\bar{U}, \hat{V}+\sigma\varepsilon^2 Y)-g(\bar{U}, \hat{V})]$ on the left side is of order $O(\varepsilon^2)$ provided $\sup|Y|<\infty$.

Let us solve:

$$\varepsilon^2 Y''+g(\bar{U}, \hat{V})-g(\hat{U}, \hat{V})=0\,,$$

or equivalently

$$\ddot{\tilde{Y}}+g(\tilde{\bar{U}}, \tilde{\hat{V}})-g(\tilde{\hat{U}}, \tilde{\hat{V}})=0$$

in terms of the fast variable $\eta=(x-x_*(\sigma))/\varepsilon$, where the tilde "$\tilde{\ }$" indicates that the fast variable η is considered as an independent variable. This equation is easily solved on each one of the half intervals $[-x^*(\sigma)/\varepsilon, 0]$ and $[0, (1-x^*(\sigma))/\varepsilon]$ in the following manner.

$$\tilde{Y}_+(\eta)=-\int_\eta^{[1-x^*(\sigma)]/\varepsilon}\int_\tau^{[1-x^*(\sigma)]/\varepsilon}[g(\tilde{\bar{U}}(s), \tilde{\hat{V}}(s))-g(\tilde{\hat{U}}(s), \tilde{\hat{V}}(s))]ds\,d\tau\,, \qquad \text{for} \quad \eta\geq 0$$

$$\tilde{Y}_-(\eta)=-\int_{-x^*(\sigma)/\varepsilon}^{\eta}\int_{-x^*(\sigma)/\varepsilon}^{\tau}[g(\tilde{\bar{U}}(s), \tilde{\hat{V}}(s))-g(\tilde{\hat{U}}(s), \tilde{\hat{V}}(s))]ds\,d\tau\,, \qquad \text{for} \quad \eta\leq 0\,.$$

Now let us define $\tilde{Y}(\eta)$ by:

$$\tilde{Y}(\eta)=\tilde{Y}_+(\eta)-[\tilde{Y}_+(0)+\eta\dot{\tilde{Y}}_+(0)]\zeta_0(\eta) \quad \text{for} \quad \eta\geq 0\,,$$
$$\tilde{Y}(\eta)=\tilde{Y}_-(\eta)-[\tilde{Y}_-(0)+\eta\dot{\tilde{Y}}_-(0)]\zeta_0(\eta)\,, \quad \text{for} \quad \eta\leq 0\,.$$

LEMMA 2.2. *The function $\tilde{Y}$ is of C^1 on $[-x^*(\sigma)/\varepsilon, (1-x^*(\sigma))/\varepsilon]$ and has a finite C^3-uniform norm on $[-x^*(\sigma)/\varepsilon, (1-x^*(\sigma))/\varepsilon]-\{0\}$, which is bounded uniformly with respect to $(\varepsilon, \sigma)\in(0, \varepsilon_0]\times(0, \sigma_0]$, where ε_0 is any fixed positive number.*

The proof of this lemma follows immediately from the construction.

Now define $Y(x)$ and $V(x, \varepsilon, \sigma)$ by

$$Y(x)=\tilde{Y}([x-x^*(\sigma)]/\varepsilon) \qquad V(x, \varepsilon, \sigma)=\hat{V}(x, \sigma)+\sigma\varepsilon^2 Y(x)\,.$$

LEMMA 2.3. *The function $V(x, \varepsilon, \sigma)$ belongs to $C^2_N[0, 1]$ for $(\varepsilon, \sigma)\in(0, \varepsilon_0]\times(0, \sigma_0]$, and $|V(\cdot, \varepsilon, \sigma)|_2$ is bounded uniformly in $(\varepsilon, \sigma)\in(0, \varepsilon_0]\times(0, \sigma_0]$.*

PROOF. We only show that $V''(x, \varepsilon, \sigma)$ is continuous at $x=x^*(\sigma)$. The remaining part of the lemma follows immediately from the construction.

Let us define

$$V''_+ = \lim_{x \downarrow x^*(\sigma)} V''(x, \varepsilon, \sigma), \quad \text{and} \quad V''_- = \lim_{x \uparrow x^*(\sigma)} V''(x, \varepsilon, \sigma).$$

Then, from the construction we have

$$V''_\pm = \lim_{x \to x^*(\sigma) \pm 0} \hat{V}''(x^*(\sigma), \sigma) + \sigma \ddot{\tilde{Y}}_\pm(0) = -\sigma g(h_\pm(v^*), v^*)$$
$$-\sigma[g(\bar{U}(x^*(\sigma), \varepsilon, \sigma), v^*) - g(h_\pm(v^*), v^*)] = -\sigma g(\bar{U}(x^*(\sigma), \varepsilon, \sigma), v^*),$$

hence $V''_+ = V''_-$, proving the continuity of $V''(x, \varepsilon, \sigma)$ at $x = x^*(\sigma)$. q.e.d.

We are now in a position to define a C^2 approximation of the u-component by

$$U(x, \varepsilon, \sigma) = H(V(x, \varepsilon, \sigma))[Z(\eta, \varepsilon)\zeta_0([x - x^*(\sigma)]/d) + \zeta_+([x - x^*(\sigma)]/d)] + h_-(V(x, \varepsilon, \sigma)). \tag{2.7}$$

Compare this with the one in (2.6). The pair $(U(x, \varepsilon, \sigma), V(x, \varepsilon, \sigma))$ is the family of approximate solutions to $(2.1)_{\varepsilon,\sigma}$ from which we will find a family of genuine solutions as a perturbation.

2.3. Perturbation from the curve $C^*_- \cup C^*_+$. We look for a family of solutions of $(2.1)_{\varepsilon,\sigma}$ in the following type of perturbation from the approximate solutions $(U(\cdot, \varepsilon, \sigma), V(\cdot, \varepsilon, \sigma))$

$$\begin{cases} u = H(V+s)[Z\zeta_0 + \zeta_+] + h_-(V+s) + r \\ v = V + s \end{cases} \tag{2.8}$$

rather than the usual type of perturbation

$$\begin{cases} u = U + r \\ v = V + s \end{cases} \tag{2.9}$$

where r, s will belong to C^2_N with small norms. The transformation (2.8) means that we are looking for a solution (u, v) whose graph is a perturbation from the curve $C^*_- \cup C^*_+$.

The Taylor expansion in (r, s) of the right hand side of the first equation in (2.8) read

$$u = U + r + U_1 s + O(|s|_0^2) \quad \text{as} \quad |s|_0 \to 0$$

where $U_1 = H'(V)[Z\zeta_0 + \zeta_+] + h'_-(V)$. It turns out that the linear transformation $u = U + r + U_1 s$, $v = V + s$ is as effective as the nonlinear one (2.8) for our purpose. We therefore transform the problem $(2.1)_{\varepsilon,\sigma}$ in terms of the following change of variables

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} 1 & U_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{2.10}$$

into the equations for the new functions (u, v)

$$(2.11)_{\varepsilon,\sigma} \qquad \begin{cases} L^{\varepsilon,\sigma}u+N^{\varepsilon,\sigma}v+R_1(\varepsilon,\sigma)+F_1(u,v,\varepsilon,\sigma)=0 \\ M^{\varepsilon,\sigma}v+g_u(U,V)u+R_2(\varepsilon,\sigma)+F_2(u,v,\varepsilon,\sigma)=0 \end{cases}$$

where $L^{\varepsilon,\sigma}$, $N^{\varepsilon,\sigma}$, $M^{\varepsilon,\sigma}$: $C_N^2 \to C^0$ (or $H_N^2 \to L^2$) are given by

$$L^{\varepsilon,\sigma}u=\varepsilon^2 u''+f_u(U,V)u$$
$$N^{\varepsilon,\sigma}v=\varepsilon^2(U_1 v)''+[f_u(U,V)U_1+f_v(U,V)]v$$
$$M^{\varepsilon,\sigma}v=\sigma^{-1}v''+[g_u(U,V)U_1+g_v(U,V)]v$$

and

$$R_1(\varepsilon,\sigma)=\varepsilon^2 U''+f(U,V)$$
$$R_2(\varepsilon,\sigma)=\sigma^{-1}V''+g(U,V)$$
$$F_1(u,v,\varepsilon,\sigma)=f(U+u+U_1 v,V+v)-f(U,V)-f_u(U,V)u \\ -[f_u(U,V)U_1+f_v(U,V)]v$$
$$F_2(u,v,\varepsilon,\sigma)=g(U+u+U_1 v,V+v)-g(U,V)-g_u(U,V)u \\ -[g_u(U,V)U_1+g_v(U,V)]v\,.$$

It is easy to verify the following (see [11, Lemma 2.1]):

LEMMA 2.4. *For each* $i=1, 2$

$|R_i(\varepsilon,\sigma)|_0=O(\varepsilon^2)$ *as* $\varepsilon\to 0$ *uniformly in* $\sigma\in(0,\sigma_0]$,

$|F_i(u,v,\varepsilon,\sigma)|_0=O(|u|_0^2+|v|_0^2)$ *as* $|u|_0+|v|_0\to 0$ *uniformly in* $(\varepsilon,\sigma)\in(0,\varepsilon_0]\times(0,\sigma_0]$.

REMARK 2.5. It was Ito [12] who first pointed out the superiority of the transformation (2.8) over (2.9) in the context of boundary layer phenomena.

3. Spectral analysis of linear operator. In the last section, the problem $(2.1)_{\varepsilon,\sigma}$ was reduced to the operator equation (2.11) on appropriate function spaces. The main subject of this section is the eigenvalue problem of the linear operator $\Gamma(\varepsilon,\sigma)$: $C_{N,\varepsilon}^2\times C_N^2\to C^0\times C^0$ (or $H_{N,\varepsilon}^2\times H_N^2\to L^2\times L^2$),

$$(\mathrm{EP})_{\varepsilon,\sigma} \qquad \Gamma(\varepsilon,\sigma)\begin{pmatrix} w \\ z \end{pmatrix}=\rho\begin{pmatrix} w \\ z \end{pmatrix}$$

where the operator $\Gamma(\varepsilon,\sigma)$ is defined as the linearization of left side of $(2.1)_{\varepsilon,\sigma}$ around the approximate solution $(U(x,\varepsilon,\sigma), V(x,\varepsilon,\sigma))$ constructed in the previous section.

THEOREM B. *There exist a constant* $\rho_0>0$ *and a unique, real, simple eigenvalue* $\rho=\rho(\varepsilon,\sigma)$ *of the problem* $(\mathrm{EP})_{\varepsilon,\sigma}$ *for* $(\varepsilon,\sigma)\in(0,\varepsilon_0]\times(0,\sigma_0]$ *in the region* $\{\rho\in C;$ $\operatorname{Re}\rho\geq-\rho_0\}$. *Moreover,* $\rho(\varepsilon,\sigma)=\varepsilon\hat{\rho}(\varepsilon,\sigma)$ *is a continuous function of* $(\varepsilon,\sigma)\in$

$(0, \varepsilon_0] \times (0, \sigma_0]$ *and*

$$\lim_{\varepsilon \to 0} \hat{\rho}(\varepsilon, \sigma) J'(v^*) > 0 .$$

In order to prove Theorem B we first analyse the operators $L^{\varepsilon,\sigma}$, $M^{\varepsilon,\sigma}$ separately.

3.1. The operator $L^{\varepsilon,\sigma}: H_N^2 \to L^2$. Let us denote by $\{\phi_n(\varepsilon, \sigma), \lambda_n(\varepsilon, \sigma)\}_{n=1}^{\infty}$ a complete orthonormal system of eigenfunctions and eigenvalues of $L^{\varepsilon,\sigma}$ arranged so that $\lambda_1 > \lambda_2 > \cdots$, $\lambda_n \to -\infty$ as $n \to \infty$.

LEMMA 3.1. (i) *There exists a constant* $\lambda_0 > 0$ *such that* $\lambda_2(\varepsilon, \sigma) \le -\lambda_0$ *for* $(\varepsilon, \sigma) \in (0, \varepsilon_0] \times (0, \sigma_0]$.

(ii) $\lim_{\varepsilon \to 0} \lambda_1(\varepsilon, \sigma) = 0$ *uniformly in* $\sigma \in (0, \sigma_0]$.

(iii) *There are constants* $k > 0$ *and* $\beta > 0$ *such that*

$$|\phi_1(x, \varepsilon, \sigma)| \le k |\phi_1(x^*(\sigma), \varepsilon, \sigma)| \exp[-\beta |x - x^*(\sigma)|/\varepsilon] .$$

(iv) $\varepsilon^{1/2}\phi_1(\varepsilon\eta + x^*(\sigma), \varepsilon, \sigma) \to K\dot{z}_0(\eta)$ *as* $\varepsilon \to 0$ *in* $C^2_{loc}(R)$ *uniformly in* $\sigma \in (0, \sigma_0]$, *where the constant* K *is given by*

$$K^{-1} := \left[\int_{-\infty}^{\infty} |\dot{z}_0(\eta)|^2 d\eta \right]^{1/2} = \|\dot{z}_0\|_{L^2(R)} .$$

(v) *The following limit exists*

$$\lim_{\varepsilon \to 0} \lambda_1(\varepsilon, \sigma)/\varepsilon = -K_*^2 J'(v^*) \hat{V}'(x^*(\sigma), \sigma) = \sigma K_*^2 J'(v^*) \int_0^{x^*(\sigma)} g(h_-(\hat{V}(x, \sigma)), \hat{V}(x, \sigma)) dx$$

$$= -\sigma K_*^2 J'(v^*) \int_{x^*(\sigma)}^1 g(h_+(\hat{V}(x, \sigma)), \hat{V}(x, \sigma)) dx ,$$

where $K_* = KH(v^*)^{-1}$.

PROOF. For the detail of proof, refer to [11, Theorem 3.1, Lemma 3.4]. We only exhibit a computation which leads us to the formula in (v).

Multiply the relation $L^{\varepsilon,\sigma}\phi_1 = \lambda_1(\varepsilon, \sigma)\phi_1$ by the function $\dot{z}_0((x - x^*(\sigma))/\varepsilon)$ and integrate the result by parts over the interval $[x^*(\sigma) - d/4, x^*(\sigma) + d/4]$ to obtain

$$\lambda_1 \int_{-d/4\varepsilon}^{d/4\varepsilon} \Phi(\eta)\dot{z}_0(\eta) d\eta = [\dot{\Phi}(\eta)\dot{z}_0(\eta) - \Phi(\eta)\ddot{z}_0(\eta)]_{-d/4\varepsilon}^{d/4\varepsilon} + \int_{-d/4\varepsilon}^{d/4\varepsilon} [\ddot{z}_* + f_u(\#) z_*] \Phi d\eta$$

where $z_* = \dot{z}_0$ and

$$\Phi(\eta) = \varepsilon^{1/2}\phi_1(\varepsilon\eta + x^*(\sigma), \varepsilon, \sigma) \to K\dot{z}_0(\eta) \qquad \text{as} \quad \varepsilon \to 0 \quad \text{in} \quad C^2_{loc}(R)$$

$$f_u(\#) = f_u(H(\tilde{V}(\eta))[z_0(\eta) + \varepsilon z_1(\eta)] + h_-(\tilde{V}(\eta)), \tilde{V}(\eta))$$

with $\tilde{V}(\eta) = V(\varepsilon\eta + x^*(\sigma), \varepsilon, \sigma)$.

It follows from (iii) of Lemma 3.1 and the estimate on $|\dot{z}_0(\eta)|$ and $|\ddot{z}_0(\eta)|$ that

$$[\dot{\Phi}(\eta)\dot{z}_0(\eta)-\Phi(\eta)\ddot{z}_0(\eta)]^{d/4\varepsilon}_{-d/4\varepsilon}=O(\exp[-\beta d/\varepsilon]) \qquad \text{as} \quad \varepsilon\to 0\,.$$

Therefore, by Lebesgue's dominated convergence theorem, one obtains

$$\text{(3.1)} \qquad \left(\lim_{\varepsilon\to 0}\lambda_1(\varepsilon,\sigma)/\varepsilon\right)\left(\int_{-\infty}^{\infty}|\dot{z}_0(\eta)|^2 d\eta\right)=\int_{-\infty}^{\infty}\{f_{uu}(\#\#)H(v^*)z_1(\eta)$$

$$+\hat{V}'(x^*(\sigma),\sigma)[f_{uu}(\#\#)(H'(v^*)\eta z_0(\eta)+h'_-(v^*)\eta)+f_{uv}(\#\#)\eta]\}\dot{z}_0(\eta)^2 d\eta$$

where $f_{uu}(\#\#)$ and $f_{uv}(\#\#)$ are evaluated at $(H(v^*)z_0(\eta)+h_-(v^*),v^*)$. By using integration by parts as well as $-f(\#\#)=H(v^*)\ddot{z}_0$, one continues (3.1) as follows:

$$K^{-2}\left(\lim_{\varepsilon\to 0}\lambda_1(\varepsilon,\sigma)/\varepsilon\right)=-H(v^*)^{-1}\int_{-\infty}^{\infty}\{H(v^*)\ddot{z}_1+f_u(\#\#)H(v^*)z_1$$

$$+[f_u(\#\#)(H'(v^*)z_0+h'_-(v^*))+f_v(\#\#)]\eta\hat{V}'(x^*(\sigma),\sigma)$$

$$+\hat{V}'(x^*(\sigma),\sigma)H'(v^*)[\eta\ddot{z}_0+2\dot{z}_0]\}\ddot{z}_0 d\eta-H(v^*)^{-1}\int_{-\infty}^{\infty}f_v(\#\#)\dot{z}_0 d\eta\,\hat{V}'(x^*(\sigma),\sigma)\,.$$

The quantity {---} in the first term under integral sign is identically equal to zero, in view of the relation (2.4), while the integral in the second term reduces to

$$\int_{-\infty}^{\infty}f_v(H(v^*)z_0(\eta)+h_-(v^*),v^*)\dot{z}_0(\eta)d\eta=H(v^*)^{-1}\int_{h_-(v^*)}^{h_+(v^*)}f_v(s,v^*)ds=H(v^*)^{-1}J'(v^*)\,.$$

Therefore the relation (3.1) gives

$$\lim_{\varepsilon\to 0}\lambda_1(\varepsilon,\sigma)/\varepsilon=-K^2 J'(v^*)\hat{V}'(x^*(\sigma),v^*)H(v^*)^{-2}=-K_*^2 J'(v^*)\hat{V}'(x^*(\sigma),v^*)\,.$$

The second and the third expressions in (v) can be obtained from

$$\hat{V}'(x^*(\sigma),\sigma)=-\sigma\int_0^{x^*(\sigma)}g(h_-(\hat{V}(x,\sigma)),\hat{V}(x,\sigma))dx=\sigma\int_{x^*(\sigma)}^1 g(h_+(\hat{V}(x,\sigma)),\hat{V}(x,\sigma))dx\,.$$

q.e.d.

COROLLARY 3.2. (i) *The statements* (i), (ii), (iii) *and* (iv) *in Lemma* 3.1 *are still valid when the potential function* $f_u(U(\cdot,\varepsilon,\sigma),V(\cdot,\varepsilon,\sigma))$ *is perturbed to* $f_u(U(\cdot,\varepsilon,\sigma)+o(1),\ V(\cdot,\varepsilon,\sigma)+o(1))$, *where* $o(1)\to 0$ *as* $\varepsilon\to 0$. *On the other hand, the formula in* (v) *remains the same as long as the potential function is perturbed to* $f_u(U(\cdot,\varepsilon,\sigma)+\ p(\cdot,\varepsilon,\sigma),V(\cdot,\varepsilon,\sigma)+o(\varepsilon))$, *where* $p(x,\varepsilon,\sigma)$ *is a continuous function such that* $\varepsilon^{-1}p(\varepsilon\eta+x^*(\sigma),\varepsilon,\sigma)\to\text{const.}\dot{z}_0(\eta)$ *as* $\varepsilon\to 0$ *in* $C^0_{loc}(R)$ *and* $o(\varepsilon)/\varepsilon\to 0$ *as* $\varepsilon\to 0$.

(ii) *Let* $Q(\varepsilon,\sigma): L^2\to L^2$ *be the orthogonal projection onto the* $\text{span}\{\phi_n(\varepsilon,\sigma)\}_{n\geq 2}$. *Then*

$$(L^{\varepsilon,\sigma})^{-1}: QL^2\to H^2_{N,\varepsilon},\ (or\ QC^0\to C^2_{N,\varepsilon})$$

is bounded uniformly in $(\varepsilon, \sigma) \in (0, \varepsilon_0] \times (0, \sigma_0]$.

3.2. The operator $M^{\varepsilon,\sigma}: H_N^2 \to L^2$. Let us denote by $\psi_n(\varepsilon, \sigma)$, $\mu_n(\varepsilon, \sigma)$ $n=1, 2, \cdots$, a complete orthonormal system of eigenfunctions and eigenvalues of $M^{\varepsilon,\sigma}$ arranged so that: $\mu_1 > \mu_2 > \cdots$, $\mu_n \to -\infty$ as $n \to \infty$.

LEMMA 3.3. (i) *There exists a constant* $\mu_0 > 0$ *such that*

$$\mu_1(\varepsilon, \sigma) \leq -\mu_0\,, \qquad \textit{for} \quad (\varepsilon, \sigma) \in (0, \varepsilon_0] \times (0, \sigma_0]\,.$$

(ii) $\sup\{|\psi_n(\varepsilon, \sigma)|_0;\ n=1, 2, \cdots, (\varepsilon, \sigma) \in (0, \varepsilon_0] \times (0, \sigma_0]\}$ *is finite.*

PROOF. Let us define the limiting operator $M^{0,\sigma}$ by

$$M^{0,\sigma} v = \sigma^{-1} v'' + B(x, \sigma) v$$

where $B(x, \sigma) = g_u(\hat{U}(x, \sigma), \hat{V}(x, \sigma)) \hat{U}_1(x, \sigma) + g_v(\hat{U}(x, \sigma), \hat{V}(x, \sigma))$ and

$$\hat{U}_1 = h'_-(\hat{V}(x, \sigma)) \quad \text{for} \quad x \in [0, x^*(\sigma))\,, \quad \hat{U}_1 = h'_+(\hat{V}(x, \sigma)) \quad \text{for} \quad x \in [x^*(\sigma), 1]\,.$$

Since $h'_\pm(v) = -f_v(h_\pm(v), v)/f_u(h_\pm(v), v)$, the conditions (A.3) and (A.4) imply the existence of a constant $\mu_0 > 0$ such that $\sup\{B(x, \sigma);\ 0 \leq x \leq 1\} < -\mu_0$.

Therefore, $(M^{0,\sigma} - \mu)^{-1}: L^2 \to L^2$ exists and is bounded uniformly for $\operatorname{Re}\mu \geq -\mu_0$. By a standard bootstrap argument, $(M^{0,\sigma} - \mu)^{-1}: L^2 \to H_N^2$ is also bounded uniformly for $\operatorname{Re}\mu \geq -\mu_0$. On the other hand, one has

$$\|g_u(U, V) U_1 + g_v(U, V) - B(\cdot, \sigma)\|_0 = O(\varepsilon^{1/2}) \qquad \text{as} \quad \varepsilon \to 0$$

and hence

$$(M^{\varepsilon,\sigma} - \mu) = (M^{0,\sigma} - \mu)\{I + (M^{0,\sigma} - \mu)^{-1}[g_u U_1 + g_v - B(\cdot, \sigma)] \cdot\}$$

is invertible uniformly with respect to μ, $\operatorname{Re}\mu \geq -\mu_0$, and $(\varepsilon, \sigma) \in (0, \varepsilon_0] \times (0, \sigma]$ for some small $\varepsilon_0 > 0$. This in particular implies that $\mu_1(\varepsilon, \sigma) < -\mu_0$.

(ii) This is a well-known result from the Sturm-Liouville theory.

REMARK 3.4. Asymptotic behaviors of eigenvalues $\mu_n(\varepsilon, \sigma)$ as $\sigma \to 0$ are given by

$$\sigma \mu_n(\varepsilon, \sigma) = -\pi^2 (n-1)^2 + O(\sigma)\,, \quad n \geq 2\,,$$

$$\lim_{\sigma \to 0} \mu_1(\varepsilon, \sigma) = x^*(0) G'_-(v^*) + (1 - x^*(0)) G'_+(v^*) + O(\varepsilon^\delta)$$

for any $\delta \in [0, 1)$. Recall here that: $G_\pm = g(h_\pm(v), v)$, and $x^*(0) = G_+(v^*)/(G_+(v^*) - G_-(v^*))$.

3.3. Combined operators. In order to analyze the eigenvalue problem for $\Gamma(\varepsilon, \sigma)$, it is necessary to consider combined operators of $M^{\varepsilon,\sigma}$, $N^{\varepsilon,\sigma}$ and $L^{\varepsilon,\sigma}$.

LEMMA 3.5. (i) $\|N^{\varepsilon,\sigma}(M^{\varepsilon,\sigma} - \mu)^{-1}\|_{L^2 \to L^2} = O(\varepsilon^{1/2})$ *as* $\varepsilon \to 0$ *uniformly in* $\sigma \in (0, \sigma_0]$ *and* $\operatorname{Re}\mu \geq -\mu_0$.

(ii) $\|(M^{\varepsilon,\sigma} - \mu)^{-1} g_u \phi_1\|_{L^\infty} = O(\varepsilon^{1/2})$ *as* $\varepsilon \to 0$ *uniformly in* $\sigma \in (0, \sigma_0]$ *and* $\operatorname{Re}\mu \geq -\mu_0$, *where* $g_u = g_u(U, V)$.

(iii) $\|N^{\varepsilon,\sigma}(M^{\varepsilon,\sigma}-\mu)^{-1}g_u\phi_1\|_0 = O(\varepsilon)$ as $\varepsilon \to 0$ *uniformly in* $\sigma \in (0, \sigma_0]$ *and* $\operatorname{Re}\mu \geq -\mu_0$.

(iv) $\|(M^{\varepsilon,\sigma}-\mu)^{-1}N_*^{\varepsilon,\sigma}\|_{H^2\to H^2} = O(\varepsilon^{1/2})$ *as* $\varepsilon \to 0$ *uniformly in* $\sigma \in (0, \sigma_0]$ *and* $\operatorname{Re}\mu \geq -\mu_0$, *where* $N_*^{\varepsilon,\sigma}$ *is the adjoint operator of* $N^{\varepsilon,\sigma}$, *i.e.*, $N_*^{\varepsilon,\sigma}v = U_1 L^{\varepsilon,\sigma}v + f_v(U, V)v$.

PROOF. (i) By a direct computation, one has

$$N^{\varepsilon,\sigma}(M^{\varepsilon,\sigma}-\mu)^{-1}w = \sigma\varepsilon^2 U_1[w + (\mu - B^{\varepsilon,\sigma})h] + 2\varepsilon^2 U_1' h' + [\varepsilon^2 U_1'' + f_u(U, V)U_1 + f_v(U, V)]h$$

where $h = (M^{\varepsilon,\sigma}-\mu)^{-1}w$ and $B^{\varepsilon,\sigma} = g_u(U, V)U_1 + g_v(U, V)$. It then follows from this that

$$\begin{aligned}\|N^{\varepsilon,\sigma}(M^{\varepsilon,\sigma}-\mu)^{-1}w\|_0 &\leq \sigma\varepsilon^2(1 + \|\mu - B^{\varepsilon,\sigma}\|_{L^\infty}\|(M^{\varepsilon,\sigma}-\mu)^{-1}\|_{L^2\to L^2})\|w\|_0 \\ &\quad + 2\varepsilon\|\varepsilon U_1'\|_{L^\infty}\|(M^{\varepsilon,\sigma}-\mu)^{-1}\|_{L^2\to H^1}\|w\|_0 + \|\varepsilon^2 U_1'' + f_u(u, V)U_1 \\ &\quad + f_v(U, V)\|_0\|(M^{\varepsilon,\sigma}-\mu)^{-1}\|_{L^2\to H^1}\|w\|_0 \\ &= [O(\varepsilon) + \|(M^{\varepsilon,\sigma}-\mu)^{-1}\|_{L^2\to H^1}\|L^{\varepsilon,\sigma}U_1 + f_v(U, V)\|_0]\|w\|_0 .\end{aligned}$$

By using the integration in terms of the fast variable $\eta = [x - x^*(\sigma)]/\varepsilon$ and the fact that $f_u(h_\pm(v), v)h_\pm'(v) + f_v(h_\pm(v), v) \equiv 0$, one can easily show that $\|L^{\varepsilon,\sigma}U_1 + f_v(U, V)\|_0 = O(\varepsilon^{1/2})$ as $\varepsilon \to 0$. This proves the statement (i).

(ii) By using the eigenfunction expansion

$$(M^{\varepsilon,\sigma}-\mu)^{-1}g_u\phi_1 = \sum_{n=1}^{\infty} \psi_n(\varepsilon, \sigma)\langle g_u\phi_1(\varepsilon, \sigma), \psi_n(\varepsilon, \sigma)\rangle/(\mu_n - \mu)$$

and Lemma 3.3 (ii), one obtains

$$\|(M^{\varepsilon,\sigma}-\mu)^{-1}g_u\phi_1\|_{L^\infty} \leq \sum_{n=1}^{\infty} \|\psi_n\|_{L^\infty}|\langle g_u\phi_1, \psi_n\rangle|/|\mu_n - \mu| \leq c\sum_{n=1}^{\infty} |\langle g_u\phi_1, \psi_n\rangle| n^{-2}$$

where the constant $c > 0$ does not depend on ε, σ and μ. On the other hand, one has

$$\begin{aligned}\langle g_u\phi_1, \psi_n\rangle &= \int_0^1 g_u(U, V)\phi_1\psi_n dx = \varepsilon^{1/2}\int_{-x^*(\sigma)/\varepsilon}^{[1-x^*(\sigma)]/\varepsilon} g_u(\tilde{U}(\eta), \tilde{V}(\eta))\varepsilon^{1/2}\tilde{\phi}_1(\eta)\tilde{\psi}_n(\eta)d\eta \\ &= \varepsilon^{1/2}\left[\int_{-\infty}^{\infty} Kg_u(H(v^*)z_0(\eta) + h_-(v^*), v^*)\dot{z}_0(\eta)\psi_n(x^*(\sigma))d\eta + o(1)\right] \\ &= \varepsilon^{1/2}K_*[G_+(v^*) - G_-(v^*)]\psi_n(x^*(\sigma)) + o(\varepsilon^{1/2}),\end{aligned}$$

which completes the proof of (ii).

(iii) This follows from the proofs of part (i) and part (ii) above.

(iv) From the expression

$$(M^{\varepsilon,\sigma}-\mu)^{-1}N_*^{\varepsilon,\sigma}w = \varepsilon^2(M^{\varepsilon,\sigma}-\mu)^{-1}(U_1 w'') + (M^{\varepsilon,\sigma}-\mu)^{-1}[(f_u U_1 + f_v)w]$$

for $w \in H_N^2$, it follows that

$$\begin{aligned}
&\|(M^{\varepsilon,\sigma}-\mu)^{-1}N_*^{\varepsilon,\sigma}w\|_0 \\
&\le \varepsilon^2\|(M^{\varepsilon,\sigma}-\mu)^{-1}\|_{L^2\to L^2}\|U_1\|_{L^\infty}\|w''\|_0+\|(M^{\varepsilon,\sigma}-\mu)^{-1}\|_{L^2\to L^2}\|f_uU_1+f_v\|_0\|w\|_{L^\infty} \\
&\le O(\varepsilon^2)\|w\|_2+\text{const.}\|f_uU_1+f_v\|_0\|w\|_1\le O(\varepsilon^{1/2})\|w\|_2\,,
\end{aligned}$$

since $\|f_uU_1+f_v\|_0=O(\varepsilon^{1/2})$ as $\varepsilon\to 0$. A similar computation also shows that

$$\|(M^{\varepsilon,\sigma}-\mu)^{-1}N_*^{\varepsilon,\sigma}w\|_2\le O(\varepsilon^{1/2})\|w\|_2\,. \qquad \text{q.e.d.}$$

3.4. The proof of Theorem B. We consider the eigenvalue problem $(\mathrm{EP})_{\varepsilon,\sigma}$ for $\operatorname{Re}\rho\ge-\rho_0$ where $\rho_0=\min(\lambda_0,\mu_0)$. The problem $(\mathrm{EP})_{\varepsilon,\sigma}$ is written as

$$(L^{\varepsilon,\sigma}-\rho)w+N^{\varepsilon,\sigma}z=0 \tag{3.3.a}$$

$$(M^{\varepsilon,\sigma}-\rho)z+g_uw=0\,. \tag{3.3.b}$$

In view of Lemma 3.3 (i), (3.3.b) can be solved in z as $z=-(M^{\varepsilon,\sigma}-\rho)^{-1}g_uw=-(M^{\varepsilon,\sigma}-\rho)^{-1}g_u(\alpha\phi_1+\bar{w})$ where $w=\alpha\phi_1+\bar{w}$ with $\alpha\in C$, $\langle\phi_1,\bar{w}\rangle=0$. Substituting this into (3.3.a) and using the decomposition $L^2=[\phi_1]\oplus[\phi_n;n\ge 2]$ one obtains

$$\alpha(\lambda_1(\varepsilon,\sigma)-\rho)-\alpha\langle(M^{\varepsilon,\sigma}-\rho)^{-1}g_u\phi_1,\bar{U}_1\phi_1\rangle-\langle(M^{\varepsilon,\sigma}-\rho)^{-1}g_u\bar{w},\bar{U}_1\phi_1\rangle=0 \tag{3.4.a}$$

$$(L^{\varepsilon,\sigma}-\rho)\bar{w}-QN^{\varepsilon,\sigma}(M^{\varepsilon,\sigma}-\rho)^{-1}g_u\bar{w}=\alpha QN^{\varepsilon,\sigma}(M^{\varepsilon,\sigma}-\rho)^{-1}g_u\phi_1 \tag{3.4.b}$$

where $\bar{U}_1=\lambda_1(\varepsilon,\sigma)U_1+f_v(U,V)$, $Q=Q(\varepsilon,\sigma)$: $L^2\to\operatorname{span}[\phi_n(\varepsilon,\sigma);n\ge 2]$ the orthogonal projection, and integration by parts is used for the second and the third terms of (3.4.a). Because of Lemma 3.1 (i) and Lemma 3.5 (i), (3.4.b) can be solved in $\bar{w}$, yielding

$$\bar{w}=\alpha K^{\varepsilon,\sigma,\rho}\phi_1 \tag{3.5}$$

where

$$K^{\varepsilon,\sigma,\rho}u=\sum_{n=0}^{\infty}[(L^{\varepsilon,\sigma}-\rho)^{\dagger}N^{\varepsilon,\sigma}(M^{\varepsilon,\sigma}-\rho)^{-1}g_u]^{n+1}u \tag{3.6}$$

with

$$(L-\rho)^{\dagger}u=\sum_{n\ge 2}\phi_n\langle u,\phi_n\rangle/(\lambda_n(\varepsilon,\sigma)-\rho)\,.$$

COROLLARY 3.6. (i) *The operator* $K^{\varepsilon,\sigma,\rho}: L^2\to H^2_{N,\varepsilon}$ *(resp.* $C^0\to C^2_{N,\varepsilon}$*) is bounded uniformly in* $(\varepsilon,\sigma)\in(0,\varepsilon_0]\times(0,\sigma_0]$ *and* $\operatorname{Re}\rho\ge-\rho_0$.

(ii) $\|K^{\varepsilon,\sigma,\rho}\phi_1\|_0=O(\varepsilon)$, $\|(M^{\varepsilon,\sigma}-\rho)^{-1}g_uK^{\varepsilon,\sigma,\rho}\phi_1\|_1=O(\varepsilon)$ *as* $\varepsilon\to 0$ *uniformly in* $(\sigma,\rho)\in(0,\sigma_0]\times\{\rho\in C;\operatorname{Re}\rho\ge-\rho_0\}$.

This corollary follows immediately from Lemmas 3.1, 3.3, and 3.5. From (3.4) and (3.5), we obtain:

LEMMA 3.7. *Any eigenvalue $\rho(\varepsilon, \sigma)$ of* $(\mathrm{EP})_{\varepsilon,\sigma}$ *with* $\operatorname{Re}\rho \geq -\rho_0$ *has to satisfy the following equation*

$$\lambda_1(\varepsilon, \sigma)-\rho-\langle (M^{\varepsilon,\sigma}-\rho)^{-1}g_u[I+K^{\varepsilon,\sigma,\rho}]\phi_1, \bar{U}_1\phi_1\rangle=0 \tag{3.7}$$

or equivalently written in terms of real and imaginary parts

$$(3.7)_R \qquad 0=\lambda_1(\varepsilon, \sigma)-\rho_R-\frac{1}{2}\langle [(M^{\varepsilon,\sigma}-\rho)^{-1}+(M^{\varepsilon,\sigma}-\bar{\rho})^{-1}](g_u\phi_1), \bar{U}_1\phi_1\rangle$$

$$-\left\langle\left\{\frac{1}{2}[(M^{\varepsilon,\sigma}-\rho)^{-1}+(M^{\varepsilon,\sigma}-\bar{\rho})^{-1}]g_uK_R^{\varepsilon,\sigma,\rho}\right.\right.$$

$$\left.\left.-\rho_I(M^{\varepsilon,\sigma}-\rho)^{-1}(M^{\varepsilon,\sigma}-\bar{\rho})^{-1}g_uK_I^{\varepsilon,\sigma,\rho}\right\}\phi_1, \bar{U}_1\phi_1\right\rangle$$

$$(3.7)_I \qquad -\rho_I\left\{1+\langle (M^{\varepsilon,\sigma}-\rho)^{-1}(M^{\varepsilon,\sigma}-\bar{\rho})^{-1}(g_u\phi_1), \bar{U}_1\phi_1\rangle+\left\langle\left\{\frac{1}{2}[(M^{\varepsilon,\sigma}-\rho)^{-1}\right.\right.\right.$$

$$\left.\left.\left.+(M^{\varepsilon,\sigma}-\bar{\rho})^{-1}]g_uK_I^{\varepsilon,\sigma,\rho}+(M^{\varepsilon,\sigma}-\rho)^{-1}(M^{\varepsilon,\sigma}-\bar{\rho})^{-1}g_uK_R^{\varepsilon,\sigma,\rho}\right\}\phi_1, \bar{U}_1\phi_1\right\rangle\right\}=0$$

where $\rho=\rho_R+\rho_I\sqrt{-1}$, ρ_R, $\rho_I\in R$, $\bar{\rho}=\rho_R-\rho_I\sqrt{-1}$, $K^{\varepsilon,\sigma,\rho}=K_R^{\varepsilon,\sigma,\rho}+\sqrt{-1}\rho_IK_I^{\varepsilon,\sigma,\rho}$, *with* $K_R^{\varepsilon,\sigma,\rho}$, $K_I^{\varepsilon,\sigma,\rho}$ *being real operators.*

From the estimates in Lemma 3.5 and Corollary 3.6, the relations in $(3.7)_R$ and $(3.7)_I$ read:

$$\lambda_1(\varepsilon, \sigma)-\rho_R-O(\varepsilon)=0\,, \qquad -\rho_I\{1+O(\varepsilon)\}=0\,.$$

This immediately implies $\rho_I=0$ for small $\varepsilon>0$, say $\varepsilon\in(0, \varepsilon_0]$, and

$$\lambda_1(\varepsilon, \sigma)-\rho-\langle (M^{\varepsilon,\sigma}-\rho)^{-1}g_u\phi_1, f_v\phi_1\rangle+O(\varepsilon^{3/2})$$

where $\rho\in R$. Since $\lambda_1(\varepsilon, \sigma)=O(\varepsilon)$ and $\langle (M^{\varepsilon,\sigma}-\rho)^{-1}g_u\phi_1, f_v\phi_1\rangle=O(\varepsilon)$ as $\varepsilon\to 0$, the eigenvalue ρ has to be of $O(\varepsilon)$, which in turn enables us to set $\rho=\varepsilon\hat{\rho}$. The relation (3.7) is therefore equivalent to

$$\hat{\lambda}_1(\varepsilon, \sigma)-\hat{\rho}-\varepsilon^{-1}\langle (M^{\varepsilon,\sigma}-\varepsilon\hat{\rho})^{-1}g_u\phi_1, f_v\phi_1\rangle+O(\varepsilon^{1/2})=0\,, \qquad \varepsilon>0 \tag{3.8}$$

where $\hat{\lambda}_1(\varepsilon, \sigma)=\varepsilon^{-1}\lambda_1(\varepsilon, \sigma)$. Recall from Lemma 3.1 that $\lambda_1(\varepsilon, \sigma)$ is a continuous function defined on $(0, \varepsilon_0]\times(0, \sigma_0]$ and the limit $\lim_{\varepsilon\to 0}\hat{\lambda}_1(\varepsilon, \sigma)$ exists. By using Lemma 3.1 and the eigenfunction expansion for $M^{\varepsilon,\sigma}$, we obtain

$$\lim_{\varepsilon\to 0}\langle (M^{\varepsilon,\sigma}-\varepsilon\hat{\rho})^{-1}g_u\phi_1, f_v\phi_1\rangle=\lim_{\varepsilon\to 0}\varepsilon^{-1}\sum_{n=1}^{\infty}\langle g_u\phi_1, \psi_n\rangle\langle f_v\phi_1, \psi_n\rangle(\mu_n(\varepsilon, \sigma)-\varepsilon\hat{\rho})^{-1}$$

$$=K_*^2[G_+(v^*)-G_-(v^*)]J'(v^*)\sum_{n=1}^{\infty}[\psi_n(x^*(\sigma), 0, \sigma)]^2\mu_n(0, \sigma)^{-1}=:\hat{\rho}_0(\sigma).$$

Let $E(\hat{\rho}, \sigma, \varepsilon)$ be the function defined by the left side of (3.8), which is a continuous function of $(\hat{\rho}, \sigma, \varepsilon)$ on $R \times (0, \sigma_0] \times (0, \varepsilon_0]$ and analytic in $\hat{\rho}$ for a fixed (ε, σ). Note that the function $E(\hat{\rho}, \sigma, \varepsilon)$ can be extended continuously to $R \times [0, \sigma_0] \times [0, \varepsilon_0]$ because of Lemma 3.1 (ii) and Remark 3.4. Applying the implicit function theorem to $E(\hat{\rho}, \sigma, \varepsilon) = 0$ around $(\hat{\rho}, \sigma, \varepsilon) = (\hat{\lambda}_1(0, \sigma) - \hat{\rho}_0(0), \sigma, 0)$, one obtains a unique solution $\hat{\rho} = \hat{\rho}(\varepsilon, \sigma)$ of (3.8) with $\hat{\rho}(0, \sigma) = \hat{\lambda}_1(0, \sigma) - \hat{\rho}_0(0)$, which is continuous in (ε, σ). Therefore the problem $(\mathrm{EP})_{\varepsilon,\sigma}$ has a unique real eigenvalue $\rho(\varepsilon, \sigma) = \varepsilon\hat{\rho}(\varepsilon, \sigma)$ in the region $\{\rho \in C;\ \mathrm{Re}\,\rho \geq -\rho_0\}$.

LEMMA 3.8. *$\hat{\rho}(0, \sigma) J'(v^*) > 0$ is true for $\sigma \in [0, \sigma_0]$.*

PROOF. Let us notice that

$$c^*(\sigma) := \sum_{n=1}^{\infty} [\psi_n(x^*(\sigma), 0, \sigma)]^2 \mu_n(0, \sigma)^{-1} = \langle (M^{0,\sigma})^{-1} \delta_\sigma^*, \delta_\sigma^* \rangle$$

where δ_σ^* is the Dirac point mass at $x = x^*(\sigma)$, and $\langle (M^{0,\sigma})^{-1} \delta_\sigma^*, \delta_\sigma^* \rangle$ is the duality paring between H^1 and H^{-1} (notice that $\delta_\sigma^* \in H^{-1}$ and $(M^{0,\sigma})^{-1} \delta_\sigma^* \in H^1$). In order to evaluate $c^*(\sigma)$, let us put $z_*(x, \sigma) = (M^{0,\sigma})^{-1} \delta_\sigma^*$. Then $c^*(\sigma) = z_*(x^*(\sigma), \sigma)$. The function z_* is the solution of the following

$$(3.9) \qquad -\sigma^{-1} \langle z'_*, \psi' \rangle + \langle B^\sigma z_*, \psi \rangle = \langle \psi, \delta_*^\sigma \rangle \qquad \text{for all} \quad \psi \in H^1$$

where $B^\sigma(x) = [f_u g_v - f_v g_u] f_u^{-1}$ evaluated at $(u, v) = (\hat{U}(x, \sigma), \hat{V}(x, \sigma))$. The function $\hat{V}'(x, \sigma) \in H^1 \cap C^2\ ([0, 1] - \{x^*(\sigma)\})$ satisfies

$$(3.10) \qquad \sigma^{-1} \hat{V}''' + B^\sigma \hat{V}' = 0 \qquad \text{on} \quad [0, 1] - \{x^*(\sigma)\}$$

in the classical sense. Substitute $\psi = \hat{V}'$ in (3.9) and use (3.10) to obtain

$$-\sigma^{-1} [\langle z_*, \hat{V}''' \rangle + \langle z'_*, \hat{V}'' \rangle] = \hat{V}'(x^*(\sigma)) .$$

By integration by parts, this gives rise to

$$\begin{aligned} z_*(x^*(\sigma), \sigma) &= [\sigma \hat{V}'(x^*(\sigma), \sigma) + z_*(1) \hat{V}''(1) - z_*(0) \hat{V}''(0)][\hat{V}''_+ - \hat{V}''_-]^{-1} \\ &= [-\hat{V}'(x^*(\sigma), \sigma) + z_*(1) g(h_+(\hat{V}(1, \sigma)), \hat{V}(1, \sigma)) \\ &\quad - z_*(0) g(h_-(\hat{V}(0, \sigma)), \hat{V}(0, \sigma))][G_+(v^*) - G_-(v^*)]^{-1} \end{aligned}$$

where $\hat{V}''_\pm = \lim_{x \to x^*(\sigma) \pm 0} \hat{V}''(x, \sigma) = -\sigma G_\pm(v^*)$.

Therefore, recalling $\hat{\lambda}_1(0, \sigma) = -K_*^2 J'(v^*) \hat{V}'(x^*(\sigma), \sigma)$ from Lemma 3.1, one obtains

$$\hat{\rho}(0, \sigma) = \hat{\lambda}_1(0, \sigma) - K_*^2 J'(v^*)[G_+(v^*) - G_-(v^*)] z_*(x^*(\sigma), \sigma) = K_*^2 J'(v^*)[z_*(0) g_0^\sigma - z_*(1) g_1^\sigma]$$

where $g_0^\sigma = g(h_-(\hat{V}(0, \sigma)), \hat{V}(0, \sigma)) < 0$, $g_1^\sigma = g(h_+(\hat{V}(1, \sigma)), \hat{V}(1, \sigma)) > 0$. We shall show that $z_*(1) < 0$, $z_*(0) < 0$. To do so, notice that z_* satisfies $\sigma^{-1} z''_* + B^\sigma z_* = 0$, on $[0, 1] - \{x^*(\sigma)\}$ in the classical sence, and that $B^\sigma < 0$ and $z_*(x^*(\sigma), \sigma) = \sum_{n=1}^{\infty} [\psi_n(x^*(\sigma), \sigma)]^2 \mu_n(0, \sigma)^{-1} < 0$ from Lemma 3.3 (i). Therefore, z_* has to be concave as long as $z_* < 0$. On account of the boundary conditions $z'_*(0) = 0 = z'_*(1)$, we obtain that $z_*(0)<0$ and $z_*(1)<0$, see Figure 3. Therefore $z_*(0)g_0^\sigma - z_*(1)g_1^\sigma > 0$ follows, concluding the proof.

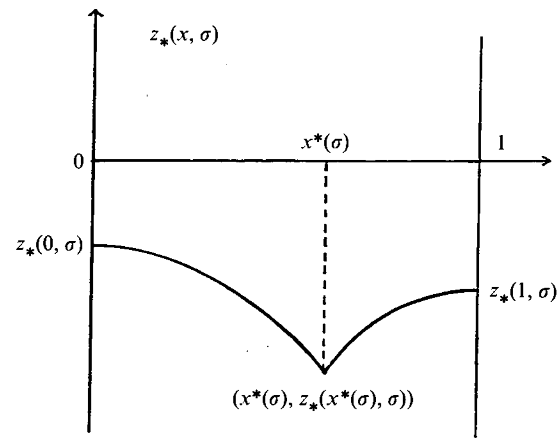

FIGURE 3.

In order to complete the proof of Theorem B, it remains to show the simplicity of the eigenvalue $\rho(\varepsilon, \sigma)$. It suffices to show an eigenfunction (ϕ, ψ) of $\Gamma(\varepsilon, \sigma)$ corresponding to $\rho(\varepsilon, \sigma)$ does not belong to the range of $\Gamma(\varepsilon, \sigma)-\rho(\varepsilon, \sigma)$. Now, one can show that the adjoint operator $\Gamma(\varepsilon, \sigma)^*$ of $\Gamma(\varepsilon, \sigma)$ has a unique real eigenvalue $\rho(\varepsilon, \sigma)^* = \rho(\varepsilon, \sigma)$ (the same value as the unique real eigenvalue of $\Gamma(\varepsilon, \sigma)$ above) in the region $\{\rho \in C; \operatorname{Re} \rho \geq -\rho_0\}$ by the same line of argument for $\Gamma(\varepsilon, \sigma)$ and integration by parts (here we use Lemma 3.5 (iv)). Let (ϕ, ψ), (ϕ^*, ψ^*) be eigenfunctions of $\Gamma(\varepsilon, \sigma)$ and $\Gamma(\varepsilon, \sigma)^*$ corresponding to $\rho(\varepsilon, \sigma)$. More specifically,

$$\phi = \phi_1 + \bar{\phi}, \qquad \bar{\phi} = K^{\varepsilon,\sigma,\rho(\varepsilon,\sigma)}\phi_1, \qquad \langle \phi_1, \bar{\phi} \rangle = 0$$

$$\phi^* = \phi_1 + \bar{\phi}^*, \qquad \bar{\phi}^* = K_*^{\varepsilon,\sigma,\rho(\varepsilon,\sigma)}\phi_1, \qquad \langle \phi_1, \bar{\phi}^* \rangle = 0$$

$$\psi = -(M^{\varepsilon,\sigma} - \rho(\varepsilon, \sigma))^{-1} g_u \phi, \qquad \psi^* = -(M^{\varepsilon,\sigma} - \rho(\varepsilon, \sigma))^{-1} N_*^{\varepsilon,\sigma} \phi^*$$

where $K_*^{\varepsilon,\sigma,\rho}$ is the counterpart of $K^{\varepsilon,\sigma,\rho}$ for $\Gamma(\varepsilon, \sigma)^*$. By using the estimates $[\|\bar{\phi}\|_0, \|\bar{\phi}^*\|_0, \|\psi\|_0, \|\psi^*\|_0] = O(\varepsilon^{1/2})$, one obtains

$$\langle (\phi, \psi), (\phi^*, \psi^*) \rangle = \|\phi_1\|_0 + O(\varepsilon) = 1 + O(\varepsilon) \neq 0\,.$$

This implies (ϕ, ψ) does not belong to the range of $\Gamma(\varepsilon, \sigma)-\rho(\varepsilon, \sigma)$ because the range of $\Gamma(\varepsilon, \sigma)-\rho(\varepsilon, \sigma)$ is characterized as the orthogonal complement of (ϕ^*, ψ^*).

4. Proof of Theorem A.

4.1. Existence via the Method of Liapunov-Schmidt. We will show the solvability of (2.11) or equivalently

$$\Gamma(\varepsilon, \sigma)W + R(\varepsilon, \sigma) + F(W, \varepsilon, \sigma) = 0 \tag{4.1}$$

where $W=(u, v)$, $R(\varepsilon, \sigma)=(R_1(\varepsilon, \sigma), R_2(\varepsilon, \sigma))$, and $F(W, \varepsilon, \sigma)=(F_1(u, v, \varepsilon, \sigma), F_2(u, v, \varepsilon, \sigma))$. Let $\Phi(\varepsilon, \sigma)=(\Phi_1(\varepsilon, \sigma), \Phi_2(\varepsilon, \sigma))$ be the eigenfunction of $\Gamma(\varepsilon, \sigma)$ corresponding to $\rho(\varepsilon, \sigma)$, given by

$$\Phi_1(\varepsilon, \sigma) = \varepsilon^{1/2}[\phi_1(\varepsilon, \sigma) + K^{\varepsilon,\sigma,\rho(\varepsilon,\sigma)}\phi_1(\varepsilon, \sigma)] \tag{4.2.a}$$

$$\Phi_2(\varepsilon, \sigma) = -(M^{\varepsilon,\sigma} - \rho(\varepsilon, \sigma))^{-1}[g_u(U, V)\Phi_1(\varepsilon, \sigma)]\,. \tag{4.2.b}$$

One should notice that

$$\|\Phi_1(\varepsilon, \sigma)\|_{L^\infty} \le \|\varepsilon^{1/2}\phi_1(\varepsilon, \sigma)\|_{L^\infty} + O(\varepsilon^{1/2})\,, \qquad \|\Phi_2(\varepsilon, \sigma)\|_{L^\infty} = O(\varepsilon) \qquad \text{as} \quad \varepsilon \to 0$$

with $\|\varepsilon^{1/2}\phi_1(\varepsilon, \sigma)\|_{L^\infty}$ being bounded uniformly in $(\varepsilon, \sigma) \in (0, \varepsilon_0] \times (0, \sigma_0]$. We designate by $P(\varepsilon, \sigma)$ the orthogonal projection onto the span of $\Phi(\varepsilon, \sigma)$ in $L^2 \times L^2$. The equation (4.1) is rewritten as

$$\alpha\rho(\varepsilon, \sigma)\Phi(\varepsilon, \sigma) + PR(\varepsilon, \sigma) + PF(\alpha\Phi + \bar{W}, \varepsilon, \sigma) = 0 \tag{4.3.a}$$

$$\bar{\Gamma}(\varepsilon, \sigma)\bar{W} + (I-P)R(\varepsilon, \sigma) + (I-P)F(\alpha\Phi + \bar{W}, \varepsilon, \sigma) = 0 \tag{4.3.b}$$

where $\bar{\Gamma}$ is Γ restricted to $(I-P)(L^2 \times L^2)$ and $W = \alpha\Phi + \bar{W}$ with $\langle \Phi, \bar{W} \rangle = 0$, $\alpha \in R$. By virtue of Lemma 2.4 and Lemma 3.1 (i), the second equation in (4.3) gives, via the implicit function theorem, $W = \bar{W}_*(\alpha, \varepsilon, \sigma)$ with $|\bar{W}_*(\alpha, \varepsilon, \sigma)|_{C^0 \times C^0} = O(|\alpha|^2 + \varepsilon^2)$. Then the first equation in (4.3) yields the bifurcation equation

$$B(\alpha, \varepsilon, \sigma) = B_0(\varepsilon, \sigma) + B_1(\varepsilon, \sigma)\alpha + B_2(\varepsilon, \sigma)\alpha^2 + O(\alpha^3) = 0 \tag{4.4}$$

where

$$B_0(\varepsilon, \sigma) = O(\varepsilon^2)\,, \quad B_1(\varepsilon, \sigma) = \varepsilon\hat{\rho}(\varepsilon, \sigma) + o(\varepsilon)\,, \quad B_2(\varepsilon, \sigma) = o(1)$$

as $\varepsilon \to 0$ uniformly in $\sigma \in (0, \sigma_0]$. (These order estimates will be proved below.) Therefore (4.4) can be solved, via the implicit function theorem again, in α as $\alpha = \alpha^*(\varepsilon, \sigma) = \varepsilon\hat{\alpha}^*(\varepsilon, \sigma) = O(\varepsilon)$ as $\varepsilon \to 0$. Let us denote by $W^* = (u^*(x, \varepsilon, \sigma), v^*(x, \varepsilon, \sigma))$ the solution of (4.1), which satisfies the following:

$$\begin{aligned} &|u^*(\cdot, \varepsilon, \sigma)|_0 = O(\varepsilon) && \text{as} \quad \varepsilon \to 0 \\ &\varepsilon^{-1}\tilde{u}^*(\eta, \varepsilon, \sigma) \to K\hat{\alpha}^*(0, \sigma)\dot{z}_0(\eta) && \text{as} \quad \varepsilon \to 0 \quad \text{in} \quad C^2_{loc}(R) \\ &|v^*(\cdot, \varepsilon, \sigma)|_0 = O(\varepsilon^2) && \text{as} \quad \varepsilon \to 0\,. \end{aligned} \tag{4.5}$$

The desired family of solutions of our original problem (1.1) is given by

$$u(x, \varepsilon, \sigma) = U(x, \varepsilon, \sigma) + U_1(x, \varepsilon, \sigma)v^*(x, \varepsilon, \sigma) + u^*(x, \varepsilon, \sigma) \tag{4.6.a}$$

$$v(x, \varepsilon, \sigma) = V(x, \varepsilon, \sigma) + v^*(x, \varepsilon, \sigma)\,. \tag{4.6.b}$$

From the construction of the functions U, U_1, and V, Theorem A (i) follows immediately.

The order estimates on B_i, $i=0, 1, 2$, are proved as follows: $B_0(\varepsilon, \sigma)= PR(\varepsilon, \sigma)+PF(\bar{W}_*(0, \varepsilon, \sigma), \varepsilon, \sigma)=O(\varepsilon^2)$ follows from Lemma 2.4, $\bar{W}_*(0, \varepsilon, \sigma)=O(\varepsilon^2)$, $F(w, \varepsilon, \sigma)=O(|w|^2)$. Since $D_wF(\bar{W}_*(0, \varepsilon, \sigma), \varepsilon, \sigma)=O(\varepsilon^2)$ we immediately obtain

$$B_1(\varepsilon, \sigma)=\rho(\varepsilon, \sigma)+\frac{\partial}{\partial\alpha}PF(\alpha\Phi+\bar{W}_*, \varepsilon, \sigma)|_{\alpha=0}=\rho(\varepsilon, \sigma)+O(\varepsilon^2)\,.$$

As for B_2, notice that

$$B_2(\varepsilon, \sigma)=\frac{1}{2}\frac{\partial^2}{\partial\alpha^2}PF(\alpha\Phi+\bar{W}_*, \varepsilon, \sigma)\ |_{\alpha=0}=\frac{1}{2}PD_w^2F(\bar{W}_*(0, \varepsilon, \sigma), \varepsilon, \sigma)\langle\Phi, \Phi\rangle+O(\varepsilon^2)$$

Since $\Phi_1(\varepsilon, \sigma)=\varepsilon^{1/2}\phi_1(\varepsilon, \sigma)+O(\varepsilon^{1/2})$ and $\Phi_2(\varepsilon, \sigma)=O(\varepsilon)$, $B_2(\varepsilon, \sigma)$ can be expressed as

$$B_2(\varepsilon, \sigma)=\frac{1}{\varepsilon}\int_0^1 f_{uu}(U(x, \varepsilon, \sigma), V(x, \varepsilon, \sigma))[\varepsilon^{1/2}\phi_1(x, \varepsilon, \sigma)]^3dx+O(\varepsilon^{1/2})$$

By using Lemma 3.1 (iii), (iv), the integral on the right side converges to $K^3\int_{-\infty}^{\infty}f_{uu}(z(\eta), v^*)\dot{z}(\eta)^3d\eta$ as $\varepsilon\to 0$. Integration by parts yields

$$\begin{aligned}\int_{-\infty}^{\infty}f_{uu}(z(\eta), v^*)\dot{z}(\eta)^3d\eta &= -\int_{-\infty}^{\infty}f_u(z(\eta), v^*)2\dot{z}[-f(z(\eta), v^*)]d\eta\\ &= -\int_{-\infty}^{\infty}\frac{d}{d\eta}[f(z(\eta), v^*]^2d\eta=0\,.\end{aligned}$$

4.2. Stability. On account of the change of variables (2.10), we have to analyze the eigenvalue problem

$$\Gamma^{\varepsilon,\sigma}\begin{pmatrix} w \\ z \end{pmatrix}=\rho\begin{pmatrix} 1 & U_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} w \\ z \end{pmatrix} \tag{4.7}$$

in order to determine stability of the solution $(u(\cdot, \varepsilon, \sigma), v(\cdot, \varepsilon, \sigma))$ of (1.1), where

$$\Gamma^{\varepsilon,\sigma}=\begin{pmatrix} L^{\varepsilon,\sigma} & N^{\varepsilon,\sigma} \\ g_u & M^{\varepsilon,\sigma} \end{pmatrix}$$

$$L^{\varepsilon,\sigma}w=\varepsilon^2w''+f_u(u(\cdot, \varepsilon, \sigma), v(\cdot, \varepsilon, \sigma))w$$

$$N^{\varepsilon,\sigma}z=L^{\varepsilon,\sigma}(U_1z)+f_v(u(\cdot, \varepsilon, \sigma), v(\cdot, \varepsilon, \sigma))z$$

$$g_u=g_u(u(\cdot, \varepsilon, \sigma), v(\cdot, \varepsilon, \sigma))$$

$$M^{\varepsilon,\sigma}z=\sigma^{-1}z''+[g_u(u(\cdot, \varepsilon, \sigma), v(\cdot, \varepsilon, \sigma))U_1+g_v(u(\cdot, \varepsilon, \sigma), v(\cdot, \varepsilon, \sigma))]z\,.$$

In order to indicate the linearization around the true solution $(u(\cdot, \varepsilon, \sigma), v(\cdot, \varepsilon, \sigma))$, we use in this section the same notation $L^{\varepsilon,\sigma}$, $M^{\varepsilon,\sigma}$, etc. as those around the approximate solution $(U(\cdot, \varepsilon, \sigma), V(\cdot, \varepsilon, \sigma))$. No confusion should arise in this regard.

The equation (4.7) now reads

(4.8) $$(L^{\varepsilon,\sigma}-\rho)w+(N^{\varepsilon,\sigma}-\rho U_1)z=0\,, \qquad (M^{\varepsilon,\sigma}-\rho)z+g_u w=0\,.$$

From (4.5) and Corollary 3.2, it follows that Lemma 3.1 remains true for $L^{\varepsilon,\sigma}$ above and that Lemmas 3.3, 3.5 and 3.6 are valid for $M^{\varepsilon,\sigma}$ and $N^{\varepsilon,\sigma}$. Following the same line of analysis as that applied to $\Gamma(\varepsilon, \sigma)$, we obtain

(4.9.a) $$\alpha(\lambda_1-\rho)-\alpha\langle(M^{\varepsilon,\sigma}-\rho)^{-1}g_u\phi_1, \bar{U}_1\phi_1\rangle-\langle(M^{\varepsilon,\sigma}-\rho)^{-1}g_u\bar{w}, \bar{U}_1\phi_1\rangle=0\,.$$

(4.9.b) $$(L^{\varepsilon,\sigma}-\rho)\bar{w}-Q(N^{\varepsilon,\sigma}-\rho U_1)(M^{\varepsilon,\sigma}-\rho)^{-1}g_u\bar{w}=\alpha Q(N^{\varepsilon,\sigma}-\rho U_1)(M^{\varepsilon,\sigma}-\rho)^{-1}g_u\phi_1\,,$$

where $\bar{U}_1=(\lambda_1-\rho)U_1+f_v(u, v)$. By virtue of Lemma 3.5 (i), there exists a constant $\delta_0>0$, which is independent of $(\varepsilon, \sigma)\in(0, \varepsilon_0]\times(0, \sigma_0]$, such that for $|\rho|\le\delta_0$, the operator on the left of (4.9.b) is invertible uniformly in $(\varepsilon, \sigma)\in(0, \varepsilon_0]\times(0, \sigma_0]$. Therefore, the statement in Theorem B is valid for (4.7) in the region $\{\rho\in C; \operatorname{Re}\rho\ge-\rho_0\}\cap\{\rho\in C; |\rho|\le\delta_0\}$.

4.3. Eigenvalues in $\{\rho\in C; \operatorname{Re}\rho\ge-\rho_0, |\rho|>\delta_0\}$. We first cite the following theorem.

THEOREM 4.1 ([8, Lemma 2.2]). *For $\rho\in\{\rho\in C; \operatorname{Re}\rho\ge-\rho_0, |\rho|>\delta_0\}$, any C^2-function $P(u, v)$ and $h\in L^2\cap L^\infty$, we have the following convergence*

$$(L^{\varepsilon,\sigma}-\rho)^{-1}P(u(\cdot, \varepsilon, \sigma), v(\cdot, \varepsilon, \sigma))h\to(f_u^\sigma-\rho)^{-1}P^\sigma h \qquad \text{as} \quad \varepsilon\to0$$

in the L^2-sense, where $f_u^\sigma=f(\hat{U}(x, \sigma), \hat{V}(x, \sigma))$ and $P^\sigma=P(\hat{U}(x, \sigma), \hat{V}(x, \sigma))$.

Fujii and Nishiura [8] proved this theorem for $(L^{\varepsilon,\sigma}-\rho)^\dagger$, but their proof works for our situation (even simpler).

The first equation of (4.8) gives

$$\begin{aligned} w&=-(L^{\varepsilon,\sigma}-\rho)^{-1}(N^{\varepsilon,\sigma}-\rho U_1)z=-(L^{\varepsilon,\sigma}-\rho)^{-1}[(L^{\varepsilon,\sigma}-\rho)U_1z+f_vz] \\ &=-U_1z-(L^{\varepsilon,\sigma}-\rho)^{-1}(f_vz)\,. \end{aligned}$$

Then the second of (4.8) yields

$$\sigma^{-1}z''+g_vz-g_u(L^{\varepsilon,\sigma}-\rho)^{-1}(f_vz)=\rho z\,.$$

Multiply this equation by the complex conjugate $\bar{z}$ of z, and integrate the result over $[0, 1]$ to obtain

(4.10) $$-\sigma^{-1}\|z'\|_0^2+\int_0^1 g_v|z|^2dx-\int_0^1 \bar{z}g_u(L^{\varepsilon,\sigma}-\rho)^{-1}(f_vz)dx=\rho\|z\|_0^2\,.$$

Let us split this into real and imaginary parts by setting $\rho=\rho_R+\sqrt{-1}\rho_I, \bar{\rho}=\rho_R-\sqrt{-1}\rho_I$, $z=z_R+\sqrt{-1}z_I$, where ρ_R, $\rho_I\in R$ and z_R, z_I are real-valued functions.

$(4.10)_R$ $$-\sigma^{-1}\|z'\|_0^2+\int_0^1 g_v|z|^2dx-\frac{1}{2}\int_0^1 \{z_Rg_u[(L^{\varepsilon,\sigma}-\rho)^{-1}+(L^{\varepsilon,\sigma}-\bar{\rho})^{-1}](f_vz_R)$$

$$+z_I g_u[(L^{\varepsilon,\sigma}-\rho)^{-1}+(L^{\varepsilon,\sigma}-\bar{\rho})^{-1}](f_v z_I)\}dx+\rho_I\int_0^1\{z_R g_u(L^{\varepsilon,\sigma}-\rho)^{-1}(L^{\varepsilon,\sigma}-\bar{\rho})^{-1}(f_v z_I)$$

$$-z_I g_u(L^{\varepsilon,\sigma}-\rho)^{-1}(L^{\varepsilon,\sigma}-\bar{\rho})^{-1}(f_v z_R)\}dx=\rho_R\|z\|_0^2,$$

$$(4.10)_I \qquad -\rho_I\int_0^1\{z_R g_u(L^{\varepsilon,\sigma}-\rho)^{-1}(L^{\varepsilon,\sigma}-\bar{\rho})^{-1}(f_v z_R)$$

$$+z_I g_u(L^{\varepsilon,\sigma}-\rho)^{-1}(L^{\varepsilon,\sigma}-\bar{\rho})^{-1}(f_v z_I)\}dx-\frac{1}{2}\int_0^1\{z_R g_u[(L^{\varepsilon,\sigma}-\rho)^{-1}$$

$$+(L^{\varepsilon,\sigma}-\bar{\rho})^{-1}](f_v z_I)-z_I g_u[(L^{\varepsilon,\sigma}-\rho)^{-1}+(L^{\varepsilon,\sigma}-\bar{\rho})^{-1}](f_v z_R)\}dx=\rho_I\|z\|_0^2.$$

We normalize z so that $\|z\|_0=1$. Since $\|(L^{\varepsilon,\sigma}-\rho)^{-1}\|\to 0$ and $\rho_I\|(L^{\varepsilon,\sigma}-\rho)^{-1}(L^{\varepsilon,\sigma}-\bar{\rho})^{-1}\|\to 0$ as $|\rho|\to\infty$ inside the region $\{\rho\in C;\, \mathrm{Re}\,\rho\geq-\rho_0\}$, the relations $(4.10)_R$ and $(4.10)_I$ imply the existence of a constant $m_0>0$ such that

$$\rho_R<m_0\,, \quad \text{and} \quad |\rho_I|<m_0$$

where the constant m_0 is independent of $(\varepsilon,\sigma)\in(0,\varepsilon_0]\times(0,\sigma_0]$. These relations also imply that there is a constant $m_1>0$ which is independent of (ε,σ), such that

$$\|z'\|_0\leq m_1\sqrt{\sigma}\,\|z\|_0=m_1\sqrt{\sigma}\,.$$

Let ε tend to zero in $(4.10)_R$ and $(4.10)_I$. By using Theorem 4.1, we obtain

$$(4.11)_R \qquad -\sigma^{-1}\|z'\|_0^2+\int_0^1 g_v^\sigma|z|^2dx-\int_0^1\frac{(f_u^\sigma-\rho_R)g_u^\sigma f_v^\sigma|z|^2}{(f_u^\sigma-\rho_R)^2+\rho_I^2}dx=\rho_R$$

$$(4.11)_I \qquad -\rho_I\int_0^1\frac{g_u^\sigma f_v^\sigma|z|^2}{(f_u^\sigma-\rho_R)^2+\rho_I^2}dx=\rho_I$$

where f_u^σ, g_u^σ etc. are evaluated at $(\hat{U}(x,\sigma),\hat{V}(x,\sigma))$. When $\rho_I=0$, $(4.11)_R$ gives

$$(4.12) \qquad -\sigma^{-1}\|z'\|_0^2+\int_0^1\frac{[\rho^2-(f_u^\sigma+g_v^\sigma)\rho+(f_u^\sigma g_v^\sigma-f_v^\sigma g_u^\sigma)]}{f_u^\sigma-\rho}|z|^2dx=0\,.$$

One should notice that $(f_u^\sigma+g_v^\sigma)<0$, $(f_u^\sigma g_v^\sigma-f_v^\sigma g_u^\sigma)>0$ from the conditions (A.3), (A.4) and (A.5). It is therefore easy to find a constant $\rho_I>0$ such that the integrand of the second term in (4.12) is strictly negative for $\rho\geq-\rho_1$. This means that real eigenvalues in the region $\{\rho\in C;\, \mathrm{Re}\,\rho\geq-\rho_0,\, |\rho|>\delta_0\}$ have to satisfy $\rho<-\rho_1$.

When $\rho_I\neq 0$, the relation $(4.11)_I$ gives

$$\int_0^1\frac{g_u^\sigma f_v^\sigma|z|^2}{(f_u^\sigma-\rho_R)^2+\rho_I^2}dx=-1$$

which together with $(4.11)_R$ gives rise to

$$(4.13)\quad 2\rho_R\left[1+\int_0^1 \frac{f_u^\sigma g_v^\sigma |z|^2}{(f_u^\sigma-\rho_R)^2+\rho_I^2}dx\right]=-\sigma^{-1}\|z'\|_0^2+\int_0^1 \frac{f_u^\sigma \det^\sigma+g_v^\sigma|\rho|^2}{(f_u^\sigma-\rho_R)^2+\rho_I^2}|z|^2dx\,.$$

where $\det^\sigma=f_u^\sigma g_v^\sigma-f_v^\sigma g_u^\sigma$. Since $f_u^\sigma g_v^\sigma\geq 0, f_u^\sigma \det^\sigma<0$ and $g_v^\sigma\leq 0$ on account of the conditions (A.3), (A.4) and (A.5), the relation (4.13) gives a constant $\rho_2>0$ independent of σ such that $\rho_R\leq-\rho_2$. We complete the proof of Theorem A (ii) by taking $\rho^*=\min\{\rho_0, \rho_1, \rho_2\}$.

5. Stability analysis for multiple transition layers. Once we know the existence of single transition layer solutions of the problem

$$(\mathrm{P})_{\varepsilon,p}\qquad \varepsilon^2u''+f(u,v)=0\,,\quad \sigma^{-1}v''+g(u,v)=0\,,\qquad \text{for}\quad x\in(0,1)\,,$$
$$u'=0=v'\quad\text{at}\quad x=0,1$$

then the folding-up principle gives families of solutions with multiple transition layers. To be more precise, let us assume $(u(x), v(x))$ is a solution of $(\mathrm{P})_{\bar\varepsilon,\bar\sigma}$. For each positive integer n, the pair of functions $(u_n(x), v_n(x))$ defined by

$$(5.1)\qquad u_n(x)=T^nu(x)\,,\qquad v_n(x)=T^nv(x)$$

solves the problem $(\mathrm{P})_{\varepsilon,\sigma}$ with $\varepsilon=\bar\varepsilon/n$, $\sigma=n^2\bar\sigma$, where for $x\in[i/n,(i+1)/n]$

$$T^nw(x)=w(nx-i)\,,\quad\text{if}\quad i\quad\text{is even}\,,\qquad T^nw(x)=w(i+1-nx)\,,\quad\text{if}\quad i\quad\text{is odd}\,.$$

Let $(u(x,\bar\varepsilon,\bar\sigma), v(x,\bar\varepsilon,\bar\sigma))$ be the family of solutions with single transition layer given in Theorem A. Then $(u_n(x,\varepsilon,\sigma), v_n(x,\varepsilon,\sigma))$ defined by (5.1) is a family of solutions of $(\mathrm{P})_{\varepsilon,\sigma}$ with n internal transition layers where $\varepsilon=\bar\varepsilon/n$, $\sigma=n^2\bar\sigma$. It is the main purpose of this section to determine the stability property of (u_n, v_n) as an equilibrium solution of the parabolic equation (1.1). We prove the following:

THEOREM C. *If $J'(v^*)$ is negative (positive), the solution $(u_n(\cdot,\varepsilon,\sigma), v_n(\cdot,\varepsilon,\sigma))$ of $(\mathrm{P})_{\varepsilon,\sigma}$ is stable (resp. unstable with index n), for $\varepsilon\in(0,\varepsilon(\sigma)]$, $\sigma\in(0,\sigma_0]$, as an equilibrium solution of* (1.1), *where $\varepsilon(\sigma)$ is a continuous function such that $\varepsilon(\sigma)\to 0$ as $\sigma\to 0$.*

Let $\Gamma^{\varepsilon,\sigma}: H_N^2\times H_N^2\to L^2\times L^2$ be given by

$$\Gamma^{\varepsilon,\sigma}=\begin{pmatrix} L^{\varepsilon,\sigma} & N^{\varepsilon,\sigma}\\ g_u & M^{\varepsilon,\sigma}\end{pmatrix}$$

where

$$L^{\varepsilon,\sigma}w=\varepsilon^2w''+f_u(u_n(\cdot,\varepsilon,\sigma), v_n(\cdot,\varepsilon,\sigma))w\,,\qquad N^{\varepsilon,\sigma}z=L^{\varepsilon,\sigma}(U_nz)+f_v(u_n,v_n)z$$

$$g_u=g_u(u_n(\varepsilon,\sigma), v_n(\varepsilon,\sigma))\,,\qquad M^{\varepsilon,\sigma}z=\sigma^{-1}z''+[g_u(u_n,v_n)U_n+g_v(u_n,v_n)]z$$

$$U_n=U_n(x,\varepsilon,\sigma)=[T^nU_1(\cdot,\bar\varepsilon,\bar\sigma)](x)\,.$$

We also defined

$$x_1^*(\sigma)=x^*(n^{-2}\sigma)/n, \quad \text{and} \quad x_j^*(\sigma)=j/n-x_1^*(\sigma), \qquad \text{for } j \text{ even}, \quad 2\le j\le n,$$

$$x_j^*(\sigma)=(j-1)/n+x_1^*(\sigma), \qquad \text{for } j \text{ odd}, \quad 3\le j\le n.$$

The function $u_n(x, \varepsilon, \sigma)$ exhibits an internal transition layer at each point $x=x_j^*(\sigma)$, $j=1, \cdots, n$, as $\varepsilon\to 0$.

LEMMA 5.1. *Let* $[\phi_j(\cdot, \varepsilon, \sigma), \lambda_j(\varepsilon, \sigma)]_{j=1}^{\infty}$ *be a complete orthonormal system of eigenfunctions and eigenvalues of* $L^{\varepsilon,\sigma}$ *such that* $\lambda_1>\lambda_2>\cdots, \lambda_m\to-\infty$ *as* $m\to\infty$.

(i) *There exists a constant* $\lambda_0>0$ *such that*

$$\lambda_{n+1}(\varepsilon, \sigma)<-\lambda_0, \quad (\varepsilon, \sigma)\in(0, \varepsilon_0/n]\times(0, n^2\sigma_0].$$

(ii) $\lim_{\varepsilon\to 0}\lambda_j(\varepsilon, \sigma)=0$ *uniformly in* $\sigma\in(0, n^2\sigma_0]$, $j=1, \cdots, n$.

(iii) *There exist constants* $k>0$ *and* $\beta>0$ *such that*

$$|\phi_j(x, \varepsilon, \sigma)|\le k|\phi_j(x_l^*(\sigma), \varepsilon, \sigma)|\exp[-\beta|x-x_l^*(\sigma)|\varepsilon^{-1}]$$

for $x\in[(l-1)/n, l/n]$, $j=1, \cdots, n$.

(iv) *There exist constants* $a_{jl}\in R$, j, $l=1, \cdots, n$, *such that as* $\varepsilon\to 0$

$$\varepsilon^{1/2}\phi_j(\varepsilon\eta+x_l^*(\sigma), \varepsilon, \sigma)\to a_{jl}K\dot{z}_0(\eta) \qquad \text{in} \quad C^2_{loc}(R)$$

uniformly in $\sigma\in(0, n^2\sigma_0]$. *Moreover, the matrix* $A=(a_{ij})_{i,j=1}^n$ *is orthogonal.*

(v) *The following limits exist*:

$$\lim_{\varepsilon\to 0}\frac{\lambda_j(\varepsilon, \sigma)}{\varepsilon}=\frac{\sigma}{n}K_*^2J'(v^*)\int_0^{x^*(\sigma n^{-2})}g(h_-(\hat{V}(x, \sigma/n^2)), \hat{V}(x, \sigma/n^2))dx=:\hat{\lambda}(\sigma)$$

for $j=1, \cdots, n$.

For the proof of this lemma we refer to [11, Lemmas 5.1, 5.2 and 5.3].

LEMMA 5.2. *There exists a constant* $\mu_0>0$ *such that the principal eigenvalue of* $M^{\varepsilon,\sigma}$ *satisfies* $\mu_1(\varepsilon, \sigma)<-\mu_0$ *for* $(\varepsilon, \sigma)\in(0, \varepsilon_0/n]\times(0, n^2\sigma_0]$.

The proof of this lemma is nearly identical to that of Lemma 3.3, and hence omitted.

In order to determine the stability property of the solution (u_n, v_n), we examine the eigenvalue problem

$$\Gamma^{\varepsilon,\sigma}\begin{pmatrix} w \\ z \end{pmatrix}=\rho\begin{pmatrix} 1 & U_n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} w \\ z \end{pmatrix} \tag{5.2}$$

which is equivalent to

$$(L^{\varepsilon,\sigma}-\rho)w+(N^{\varepsilon,\sigma}-\rho U_n)z=0, \quad (M^{\varepsilon,\sigma}-\rho)z+g_u w=0. \tag{5.3}$$

We are interested in the eigenvalues ρ of (5.2) in the region $C_{\rho_0}:=\{\rho\in C; \operatorname{Re}\rho>\rho_0\}$ with $\rho_0=\min\{\lambda_0, \mu_0\}$.

For $\rho \in C_{\rho_0}$, Lemma 5.2 implies that $(M^{\varepsilon,\sigma}-\rho)^{-1}$ is bounded uniformly in ρ, $(\varepsilon, \sigma) \in (0, \varepsilon_0/n] \times (0, n^2\sigma_0]$, hence the second of (5.3) gives

$$z = -(M^{\varepsilon,\sigma}-\rho)^{-1}g_u w = -(M^{\varepsilon,\sigma}-\rho)^{-1}g_u\left(\sum_{j=1}^{n} \alpha_j\phi_j + \bar{w}\right)$$

where $\alpha_j \in C$, $j=1, \cdots, n$, $\langle\phi_j, \bar{w}\rangle = 0$, $j=1, \cdots, n$. By using decomposition $L^2 = [\phi_1, \cdots, \phi_n] \oplus QL^2$ and integration by parts, we obtain from (5.3)

$$\alpha_i(\lambda_i-\rho) - \sum_{j=1}^{n} \alpha_j\langle(M^{\varepsilon,\sigma}-\rho)^{-1}g_u\phi_j, \bar{U}_{n,i}\phi_i\rangle - \langle(M^{\varepsilon,\sigma}-\rho)^{-1}g_u\bar{w}, \bar{U}_{ni}\phi_i\rangle, \quad i=1, \cdots, n, \tag{5.4}$$

$$(L^{\varepsilon,\sigma}-\rho)\bar{w} - Q(N^{\varepsilon,\sigma}-\rho U_n)(M^{\varepsilon,\sigma}-\rho)^{-1}g_u\bar{w} = \sum_{j=1}^{n} \alpha_j Q(M^{\varepsilon,\sigma}-\rho U_n)(N^{\varepsilon,\sigma}-\rho)^{-1}g_u\phi_j \tag{5.5}$$

where $\bar{U}_{n,i} = (\lambda_i-\rho)U_n + f_v(u_n, v_n)$, $i=1, \cdots, n$. Since $\|N^{\varepsilon,\sigma}(N^{\varepsilon,\sigma}-\rho)^{-1}g_u\cdot\|_{L^2\to L^2} = O(\varepsilon^{1/2})$ as $\varepsilon\to 0$ (see Lemma 3.5 (i)), there is a constant $\delta_0>0$ such that for ρ in $\{\rho\in C; |\rho|<\delta_0\} \cap C_{\rho_0}$ the operator on the left of (5.5) is uniformly invertible. For eigenvalues ρ of (5.2) in $\{|\rho|>\delta_0\} \cap C_{\rho_0}$, we can follow the procedure in section 4.3 to show that there is a positive constant $\bar{\rho}_0$ such that $\operatorname{Re}\rho \leq -\bar{\rho}_0$. We therefore concentrate on the eigenvalues in $\{\rho\in C; |\rho|<\delta_0\} \cap C_{\rho_0}$. For such ρ, the equation (5.5) can be solved in $\bar{w}$ as a function of $(\alpha_1, \cdots, \alpha_n)$

$$\bar{w} = K^{\varepsilon,\sigma,\rho}\left(\sum_{j=1}^{n} \alpha_j\phi_j\right)$$

which together with (5.4) gives an equation for ρ to satisfy as an eigenvalue of (5.2):

$$\det[\operatorname{diag}(\lambda_1-\rho, \cdots, \lambda_n-\rho) - \varepsilon\Phi^0(\varepsilon, \sigma) - \varepsilon^{3/2}\Phi^1(\varepsilon, \sigma)] = 0 \tag{5.6}$$

where $\Phi^0 = (\Phi^0_{ij})^n_{i,j=1}$, $\Phi^1 = (\Phi^1_{ij})^n_{i,j=1}$ are matrices defined by

$$\Phi^0_{ji} = \varepsilon^{-1}\langle(M^{\varepsilon,\sigma}-\rho)^{-1}g_u\phi_j, \bar{U}_{n,i}\phi_i\rangle,\ \Phi^1_{ji} = \varepsilon^{-3/2}\langle(M^{\varepsilon,\sigma}-\rho)^{-1}g_uK^{\varepsilon,\sigma,\rho}\phi_j, \bar{U}_{n,i}\phi_i\rangle.$$

It follows from the result in section 3 that Φ^0, $\Phi^1 = O(1)$ as $\varepsilon\to 0$. Since $\lambda_j(\varepsilon, \sigma) = O(\varepsilon)$ as $\varepsilon\to 0$, $j=1, \cdots, n$, the equation (5.6) shows that $\rho = O(\varepsilon)$. Let us set $\rho(\varepsilon, \sigma) = \varepsilon\hat{\rho}(\varepsilon, \sigma)$, $\lambda_j(\varepsilon, \sigma) = \varepsilon\hat{\lambda}_j(\varepsilon, \sigma)$, $j=1, \cdots, n$, which reduces (5.6) to

$$\det[\operatorname{diag}(\hat{\lambda}_1-\hat{\rho}, \cdots, \hat{\lambda}_n-\hat{\rho}) - \Phi^0(\varepsilon, \sigma) - \varepsilon^{1/2}\Phi^1(\varepsilon, \sigma)] = 0. \tag{5.7}$$

To obtain an estimate for $\hat{\rho}(0, \sigma)$, we first compute Φ^0_{ij}. Notice that $\lim_{\varepsilon\to 0}\hat{\lambda}_j(\varepsilon, \sigma) = \hat{\lambda}(\sigma)$, $j=1, \cdots, n$, from Lemma 5.1 (v). A computation similar to that which follows Lemma 3.7 gives

$$\lim_{\varepsilon\to 0}\Phi^0_{ji}(\varepsilon, \sigma) = C_* \sum_{k,m=1}^{n} a_{jk}a_{im}\langle(M^{0,\sigma})^{-1}\delta^*_k, \delta^*_m\rangle$$

where $C_* = K_*^2 J_v(v^*)[G_+(v^*) - G_-(v^*)]$ and δ_k^* represents the Dirac δ-function at $x = x_k^*(\sigma)$, and $\langle (M^{0,\sigma})^{-1}\delta_k^*, \delta_m^*\rangle$ is the duality pairing in $H^1 \times H^{-1}$. Therefore $\hat{\rho}(0, \sigma)$ satisfies

$$\det[\hat{\lambda}(\sigma) - \hat{\rho})1_n - A^T M(\sigma) A] = 0$$

where $M(\sigma) = M_{ij}(\sigma))_{i,j=1}^n$ is a matrix defined by

$$M_{ij}(\sigma) = C_* \langle (M^{0,\sigma})^{-1}\delta_i^*, \delta_j^*\rangle .$$

Since the matrix A is an orthogonal matrix, the equation for $\hat{\rho}$ is equivalent to

$$\det[\hat{\lambda}(\sigma) - \hat{\rho})1_n - M(\sigma)] = 0 . \tag{5.8}$$

For the equation (5.8), Fujii and Nishiura [16] gives the following:

LEMMA 5.3. *There exist n continuous functions $\tau_j^n(\sigma)$, $j = 1, \cdots, n$, of $\sigma \in (0, n^2\sigma_0]$ such that*

$$\tau_1^n(\sigma) > \tau_2^n(\sigma > \cdots > \tau_n^n(\sigma) > 0, \ |\hat{\lambda}(\sigma)| < |J'(v^*)\tau_n^n(\sigma)|$$

and the solution $\hat{\rho}$ of (5.8) are given by

$$\hat{\rho}(\sigma) = \hat{\lambda}(\sigma) + J'(v^*)\tau_j^n(\sigma) , \qquad j = 1, \cdots, n .$$

Moreover,

$$\lim_{\sigma \to 0} \tau_j^n(\sigma) = 0 , \quad j = 2, \cdots, n , \qquad \lim_{\sigma \to 0} \tau_1^n(\sigma) = \gamma_* K_*^2 [G_+(v^*) - G_-(v^*)]$$

where

$$-\frac{1}{\gamma_*} = x^*(0)[[f_u g_v - f_v g_u]/f_u]_{v=v^*, u=h_-(v^*)} + (1 - x^*(0))[[f_u g_v - f_v g_u]/f_u]_{v=v^*, u=h_+(v^*)} .$$

This completes the proof of Theorem C.

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