Construction of approximative and almost periodic solutions of perturbed linear Schrödinger and wave equations

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Août 1995

IHES/M/95/80

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Summary.

Consider $1D$ nonlinear Schrödinger equation

$$i\,u_t - u_{xx} + V(x)\,u + \varepsilon\,\frac{\partial H}{\partial \overline{u}} = 0 \tag{0.1}$$

and nonlinear wave equation

$$y_{tt} - y_{xx} + \rho\,y + \varepsilon\,F'(y) = 0 \tag{0.2}$$

under Dirichlet boundary conditions. We assume here $H(u,\overline{u})$ and $F(y)$ polynomials.

It is proved that for "typical" periodic potential V in $(0,1)$ and typical $\rho \in \mathbf{R}$ in (0.2) the following is true. Let $u(0)$ (resp. $y(0), y'(0)$) be smooth initial data for $t=0$. Then the corresponding solution $u(t)$ of (0.1) (resp. $y(t)$ of (0.2)) will be ε^M-close to a quasi-periodic function of time, for times $|t| < \varepsilon^{-M}$ where M may be any choosen number (letting $\varepsilon \to 0$) (see Proposition 4.18 and Proposition 5.13)(**).

In the second part of the paper, we use the technique from [B] (see also references in [B] on earlier works such as [C-W]) to construct almost periodic (in time) solutions of say a wave equation

$$y_{tt} - y_{xx} + V(x)\,y + \varepsilon\,F'(y) = 0$$

under Dirichlet boundary conditions. Here V is a "typical" real analytic periodic potential. The frequencies of these solutions form a full set, i.e. $\lambda'_j \approx \lambda_j = \sqrt{\mu_j}$

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(**) This result may be seen as a Nekhovoshev type result (cf. [N]) for Hamiltonian PDE, in the nonresonant regime (which is the easiest to study). In this spirit, results in finite dimensional phase space have been obtained by various authors, but for different interactions, essentially finite range, which does not cover natural PDE models. See for instance [B-F-G].

where $\{\mu_j\}$ is the Dirichlet spectrum of $-\frac{d^2}{dx^2} + V(x)$. However, they are obtained starting from an unperturbed solution $u_0(x,t) = \sum_{j=1}^{\infty} a_j \cos \lambda_j t \cdot \varphi_j(x)$, subject to a strong decay assumption $|a_j| \to 0$ on the initial amplitudes $\{a_j\}$. The argument would need to be considerably refined to reach a more realistic decay. Again, the construction of invariant tori of infinite dimension (via usual KAM techniques) is achieved for certain models with finite range interactions (see [F-S-W]). There are also the results of [C-P], but they require a very rapidly increasing frequency sequence $\{\lambda_j\}$.

Acknowledgement. The author benefitted from several discussions with S. Kuksin related to the first part of the paper.

1. Approximative solution of equation.

Consider an equation of the form (written as perturbed linear Schrödinger equation)

$$i\, u_t + A\, u + \varepsilon \frac{\partial H}{\partial \overline{u}} = 0 \tag{1.1}$$

where $H = H(u, \overline{u})$ is a polynomial in $u, \overline{u}$ with real coefficients.

Here A is a selfadjoint matrix with spectrum $\lambda_j \simeq j^2$ and certain diophantine properties (to be specified). Let $\{\varphi_j\}$ be the eigenfunction basis which we assume well-localized wrt the system of exponentials.

Fix modes $1, 2, \ldots, j_0$ and $j_1 > j_0$. Let r be a fixed positive integer.

Let $a_1, \ldots, a_{j_0}, a_{j_1} \in \mathbf{R}_+^*$ and denote $a = (a_1, \ldots, a_{j_0}, a_{j_1})$, $\underline{a} = (a_1, \ldots, a_{j_0})$. Eventually j_0 and $\underline{a}$ will be fixed ($\underline{a}$ with first decay estimates) and $a_{j_1} \to 0$.

We construct an "approximative" solution u_a of (1) which is quasi-periodic in time with frequencies

$$\lambda' = (\lambda'_1, \ldots, \lambda'_{j_0}, \lambda'_{j_1}) \,. \tag{1.2}$$

This will essentially be achieved by a finite expansion in an ε series.

We first construct a periodic function $F = F^a(\theta_1, \ldots, \theta_{j_0}, \theta_{j_1})$ on $\mathbf{T}^{j_0+1}$ and $\lambda' = \lambda'(a)$ satisfying

$$i\,\langle \lambda', \nabla F\rangle + A\,F + \varepsilon \frac{\partial H}{\partial \overline{u}}\,(F, \overline{F}) = \mathcal{E}_a \tag{1.3}$$

where $\mathcal{E}_a = \mathcal{E}_a(\theta)$ will satisfy in particular ($r_1 \sim r$)

$$\|\mathcal{E}_a\| < \varepsilon^{r_1} + 0(|a_{j_1}|^2) \tag{1.4}$$

$$||\partial^{\alpha}_{a_{j_1}}\,\partial^{\beta}_{\underline{a}}\,\partial^{\gamma}_{\lambda'}\,\mathcal{E}_a|| < \varepsilon^{r_1} + 0(|a_{j_1}|^{2-\alpha}) \tag{1.5}$$

for $\alpha \leq 2$ and β, γ bounded

$$\partial_{\theta_j}\,\mathcal{E}_a\,|_{a_j=0} = 0\,. \tag{1.6}$$

Assume following properties satisfied for the spectrum of A

$$||k_1\,\lambda_1 + \cdots + k_{j_0}\,\lambda_{j_0} + k_{j_1}\,\lambda_{j_1} + k'_{j_2}\,\lambda_{j_2}| > \varepsilon^{1/10} \tag{1.7}$$

for $|k_1| + \cdots + |k_{j_0}| < r$, $|k_{j_1}| \leq 1$, $|k'_{j_2}| \leq 1$ (j_2 arbitrary) excluding the resonnant cases

$$\sum_{j=1}^{j_0} |k_j| + |k_{j_1}| + |k'_{j_2}| = 0 \tag{1.8}$$

and

$$j_2 \in \{1, \ldots, j_0, j_1\}\ ,\ k'_{j_2} = -k_{j_2}\ ,\ k_j = 0 \quad \text{for} \quad j \neq j_2\,. \tag{1.9}$$

Observe that one has $\sum\limits_{j=1}^{j_0} |k_j|\,|\lambda_j| \lesssim r\,j_0^2$ and for $j_1 \neq j_2$ sufficiently large $|\lambda_{j_1} - \lambda_{j_2}| > \max(|j_1|, |j_2|)$, from the spectral asymptotics. Hence (1.7) need only be verified for $|j_1|, |j_2| \lesssim j_0^2\,r$ and this leads clearly to $j_0^r\,(j_0^2\,r)^2 < j_0^{2r}$ conditions. Thus the expected restriction on j_0 and r is of the form

$$r\,\log j_0 \ll \log \frac{1}{\varepsilon}\,. \tag{1.10}$$

The new frequency $\lambda' = (\lambda'_1, \ldots, \lambda'_{j_0}, \lambda'_{j_1}) = \lambda'(a)$ will satisfy

$$|\lambda'_j - \lambda_j| \leq \varepsilon \quad \text{for} \quad j \in \{1, \ldots, j_0, j_1\}\,. \tag{1.11}$$

Hence (1.7) will remain valid after replacement of $\lambda_1, \ldots, \lambda_{j_0}, \lambda_{j_1}$ by λ'.

We will define

$$u_{a,\theta}(t) = F^a(\theta_1 + \lambda'_1\,t, \ldots, \theta_{j_0} + \lambda'_{j_0}\,t\ ,\ \theta_{j_1} + \lambda'_{j_1}\,t)$$

that will clearly satisfy

$$i\,\dot{u}_{a,\theta} + A\,u_{a,\theta} + \varepsilon\,\frac{\partial H}{\partial\,\overline{u}}\,(u_{a,\theta}, \overline{u}_{a,\theta}) = \mathcal{E}_a(\theta_1 + \lambda'_1\,t, \ldots, \theta_{j_0} + \lambda'_{j_0}\,t, \theta_{j_1} + \lambda'_{j_1}\,t)\,. \tag{1.12}$$

We construct F^a as a standard perturbation series in ε truncated to order $r_1 \sim r$.

Rewrite the equation

$$(-\langle\lambda', k\rangle + \lambda_j)\ \widehat{F}_j(k) + \varepsilon\,\widehat{\frac{\partial H}{\partial\,\overline{u}_j}}\,(k) = 0\,. \tag{1.13}$$

Here $k = (k_1, \ldots, k_{j_0}, k_{j_1})$. Recall that H is a polynomial in $u, \overline{u}$ of degree d.

We delete the set $\mathcal{R}$ of resonnant sites $(j,k) = (j, e_j)$ for $j \in \{1, \ldots, j_0, j_1\}$ and e_j the j-unit vector in $\mathbb{Z}^{j_0+1}$ (they correspond to the Q-equations). Define for $(j,k) \in \mathcal{R}$

$$\widehat{F}_j\,(e_j) = a_j \qquad j \in \{1, \ldots, j_0, j_1\}\,. \tag{1.14}$$

Put also

$$\widehat{F}_j\,(k) = 0 \quad \text{if} \quad |k| > r+1 \quad \text{or} \quad |k_{j_1}| > 2\,. \tag{1.15}$$

Determine remaining $\widehat{F}_j\,(k)$ inductively as series in ε

$$\widehat{F}_j\,(k) = \sum_s F_s(j,k)\,\varepsilon^s \tag{1.16}$$

using equation (1.13) and starting from

$$\begin{aligned} F_j &= a_j\, e^{i\theta_j} \quad \text{for} \quad j \in \{1, \ldots, j_0, j_1\} \\ &= 0 \quad \text{otherwise}\,. \end{aligned} \tag{1.17}$$

Thus

$$F_{s+1}\,(j,k) = \frac{1}{\langle \lambda', k\rangle - \lambda_j}\, \frac{1}{s!}\, \frac{\partial^s}{\partial\,\varepsilon^s} \left[\widehat{\frac{\partial H}{\partial\,\overline{u}_j}}\, (F_{(s)}, \overline{F}_{(s)}) \right] \Bigg|_{\varepsilon=0} \tag{1.18}$$

where

$$F_{(s)} = \sum_{j,k,s' \leq s} \varepsilon^{s'}\, F_{s'}\,(j,k)\, \varphi_j(x)\, e^{i\langle k,\theta\rangle}\,. \tag{1.19}$$

By induction, it is easily seen that in $F_s(j,k)$ the multi index k satisfies $|k| \leq s(d-1)$ where d is the degree of H. Hence, the first constraint in (1.15) may be ignored up to order $r_1 \sim \frac{r}{d-1}$.

By (1.7), the divisor in (1.18) remains at least $\varepsilon^{1/10}$. By induction, one gets therefore bounds

$$|F_s(j,k)| \leq \sum_{j,k} |F_s(j,k)| \leq \|F_s\| < s^{2sd}\, \varepsilon^{-\frac{s}{10}}\,. \tag{1.20}$$

Similar bounds hold for derivatives in a and λ' of bounded order (only order $= 2$ will be considered). We disregard at this point the dependence of λ' on a which will be established afterwards. From (1.20), the s-term in (1.16) is controlled by $\varepsilon^{s/2}$.

By construction, equation (1.13) will be satisfied up to an error of order $\varepsilon^{r_1} + 0(|a_{j_1}|^2)$, except for $(j,k) \in \mathcal{R}$. The $|a_{j_1}|^2$-term comes from the second restriction $|k_{j_1}| \leq 1$ in (1.15).

Projecting (1.13) on $\mathcal{R}$, one determines the new frequencies $\lambda' = (\lambda'_1, \dots, \lambda'_{j_0}, \lambda'_{j_1})$, thus

$$\lambda'_j = \lambda_j + \frac{\varepsilon}{a_j} \widehat{\frac{\partial H}{\partial \overline{u}_j}} (e_j) \qquad j = 1, 2, \dots, j_0, j_1 \tag{1.21}$$

solved as implicit equation in λ' as function of a (λ' is real).

Observe that in particular $\widehat{\frac{\partial H}{\partial \overline{u}_j}} (e_j) \mid_{a_j=0} = 0$ so that the dependence of λ' on a remains smooth up to 0. In particular, for $a_{j_1} \to 0$, we have

$$\lambda'_{j_1}(\underline{a}) = \lambda_{j_1} + \varepsilon \, \partial_{a_{j_1}} \widehat{\frac{\partial H}{\partial \overline{u}_j}} (e_j) \mid_{a_{j_1}=0} . \tag{1.22}$$

From the preceding, the function $u_{a,\theta}$ introduced above has a smooth dependence on a and satisfies equation (1.12) with right member error term

$$0\,(\varepsilon^{r_1} + |a_{j_1}|^2) . \tag{1.23}$$

this approximative solution $u_{\theta,a}$ has the form

$$u_{\theta,a} = \sum_{j=1,\dots,j_0,j_1} a_j \, e^{i\theta_j} \, e^{i\lambda'_j t} \, \varphi_j(x) + \varepsilon^{1/2} \, u'_{\theta,a} \tag{1.24}$$

with in particular

$$\frac{\partial \, u'_{\theta,a}}{\partial \, \theta_j} \mid_{a_j=0} = 0 . \tag{1.25}$$

One has

$$\left| \frac{\partial \, \lambda'_j}{\partial \, a_k} \right| < \varepsilon \tag{1.26}$$

$$\frac{\partial \, \lambda'_j}{\partial \, a_{j_1}} \mid_{a_{j_1}=0} = 0 . \tag{1.27}$$

To verify (1.27), distinguish the cases $j = 1, \dots, j_0$ and $j = j_1$. Use the fact that for fixed λ', $\widehat{F}^a(k)$ is even (resp. odd) in a_{j_1} if $k_{j_1} = 0$ (resp. $k_{j_1} = \pm 1$), as verified from the inductive construction.

2. Linearized equation.

Differentiate (1.24) in θ_j and a_j. Taking (1.25) into account, one gets

$$a_j \; U_{j,a,\theta} = i \; a_j \; e^{i\theta_j} \; e^{i\lambda'_j t} \; \varphi_j(x) + 0\,(\varepsilon^{1/2} \; |a_j|) \tag{2.1}$$

and

$$V_{j,a,\theta} = e^{i\theta_j} \; e^{i\lambda'_j t} \; \varphi_j(x) + 0(\varepsilon^{1/2}) + 0\left(\left| \frac{\partial \, \lambda'}{\partial \, a_j} \right| \; |t| \right) \tag{2.2}$$

satisfying resp.

$$i\,U_t + A\,U + \varepsilon\,\frac{\partial^2 H}{\partial\,\overline{u}_{a,\theta}\,\partial u_{a,\theta}}\,U + \varepsilon\,\frac{\partial^2 H}{\partial\,\overline{u}^2_{a,\theta}}\,\overline{U} =$$

$$\frac{1}{a_j}\,\frac{\partial}{\partial\,\theta_j}\,\mathcal{E}_a(\theta_1 + \lambda'_1 t, \ldots, \theta_{j_0} + \lambda'_{j_0} t, \theta_{j_1} + \lambda'_{j_1} t) \tag{2.3}$$

and

$$i\,V_t + A\,V + \varepsilon\,\partial\overline{\partial}\,H\cdot V + \varepsilon\,\overline{\partial}\,\overline{\partial}\,H\cdot\overline{V} =$$

$$\frac{\partial\,\mathcal{E}_a}{\partial\,a_j} + \left\langle \nabla_{\lambda'}\,\mathcal{E}_a, \frac{\partial\,\lambda'}{\partial\,a_j}\right\rangle + \left\langle \nabla_{\theta}\,\mathcal{E}_a, \frac{\partial\,\lambda'}{\partial\,a_j}\right\rangle\,t\,. \tag{2.4}$$

Here the left side is the linearized equation at $u = u_{a,\theta}$.

We take $a_j = c_j \neq 0$ for $j = 1, \ldots, j_0$ (fixed) and let $a_{j_1} \to 0$ for $j_1 > j_0$ varying over all $j > j_0$. Denote $u_{\theta,(c,0)}$ by u_θ and denote $\lambda'_j = \lambda'_j(c_1, \ldots, c_{j_0})$ for all $j = 1, \ldots, j_0 \cdot j_0 + 1, \ldots$.

(i) *Case* $j = 1, \ldots, j_0$

By (1.26)

$$U_{j,\theta} = i\,e^{i\theta_j}\,e^{i\lambda'_j t}\,\varphi_j + 0(\varepsilon^{1/2}) \tag{2.5}$$

$$V_{j,\theta} = e^{i\theta_j}\,e^{i\lambda'_j t}\,\varphi_j + 0(\varepsilon^{1/2}) + 0(\varepsilon\,|t|) \tag{2.6}$$

satisfying an approximative linearized equation at $u = u_\theta$

$$i\,U_t + A\,U + \varepsilon\,\partial\overline{\partial}\,H\cdot U + \varepsilon\,\overline{\partial}^2\,H\,\overline{U} = \mathcal{E}_j(\theta, t) \tag{2.7}$$

where error term $\mathcal{E}_j$ satisfying by (1.5), (1.6)

$$\|\mathcal{E}_j\| < \varepsilon^{r_1}\,(1 + |t|)\,. \tag{2.8}$$

(ii) *Case* $j = j_1$

$$U_{j_1,\theta} = i\,e^{i\theta_{j_1}}\,e^{i\lambda'_{j_1} t}\,\varphi_{j_1} + 0(\varepsilon^{1/2}) \tag{2.9}$$

$$V_{j_1,\theta} = e^{i\theta_{j_1}}\,e^{i\lambda'_{j_1} t}\,\varphi_{j_1} + 0(\varepsilon^{1/2}) \tag{2.10}$$

by (1.27) and satisfying (2.7) with error term $\mathcal{E}_j$ satisfying

$$\|\mathcal{E}_j\| < \varepsilon^{r_1} \tag{2.11}$$

invoking (1.5), (1.6), (1.27).

Thus for $j > j_0$, the error terms remain small for all time. For $j = 1, \dots, j_0$, there is a linear growth in $|t|$.

Observe that the expression (linearization at $u = u_\theta$)

$$i\,U_t + A\,U + \varepsilon\,\partial\overline{\partial} H \cdot U + \varepsilon\,\overline{\partial}^2\,H \cdot \overline{U} \tag{2.12}$$

is $\mathbf{R}$-linear in U.

Fixing θ, define

$$\begin{cases} U_j = \cos\theta_j \cdot V_{j,\theta} - \sin\theta_j\;U_{j,\theta} = e^{i\lambda_j' t}\;\varphi_j + 0(\varepsilon^{1/2}) + 0(\varepsilon\,|t|) & (2.13) \\ V_j = \sin\theta_j \cdot V_{j,\theta} + \cos\theta_j\;U_{j,\theta} = i\;e^{i\lambda_j' t}\;\varphi_j + 0(\varepsilon^{1/2}) + 0(\varepsilon\,|t|)\,. & (2.14) \end{cases}$$

From the preceding $\{(U_j, V_j)\ |_{j=1,2,\dots,j_0,j_0+1,\dots}\}$ are approximative solutions of (2.12) $= 0$, in the sense that the equation is satisfied up to error $(1 + |t|)\;\varepsilon^{r_1}$.

For $t = 0$, one has

$$U_j^0 = U_j(x, 0) = \varphi_j(x) + 0(\varepsilon^{1/2}) \tag{2.15}$$

$$V_j^0 = V_j(x, 0) = i\;\varphi_j(x) + 0(\varepsilon^{1/2}) \tag{2.16}$$

and hence a perturbation of the sequence $\{(\varphi_j, i\,\varphi_j)\}$. From a more careful analysis of the error terms in (2.15), (2.16) one verifies that $\{(U_j^0, V_j^0)\}$ is in fact a perturbed basis.

It follows indeed from the construction of $F^{a_1,\dots,a_{j_0},a_{j_1}}\;(\theta_1, \dots, \theta_{j_0}, \theta_{j_1})$ for $j_1 \gg j_0$ and (1.27) $\frac{\partial\lambda'}{\partial a_{j_1}}\;|_{a_{j_1}=0} =\; 0$ that $\partial_{a_{j_1}}\;u_{a,\theta}\;|_{a_{j_1}=0}$ has a good localization wrt the j_1-mode in x-space. Hence, given any smooth φ, $||\varphi|| \le 1$ ($||\;||$ referring to a sufficiently smooth Sobolev norm, there is an expansion

$$\varphi = \varphi(x) = \sum\,(\alpha_j\;U_j^0 + \beta_j\;V_j^0) \qquad (\alpha_j, \beta_j \in \mathbf{R})\,. \tag{2.17}$$

The function

$$\Phi = \sum\,(\alpha_j\;U_j + \beta_j\;V_j) \tag{2.18}$$

satisfies in particular by (2.5), (2.6), (2.9), (2.10)

$$\Phi(0) = \varphi \qquad ||\Phi(t)|| \le (1 + \varepsilon\,|t|)\;||\varphi|| \le 1 + |t| \tag{2.19}$$

and

$$i\,\Phi_t + A\,\Phi + \varepsilon\,\frac{\partial^2 H}{\partial\,\overline{u}_\theta\,\partial u_\theta}\;\Phi + \varepsilon\,\frac{\partial^2 H}{\partial\,\overline{u}_\theta^2}\,\overline{\Phi} = w \tag{2.20}$$

where by (2.8), (2.11)

$$\|w\| = 0\,(\varepsilon^{r_1}\,(1+|t|))\,. \tag{2.21}$$

Denote $S_\theta\,(t)\,\varphi = U(x,t)$ the solution of the IVP

$$\begin{cases} i\,U_t + A\,U + \varepsilon\,\frac{\partial^2 H}{\partial\,\overline{u}_\theta\,\partial u_\theta}\;U + \varepsilon\,\frac{\partial^2 H}{\partial\,\overline{u}_\theta^2}\,\overline{U} = 0 \\ U(0) = \varphi\,. \end{cases} \tag{2.22}$$

Then $S_\theta(T)^{-1}\,\psi = V(x,T)$, where $V(x,t)$ is obtained by solving

$$\begin{cases} -i\,V_t + A\,V + \varepsilon\,\frac{\partial^2 H}{\partial\,\overline{u}_\theta\,\partial u_\theta}\;(T-t)\,V + \varepsilon\,\frac{\partial^2 H}{\partial\,\overline{u}_\theta^2}\;(T-t)\,\overline{V} = 0 \\ V(0) = \psi\,. \end{cases} \tag{2.23}$$

It follows from definition of u_θ that

$$u_\theta\,(T-t) = F^c\,(\theta_1 + \lambda_1'\,(T-t),\ldots,\theta_{j_0} + \lambda_{j_0}'\,(T-t)) = u_{\theta+\lambda' T}\,(-t) \tag{2.24}$$

and thus $V(t) = S_{\theta+\lambda' T}\,(-t)\,\psi$. The conclusion is that

$$S_\theta\,(T)^{-1} = S_{\theta+\lambda' T}\,(-T)\,. \tag{2.25}$$

Our aim is to establish a bound on the flow map $S_\theta(t)$, for $|t| < \varepsilon^{-r_2}$, $r_2 \sim r$. Given φ, let Φ be as above satisfying (2.20). Define $U_1 = U - \Phi$ satisfying the IVP

$$\begin{cases} i\,(U_1)_t + A\,U_1 + \varepsilon\,\frac{\partial^2 H}{\partial\,\overline{u}_\theta\,\partial u_\theta}\;U_1 + \varepsilon\,\frac{\partial^2 H}{\partial\,\overline{u}_\theta^2}\;\overline{U}_1 = -w \\ U_1(0) = 0\,. \end{cases} \tag{2.26}$$

Hence

$$U_1(t) = \int_0^t S_\theta(t)\;S_\theta(\tau)^{-1}\;(i\,w(\tau))\,d\tau \tag{2.27}$$

and from (2.25)

$$S_\theta(t)\,\varphi = U = \Phi + \int_0^t S_\theta(t)\;S_{\theta+\lambda'\tau}\,(-\tau)\;(i\,w(\tau))\,d\tau\,. \tag{2.28}$$

It follows now from (2.19), (2.20) that

$$\|S_\theta(t)\,\varphi\| \le (1+|t|) + |t|\left(\max_{\psi\in\mathbf{T}^{j_0},|\tau|\le|t|}\|S_\psi(\tau)\|^2\right)\;(\varepsilon^{r_1}\,(1+|t|))\,. \tag{2.29}$$

Hence

$$\|S_\theta(t)\| \le 1 + |t|\;\left\{1 + \varepsilon^{r_1}\,(1+|t|)\;\left[\max_{\psi\in\mathbf{T}^{j_0},|\tau|\le|t|}\|S_\psi(\tau)\|^2\right]\right\} \tag{2.30}$$

implying that

$$\|S_\theta(t)\| < 1 + 2\,|t| \quad \text{for} \quad |t| < \varepsilon^{-r_2} \quad \text{with} \quad r_2 \sim r_1 \sim r\,. \tag{2.31}$$

3. Estimating solutions of the perturbed equation.

Return to the original IVP

$$\begin{cases} i\,u_t + A\,u + \varepsilon\,\frac{\partial H}{\partial \bar{u}} = 0 \\ u(0) = \varphi\,. \end{cases} \tag{3.1}$$

We consider a sufficiently smooth Sobolev norm

$$\|\varphi\|_s = \sum (1+|j|)^s\,|\alpha_j| \tag{3.2}$$

for

$$\varphi = \sum \alpha_j\,\varphi_j\,. \tag{3.3}$$

Given a sequence $\alpha = \{\alpha_j\}_{j\le j_0}, \alpha_j \neq 0, \|\alpha\|_s \le 1$, write $\alpha_j = c_j\,e^{i\theta_j}$, $c_j > 0$. One obtains a quasi-periodic approximative solution $u_\alpha = u_{c,\theta}$, i.e.

$$i\,\dot{u}_\alpha + A\,u_\alpha + \varepsilon\,\frac{\partial H}{\partial \bar{u}_\alpha} = w \tag{3.4}$$

with

$$\|w\| < \varepsilon^{r_1} \tag{3.5}$$

and

$$\Big\| u_\alpha(0) - \sum_{j\le j_0} \alpha_j\,\varphi_j \Big\|_s < \sqrt{\varepsilon}\,. \tag{3.6}$$

Observe that F^α, u_α (resp. $\lambda'(|\alpha|)$) depend smoothly on α (resp. $|\alpha_j|^2, j = 1,\dots,$ j_0). Hence, the map

$$\Omega : B_{\|\ \|_s}(1) \to \mathbb{C}^{j_0}\ ,\ \|\ \|_s : \alpha = (\alpha_j)_{j=1,\dots j_0} \mapsto \alpha - (\langle u_\alpha(0), \varphi_j\rangle)_{j=1,\dots,j_0} \tag{3.7}$$

satisfies $\|\Omega\| < \sqrt{\varepsilon}$ and similarly a contractive estimate.

It follows from the open map principle that there is $(c_j)_{j=1,\dots,j_0}$, $c_j > 0$ and $\theta \in \mathbb{T}^{j_0}$ so that

$$\langle u_{c,\theta}(0), \varphi_j\rangle = \alpha_j \qquad j = 1,\dots,j_0 \tag{3.8}$$

and

$$\|c\|_s \le 2\,. \tag{3.9}$$

Denote $U = u - u_{c,\theta}$ satisfying the difference equation

$$i U_t + A U + \varepsilon \frac{\partial^2 H}{\partial \overline{u}_{c,\theta} \partial u_{c,\theta}} U + \varepsilon \frac{\partial^2 H}{\partial \overline{u}^2_{c,\theta}} \overline{U} + 0(|U|^2) = -w \tag{3.10}$$

$$U(0) = \varphi - u_{c,\theta}(0) = \sum_{j > j_0} (\alpha_j - \langle u_{c,\theta}(0), \varphi_j \rangle) \varphi_j \tag{3.11}$$

hence

$$\|U(0)\|_{s_1} < j_0^{-(s-s_1)} \quad \text{for} \quad s > s_1 . \tag{3.12}$$

Define $U = \gamma U'$, $\gamma > 0$ to be specified, satisfying

$$i U'_t + A U' + \varepsilon \, \partial \overline{\partial} H \cdot U' + \varepsilon \, \overline{\partial}^2 H \cdot \overline{U}' + \gamma \, 0(|U'|^2) = w' \tag{3.13}$$

where by (3.12), (3.5)

$$\|U'(0)\|_{s_1} < \gamma^{-1} j_0^{-(s-s_1)} \quad \text{and} \quad \|w'\| < \gamma^{-1} \varepsilon^{r_1} . \tag{3.14}$$

One has then from the integral equation

$$U'(t) = S_\theta(t) U'(0) - \int_0^t S_\theta(t) S_\theta(\tau)^{-1} [i w'(\tau) - i \gamma \, 0(|U'(\tau)|^2)] d\tau \tag{3.15}$$

by (3.14), (2.31) following estimate

$$\|U'(t)\|_{s_1} \leq (1 + 2|t|) \gamma^{-1} j_0^{-(s-s_1)} + (1 + 2|t|)^3 (\gamma^{-1} \varepsilon^{r_1} + \gamma) \tag{3.16}$$

provided

$$\|U'(\tau)\|_{s_1} < 1 \quad \text{for} \quad |\tau| \leq |t| . \tag{3.17}$$

Taking $\gamma = \varepsilon^{r_1/2}$, this yields (3.17) ad hence

$$\|U(t)\|_{s_1} < \varepsilon^{r_1/2} \tag{3.18}$$

provided

$$|t| < \varepsilon^{r_1/2} j_0^{s-s_1} + \varepsilon^{r_1/10} . \tag{3.19}$$

Recall (1.10), hence $j_0 < \varepsilon^{-c/r}$. This yields for appropriate $r \sim \sqrt{s}$ the bound

$$\|U(t)\|_{\frac{s}{2}} < \varepsilon^{c\sqrt{s}} \quad \text{for} \quad |t| < \varepsilon^{-c\sqrt{s}} \tag{3.20}$$

controlling the difference beween u and the quasi-periodic $u_{c,\theta}$.

4. Application to perturbed linear SE with potential.

Consider for A a Sturm-Liouville operator $-\frac{d^2}{dx^2} + V(x)$ where V is a real smooth even periodic potential. Denote by $\{\lambda_j\}$ the Dirichlet spectrum. Our aim is to verify for "generic" V nonresonance properties as required in the preceding argument.

Lemma 4.1. *Fix $r \in \mathbb{Z}$, $r \geq 1$. Then one has "typically"*

$$|a_{j_1} \lambda_{j_1} + \cdots + a_{j_r} \lambda_{j_r}| \gtrsim \max(j_1^{-C(r)}, J^{-10r}) \tag{4.2}$$

for all J, $j_1 < j_2 < \cdots < j_r < J$, $a_{j_i} \in \mathbb{Z}$ $(i = 1, \ldots, r)$, $a_{j_1} \neq 0$, $\sum_{i=1}^{r} |a_{j_i}| \leq r$ and j_1 sufficiently large.

Proof. For j large, there is an asymptotic expansion

$$\lambda_j = j^2 \pi^2 + c_0(V) + c_1(V) j^{-1} + \cdots + c_R(V) j^{-R} + 0(j^{-R-1}). \tag{4.3}$$

Here $c_0(V) = \int_0^1 V(x)\,dx$ and the $c_s(V)$ are certain multilinear expressions in V (cf. [P-T]). Consider $\{c_s(V) \mid s = 1, \ldots, R\}$ as independent random variables. The integer R will depend on r. Write

$$\sum_{i=1}^{r} a_{j_i} \lambda_{j_i} = \pi^2 \left(\sum a_{j_i} j_i^2\right) + \sum_{s=1}^{R} c_s(V) \left(\sum_{i=1}^{r} a_{j_i} j_i^{-s}\right) + 0(j_1^{-R-1}). \tag{4.4}$$

We may assume $\sum a_{j_i} j_i^2 = 0$, since otherwise $|(4.4)| > 1 - 0(1) > \frac{1}{2}$. Taking $R > C(r)$, it will thus suffice to show that for typical $c = (c_s(V))_{s=1,\ldots,R}$

$$\left|\sum_{s=1}^{R} c_s \left(\sum_{i=1}^{r} a_{j_i} j_i^{-s}\right)\right| > \max(j_1^{-C(r)}, J^{-10r}) \tag{4.5}$$

where $j_1 < j_2 < \cdots < j_r < J$, $a_{j_1} \neq 0$, $\sum_{i=1}^{r} |a_{j_i}| \leq r$.

We proceed by induction on r. Given $(a_{j_1}, \ldots, a_{j_{r+1}})$, let $r_* = \max\{i = 1, \ldots, r+1 \mid a_{j_i} \neq 0\}$. We may clearly assume $r_* > 1$. From hypothesis, (4.5) holds for typical $c = (c_s)_{s=1,\ldots,R}$. Hence in particular

$$\left|\sum_{s=1}^{R} c_s \left(\sum_{i=1}^{r_*-1} a_{j_i} j_i^{-s}\right)\right| > \max\left(j_1^{-C(r)}, J^{-10r}\right). \tag{4.6}$$

It follows thus that also

$$\left|\sum_{s=1}^{R} c_s \left(\sum_{i=1}^{r_*} a_{j_i} j_i^{-s}\right)\right| > \frac{1}{2} \max\left(j_1^{-C(r)}, J^{-10r}\right) \tag{4.7}$$

if j_{r_*} is large enough to ensure that $(r+1)\ j_{r_*}^{-1} < \frac{1}{10}\ j_1^{-C(r)}$. Hence (4.7) needs only be verified for

$$j_1 < j_2 < \cdots < j_{r+1} < \min\left(20(r+1)\ j_1^{C(r)}, J\right). \tag{4.8}$$

Consider

$$\max_{s=1,\dots,r+1} \left|\sum_{i=1}^{r+1} a_{j_i}\ j_i^{-s}\right|. \tag{4.9}$$

Considering the usual Vandermonde determinants, it follows that

$$(4.9) \leq \frac{\prod_{i=2}^{r+1}\left(\frac{1}{j_1}-\frac{1}{j_i}\right)}{\prod_{i=2}^{r+1}\left(1+\frac{1}{j_i}\right)} \gtrsim j_1^{-2r}. \tag{4.10}$$

By straight probabilistic considerations, it follows that for typical $c=(c_s)_{s=1,\dots,R}$ and all $j_1 < j_2 < \cdots < j_{r+1}$ satisfying (4.8), $a_{j_1} \neq 0$, $\sum\limits_{i=1}^{r+1} |a_{j_i}| \leq r+1$

$$\left|\sum_{s=1}^{R} c_s \left(\sum_{i=1}^{r+1} a_{j_i}\ j_i^{-s}\right)\right| > j_1^{-2r}\ \left[(20(r+1)\ j_1^{C(r)})^{-r-1}, J^{-r-1}\right]. \tag{4.11}$$

Thus, from the preceding, (4.11) will hold in general if $a_{j_1} \neq 0$ and we may take

$$C(r+1) = 3r + (r+1)\,C(r).$$

Hence $C(r) = 2r!$ will satisfy (4.2).

Lemma 4.12. *Fix $r \geq 1$. Then the spectrum (λ_j) of V satisfies typically*

$$|a_{j_1}\ \lambda_{j_1} + \cdots + a_{j_r}\ \lambda_{j_r}| > J^{-30r} \tag{4.13}$$

for $j_1 < j_2 < \cdots < j_r < J$, J large, $a_{j_i} \in \mathbb{Z}$, $0 < \sum\limits_{i=1}^{r} |a_{j_i}| < r$.

Proof. From Lemma 4.1, we get the statement for $j_1 > j_* = j_*(r)$; we replace here r by $2r$ in the statement. Hence, it remains to fulfil, given $j_1 < \cdots < j_{r'} < j_*$ and $a_{j_1}, \dots, a_{j_{r'}} \in \mathbb{Z}$, $0 < \sum\limits_{i=1}^{r'} |a_{j_i}| \leq r+1$

$$\left|\sum_{i=1}^{r'} a_{j_i}\ \lambda_{j_i}\right| > J^{-30r} \tag{4.14}$$

and

$$\left| \sum_{i=1}^{r'} a_{j_i}\, \lambda_{j_i} + \sum_{i=r'+1}^{r+1} a'_{j_i}\, \lambda_{j_i} \right| > J^{-30r} \tag{4.15}$$

for at most one system $\{a'_{j_i} \mid i = r'+1, \ldots, r+1\}$ (depending on $\{a_{j_i} \mid i = 1, \ldots, r'\}$). Thus this amounts to a bounded number $B(r)$ of extra conditions on the λ_j's. Since $\sum_{i=1}^{r'} |a_{j_i}| \neq 0$, they may be satisfied by a small variation $0(J^{-30r})$ of the initial Fourier coefficients of V, small enough to preserve the other inequalities already valid. Writing for instance

$$V(x) = \sum_{k \geq 1} \widehat{V}(k) \ \cos 4\pi kx \tag{4.16}$$

we will use here the fact that (cf. [P-T])

$$\begin{aligned} \frac{\partial \lambda_j}{\partial \widehat{V}(k)} &= \int_0^1 \varphi_j^2(V, x) \ \cos 4\pi kx \\ &= \int_0^1 \sin^2\, 2\pi jx \cdot \cos 4\pi kx + 0\left(\frac{1}{j}\right) = \frac{1}{2}\, \delta_{jk} + 0\left(\frac{1}{j}\right) . \end{aligned} \tag{4.17}$$

This yields Lemma 4.12.

For the application of section (1), we may take $J = j_0^2\, r$, so that (4.13) is compatible with (1.10). This yields following result

Proposition 4.18. *Consider a 1D NLSE of the form*

$$i\, u_t - u_{xx} + V(x)\, u + \varepsilon\, \frac{\partial H}{\partial \overline{u}} = 0 \tag{4.19}$$

under Dirichlet boundary conditions. Here V is an even real periodic smooth potential and H is a polynomial of the form $H(|u|^2)$. Thus for "typical" V (in the sense described above), the following is true. Let $u(0)$ be a smooth initial data for $t = 0$. Then the solution $u(t)$ of (4.19) will be for times $|t| < \varepsilon^{-M}$ an ε^M-perturbation of a quasi-periodic function of time. Here $M > 0$ may be taken to be any fixed number.

5. Further comments.

(i) Consider the case of a finite dimensional phase space, i.e.

$$A = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & 0 \\ & & \ddots & \\ & 0 & & \lambda_d \end{pmatrix} \tag{5.1}$$

with $\lambda = (\lambda_1, \ldots, \lambda_d)$ diophantine, i.e.

$$\left| \sum_{i=1}^{d} k_i \, \lambda_i \right| > c \, \left(\sum |k_i| \right)^{-C} . \tag{5.2}$$

Thus the required nonresonance condition

$$\left| \sum_{i=1}^{d} k_i \, \lambda_i \right| > \varepsilon^{1/10} \tag{5.3}$$

will be fulfilled with $\sum |k_i| = r$ up to $\log r \sim \log \frac{1}{\varepsilon}$. Thus by truncating the ε-series to order $r \sim \frac{1}{\varepsilon^\alpha}$ ($\alpha > 0$ appropriate), one gets approximate solutions of accuracy $\exp\left(-\frac{1}{\varepsilon^\alpha}\right)$. This enables to get the flow of the perturbed linear system close to a quasi-periodic one, up to times $T \sim \exp \frac{1}{\varepsilon^\beta}$, for some $\beta > 0$. Both α, β depend on dimension d. Compare with Nekhorochev's estimate [N].

(ii) Instead of considering the NLS

$$i\, u_t - u_{xx} + V\, u + \varepsilon \, \frac{\partial H}{\partial \overline{u}} \, (u, \overline{u}) = 0 \tag{5.4}$$

one may consider a NLW

$$y_{tt} - y_{uu} + V_y + \varepsilon \; F'(y) = 0 \, . \tag{5.5}$$

We consider again Dirichlet boundary conditions. Let V be an even potential and $\phi'(y)$ an odd (polynomial) function of y. Thus the eigenfunctions $\{\varphi_j\}_{j \geq 1}$ of $-\frac{d^2}{dx^2} + V$ span the space of odd periodic functions. For the wave equation, the IVP relates to both $y(0)$, $y'(0)$ and the phase space is $H_0^s(\mathbf{T}) \times H_0^{s-1}(\mathbf{T})$ (s large), denoting $H_0^s(\mathbf{T})$ the subspace of $H^s(\mathbf{T})$ of real odd functions.

Replace equation (5.5) by

$$\begin{cases} y_t = Bz \\ z_t = -By - \varepsilon \; B^{-1} \; F'(y) \end{cases} \tag{5.6}$$

where $B = \left(-\frac{d^2}{dx^2} + V\right)^{1/2}$.

Denoting

$$u = y + i\, z \tag{5.7}$$

one gets an equation of the form (1.1)

$$i\, u_t = B\, u + \varepsilon \; B^{-1} \; F'(Re\, u) \tag{5.8}$$

where B is replacing the operator A. The small divisors are in this case given by (cf. (1.7))

$$\sqrt{\lambda_{j_2}}\left[\left(\sum_{i=1}^{j_0} k_i\sqrt{\lambda_i}+k_{j_1}\sqrt{\lambda_{j_1}}\right)-\sqrt{\lambda_{j_2}}\right] \tag{5.9}$$

with $\sum_{i=1}^{j_0}|k_i|\le r$, $|k_{j_1}|\le 1$. In particular, the expressions

$$\sum_{i=1}^{j_0} k_i\sqrt{\lambda_i}+k_{j_1}\sqrt{\lambda_{j_1}}-\sqrt{\lambda_{j_2}} \tag{5.10}$$

will be kept from zero (except in the resonant case).

Observe that $\sqrt{\lambda_j}=\pi j+0\left(\frac{1}{j}\right)$ for $j\to\infty$. The problem therefore reduces to keeping

$$\left\|a_{j_1}\sqrt{\lambda_{j_1}}+\cdots+a_{j_r}\sqrt{\lambda_{j_r}}\right\| \tag{5.11}$$

away from zero for $1\le j_1<\cdots<j_r<J=j_0^{C(r)}$.

For large j, one has again the asymptotic expansion

$$\sqrt{\lambda_j}=\pi j\left(1+\frac{\lambda_j-\pi^2 j^2}{\pi^2 j^2}\right)^{1/2}=\pi j+\frac{1}{2\pi j}\,c_0(V)+\sum_{1<s\le R}c_3'\,j^{-s}+0\left(j^{-R-1}\right) \tag{5.12}$$

where the c_s' may be computed from the $c_j(V)$.

One may therefore reproduce the same reasonning as in the NLS case to obtain the required diophantine conditions.

Observe that in this case, one may already proceed considering a constant $V(x)=\rho$, since $\left\{\sqrt{\pi^2 j^2+\rho}\right\}_{j\ge1}$ are linearly independent functions of ρ. The reader will easily check details. Thus one obtains following result

Proposition 5.13. *Consider a NLW*

$$y_{tt}-y_{xx}+\rho y+F'(y)=0 \tag{5.14}$$

with $F'(y)$ odd and $0(|y|^3)$. Let ρ be typical. Then for smooth, odd periodical data $y(0)$, $y'(0)$ of size ε, the solution $y(t)$ of (5.14) will be ε^M-close to a quasi-periodic function of time, for $|t|<T\sim\varepsilon^{-M}$.

$M>0$ is again an arbitrarily choosen number here.

6. Construction of infinite dimensional invariant tori for nonlinear perturbations of 1D linear wave and Schrödinger equations under Dirichlet boundary conditions.

(i) We consider the model of a wave equation

$$u_{tt} - u_{xx} + V\,u + \varepsilon\, f(u) = 0\,. \tag{6.1}$$

Here V will be a periodic even real analytic potential and $f(u)$ an odd polynomial function of u, $f(u) = 0(|u|^3)$. Denote $\{\mu_j\}$ and $\{\varphi_j\}$ the Dirichlet spectrum and eigenfunctions of $-\frac{d^2}{dx^2} + V$. Write

$$\mu_j = \lambda_j^2\,. \tag{6.2}$$

We will consider $\{\lambda_j\}$ as parameters, by variation of the potential.

For $\varepsilon = 0$, the (unperturbed) solution u_0 of (6.1) will be

$$u_0\,(x,t) = \sum_{j=1}^{\infty} a_j\ \varphi_j(x)\ \cos\lambda_j\, t \tag{6.3}$$

where $\{a_j\}$, $a_j > 0$ will be a very rapidly decreasing sequence.

The perturbed solution has the form

$$u_\varepsilon(x,t) = \sum_{j=1}^{\infty} \sum_{n\in\prod_\infty \mathbf{Z}} \widehat{u}\,(j,n)\ \varphi_j(x)\ e^{i(n,\lambda')t} \tag{6.4}$$

where $\prod_\infty \mathbf{Z}$ stands for the space of finite sequences of integers $n = \{n_k\}$ and

$$\widehat{u}\,(j,n) = \widehat{u}\,(j,-n) \tag{6.5}$$

$$\lambda_j' = \lambda_j + 0\left(\frac{\varepsilon}{j}\right) \qquad \text{(uniformly in } j\text{) is the perturbed frequency} \tag{6.6}$$

$$\widehat{u}\,(j,e_j) = \widehat{u}\,(j,-e_j) = \frac{1}{2}\,a_j \qquad (e_j = j\text{-unit vector in } \prod_\infty \mathbf{Z}) \tag{6.7}$$

$$\sum_{(j,n)\notin S} e^{j^c + \sum_k A_k |n_k|^c} \cdot |\widehat{u}\,(j,n)| < \sqrt{\varepsilon} \tag{6.8}$$

where

$$S = \{(j,\pm e_j)\} \tag{6.9}$$

is the resonant set, $c > 0$ and $\{A_k\}$ are increasing weights, depending on the decay of the sequence $\{a_k\}$ in (6.3).

Recall that since $\mu_j = \pi^2 j^2 + 0(1)$, $\lambda_j = \pi j + 0\left(\frac{1}{j}\right)$.

(ii) The lattice representation of the linearized operator after passing to Fourier transform is

$$(-\langle n, \lambda'\rangle^2 + \lambda_j^2)\, \widehat{u}(j,n) + \varepsilon\, S_{f'(u_1)}\, \widehat{u} = 0 \tag{6.10}$$

where S_ϕ corresponds to the ϕ-multiplication operator and u_1 is a previous approximation to the solution u. Thus

$$T = D + T' \tag{6.11}$$

where

$$D = \text{diagonal } \lambda_j^2 - \langle n, \lambda'\rangle^2 \quad \text{and} \quad T' = \varepsilon\, S_{f'(u_1)} = \text{nondiagonal perturbation.} \tag{6.12}$$

Introducing as in [B] an extra parameter σ to exploit translation in time frequency, denote

$$T^\sigma = D^\sigma + T' \tag{6.13}$$

where

$$D^\sigma(j,n) = \lambda_j^2 - (\langle n, \lambda'\rangle + \sigma)^2\,. \tag{6.14}$$

Denoting $n = \{n_k\}_{k=1}^\infty$, let T_N^σ be the restriction of T^σ for n such that

$$\begin{cases} |n_k| \le N \\ n_k = 0 \quad \text{for} \quad k > k(N)\,. \end{cases} \tag{6.15}$$

Here $k(N)$ will be a very slowly increasing function of N.

In order to solve the P-equation, we need to discuss the control of the inverse of T_N^σ and T_N. The method is the same as in [B] except for an unbounded number of frequencies $\{\lambda'_j\} = \lambda'$. The number of λ'_j-frequencies will be increased slowly and the essential point is to make the process work by letting the a_j's go fast enough to zero, without having to decrease the ε (perturbation).

(iii) The first step consists in controlling the inverse of $T_{N_1}^\sigma$. Denote $k(N_1)$ by $k(1)$.

Consider sites (j_1, n_1), (j_2, n_2) where the diagonal does not control the perturbation (the remainder is taken care off by a Neumann series, cf. [B]).

Thus assume

$$|(\langle n_1, \lambda'\rangle + \sigma)^2 - \lambda_{j_1}^2| < \varepsilon^{1/10} \quad \text{and} \quad |(\langle n_2, \lambda'\rangle + \sigma)^2 - \lambda_{j_2}^2| < \varepsilon^{1/10} \tag{6.16}$$

hence

$$|\sigma + \langle n_1, \lambda'\rangle \pm \lambda_{j_1}| < \varepsilon^{1/10} \quad \text{and} \quad |\sigma + \langle n_2, \lambda'\rangle \pm \lambda_{j_2}| < \varepsilon^{1/10} \tag{6.17}$$

and thus

$$|\langle n_1 - n_2, \lambda' \rangle \pm \lambda_{j_1} \pm \lambda_{j_2}| < 2\,\varepsilon^{1/10}\,. \tag{6.18}$$

Recall that $n_1(k) = 0 = n_2(k)$ for $k > k(1)$. If $k(1)$, N_1 are sufficiently small compared with ε, (6.18) may be reduced to an identical vanishing of the expression

$$\langle n_1 - n_2, \lambda \rangle \pm \lambda_{j_1} \pm \lambda_{j_2}\,. \tag{6.19}$$

(We use here the fact that $\lambda_j = \pi j + 0\left(\frac{1}{j}\right)$ for $j \to \infty$.)

It follows that in particular $|n_1 - n_2| < B_1$, where B_1 depends on $k(1)$. As in [B], the inverse of $T^\sigma_{N_1}$ will be therefore controlled by a "local determinant" of the form

$$\prod_{|n-n_0|<B_1, |j-j_0|<B_1} \left((\sigma + \langle n, \lambda' \rangle)^2 - \lambda_j^2\right) + 0\left(\sqrt{\varepsilon}\right)\,. \tag{6.20}$$

In fact, in (6.20) either $j \lesssim k(1)$ or (6.20) may be replaced by

$$(\sigma + \langle n, \lambda' \rangle)^2 - \lambda_j^2 + 0\left(\sqrt{\varepsilon}\right) \tag{6.21}$$

since at most one factor in the (6.20)-product may be small. Observe that (6.21) yields further

$$\sigma + \langle n, \lambda' \rangle \pm \lambda_j + 0\left(\frac{\sqrt{\varepsilon}}{j}\right)\,. \tag{6.22}$$

In both cases, application of the preparation theorem allows to control $(T^\sigma_{N_1})^{-1}$ by reciprocals of polynomials

$$p(\sigma_1) = \sigma_1^d + \sum_{s<d} a_s\, \sigma_1^s \tag{6.23}$$

where

$$d \le 2k(1) \tag{6.24}$$

$$\sigma_1 = \sigma + \langle n, \lambda' \rangle \pm \lambda_j \text{ for some } n \text{ in the range (6.15) and } j\,. \tag{6.25}$$

The coefficients a_s are smooth functions $a_s(\lambda, \lambda')$ and their dependence on λ_k for large k (mainly depending on $k(1)$), goes to zero for $k \to \infty$.

Remark. For $\sigma = 0$, the resonant pairs $(j, n) \in S$ have to be excluded when defining T_{N_1}. In discussing $(T^\sigma_{N_1})^{-1}$ however, the full region of sites (j, n) needs to be considered since after translation in time frequency (in the inductive process), there is no exclusion anymore of the sites corresponding to $(j, \pm e_j)$, $j \le k(1)$.

(iv) The main step consists in controlling $(T_N^{\sigma,\lambda'})^{-1}$, $\lambda' = (\lambda_1', \dots, \lambda_r')$, r fixed and $N > N_r$, $r < k(N_r)$, arbitrary. We essentially proceed by induction on r.

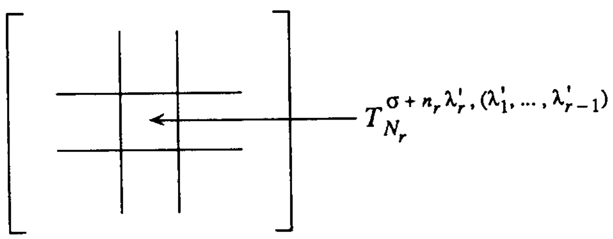
As a consequence of the behaviour of $\widehat{u}_1$, u_1 approximative solution, more precisely (cf. (6.3))

$$|\widehat{u}_1(j,n)| = 0(|a_s|) \quad \text{for} \quad a_s \to 0 \quad \text{and} \quad n_s \neq 0 \tag{6.26}$$

one has the off-diagonal estimate on the matrix elements

$$\|T'(n',n'')\| < \gamma_s = 0\,(|a_s|) \quad \text{if} \quad n'_s \neq n''_s \tag{6.27}$$

for $s = 1, \ldots, r$. In (6.27), $\|\ \|$ refers to the operator norm of the matrix wrt the j-index. Fix N_r and write $T_{N_r}^{\sigma,(\lambda'_1,\ldots,\lambda'_r)}$ as a block-matrix



where n_r is indexing the blocks (thus their number is the range of n_r, i.e. N_r).

For the non-diagonal blocks, $n'_r \neq n''_r$ and hence, by (6.27), their norm is $0(\gamma_r) = 0(|a_r|)$ for $a_r \to 0$, hence may assumed arbitrarily small. From the induction hypothesis, the diagonal matrices are controlled by (reciprocals of) polynomials

$$p(\sigma_1) = \sigma_1^d + \sum_{s<d} a_s\,\sigma_1^s \tag{6.28}$$

where $d \leq d(r-1)$ is an increasing function of r and

$$\sigma_1 = \sigma + n_r\,\lambda'_r + [n_1\,\lambda'_1 + \cdots + n_{r-1}\,\lambda'_{r-1}] \pm \lambda_j\,. \tag{6.29}$$

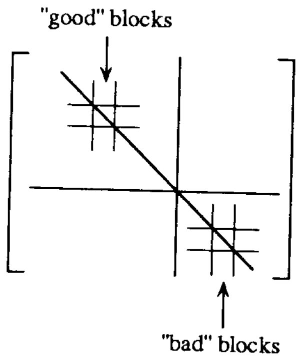
Proceeding by algebraic elimination as in [B] and separation considerations, this permits us to localize those blocks with "badly" behaving inverse to an interval wrt the n_r-index

$$|n_r - m| < B = B(d(r-1)) \tag{6.30}$$

of size B, depending on the degree $\leq d(r-1)$ of the polynomials above.

Next, we describe the procedure to obtain the local determinant corresponding to $T_{N_r}^{\sigma,(\lambda'_1,\ldots,\lambda'_r)}$. Write

$$T_{N_r}^{\sigma,\lambda'} =$$



$$= \left[\begin{array}{c|c} A_0 & \\ \hline & A_1 \end{array}\right] \tag{6.31}$$

where "good" and "bad" refers to bounds on the inverse. Thus A_1 corresponds to the blocks with n_r satisfying (6.30). Denote $T_\alpha = T_{N_r}^{\sigma+n_r\lambda'_r,(\lambda'_1,\ldots,\lambda'_{r-1})}$ for n_r in the interval (6.30). By induction hypothesis, one has for each T_α a "local determinant", controlling T_α^{-1}, obtained as

$$\det\left(\overline{\overline{T}}_\alpha - U_\alpha\,(\overline{T}_\alpha)^{-1}\,U_\alpha^*\right) \tag{6.32}$$

where

$$T_\alpha = \begin{array}{c} \overbrace{\quad\overline{\Omega}_\alpha\quad} \; \overbrace{\;\overline{\overline{\Omega}}_\alpha\;} \\ \left[\begin{array}{c|c} \overline{T}_\alpha & U_\alpha^* \\ \hline U_\alpha & \overline{\overline{T}}_\alpha \end{array}\right] \end{array} \tag{6.33}$$

$\overline{T}_\alpha$ has a well-controlled inverse, $\overline{\overline{\Omega}}_\alpha$ is of size bounded by $d(r-1)$. Denote

$$\overline{\Omega} = \bigcup_\alpha \overline{\Omega}_\alpha \tag{6.34}$$

and

$$\overline{T} = T\,|_{A_0\cup\overline{\Omega}} = \left(T\,|_{A_0} \oplus \bigoplus_\alpha T_\alpha\,|_{\overline{\Omega}_\alpha}\right) + 0\,(\gamma_r)\,. \tag{6.35}$$

Hence $\overline{T}$ has a controlled inverse by the preceding, provided γ_r is sufficiently small. Let now

$$
T_{N_r}^{\sigma,\lambda'} = \left[\begin{array}{c|c} \overset{\overline{\Omega}}{\overline{T}} & \overset{\overline{\overline{\Omega}}}{U^*} \\ \hline U & \overline{\overline{T}} \end{array}\right] \tag{6.36}
$$

where thus

$$
\overline{\overline{\Omega}} = \bigcup_\alpha \overline{\overline{\Omega}}_\alpha \tag{6.37}
$$

$$
U = \left(\bigoplus_\alpha U_\alpha\right) + 0(\gamma_r)\,. \tag{6.38}
$$

Then $(T_{N_r}^{\sigma,\lambda'})^{-1}$ will be controlled by

$$
\det\left(\overline{\overline{T}} - U(\overline{T})^{-1}\, U^*\right) =
$$

$$
\prod_\alpha \det\left(\overline{\overline{T}}_\alpha - U_\alpha\,(\overline{T})^{-1}\, U_\alpha^*\right) + 0\,(\gamma_r)\,. \tag{6.39}
$$

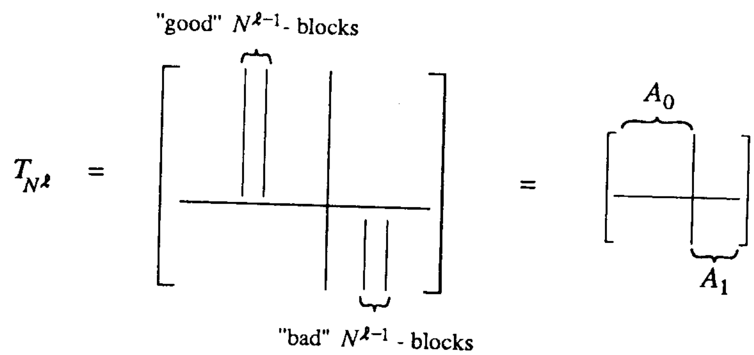
There are at most $B(d(r-1))$ α-factors, which (using the preparation theorem) may be replaced by polynomials in σ of degree $\leq d(r-1)$ (from the induction hypothesis say). The first term in (6.39) yields thus a polynomial in σ of degree $< B(d(r-1))\, d(r-1) = d(r)$. For γ_r small enough another application of the preparation theorem will enable to substitute (6.39) for a σ-polynomial of degree $d(r)$.

Next, one needs to consider general size scales $N > N_r$, still considering r frequencies $\lambda'_1, \ldots, \lambda'_r$. We set up an induction on the size of N. Consider thus a rapidly increasing scale

$$
N_r = N^1 \ll N^2 \ll \cdots \ll N^{\ell-1} \ll N^\ell \ll \cdots
$$

say with $\log N^\ell \sim (\log N^{\ell-1})^2$. On the other hand, as will be clear below, some growth restriction is necessary, at least $(\log N^\ell)^C < N^{\ell-1}$.

We describe the procedure to control $(T_{N^\ell}^\sigma)^{-1}$. Write

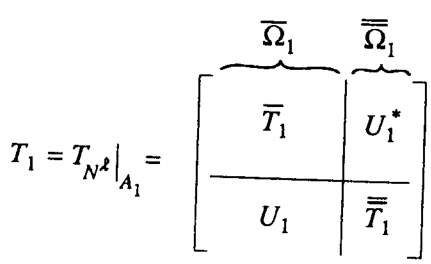


(6.40)

where "good" and "bad" again refers to control of the inverse operator after corresponding restriction. Thus the inverse of $T_{N^\ell}\ |_{A_0}$ is controlled because A_0 is constructed from "good" $N^{\ell-1}$-blocks (using the localization identity for the inverse$^{(*)}$) and A_1 is obtained from restriction to an interval of size $C_r\, N^{\ell-1}$, $C_r = C(d_r)$. At this point, some restriction on the size of N^ℓ matters. This construction is again based on controlling the $N^{\ell-1}$-block restrictions by polynomials (induction hypothesis) and algebraic elimination, analogously to the one made above.

We now treat $T_{N^\ell}\ |_{A_1} = T^{\overline{\sigma},\lambda'}_{CN^{\ell-1}}$ according to the inductive hypothesis, since A_1 may be viewed of previous size scale $N^{\ell-1}$. Thus there is a splitting of the index set A_1

$$A_1 = \overline{\Omega}_1 \cup \overline{\overline{\Omega}}_1 \tag{6.41}$$



(6.42)

where

$$\#\, \overline{\overline{\Omega}}_1 \le d(r) \tag{6.43}$$

$$(\overline{T}_1)^{-1} \text{ is well-controlled} \tag{6.44}$$

$$\det\left(\overline{\overline{T}}_1 - U_1^*(\overline{T}_1)^{-1}\ U_1\right) \text{ is equivalent with polynomial in } \sigma \text{ of degree } d \le d(r)\,. \tag{6.45}$$

$^{(*)}$ $(T_{\Lambda_1 \cup \Lambda_2})^{-1} = (T^{-1}_{\Lambda_1} + T^{-1}_{\Lambda_2}) - (T^{-1}_{\Lambda_1} + T^{-1}_{\Lambda_2})(T_{\Lambda_1\cup\Lambda_2} - T_{\Lambda_1} - T_{\Lambda_2})(T_{\Lambda_1\cup\Lambda_2})^{-1}$

Define

$$\overline{\Omega} = A_0 \cup \overline{\Omega}_1 \ , \ \overline{\overline{\Omega}} = \overline{\overline{\Omega}}_1 \, . \tag{6.46}$$

Again from the localization identity for the inverse, one gets a control on the inverse of $T\,|_{\overline{\Omega}}$ from information relative to A_0 and $\overline{\Omega}_1$. Write next

$$T_{N^\ell} = \begin{array}{c} \overbrace{\qquad\ \ }^{\overline{\Omega}}\ \overbrace{\quad\ }^{\overline{\overline{\Omega}}} \\ \left[\begin{array}{c|c} \overline{T} & U^* \\ \hline U & \overline{\overline{T}} \end{array}\right] \end{array} \tag{6.47}$$

which inverse is controlled by

$$\det\left(\overline{\overline{T}} - U^*(\overline{T})^{-1}\, U\right) . \tag{6.48}$$

Enlarge A_1 to $\widetilde{A}_1$ considering an $N^{\ell-1}$-size neighborhood (in $n = (n_1, \dots, n_r)$) of A_1. Then one has

$$U^*(\overline{T})^{-1}\,U = \left(U\,|_{\overline{\overline{\Omega}}\times(\widetilde{A}_1\backslash\overline{\Omega})}\right)^* \ (\overline{T})^{-1} \ \left(U\,|_{\overline{\overline{\Omega}}\times(\widetilde{A}_1\backslash\overline{\Omega})}\right) + 0(e^{-(N^{\ell-1})^c}) \tag{6.49}$$

$$= \left(U\,|_{\overline{\overline{\Omega}}\times(\widetilde{A}_1\backslash\overline{\Omega})}\right)^* \ (\overline{T}\,|_{\widetilde{A}_1\backslash\overline{\overline{\Omega}}})^{-1} \ \left(U\,|_{\overline{\overline{\Omega}}\times(\widetilde{A}_1\backslash\overline{\Omega})}\right) + 0(e^{-(N^{\ell-1})^c}) \tag{6.50}$$

as a consequence of the off-diagonal decay of T_1 and U, the bound on $(\overline{T})^{-1}$ and the off-diagonal decay of $(\overline{T})^{-1}$.

The discussion relative to A_1 holds also for $\widetilde{A}_1$ (size scale $N^{\ell-1}$). Thus the inverse of $T\,|_{\widetilde{A}_1}$ is controlled by

$$\det\left(\overline{\overline{T}} - (T\,|_{\overline{\overline{\Omega}}\times(\widetilde{A}_1\backslash\overline{\Omega})})^* \ (T\,|_{\widetilde{A}_1\backslash\overline{\overline{\Omega}}})^{-1} \ (T\,|_{\overline{\overline{\Omega}}\times(\widetilde{A}_1\backslash\overline{\Omega})})\right) \tag{6.51}$$

equivalent with a polynomial in σ of degree $\leq d(r)$. The perturbative term in (6.50) may then again be absorbed using the preparation theorem, provided $N^{\ell-1} \geq N^r$ is sufficiently large wrt $d(r)$ (also derivative estimates need to be considered here). This finally permits to replace (6.48) by a polynomial of degree $\leq d(r)$ in σ, more precisely $(T^\sigma_{N^\ell})^{-1}$ will be again controlled by reciprocals of polynomials

$$p(\sigma_1) = \sigma_1^d + \sum_{s<d} a_s\ \sigma_1^s \tag{6.52}$$

where $d \leq d(r)$, $\sigma_1 = \langle n, \lambda' \rangle + \sigma \pm \lambda_j$ with $n = (n_1, \ldots, n_r)$ in the N^ℓ-range $|n_k| \leq N^\ell$. The coefficients $a_s = a_s(\lambda, \lambda')$ are smooth (with bounds depending on r) and with asymptotic independence on λ_k for $k \to \infty$.

This concludes the inductive procedure for constructing polynomials controlling the inverses $(T_N^{\sigma,(\lambda'_1,\ldots,\lambda'_r)})^{-1}$.

(v) We need estimates relative to $[T_N^{\sigma,(\lambda'_1,\ldots,\lambda'_r)}]^{-1}$, $N \geq N_r$ for $\sigma \in \{0\} \cup \{\pm\lambda'_s; s > r\}$. More precisely, estimates on the norm of the inverse of the restriction and off-diagonal decay estimates. Of course, in defining the restriction of the operators, the exclusion of the "resonant" sites is assumed, thus

For $T_N^{\sigma=0}$, exclude the sites $(j, n) \in S$, i.e. $n = \pm e_j$

For $T_N^{\sigma=\lambda'_s,(\lambda'_1,\ldots,\lambda'_r)}$, $s > r$, exclude the site $n = 0$, $j = s$.

Estimates on the inverses will be polynomial in $N \geq N_r$, with exponent dependent on r. Off-diagonal bounds $e^{-|x-x'|^c}$ will appear for $|x - x'| > (C_r \log N)^C$.

We set up an induction and will aim to control more generally

$$[T_N^{\sigma,(\lambda'_1,\ldots,\lambda'_r)}]^{-1} \tag{6.53}$$

for $N \geq N_r$ and σ of the form

$$\sigma = \sum_{r' \geq s > r} c_s \, \lambda'_s \pm \lambda'_{s_1} \tag{6.54}$$

where $r' = r'(r)$, $c_s \in \mathbb{Z}$ for $s = r+1, \ldots, r'$ and

$$\sum |c_s| < A_r \tag{6.55}$$

while s_1 is arbitrary.

Again here the resonant sites are excluded, i.e. we assume that (j, n) are such that

$$\left(\sum_{k=1}^{r} n_k \, \lambda'_k + \sigma \right)^2 - (\lambda'_j)^2 \tag{6.56}$$

with σ given by (6.54) does not vanish identically.

Recall that (6.53) is controlled by reciprocals of polynomials

$$p(\sigma_1) = \sigma_1^d + \sum_{s<d} a_s \, \sigma_1^s \tag{6.57}$$

with $a_s = a_s(\lambda, (\lambda'_1, \ldots, \lambda'_r))$, $d \leq d(r)$ and σ_1 of the form

$$\sigma_1 = \sigma + \langle n, \lambda' \rangle \pm \lambda_j \tag{6.58}$$

where $\max_{1 \leq k \leq r} |n_k| \leq N$ and j is arbitrary.

The different polynomials are gotten for restrictions in the (λ, λ', r)-parameter set to sufficiently small intervals (of size N^{-C_r}).

Rewrite by (6.54), (6.58) as

$$\sigma_1 = \sum_{s=1}^{r} n_s \lambda'_s + \sum_{s=r+1}^{r'} c_s \lambda'_s \pm \lambda'_{s_1} \pm \lambda_j . \tag{6.59}$$

One may then keep (6.57) away from zero considering $\frac{\partial^d}{(\partial \lambda'_s)^d} \{p(\sigma_1)\}$ for $s = 1, \ldots, r'$, provided $\max\{|n_k|, k = 1, \ldots, r \ ; \ |c_s|, s = r+1, \ldots, r'\}$ is sufficiently large (depending on $d \leq d(r)$ and derivative estimates on the coefficients a_s of the polynomials (6.57), which are also r-dependent). Thus in the opposite situation, we may in particular assume that

$$\max(|n_k| \ ; \ k = 1, \ldots, r) < N_{r-1} \tag{6.60}$$

$$\sum_{s=r+1}^{r'} |c_s| < \frac{1}{2} A_{r-1} . \tag{6.61}$$

Recall also that the a_s-coefficients in (6.57) have a weak dependence on λ_k for $k \to \infty$. Given r, we choose r' such that the a_s-coefficients in the polynomials controlling (6.53) may be essentially assumed independent of λ_k for $k \geq r'$ when invoking derivative considerations. In order to control (6.53), we split the n-index set as $\Omega = \Omega' \cup \Omega''$ where

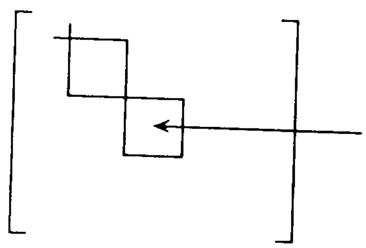
$$\Omega = \{n = (n_k)_{1 \leq k \leq r} \mid |n_k| \leq N\} \qquad (N \geq N_r) \tag{6.62}$$

and

$$\Omega' = \{n = (n_k)_{1 \leq k \leq r} \mid |n_k| \leq N_{r-1}\} \tag{6.63}$$

and control $[T_{\Omega}^{\sigma,(\lambda'_1,\ldots,\lambda'_r)}]^{-1}$ from $[T_{\Omega'}^{\sigma,(\lambda'_1,\ldots,\lambda'_r)}]^{-1}$ and $[T_{\Omega''}^{\sigma,(\lambda'_1,\ldots,\lambda'_r)}]^{-1}$ using the localization identity for the inverse, i.e. $T_{\Omega}^{-1} = (T_{\Omega'}^{-1} + T_{\Omega''}^{-1}) - (T_{\Omega'}^{-1} + T_{\Omega''}^{-1})(T_{\Omega} - T_{\Omega'} - T_{\Omega''}) T_{\Omega}^{-1}$. In the case of $T_{\Omega''}^{\sigma,(\lambda',\ldots,\lambda'_r)}$, the index n in (6.59) will satisfy $|n| > N_{r-1}$ and hence $[T_{\Omega''}^{\sigma,(\lambda'_1,\ldots,\lambda'_r)}]^{-1}$ may be controlled by considering derivatives of polynomials (6.57) in $(\lambda'_1, \ldots, \lambda'_r)$.

Consider now $T_{\Omega'}^{\sigma,(\lambda'_1,\ldots,\lambda'_r)}$ which has clearly the structure



$T_{N_{r-1}}^{\sigma+n_r\lambda'_r,(\lambda'_1,\ldots,\lambda'_{r-1})}$

where $|n_r| < N_{r-1}$ and the complement of the diagonal blocks is $0(\gamma_r)$. Thus for γ_r sufficiently small, it will suffice to get bounds on

$$\left[T_{N_{r-1}}^{\sigma+n_r\lambda'_r,(\lambda'_1,\ldots,\lambda'_{r-1})}\right]^{-1} \quad \text{for} \quad |n_r| < N_{r-1}\,. \tag{6.64}$$

Thus we aim to apply the induction hypothesis with now σ replaced by

$$\overline{\sigma} = n_r\ \lambda'_r + \sum_{r<s\leq r'} c_s\,\lambda'_s + \omega\ \lambda'_{s_1} \qquad (s_1 > r', \omega = 0, 1, -1)\,. \tag{6.65}$$

Consider the cases in which the induction hypothesis is not applicable.

If $|n_r| + \sum_{r<s\leq r'} |c_s| > A_{r-1}$, (6.61) implies that $|n_r| > \frac{1}{2}\ A_{r-1}$ and one may keep (6.57) away from zero considering $\frac{\partial^d}{(\partial\,\lambda'_r)^d}\ \{p(\sigma_1)\}$.

Assume $\omega \neq 0$ and $c_{s_0} \neq 0$ for some $(r-1)' < s_0 \leq r'$. Write

$$\sigma_1 = \sum_{s=1}^{r} n_s\ \lambda'_s + \sum_{r<s\leq(r-1)'} c_s\ \lambda'_s + \sum_{(r-1)'<s\leq r'} c_s\ \lambda'_s + \omega\ \lambda'_{s_1} \pm \lambda_j\,. \tag{6.66}$$

Assume first $j \neq s_0$. We may then proceed by differentiation wrt λ'_{s_0}, since the polynomial coefficients in (6.57) are essentially independent of λ_k for $k > (r-1)'$. (Keep in mind that eventually λ_k and λ'_k are subject to a relation (6.6), i.e. $\lambda' = \lambda + \varepsilon(\lambda)$.)

If $j = s_0$, s_1 has to be bounded in terms of v and we proceed by differentiation in λ'_{s_1}.

If now $\omega = 0$, necessarily $\sum_{(r-1)'<s\leq r'} |c_s| \geq 2$ (otherwise the induction hypothesis applies) and hence, either $c_{s_0} \neq 0$ for some $(r-1)' < s_0 \leq r'$, $s_0 \neq j$ or $|c_{s_0}| \geq 2$ for some $(r-1)' < s_0 \leq r'$. In both cases, we may again differentiate wrt λ'_{s_0}.

This concludes the induction and in particular the control of $[T_N^{\sigma,(\lambda'_1,\ldots,\lambda'_r)}]^{-1}$ for $\sigma \in \{0\} \cup \{\pm\lambda'_s \mid s > r\}$.

(vi) *Q-equations*

It remains to determine the new frequencies $\lambda' = (\lambda'_j)_{j=1,2,\dots}$ by projecting the equation (6.1) on the resonant set $S = \{(j, \pm e_j)\}$. Thus $\lambda' = \lambda'(\lambda)$ has to be determined from the infinite system

$$(\lambda_j^2 - (\lambda'_j)^2)\, a_j + \varepsilon\, \widehat{f(u)}\, (j, e_j) = 0\,. \tag{6.67}$$

Here $\widehat{u}$ is constructed from solving the P-equation, hence depends also on the pair (λ, λ').

It is however clear from the construction that the dependence on λ'_k is $0(|a_k|)$ for $k \to \infty$. Rewrite (6.67) as

$$\lambda'_j = \lambda_j + \frac{\varepsilon}{(\lambda_j + \lambda'_j)\, a_j}\, \widehat{f(u)}\, (j, e_j)\,. \tag{6.68}$$

Our aim is to solve (6.68) in $\lambda' = \lambda + \varepsilon(\lambda)$ by the standard implicit function theorem. This will be possible provided we show that $\frac{\widehat{f(u)}\,(j,e_j)}{a_j}$ has derivatives in $\lambda'_k \to 0$ for increasing k (independently on how small a_j is and uniformly in j). Recall that (for simplicity) $f(u)$ was assumed to be a polynomial in u. Also, from construction of u

$$\widehat{u}\, (j', n)\, |_{a_j = 0} = 0 \quad \text{if} \quad n_j \neq 0\,. \tag{6.69}$$

Hence $\widehat{f(u)}\, (j, e_j)\, |_{a_j=0} = 0$ and thus

$$\frac{\widehat{f(u)}\, (j, e_j)}{a_j} = \partial_{a_j}\, \widehat{f(u)}\, (j, e_j) + 0(a_j)\,. \tag{6.70}$$

Thus it amounts to control $\frac{\partial \widehat{u}}{\partial a_j}\, (j', n)$, $n_j \neq 0$. Recall that the function u is obtained from a Newton iteration scheme. At a given stage in this construction, one introduces the new frequency λ'_j and considers an approximative solution

$$u = u_0 + a_j\, \varphi_j(x)\, \cos \lambda'_j\, t + U \quad , \quad U = U_0 + U_1 + U_{-1} \tag{6.71}$$

where

$$U_0(t) = \sum{}'\, \widehat{U}_0\, (n)\, e^{i(n,\lambda')t} \tag{6.72}$$

$$U_1(t) = \sum{}'\, \widehat{U}_1\, (n)\, e^{i(n,\lambda')t}\, e^{i\lambda'_j t} \tag{6.73}$$

$$U_{-1}(t) = \sum{}'\, \widehat{U}_{-1}\, (n)\, e^{i(n,\lambda')t}\, e^{-i\lambda'_j t} \tag{6.74}$$

u_0 is the approximative solution obtained at previous stage of the Newton iteration $n = (n_1, \dots, n_{j-1})$ ranges in some bounded domain and $\widehat{U}_{-1}\,(n) = \overline{\widehat{U}_1(-n)}$.

Write equation (6.1) projected on the complement of S (P-equations) as

$$F(u)=0\,. \tag{6.75}$$

One has thus

$$F(u)=F(u_0+a_j\ \varphi_j(x)\ \cos\lambda_j'\,t)+F'(u_0)\ U+0\,(\|U\|^2) \tag{6.76}$$

$$=F(u_0)+a_j\ F'(u_0)\ (\varphi_j(x)\ \cos\lambda_j'\,t)+F'(u_0)\ U+0\,(\|U\|^2)+0\,(|a_j|^2) \tag{6.77}$$

and determines U from the equations

$$\begin{cases} F(u_0)+F'(u_0)\ U_0=0 \\ \frac{1}{2}\ a_j\ F'(u_0)\ (\varphi_j(x)\ e^{i\lambda_j' t})+F'(u_0)\ U_1=0 \\ \frac{1}{2}\ a_j\ F'(u_0)\ (\varphi_j(x)\ e^{-i\lambda_j' t})+F'(u_0)\ U_{-1}=0\,. \end{cases} \tag{6.78}$$

After passing to Fourier transform this corresponds to

$$\widehat{U}_0=-\,[T_N^{0,(\lambda_1',\ldots,\lambda_{j-1}')}]^{-1}\ [\widehat{F(u_0)}] \tag{6.79}$$

$$\widehat{U}_1=-\frac{1}{2}\ a_j\ [T_N^{\lambda_j',(\lambda_1',\ldots,\lambda_{j-1}')}]^{-1}\ [F'(u_0)\,(\varphi_j\ e^{i\lambda_j' t})]^\wedge \tag{6.80}$$

$$\widehat{U}_{-1}=-\frac{1}{2}\ a_j\ [T_N^{-\lambda_j',(\lambda_1',\ldots,\lambda_{j-1}')}]^{-1}\ [F'(u_0)\ (\varphi_j\ e^{-i\lambda_j' t})]^\wedge \tag{6.81}$$

(for some value of N, $N\sim B^s$ if λ_j' is introduced at stage s of the Newton iteration). Thus

$$\|U\|\le N^{c_j}\ (\|\widehat{F(u_0)}\|+|a_j|) \tag{6.82}$$

and from (6.77) u will be an approximative solution up to

$$\|U\|^2<\|\widehat{F}(u_0)\|^{2-\delta}+|a_j|^{2-\delta}\,. \tag{6.83}$$

Coming back to (6.68), the main contribution to $\widehat{f(u)}\ (j,e_j)$ will be given by

$$\left(f'(u_0)\ \left(\frac{1}{2}\ a_j\ \varphi_j(x)\ e^{i\lambda_j' t}+U_1\right)\right)^\wedge\ (j,e_j)=$$
$$\frac{1}{2}\ a_j\ \widehat{f'(u_0)\varphi_j}\ (j,0)+(f'(u_0)\cdot U_1)^\wedge\ (j,e_j)\,. \tag{6.84}$$

Thus the second term in (6.68) yields, by (6.80)

$$0\left(\frac{\varepsilon}{j}\right)\ \widehat{f'(u_0)\varphi_j}\ (j,0)+0\left(\frac{\varepsilon}{j}\right)\ \left\{\widehat{f'(u_0)}*\left[(T_N^{\lambda_j',(\lambda_1',\ldots,\lambda_{j-1}')})^{-1}\ \widehat{v}\right]\right\}\ (j,0) \tag{6.85}$$

where, since $\widehat{U}$ is obtained from solving the P-equation,

$$\widehat{v} = P_{\{(j,0)\}^c}\ \left[(\lambda_j^2 - (\lambda_j')^2)\,\varphi_j + \varepsilon\ f'(u_0)\ \varphi_j\right]^\wedge = \varepsilon\ P_{\{(j,0)\}^c}\ [f'(u_0)\ \varphi_j]^\wedge \quad (6.86)$$

and $*$ with respect to the j'-index needs to be interpreted according to the multiplication of $\varphi_{j'}$ eigenfunctions.

We need to analyze the dependence of the expression between { } in (6.85) on $\lambda_1', \lambda_2', \ldots$. Recall that $\widehat{u}_0$, produced along the Newton iterative process in solving the P-equation, has a λ_{j_0}'-dependence $0(|a_{j_0}|)$, for $a_{j_0} \to 0$.

Next, consider the operator $T = T_N^{\lambda_j', (\lambda_1', \ldots, \lambda_{j-1}')}$. The estimates on $[T_{N'}^{\lambda_j', \lambda'}]^{-1}$ obtained in previous section (which are uniform in j ($\sigma = \lambda_j'$)) together with the localization identity for the inverse permit to control T^{-1}.

Assume first $j_0 < j$. Decompose the index set $\Omega = \{(n_k)_{1 \le k \le j-1} \mid |n_k| < N\}$ as

$$\Omega = \Omega_1 \cup \Omega_2 \qquad (6.87)$$

where

$$\Omega_1 = \{n \in \Omega \mid n_{j_0} = n_{j_0+1} = \cdots = n_{j-1} = 0\}\,. \qquad (6.88)$$

Write further (cf. (6.85))

$$\widehat{v} = \xi + \eta \qquad (6.89)$$

where $\xi = \{\xi(j', n)\}$ with $n = (n_1, \ldots, n_{j_0-1})$ is λ_{j_0}'-independent and

$$\|\eta\| = 0(a_{j_0})\,. \qquad (6.90)$$

One has formally

$$\frac{\partial(T^{-1}\,\widehat{v})}{\partial\lambda_{j_0}'} = -T^{-1}\ \frac{\partial T}{\partial\lambda_{j_0}'}\ T^{-1}\,\widehat{v} + T^{-1}\ \frac{\partial\eta}{\partial\lambda_{j_0}'}\,. \qquad (6.91)$$

The second term in (6.91) is $0(|a_{j_0}|^{1-\delta})$, from (6.90).

Consider $\frac{\partial T}{\partial\lambda_{j_0}'}$ and split T in its diagonal and off-diagonal part. For the off-diagonal part

$$\frac{\partial}{\partial\lambda_{j_0}'}\ S_{f'(u_0)} = 0(a_{j_0}) \qquad (6.92)$$

and hence so is the corresponding contribution to the first term in (6.91).

It remains to consider

$$T^{-1}\ \frac{\partial D}{\partial\lambda_{j_0}'}\ T^{-1}\,\widehat{v} = T^{-1}\ \frac{\partial D_{\Omega_2}}{\partial\lambda_{j_0}'}\ T^{-1}\,\widehat{v} \qquad (6.93)$$

where $\frac{\partial D_{\Omega_2}}{\partial \lambda'_{j_0}}$ is the diagonal operator with elements

$$2\left(\sum_{k=1}^{j_1} n_k\,\lambda'_k + \lambda'_j\right) n_{j_0} \quad \text{for} \quad n \in \Omega_2\,. \tag{6.94}$$

Write

$$(6.93) = T^{-1}\,\frac{\partial D_{\Omega_2}}{\partial \lambda'_{j_0}}\,T^{-1}\,\xi + T^{-1}\,\frac{\partial D_{\Omega_2}}{\partial \lambda'_{j_0}}\,T^{-1}\,\eta\,. \tag{6.95}$$

The contribution of the second term in (6.95) is $0(|a_{j_0}|^{1-\delta})$, by (6.90) and decay (resp. off diagonal decay) estimates for η (resp. T^{-1}).

For the first term in (6.95), write from the localization identity

$$\begin{aligned} T^{-1}\,\frac{\partial D_{\Omega_2}}{\partial \lambda'_{j_0}}\,T^{-1}\,\xi &= -T^{-1}\,\frac{\partial D_{\Omega_2}}{\partial \lambda'_{j_0}}\,T^{-1}_{\Omega_2}\,(T - T_{\Omega_1} - T_{\Omega_2})\,T^{-1}\,\xi \\ &= -\,\varepsilon\,T^{-1}\,\frac{\partial D_{\Omega_2}}{\partial \lambda'_{j_0}}\,T^{-1}_{\Omega_2}\,(S_{f'(u_0)|\Omega_2\times\Omega_1})\,T^{-1}\,\xi\,. \end{aligned} \tag{6.96}$$

Since

$$\|S_{f'(u_0)|\Omega_2\times\Omega_1}\| = 0(a_{j_0}) \tag{6.97}$$

also (6.96) is bounded by $0(|a_{j_0}|^{1-\delta})$.

Summarizing the preceding, we get a dependence of $(T_N^{\lambda'_j,(\lambda'_1,\ldots,\lambda'_{j-1})})^{-1}\,\widehat{v}$ and hence (6.85) on λ'_{j_0} at most $0(|a_{j_0}|^{1-\delta})$, if $j_0 < j$ (with estimates uniform in j).

If now j_0, j, in skip the decomposition $\Omega = \Omega_1 \cup \Omega_2$, letting $\Omega_1 = \Omega$ dependence of $T = T_N^{\lambda'_j,(\lambda'_1,\ldots,\lambda'_{j-1})}$ on λ'_j appears from the diagonal, since the diagonal of T is

$$\left(\sum_{k=1}^{j-1} n_k\;\lambda'_k + \lambda'_j\right)^2 - (\lambda_{j'})^2\,. \tag{6.98}$$

Writing according to (6.86)

$$(T_N^{\lambda'_j,(\lambda'_1,\ldots,\lambda'_{j-1})})^{-1}\;\widehat{v} = \varepsilon\;T^{-1}\{P\,[f'(u_0)\;\varphi_j]^\wedge\} \tag{6.99}$$

and since formally

$$\frac{\partial\,T^{-1}}{\partial\,\lambda'_j} = T^{-1}\,\frac{\partial\,T}{\partial\,\lambda'_j}\;T^{-1} \tag{6.100}$$

it follows from (6.98) that [(*)]

$$\left\|\frac{\partial}{\partial\,\lambda'_j}\;(T^{-1}(P\,(f(u_0)\;\varphi_j)^\wedge))\right\| = 0(j)\,. \tag{6.101}$$

(*) In fact this bound may be improved to $\frac{1}{j^{1-\delta}}$ by a more careful analysis of T^{-1}.

Hence (6.99) has a dependence on λ'_j which is $0(\varepsilon\, j)$ and for (6.85) we get $0(\varepsilon^2)$-dependence.

Thus in conclusion, one may affirm that

$$\begin{cases} \frac{\partial}{\partial \lambda'_{j_0}}\,(6.85) & = \quad 0(|a_{j_0}|^{1-\delta}) \quad \text{for} \quad j_0 \neq j \\ & = \quad 0(\varepsilon) \qquad\qquad \text{for} \quad j_0 = j\,. \end{cases} \tag{6.102}$$

The same holds for the second term in (6.68). Consequently, the implicit function theorem is clearly applicable and yields

$$\lambda' = \lambda + \varepsilon(\lambda)\,. \tag{6.103}$$

Finally, it remains to observe that the conditions on (λ, λ') appearing in solving the P-equation will be fulfilled for the pair $(\lambda, \lambda + \varepsilon(\lambda))$ and λ in a Cantor type of positive measure. These considerations are similar to [B].

This completes the construction of a quasi-periodic solution (6.4) of (6.1). By letting the sequence of coefficients $\{a_j\}$ in (6.3) go fast enough to zero, one obtains the "admissible" set of frequencies $\lambda = (\lambda_j)$, $\lambda_j = \pi\, j + 0\left(\frac{1}{j}\right)$ by an excision of a subset of the parameter set which may be given asymptotically arbitrarily small measure. Hence, one may claim that for given sufficiently fast decaying sequence $\{a_j\}$, the sequence $\lambda_j = \sqrt{\mu_j}$ obtained from the Dirichlet spectrum $\{\mu_j\}$ of $-\frac{d^2}{dx^2} + V(x)$, V a "generic" real analytic periodic potential, will be admissible.

Proposition 6.104. *Given a sequence* $\{a_j\}$ *of reals such that* $|a_j| \to 0$ *sufficiently fast, one may for typical real analytic* V *obtain an almost periodic solution of* (6.1) *satisfying conditions* (6.5)-(6.8).

Remark. In [C-P] almost periodic solutions of infinite dimensional KAM problems are constructed, but under strong restrictions on the frequencies $(\lambda_j \sim |j|)^c$. In the present context, one obtains a "full" set of frequencies. However, there are strong decay conditions on the initial sequence of amplitudes $\{a_j\}$ in (6.3), which we did not attempt to make more explicit in previous argument. It looks worthly to try to reorganize it in order to get results for a less restrictive assumption.

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