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CONSTRUCTION OF β-CONTENT TOLERANCE REGIONS AT CONFIDENCE LEVEL γ FOR LARGE SAMPLES FROM THE k-VARIATE NORMAL DISTRIBUTION[1]

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1. Introduction. In this paper, we examine the following problem. Suppose n independent observations $\mathbf{X}_1, \cdots, \mathbf{X}_n$, are taken on a vector-random variable $\mathbf{Y}$, where the k-dimensional vector-random variable $\mathbf{Y} = N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, that is, $\mathbf{Y}$ is a normal k-variate random variable with mean vector $\boldsymbol{\mu}$ and variance-covariance matrix $\boldsymbol{\Sigma}$, assumed to be positive definite.

Suppose further that we wish to construct a tolerance region $S(\mathbf{X}_1, \cdots, \mathbf{X}_n) \subset R_k$ which is of β-content at confidence level γ, that is,

$$\Pr\{P_{\mu,\Sigma}[S(\mathbf{X}_1, \cdots, \mathbf{X}_n)] \geqq \beta\} \geqq \gamma \tag{1.1}$$

where

$$C = P_{\mu,\Sigma}[S] = \Pr(\mathbf{Y} \in S \mid \mathbf{Y} = N(\boldsymbol{\mu}, \boldsymbol{\Sigma})) \tag{1.2}$$

denotes the *coverage* of the region S. A region S that is used in many practical applications (see Shewhart (1939), Paulson (1943), and Owen (1963)) is of the form

$$S(\mathbf{X}_1, \cdots, \mathbf{X}_n) = \{\mathbf{Y} \mid (\mathbf{Y}-\overline{\mathbf{X}})'V^{-1}(\mathbf{Y}-\overline{\mathbf{X}}) \leqq K^{(k)}\} \tag{1.3}$$

where

$$\overline{\mathbf{X}} = (\overline{X}_1, \cdots, \overline{X}_k)' = n^{-1}(\textstyle\sum_{i=1}^n X_{i1}, \cdots, \sum_{i=1}^n X_{ik})' = n^{-1}\sum_{i=1}^n \mathbf{X}_i$$

$$V = (n-1)^{-1}\textstyle\sum_{i=1}^n (\mathbf{X}_i-\overline{\mathbf{X}})(\mathbf{X}_i-\overline{\mathbf{X}})',$$

and $X_{ij} =$ the jth component of $\mathbf{X}_i$, $j = 1, \cdots, k$, $i = 1, \cdots, n$, with $K^{(k)} =$ a positive constant chosen to give the region defined by (1.3) the property (1.1).

As a matter of fact, it is well known (see for example Fraser and Guttman (1956)) that for $n > k$, that setting

$$K^{(k)} = [(n-1)k/(n-k)][1+n^{-1}]F_{k,n-k,1-\delta} \tag{1.4}$$

where $F_{m,n;\alpha}$ is the point exceeded with probability α when using the F-distribution with m and n degrees of freedom, gives the tolerance region (1.3) the property of δ-expectation; that is

$$E[P_{\mu,\Sigma}[S]] = \delta. \tag{1.5}$$

If $K^{(k)}$ is so chosen we may interpret (1.3), (1.4) and (1.5) in the usual fashion, *viz.*, S serves as an *estimator* of the "central" δ-set $A_c^{(k)}$ of the normal population being sampled, where

$$A_c^{(k)} = \{\mathbf{Y} \mid (\mathbf{Y}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{Y}-\boldsymbol{\mu}) \leqq \chi^2_{k;1-\delta}\} \tag{1.6}$$

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and where $\chi^2_{k;\alpha}$ is the point exceeded with probability α when using the Chi-Square distribution with k degrees of freedom; we note then (see (1.2)) that

$$P_{\mu,\Sigma}[A_c^{(k)}] = \delta. \tag{1.7}$$

As is usual in problems of estimation, we may now wish to make statements of the form (1.1), or of the form

$$\Pr\{\beta_1 \leqq P_{\mu,\Sigma}[S] \leqq \beta_2\} \geqq \gamma_0; \tag{1.8}$$

this implies that we must find the distribution (and its properties) of $P_{\mu,\Sigma}[S]$, the probability content or coverage of the random region S.

For the case $k = 1$, the distributional problem may be by-passed by the use of an approximation due to Wald and Wolfowitz (1946). Their approximation for the value of $K^{(1)}$ needed to make the interval S (defined by (1.3) with $k = 1$) β-content at confidence level γ works well even for n as low as 2. Unfortunately, the Wald and Wolfowitz approximation does not carry over for the cases of $k \geqq 2$. In this paper, then, we develop an approximation for $K^{(k)}$, specifically for large n and $k \geqq 2$. When the approximation method is applied for the case $k = 1$, satisfactory agreement results with the Wald–Wolfowitz approximation.

2. A useful theorem. Using the notation of the previous section of this paper, we now state the following theorem.

THEOREM. *Based upon samples from the k-variate normal distribution, the tolerance region S, given by* (1.3), *has coverage C whose mean and variance are, to terms of order* n^{-1},

$$\mu_C = \Psi_k(K^{(k)}) - [K^{(k)}]^{\frac{1}{2}k}(\exp\{-\tfrac{1}{2}K^{(k)}\})[2^{\frac{1}{2}k+1}\Gamma(\tfrac{1}{2}k)n]^{-1} \quad \textit{and} \tag{2.1a}$$

$$\sigma_C^2 = [K^{(k)}]^k(\exp\{-K^{(k)}\})[k2^{k-1}\Gamma^2(\tfrac{1}{2}k)n]^{-1} \quad \textit{respectively where} \tag{2.1b}$$

$$\Psi_k(K^{(k)}) = \Pr(\chi_k^2 \leqq K^{(k)}). \tag{2.2}$$

PROOF. We shall be concerned with the coverage C of the tolerance region $S = \{\mathbf{Y} \mid U^2 = (\mathbf{Y}-\bar{\mathbf{X}})'V^{-1}(\mathbf{Y}-\bar{\mathbf{X}}) \leqq K^{(k)}\}$, that is, with the random variable C which is given by

$$C = \int_{U^2 \leqq K^{(k)}} (2\pi)^{-\frac{1}{2}k}|\Sigma^{-1}|^{\frac{1}{2}}\exp\{-\tfrac{1}{2}(\mathbf{Y}-\boldsymbol{\mu})'\Sigma^{-1}(\mathbf{Y}-\boldsymbol{\mu})\}\,dY_k \cdots dY_1 \tag{2.3}$$

where we assume Σ to be positive definite. This of course implies that Σ^{-1} is also positive definite, and it is easily shown that we may write C as

$$C = \int_{U^{*2} \leqq K^{(k)}} (2\pi)^{-\frac{1}{2}k}\exp\{-\tfrac{1}{2}\mathbf{Y}^{*\prime}\mathbf{Y}^*\}\,dY_k^* \cdots dY_1^* \tag{2.4}$$

with $U^{*2} = (\mathbf{Y}^*-\bar{\mathbf{X}}^*)'V^{*-1}(\mathbf{Y}^*-\bar{\mathbf{X}}^*)$, and where

$$\begin{aligned} \mathbf{Y}^* &= N(\mathbf{O}, I_k) \\ \bar{\mathbf{X}}^* &= N(\mathbf{O}, n^{-1}I_k) \\ V^* &= W[(n-1)^{-1}I_k, n-1] \end{aligned} \tag{2.5}$$

(that is, V^* is distributed as the Wishart variable with variance-covariance matrix $(n-1)^{-1}I_k$, and degrees of freedom $n-1$ (we have tacitly assumed that $n > k+1$)).

We denote the (i, j)th element of V^* by V_{ij}^* if $i \neq j$, and V_i^{*2} if $i = j$. Further, we let

$$W_i = Y_i^* - \bar{X}_i^*, \qquad V^{*-1} = (c_{ij}), \qquad c_{ij} = A_{ji}/|V| \tag{2.6}$$

where A_{rs} = co-factor of $V_{rs}^* = A_{sr}$.

We may thus write

$$U^{*2} = \mathbf{W}'V^{*-1}\mathbf{W} = \textstyle\sum_{i=1}^k \sum_{j=1}^k c_{ij} W_i W_j \tag{2.7}$$

and it is easy to see that we may write (2.7) in turn as

$$U^{*2} = \textstyle\sum_{t=2}^k c_{tt}^{(k-t)}[W_t + \sum_{j=1}^{t-1} c_{tj}^{(k-t)} W_j / c_{tt}^{(k-t)}]^2 + c_{11}^{(k-1)} W_1^2 \qquad \text{where} \tag{2.8}$$

$$c_{ij}^{(0)} = c_{ij}, \qquad i,j = 1, \cdots, k \tag{2.8a}$$

$$c_{ij}^{(r)} = c_{ij}^{(r-1)} - c_{k-r+1,i}^{(r-1)} c_{k-r+1,j}^{(r-1)} / c_{k-r+1,k-r+1}^{(r-1)},$$

when $r = 1, \cdots, k-1$ and $i, j = 1, \cdots, k-r$.

Using the above we may now write (2.3) or (2.4) as

$$C = \textstyle\int_{l_{11}}^{l_{21}} \cdots \int_{l_{1r}}^{l_{2r}} \cdots \int_{l_{1k}}^{l_{2k}} [\prod_{j=1}^k \phi(Y_j^*)]\, dY_k^* \cdots dY_r^* \cdots dY_1^* \tag{2.9}$$

where $\phi(z) = (2\pi)^{-\frac{1}{2}} \exp\{-\frac{1}{2}z^2\}$, with

$$\begin{aligned} l_{ir} = \bar{X}_r^* - \textstyle\sum_{j=1}^{r-1} c_{rj}^{(k-r)}(Y_j^* - \bar{X}_j^*)/c_{rr}^{(k-r)} \pm (c_{rr}^{(k-r)})^{-\frac{1}{2}} \\ \cdot \{K^{(k)} - [\textstyle\sum_{t=2}^{r-1} c_{tt}^{(k-t)}(Y_t^* - \bar{X}_t^* + \sum_{j=1}^{t-1} c_{tj}^{(k-t)}(Y_j^* - \bar{X}_j^*)/c_{tt}^{(k-t)})^2] \\ - c_{11}^{(k-1)}(Y_1^* - \bar{X}_1^*)^2\}^{\frac{1}{2}} \end{aligned} \tag{2.10}$$

for $r = 3, \cdots, k$, and

$$\begin{aligned} l_{i2} = \bar{X}_2^* - c_{21}^{(k-2)}(Y_1^* - \bar{X}_1^*)/c_{22}^{(k-2)} \\ \pm (c_{22}^{(k-2)})^{-\frac{1}{2}}\{K^{(k)} - c_{11}^{(k-1)}(Y_1^* - \bar{X}_1^*)^2\}^{\frac{1}{2}} \\ l_{i1} = \bar{X}_1^* \pm (c_{11}^{(k-1)})^{-\frac{1}{2}}[K^{(k)}]^{\frac{1}{2}}, \end{aligned} \tag{2.11}$$

where we associate the plus sign in the above formulas with $i = 2$, and the minus sign for $i = 1$. We note here that l_{ir} does not involve $\bar{X}_{r+1}^*, \cdots, \bar{X}_k^*$.

Now C is a function of

$$(\bar{\mathbf{X}}^{*\prime}, \mathbf{V}_j^{*\prime}, V_{ij}^{*\prime}) = (\bar{X}_1^*, \cdots, \bar{X}_k^*, V_1^*, \cdots, V_k^*, V_{12}^*, V_{13}^*, \cdots, V_{k-1,k}^*) \tag{2.12}$$

where we recall that $V_i^* = +(V_{ii}^*)^{\frac{1}{2}}$. We now proceed by expanding C in a Taylor Series about the point

$$(\mathbf{X}^{*\prime}, \mathbf{V}_i^{*\prime}, \mathbf{V}_{ij}') = (0, \cdots, 0, 1, \cdots, 1, 0, \cdots, 0). \tag{2.13}$$

We will denote this point by (*). It is easy to see that C evaluated at (*) is given by

$$\begin{aligned} C((*)) &= \textstyle\int_{Y^* \cdot Y^* \leqq K^{(k)}} (2\pi)^{-\frac{1}{2}k} \exp\{-\frac{1}{2}\mathbf{Y}^* \cdot \mathbf{Y}^*\}\, dY_k^* \cdots dY_1^* \\ &= \Pr(\chi_k^2 \leqq K^{(k)}) = \Psi_k(K^{(k)}). \end{aligned} \tag{2.14}$$

Now we show in the Appendix that

$$\text{(2.15)}\qquad \left.\frac{\partial C}{\partial \bar{X}_\alpha^*}\right|_{(*)} = 0; \qquad \left.\frac{\partial C}{\partial V_{\alpha\beta}^*}\right|_{(*)} = 0, \qquad \alpha \neq \beta;$$

$$\left.\frac{\partial C}{\partial V_\alpha^*}\right|_{(*)} = (\exp\{-\tfrac{1}{2}K^{(k)}\})[K^{(k)}]^{\frac{1}{2}k}[2^{\frac{1}{2}k-1}\Gamma(\tfrac{1}{2}k)k]^{-1}.$$

(The relations (2.15) above are established with the aid of a theorem due to Coxeter (1948)—see the Appendix.) From (2.14) and (2.15), we have that, approximately,

$$\text{(2.16)}\quad C = \Psi_k(K^{(k)}) + [K^{(k)}]^{\frac{1}{2}k}(\exp\{-\tfrac{1}{2}K^{(k)}\})[k2^{\frac{1}{2}k-1}\Gamma(\tfrac{1}{2}k)]^{-1} \cdot [(V_1^*-1)+\cdots+(V_k^*-1)].$$

Now using results of Romanovsky (1925), we can show that

$$\text{(2.17)}\qquad E(V_j^*) = 1-4n^{-1}+O(n^{-2})$$
$$\operatorname{Var}(V_j^*) = \tfrac{1}{2}n+O(n^{-2}).$$

From (2.16), then, we have to terms of order n^{-1},

$$\text{(2.18)}\qquad \begin{aligned} E(C) &= \Psi_k(K^{(k)}) + [K^{(k)}]^{\frac{1}{2}k}(\exp\{-\tfrac{1}{2}K^{(k)}\})[k2^{\frac{1}{2}k-1}\Gamma(\tfrac{1}{2}k)]^{-1} \\ &\qquad \cdot\{E(V_1^*-1)+\cdots+E(V_k^*-1)\}, \\ &= \Psi_k(K^{(k)}) - [K^{(k)}]^{\frac{1}{2}k}(\exp\{-\tfrac{1}{2}K^{(k)}\})[2^{\frac{1}{2}k+1}\Gamma(\tfrac{1}{2}k)n]^{-1}. \end{aligned}$$

If we now square (2.16), take expectations, and subtract from this the square of $E(C)$ where $E(C)$ is given by the first line of (2.18), we find

$$\text{(2.19)}\quad \begin{aligned} \operatorname{Var}(C) &= E(C^2)-(E(C))^2 \\ &= (K^{(k)})^k e^{-K^{(k)}}[k^2 2^{k-2}\Gamma^2(\tfrac{1}{2}k)]^{-1}[\operatorname{Var}(V_1^*)+\cdots+\operatorname{Var}(V_k^*)] \end{aligned}$$

and using (2.17) we have, to terms of order n^{-1},

$$\text{(2.20)}\qquad \operatorname{Var}(C) = (K^{(k)})^k e^{-K^{(k)}}[k2^{k-1}\Gamma^2(\tfrac{1}{2}k)n]^{-1}$$

and the theorem is proved.

3. Approximation of $K^{(k)}$. Now that we have the mean and variance of the coverage C of the region S to terms of order n^{-1}, we may use this information to approximate the distribution of C by $I_{(p,q)}$, the Incomplete Beta with parameters (p, q). Such a distribution, as is well known, has mean and variance given by, respectively,

$$\text{(3.1)}\qquad \mu_1' = p/(p+q) \quad \text{and} \quad \mu_2 = pq/(p+q)^2(p+q+1).$$

Equating μ_1' and μ_2 to μ_C and σ_C^2 of (2.1), respectively, gives

$$\text{(3.2)}\qquad \begin{aligned} p &= [\mu_C^2(1-\mu_C)-\mu_C\sigma_C^2]/\sigma_C^2 \\ q &= [\mu_C(1-\mu_C)^2-(1-\mu_C)\sigma_C^2]/\sigma_C^2 \end{aligned}$$

that is, we approximate for large n, the distribution of C by $I(p, q)$, with (p, q) given by (3.2), and where (μ_C, σ_C^2) are defined by (2.1). Of course, this now means that (p, q) are functions of $K^{(k)}$, and so we may approximate for (large) given n, the value of $K^{(k)}$ needed to make the tolerance region S, given by (1.3), β-content at level γ, where

$$\gamma = \int_\beta^1 \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} t^{p-1}(1-t)^{q-1}\,dt \tag{3.3}$$

Tables 3.1(a, b, and c) give values of $K^{(k)}$, $k = 2$, 3 and 4, for selected $n \geqq 100$, that satisfy (3.3), with γ and β set equal to .75, .90, .95 and .99. Starting guesses were made by setting $K^{(k)} = K_0^{(k)} = \chi^2_{k;1-\beta}$ and iterating until the desired γ is achieved. The incomplete beta function calculations necessary for evaluating (3.3) were done using a subroutine of Milton (1967) called BETA INC. The basic method of evaluation is by a continued fraction expansion (see for example, Abramowitz and Stegun (1964), Section 26.5.8, page 944).

As a check on these computations, a table of constants $K^{(1)}$ was computed for the same set of n, β and γ as above and by the same method. The square root was then extracted and the resulting set of constants were compared to the answers provided by the Wald–Wolfowitz approximation as tabulated by Bowker (1947). The largest percentage error occurs for the $n = 100$, $\gamma = \beta = .99$ entry and equals 4.1%. The percentage error for increasing n then drops off quite quickly—for example, if $n = 1{,}000$ and $\beta = \gamma = .99$, the percentage error is only 0.72%.

4. The β-content of δ-expectation tolerance regions. As mentioned in Section 1, if we set $K^{(k)}$ equal to the value given by (1.4), the tolerance region S defined by (1.3) is of δ-expectation, that is, its coverage C is such that $E[C] = \delta$. It is of interest to find the confidence γ that we have in such a region having coverage at least equal to β. To help us answer this question, we may proceed as in the previous section and approximate the distribution of the coverage C of such a region by the Incomplete beta distribution, with parameters $p = p(\mu_C, \sigma_C^2)$ and $q = q(\mu_C, \sigma_C^2)$ given by (3.2), where μ_C and σ_C^2 are defined by (2.1), with $K^{(k)}$ of course, specified by (1.4). But since $E(C) = \delta$ when $K^{(k)}$ is indeed given by (1.4), the value of μ_C given by (2.1 a) is in error. It turns out however, that for $k = 2$, 3 and 4, with $\delta = .75$ and $n \geqq 100$, that the percentage error is less than 0.70%, and for $\delta \geqq .90$, less that 0.60% (the largest errors occur for $k = 4$). Because of this, we modify the above approximation and approximate the distribution of C by the Incomplete beta with p and q given by

$$p = p(\delta; \sigma_C^2) \quad \text{and} \quad q = q(\delta; \sigma_C^2) \tag{4.1}$$

where p and q are defined in (3.2), and σ_C^2 is given by (2.1 b); we can now compute γ, where, once again,

$$\gamma = \Pr(C \geqq \beta) \tag{4.2}$$

with $E(C) = \delta$, by using this modified approximation. Tables 4.1 (a, b, and c) gives the values of (4.2) so obtained for $\beta = \delta = .75$, .90, .95 and .99, for selected values of $n \geqq 100$, and $k = 2$, 3 and 4, respectively.

5. Some efficiency considerations. Suppose we are sampling on a k-dimensional random variable and that we wish to construct a tolerance region with ability to pick up the center $100\delta\%$ of the population being sampled, and also, that it is a δ-expectation region. If the functional form of the distribution of the population is assumed continuous but otherwise unknown, then we may construct a distribution-free tolerance region (see for example, Tukey (1947), and/or Fraser (1951)) composed of $p = n-m+1$ "inner blocks", which means discarding $q = m$ outer blocks.

For example, if $k = 1$, and $(X_1, \cdots, X_n)$ is the sample of n independent observations from a population with continuous cumulative distribution function $F(x)$, with $(X_{(1)}, \cdots, X_{(n)})$ the order statistics, then we could use as the tolerance region the interval

$$S(X_1, \cdots, X_n) = (X_{(r)}, X_{(n-r+1)}) \tag{5.1}$$

where $2r = m$, and m satisfies condition (5.2) below, so that S is of δ-expectation.

Now as is well known, the coverage C_{DF} of such a distribution-free tolerance region, has as its distribution, the Incomplete Beta with parameters $(p, q) = (n-m+1, m)$. Hence, if we wish such a region to be of δ-expectation we now impose the restriction

$$n-m+1 = (n+1)\delta, \quad \text{or} \quad m = (n+1)(1+\delta) \tag{5.2}$$

since the mean of the distribution is $p/(p+q) = (n-m+1)/(n+1) = \delta$. Now the variance of the distribution is $pq/(p+q)^2(p+q+1)$, that is

$$\mathrm{Var}(C_{DF}) = (n-m+1)m/(n+1)^2(n+2) = \delta(1-\delta)/(n+2). \tag{5.3}$$

Now if the distribution function has known functional form, and if this known functional form is that of the k-variate normal, then we would construct a tolerance region of the form (1.3), with $K^{(k)}$ given by (1.4). We have from the theorem of Section 2 that the coverage, say C_N, of this tolerance region has variance given by

$$\mathrm{Var}(C_N) = [K^{(k)}]^k(\exp\{-K^{(k)}\})[k2^{k-1}\Gamma^2(\tfrac{1}{2}k)n]^{-1} \tag{5.4}$$

to terms of order n^{-1}. Hence, the relative efficiency (large sample) of the δ-expectation distribution-free regions with respect to the δ-expectation region (1.3), with $K^{(k)}$ given by (1.4), is the ratio of (5.4) to (5.3). We note then that the limiting relative efficiency is

$$\begin{aligned} \text{Lim. Rel. Eff.} &= \lim_{n\to\infty} \mathrm{Var}(C_N)/\mathrm{Var}(C_{DF}) \\ &= [\chi^2_{k;1-\delta}]^k(\exp\{-\chi^2_{k;1-\delta}\})[\delta(1-\delta)k2^{k-1}\Gamma^2(\tfrac{1}{2}k)]^{-1}. \end{aligned} \tag{5.5}$$

Table 5.1 gives values of the limiting relative efficiency (5.5) for $\delta = .75$, .90, .95, and .99 for $k = 1$, 2, 3 and 4. The result (5.5) for $k = 1$ was obtained by Wilks (1941).

TABLE 3.1a

Approximate values of $K^{(2)}$ needed to make the tolerance region (1.3), with $k = 2$, β-content at confidence level γ, when sampling from the bivariate normal

	β							
	$\gamma = .75$				$\gamma = .90$			
n	.75	.90	.95	.99	.75	.90	.95	.99
100	2.9724	4.9146	6.3737	9.7329	3.1612	5.2181	6.7582	10.2901
120	2.9540	4.8880	6.3426	9.6958	3.1248	5.1636	6.6929	10.2063
140	2.9398	4.8673	6.3181	9.6659	3.0968	5.1213	6.6417	10.1396
160	2.9285	4.8505	6.2982	9.6410	3.0744	5.0873	6.6003	10.0849
180	2.9191	4.8366	6.2815	9.6199	3.0560	5.0591	6.5658	10.0389
200	2.9113	4.8248	6.2673	9.6017	3.0406	5.0353	6.5366	9.9996
220	2.9045	4.8146	6.2550	9.5858	3.0274	5.0149	6.5114	9.9654
240	2.8987	4.8058	6.2442	9.5717	3.0159	4.9971	6.4894	9.9354
260	2.8935	4.7979	6.2347	9.5592	3.0058	4.9814	6.4700	9.9087
280	2.8890	4.7909	6.2261	9.5478	2.9969	4.9674	6.4527	9.8848
300	2.8849	4.7847	6.2185	9.5376	2.9889	4.9549	6.4371	9.8632
320	2.8812	4.7790	6.2115	9.5282	2.9817	4.9436	6.4230	9.8436
340	2.8778	4.7738	6.2051	9.5197	2.9751	4.9332	6.4101	9.8256
360	2.8748	4.7691	6.1993	9.5118	2.9692	4.9238	6.3983	9.8091
380	2.8719	4.7647	6.1939	9.5044	2.9637	4.9152	6.3875	9.7939
400	2.8693	4.7607	6.1889	9.4977	2.9586	4.9072	6.3775	9.7798
420	2.8669	4.7569	6.1843	9.4913	2.9539	4.8997	6.3682	9.7667
440	2.8647	4.7534	6.1800	9.4854	2.9496	4.8929	6.3595	9.7544
460	2.8626	4.7502	6.1759	9.4798	2.9455	4.8864	6.3514	9.7430
480	2.8607	4.7471	6.1722	9.4746	2.9417	4.8804	6.3438	9.7322
500	2.8588	4.7443	6.1686	9.4697	2.9382	4.8747	6.3367	9.7221
520	2.8571	4.7416	6.1652	9.4650	2.9348	4.8694	6.3300	9.7125
540	2.8555	4.7390	6.1621	9.4606	2.9317	4.8644	6.3237	9.7035
560	2.8539	4.7366	6.1591	9.4564	2.9287	4.8596	6.3177	9.6949
580	2.8525	4.7343	6.1562	9.4524	2.9259	4.8551	6.3120	9.6868
600	2.8511	4.7322	6.1535	9.4486	2.9232	4.8508	6.3066	9.6790
620	2.8498	4.7301	6.1509	9.4450	2.9207	4.8468	6.3015	9.6717
640	2.8485	4.7281	6.1485	9.4416	2.9183	4.8429	6.2966	9.6646
660	2.8474	4.7263	6.1461	9.4383	2.9160	4.8392	6.2919	9.6579
680	2.8462	4.7245	6.1439	9.4351	9.9138	4.8357	6.2875	9.6515
700	2.8451	4.7228	6.1417	9.4321	2.9117	4.8323	6.2832	9.6453
720	2.8441	4.7211	6.1396	9.4292	2.9097	4.8291	6.2791	9.6394
740	2.8431	4.7195	6.1377	9.4264	2.9077	4.8260	6.2752	9.6338
760	2.8421	4.7180	6.1358	9.4237	2.9059	4.8230	6.2714	9.6283
780	2.8412	4.7166	6.1339	9.4211	2.9041	4.8202	6.2678	9.6231
800	2.8403	4.7152	6.1322	9.4186	2.9024	4.8174	4.2643	9.6181
820	2.8395	4.7138	6.1305	9.4162	2.9008	4.8148	6.2610	9.6132
840	2.8387	4.7125	6.1288	9.4139	2.8992	4.8123	6.2577	9.6085
860	2.8379	4.7113	6.1273	9.4116	2.8976	4.8098	6.2546	9.6040
880	2.8371	4.7100	6.1257	9.4094	2.8962	4.8074	6.2516	9.5996
900	2.8364	4.7089	6.1243	9.4073	2.8947	4.8051	6.2487	9.5953
920	2.8357	4.7077	6.1228	9.4053	2.8934	4.8029	6.2459	9.5912
940	2.8350	4.7066	6.1215	9.4033	2.8920	4.8008	6.2431	9.5873
960	2.8343	4.7056	6.1201	9.4014	2.8908	4.7987	6.2405	9.5834
980	2.8337	4.7045	6.1188	9.3996	2.8895	4.7967	6.2379	9.5797
1000	2.8330	4.7035	6.1176	9.3977	2.8883	4.7947	6.2354	9.5761
∞	2.7726	4.6052	5.9915	9.2103	2.7726	4.6052	5.9915	9.2103

TABLE 3.1a (Continued)

Approximate values of $K^{(2)}$ needed to make the tolerance region (1.3), with $k = 2$, β-content at confidence level γ, when sampling from the bivariate normal

	β							
	$\gamma = .95$				$\gamma = .99$			
n	.75	.90	.95	.99	.75	.90	.95	.99
100	3.2807	5.4072	6.9948	10.6230	3.5182	5.7757	7.4486	11.2396
120	3.2323	5.3349	6.9081	10.5118	3.4453	5.6683	7.3211	11.0795
140	3.1953	5.2789	6.8404	10.4235	3.3896	5.5852	7.2216	10.9521
160	3.1657	5.2339	7.7856	10.3511	3.3454	5.5184	7.1411	10.8477
180	3.1415	5.1967	6.7401	10.2903	3.3092	5.4634	7.0743	10.7601
200	3.1211	5.1653	6.7016	10.2384	3.2789	5.4170	7.0177	10.6852
220	3.1038	5.1384	6.6684	10.1934	3.2531	5.3772	6.9691	10.6201
240	3.0887	5.1150	6.6394	10.1538	3.2308	5.3427	6.9266	10.5630
260	3.0755	5.0943	6.6138	10.1187	3.2112	5.3123	6.8892	10.5123
280	3.0638	5.0759	6.5910	10.0872	3.1939	5.2853	6.8558	10.4669
300	3.0533	5.0595	6.5705	10.0588	3.1784	5.2611	6.8259	10.4259
320	3.0438	5.0446	6.5520	10.0330	3.1645	5.2393	6.7988	10.3887
340	3.0352	5.0311	6.5351	10.0095	3.1518	5.2194	6.7742	10.3547
360	3.0274	5.0187	6.5197	9.9879	3.1403	5.2013	6.7516	10.3235
380	3.0202	5.0074	6.5054	9.9679	3.1298	5.1847	6.7309	10.2947
400	3.0136	4.9969	6.4923	9.9494	3.1201	5.1694	6.7117	10.2681
420	3.0075	4.9872	6.4801	9.9322	3.1111	5.1552	6.6940	10.2433
440	3.0018	4.9782	6.4688	9.9162	3.1028	5.1420	6.6775	10.2201
460	2.9965	4.9697	6.4582	9.9011	3.0950	5.1297	6.6621	10.1985
480	2.9916	4.9619	6.4483	9.8871	3.0878	5.1182	6.6476	10.1782
500	2.9870	4.9545	6.4390	9.8738	3.0810	5.1074	6.6341	10.1591
520	2.9826	4.9475	6.4302	9.8613	3.0746	5.0972	6.6213	10.1411
540	2.9785	4.9409	6.4219	9.8495	3.0686	5.0876	6.6093	10.1240
560	2.9746	4.9347	6.4141	9.8383	3.0630	5.0786	6.5979	10.1079
580	2.9709	4.9289	6.4067	9.8276	3.0576	5.0700	6.5871	10.0926
600	2.9674	4.9233	6.3996	9.8175	3.0525	5.0619	6.5769	10.0780
620	2.9641	4.9180	6.3929	9.8079	3.0477	5.0542	6.5672	10.0642
640	2.9610	4.9129	6.3865	9.7987	3.0431	5.0469	6.5579	10.0510
660	2.9580	4.9081	6.3804	9.7900	3.0387	5.0399	6.5491	10.0384
680	2.9551	4.9035	6.3746	9.7816	3.0346	5.0332	6.5407	10.0263
700	2.9524	4.8991	6.3690	9.7735	3.0306	5.0268	6.5326	10.0147
720	2.9498	4.8949	6.3637	9.7658	3.0268	5.0207	6.5249	10.0037
740	2.9473	4.8909	6.3586	9.7585	3.0231	5.0148	6.5175	9.9930
760	2.9449	4.8870	6.3537	9.7514	3.0196	5.0092	6.5103	9.9828
780	2.9425	4.8833	6.3490	9.7445	3.0163	5.0038	6.5035	9.9730
800	2.9403	4.8797	6.3444	9.7379	3.0131	4.9986	6.4969	9.9635
820	2.9382	4.8763	6.3401	9.7316	3.0100	4.9936	6.4906	9.9544
840	2.9361	4.8730	6.3358	9.7255	3.0070	4.9888	6.4845	9.9456
860	2.9341	4.8698	6.3318	9.7196	3.0041	4.9842	6.4786	9.9371
880	2.9322	4.8667	6.3278	9.7139	3.0013	4.9797	6.4730	9.9289
900	2.9304	4.8637	6.3241	9.7083	2.9986	4.9754	6.4675	9.9210
920	2.9286	4.8608	6.3204	7.9030	2.9960	4.9712	6.4622	9.9133
940	2.9269	4.8580	6.3168	9.6978	2.9935	4.9672	6.4570	9.9059
960	2.9252	4.8553	6.3134	9.6928	2.9911	4.9633	6.4521	9.8987
980	2.9236	4.8527	6.3101	9.6880	2.9888	4.9595	6.4472	9.8917
1000	2.9220	4.8501	6.3068	9.6832	2.9865	4.9558	6.4426	9.8849
∞	2.7726	4.6052	5.9915	9.2103	2.7726	4.6052	5.9915	9.2103

TABLE 3.1b

Approximate values of $K^{(3)}$ needed to make the tolerance region (1.3), with $k = 3$, β-content at confidence level γ, when sampling from the trivariate normal

	β							
	$\gamma = .75$				$\gamma = .90$			
n	.75	.90	.95	.99	.75	.90	.95	.99
100	4.3518	6.5985	8.2290	11.8881	4.5746	6.9271	8.6300	12.4404
120	4.3293	6.5683	8.1946	11.8477	4.5312	6.8673	8.5605	12.3543
140	4.3119	6.5448	8.1676	11.8155	4.4978	6.8209	8.5061	12.2859
160	4.2981	6.5259	8.1457	11.7888	4.4712	6.7835	8.4621	12.2299
180	4.2866	6.5101	8.1274	11.7664	4.4492	6.7525	8.4255	12.1830
200	4.2770	6.4968	8.1119	11.7471	4.4307	6.7263	8.3944	12.1429
220	4.2688	6.4854	8.0985	11.7303	4.4149	6.7038	8.3677	12.1080
240	4.2617	6.4754	8.0868	11.7155	4.4012	6.6842	8.3443	12.0775
260	4.2554	6.4666	8.0764	11.7023	4.3892	6.6669	8.3236	12.0503
280	4.2498	6.4587	8.0671	11.6904	4.3784	6.6516	8.3052	12.0260
300	4.2448	6.4517	8.0587	11.6797	4.3689	6.6378	8.2886	12.0041
320	4.2403	6.4453	8.0512	11.6699	4.3602	6.6253	8.2736	11.9841
340	4.2362	6.4395	8.0443	11.6610	4.3524	6.6139	8.2600	11.9659
360	4.2324	6.4342	8.0379	11.6527	4.3452	6.6036	8.2475	11.9492
380	4.2290	6.4293	8.0321	11.6451	4.3386	6.5940	8.2359	11.9338
400	4.2258	6.4247	8.0267	11.6381	4.3325	6.5852	8.2253	11.9195
420	4.2229	6.4205	8.0217	11.6315	4.3269	6.5770	8.2154	11.9062
440	4.2202	6.4166	8.0170	11.6253	4.3217	6.5694	8.2062	11.8938
460	4.2176	6.4130	8.0127	11.6196	4.3168	6.5623	8.1976	11.8821
480	4.2153	6.4096	8.0086	11.6142	4.3123	6.5557	8.1896	11.8712
500	4.2130	6.4064	8.0047	11.6091	4.3080	6.5494	8.1820	11.8610
520	4.2109	6.4033	8.0011	11.6042	4.3040	6.5435	8.1748	11.8513
540	4.2090	6.4005	7.9977	11.5997	4.3002	6.5380	8.1681	11.8421
560	4.2071	6.3978	7.9944	11.5954	4.2966	6.5328	8.1617	11.8335
580	4.2053	6.3952	7.9913	11.5912	4.2932	6.5278	8.1557	11.8252
600	4.2036	6.3928	7.9884	11.5873	4.2900	6.5231	8.1500	11.8174
620	4.2020	6.3905	7.9856	11.5836	4.2870	6.5186	8.1445	11.8100
640	4.2005	6.3883	7.9830	11.5801	4.2841	6.5143	8.1393	11.8029
660	4.1991	6.3862	7.9804	11.5766	4.2813	6.5103	8.1344	11.7961
680	4.1977	6.3842	7.9780	11.5734	4.2787	6.5064	8.1296	11.7896
700	4.1964	6.3823	7.9757	11.5703	4.2761	6.5026	8.1251	11.7834
720	4.1951	6.3804	7.9735	11.5673	4.2737	6.4991	8.1207	11.7774
740	4.1939	6.3787	7.9713	11.5644	4.2714	6.4957	8.1166	11.7717
760	4.1927	6.3770	7.9693	11.5616	4.2692	6.4924	8.1126	11.7662
780	4.1916	6.3754	7.9673	11.5590	4.2670	6.4892	8.1087	11.7609
800	4.1905	6.3738	7.9654	11.5564	4.2650	6.4862	8.1050	11.7558
820	4.1895	6.3723	7.9636	11.5539	4.2630	6.4833	8.1015	11.7509
840	4.1885	6.3708	7.9618	11.5515	4.2611	6.4805	8.0980	11.7461
860	4.1875	6.3694	7.9601	11.5492	4.2593	6.4778	8.0947	11.7415
880	4.1866	6.3681	7.9585	11.5470	4.2575	6.4751	8.0915	11.7371
900	4.1857	6.3668	7.9569	11.5448	4.2558	6.4726	8.0884	11.7328
920	4.1848	6.3655	7.9554	11.5428	4.2541	6.4702	8.0854	11.7287
940	4.1840	6.3643	7.9539	11.5407	4.2525	6.4678	8.0825	11.7247
960	4.1832	6.3631	7.9524	11.5388	4.2510	6.4655	8.0797	11.7208
980	4.1824	6.3619	7.9510	11.5369	4.2495	6.4633	8.0770	11.7170
1000	4.1817	6.3608	7.9497	11.5350	4.2480	6.4611	8.0743	11.7134
∞	4.1084	6.2514	7.8147	11.3449	4.1084	6.2514	7.8147	11.3449

TABLE 3.1b (Continued)

Approximate values of $K^{(3)}$ needed to make the tolerance region (1.3), with $k=3$, β-content at confidence level γ, when sampling from the trivariate normal

	β							
	$\gamma = .95$				$\gamma = .99$			
n	.75	.90	.95	.99	.75	.90	.95	.99
100	4.7138	7.1297	8.8744	12.7686	4.9866	7.5198	9.3390	13.3740
120	4.6570	7.0513	8.7834	12.6557	4.9027	7.4055	9.2073	13.2138
140	4.6133	6.9905	8.7122	12.5663	4.8384	7.3168	9.1043	13.0865
160	4.5784	6.9416	8.6546	12.4931	4.7872	7.2454	9.0209	12.9822
180	4.5498	6.9011	8.6067	12.4317	4.7451	7.1864	8.9517	12.8946
200	4.5257	6.8669	8.5662	12.3793	4.7098	7.1366	8.8930	12.8198
220	4.5051	6.8375	8.5312	12.3338	4.6796	7.0939	8.8424	12.7549
240	4.4872	6.8119	8.5007	12.2939	4.6535	7.0567	8.7982	12.6979
260	4.4715	6.7894	8.4737	12.2585	4.6306	7.0239	8.7593	12.6473
280	4.4576	6.7694	8.4497	12.2268	4.6103	6.9948	8.7245	12.6020
300	4.4451	6.7514	8.4281	12.1982	4.5921	6.9687	8.6933	12.5611
320	4.4338	6.7351	8.4086	12.1722	4.5757	6.9451	8.6651	12.5240
340	4.4236	6.7203	8.3908	12.1485	4.5609	6.9237	8.6394	12.4901
360	4.4143	6.7068	8.3745	12.1267	4.5473	6.9041	8.6158	12.4589
380	4.4058	6.6944	8.3595	12.1066	4.5349	6.8861	8.5942	12.4302
400	4.3979	6.6829	8.3456	12.0880	4.5235	6.8695	8.5742	12.4036
420	4.3906	6.6723	8.3327	12.0707	4.5129	6.8541	8.5556	12.3788
440	4.3838	6.6624	8.3208	12.0545	4.5030	6.8398	8.5384	12.3557
460	4.3775	6.6532	8.3096	12.0394	4.4939	6.8265	8.5223	12.3341
480	4.3716	6.6445	8.2991	12.0252	4.4853	6.8140	8.5072	12.3138
500	4.3660	6.6364	8.2893	12.0119	4.4773	6.8022	8.4930	12.2948
520	4.3608	6.6288	8.2800	11.9993	4.4697	6.7912	8.4796	12.2768
540	4.3559	6.6216	8.2713	11.9874	4.4626	6.7808	8.4670	12.2598
560	4.3513	6.6148	8.2630	11.9762	4.4559	6.7710	8.4551	12.2437
580	4.3469	6.6083	8.2551	11.9655	4.4496	6.7617	8.4438	12.2284
600	4.3427	6.6022	8.2477	11.9553	4.4436	6.7529	8.4331	12.2138
620	4.3388	6.5964	8.2406	11.9456	4.4378	6.7445	8.4229	12.2000
640	4.3350	6.5908	8.2339	11.9364	4.4324	6.7365	8.4132	12.1868
660	4.3314	6.5855	8.2274	11.9276	4.4272	6.7289	8.4039	12.1742
680	4.3280	6.5805	8.2213	11.9191	4.4223	6.7216	8.3951	12.1622
700	4.3247	6.5757	8.2154	11.9111	4.4176	6.7146	8.3866	12.1506
720	4.3216	6.5711	8.2097	11.9033	4.4130	6.7080	8.3785	12.1396
740	4.3186	6.5666	8.2043	11.8959	4.4087	6.7016	8.3707	12.1289
760	4.3157	6.5624	8.1991	11.8887	4.4045	6.6955	8.3633	12.1187
780	4.3129	6.5583	8.1941	11.8819	4.4006	6.6896	8.3561	12.1089
800	4.3103	6.5544	8.1893	11.8573	4.3967	6.6839	8.3492	12.0995
820	4.3077	6.5506	8.1847	11.8689	4.3930	6.6785	8.3425	12.0904
840	4.3052	6.5469	8.1803	11.8627	4.3895	6.6732	8.3361	12.0816
860	4.3029	6.5434	8.1759	11.8568	4.3861	6.6682	8.3299	12.0731
880	4.3006	6.5400	8.1718	11.8511	4.3827	6.6633	8.3240	12.0649
900	4.2983	6.5367	8.1678	11.8455	4.3796	6.6586	8.3182	12.0570
920	4.2962	6.5336	8.1639	11.8401	4.3765	6.6540	8.3126	12.0493
940	4.2941	6.5305	8.1601	11.8349	4.3735	6.6496	8.3072	12.0419
960	4.2921	6.5275	8.1565	11.8299	4.3706	6.6453	8.3020	12.0347
980	4.2902	6.5246	8.1530	11.8250	4.3678	6.6412	8.2970	12.0277
1000	4.2883	6.5218	8.1495	11.8203	4.3651	6.6371	8.2920	12.0209
∞	4.1084	6.2514	7.8147	11.3449	4.1084	6.2514	7.8147	11.3449

TABLE 3.1c

Approximate values of $K^{(4)}$ needed to make the region (1.3), with $k = 4$, β-content at confidence level γ, when sampling from the quadrinormal

	β							
	$\gamma = .75$				$\gamma = .90$			
n	.75	.90	.95	.99	.75	.90	.95	.99
100	5.6639	8.1577	9.9296	13.8403	5.9131	8.5070	10.3460	14.3949
120	5.6380	8.1245	9.8924	13.7972	5.8641	8.4426	10.2726	14.3062
140	5.6180	8.0987	9.8632	13.7630	5.8264	8.3927	10.2153	14.2359
160	5.6021	8.0778	9.8396	13.7348	5.7962	8.3524	10.1689	14.1784
180	5.5889	8.0606	9.8200	13.7111	5.7714	8.3191	10.1303	14.1303
200	5.5779	8.0460	9.8033	13.6908	5.7506	8.2909	10.0976	14.0892
220	5.5685	8.0335	9.7889	13.6732	5.7327	8.2667	10.0695	14.0535
240	5.5603	8.0226	9.7763	13.6577	5.7172	8.2456	10.0448	14.0222
260	5.5531	8.0130	9.7652	13.6439	5.7035	8.2270	10.0231	13.9945
280	5.5467	8.0044	9.7553	13.6315	5.6914	8.2105	10.0037	13.9696
300	5.5409	7.9967	9.7464	13.6203	5.6806	8.1956	9.9862	13.9472
320	5.5358	7.9897	9.7383	13.6102	5.6708	8.1822	9.9705	13.9269
340	5.5311	7.9834	9.7309	13.6009	5.6619	8.1699	9.9561	13.9083
360	5.5268	7.9776	9.7241	13.5923	5.6538	8.1588	9.9429	13.8912
380	5.5229	7.9722	9.7179	13.5844	5.6463	8.1485	9.9308	13.8754
400	5.5192	7.9673	9.7122	13.5771	5.6395	8.1390	9.9196	13.8608
420	5.5159	7.9627	9.7068	13.5703	5.6331	8.1302	9.9092	13.8473
440	5.5128	7.9585	9.7018	13.5639	5.6272	8.1220	9.8995	13.8346
460	5.5098	7.9545	9.6972	13.5579	5.6217	8.1144	9.8904	13.8228
480	5.5071	7.9508	9.6928	13.5523	5.6165	8.1072	9.8820	13.8116
500	5.5046	7.9473	9.6887	13.5470	5.6117	8.1005	9.8740	13.8012
520	5.5022	7.9440	9.6849	13.5420	5.6071	8.0942	9.8665	13.7913
540	5.4999	7.9409	9.6812	13.5373	5.6028	8.0882	9.8594	13.7820
560	5.4978	7.9380	9.6777	13.5329	5.5988	8.0825	9.8527	13.7732
580	5.4957	7.9352	9.6745	13.5286	5.5950	8.0772	9.8464	13.7648
600	5.4938	7.9325	9.6714	13.5246	5.5913	8.0721	9.8403	13.7568
620	5.4920	7.9300	9.6684	13.5207	5.5879	8.0673	9.8346	13.7492
640	5.4903	7.9276	9.6656	13.5171	5.5846	8.0627	9.8291	1.37420
660	5.4886	7.9254	9.6629	13.5136	5.5814	8.0583	9.8239	13.7351
680	5.4870	7.9232	9.6603	13.5102	5.5784	8.0541	9.8189	13.7284
700	5.4855	7.9211	9.6578	13.5070	5.5756	8.0501	9.8141	13.7221
720	5.4841	7.9191	9.6555	13.5039	5.5728	8.0463	9.8096	13.7160
740	5.4827	7.9172	9.6532	13.5009	5.5702	8.0426	9.8052	13.7102
760	5.4814	7.9153	9.6510	13.4981	5.5677	8.0391	9.8010	13.7046
780	5.4801	7.9136	9.6490	13.4953	5.5653	8.0357	9.7969	13.6992
800	5.4789	7.9119	9.6469	13.4927	5.5629	8.0324	9.7930	13.6940
820	5.4777	7.9102	9.6450	13.4902	5.5607	8.0293	9.7893	13.6890
840	5.4765	7.9087	9.6431	13.4877	5.5585	8.0262	9.7857	13.6842
860	5.4754	7.9071	9.6413	13.4853	5.5564	8.0233	9.7822	13.6795
880	5.4744	7.9057	9.6396	13.4830	5.5544	8.0205	9.7788	13.6750
900	5.4734	7.9042	9.6379	13.4808	5.5525	8.0178	9.7755	13.6707
920	5.4724	7.9029	9.6363	13.4787	5.5506	8.0151	9.7724	13.6664
940	5.4714	7.9015	9.6347	13.4766	5.5488	8.0126	9.7693	13.6624
960	5.4705	7.9002	9.6332	13.4746	5.5470	8.0101	9.7664	13.6584
980	5.4696	7.8990	9.6317	13.4726	5.5453	8.0077	9.7635	13.6546
1000	5.4687	7.8978	9.6302	13.4707	5.5437	8.0054	9.7607	13.6509
∞	5.3853	7.7794	9.4877	13.2767	5.3853	7.7794	9.4877	13.2767

TABLE 3.1c (continued)

Approximate values of $K^{(4)}$ needed to make the region (1.3), with $k = 4$, β-content at confidence level γ, when sampling from the quadrinormal

	β							
	$\gamma = .95$				$\gamma = .99$			
n	.75	.90	.95	.99	.75	.90	.95	.99
100	6.0675	8.7208	10.5983	14.7233	6.3679	9.1300	11.0753	15.3278
120	6.0039	8.6372	10.5030	14.6081	6.2753	9.0094	10.9391	15.1658
140	5.9550	8.5723	10.4286	14.5169	6.2041	8.9158	10.8325	15.0371
160	5.9159	8.5200	10.3683	14.4423	6.1472	8.8403	10.7461	14.9317
180	5.8837	8.4767	10.3183	14.3798	6.1005	8.7779	10.6743	14.8433
200	5.8567	8.4402	10.2758	14.3265	6.0612	8.7252	10.6134	14.7677
220	5.8335	8.4088	10.2392	14.2802	6.0277	8.6799	10.5609	14.7021
240	5.8134	8.3814	10.2073	14.2397	5.9985	8.6404	10.5150	14.6445
260	5.7957	8.3573	10.1790	14.2036	5.9730	8.6057	10.4745	14.5934
280	5.7800	8.3358	10.1539	14.1714	5.9503	8.5747	10.4384	14.5477
300	5.7660	8.3165	10.1313	14.1423	5.9300	8.5470	10.4060	14.5064
320	5.7533	8.2991	10.1108	14.1160	5.9117	8.5219	10.3766	14.4689
340	5.7418	8.2833	10.0922	14.0919	5.8951	8.4991	10.3499	14.4346
360	5.7313	8.2688	10.0751	14.0697	5.8800	8.4783	10.3254	14.4032
380	5.7217	8.2555	10.0594	14.0493	5.8661	8.4591	10.3029	14.3741
400	5.7128	8.2432	10.0449	14.0304	5.8533	8.4415	10.2821	14.3473
420	5.7046	8.2318	10.0314	14.0128	5.8414	8.4251	10.2628	14.3223
440	5.6970	8.2212	10.0188	13.9964	5.8304	8.4098	10.2448	14.2990
460	5.6898	8.2113	10.0071	13.9811	5.8201	8.3956	10.2280	14.2771
480	5.6832	8.2020	9.9961	13.9667	5.8105	8.3823	10.2123	14.2567
500	5.6769	8.1933	9.9858	13.9531	5.8015	8.3698	10.1975	14.2374
520	5.6710	8.1852	9.9761	13.9404	5.7931	8.3581	10.1836	14.2192
540	5.6655	8.1774	9.9669	13.9283	5.7851	8.3470	10.1705	14.2020
560	5.6603	8.1701	9.9583	13.9169	5.7776	8.3365	10.1580	14.1858
580	5.6553	8.1632	9.9500	13.9060	5.7704	8.3266	10.1463	14.1703
600	5.6506	8.1566	9.9422	13.8957	5.7637	8.3171	10.1351	14.1557
620	5.6461	8.1504	9.9348	13.8859	5.7573	8.3082	10.1245	14.1417
640	5.6419	8.1445	9.9277	13.8765	5.7512	8.2997	10.1143	14.1284
660	5.6378	8.1388	9.9210	13.8675	5.7453	8.2915	10.1047	14.1156
680	5.6340	8.1334	9.9145	13.8590	5.7398	8.2838	10.0954	14.1035
700	5.6303	8.1282	9.9084	13.8508	5.7345	8.2763	10.0866	14.0918
720	5.6267	8.1232	9.9025	13.8429	5.7294	8.2692	10.0781	14.0806
740	5.6233	8.1185	9.8968	13.8354	5.7245	8.2624	10.0700	14.0699
760	5.6201	8.1139	9.8913	13.8281	5.7199	8.2559	10.0622	14.0596
780	5.6169	8.1095	9.8861	13.8212	5.7154	8.2496	10.0548	14.0497
800	5.6139	8.1053	9.8811	13.8145	5.7111	8.2435	10.0475	14.0401
820	5.6111	8.1013	9.8762	13.8080	5.7069	8.2377	10.0406	14.0309
840	5.6083	8.0973	9.8715	13.8018	5.7029	8.2321	10.0339	14.0220
860	5.6056	8.0936	9.8670	13.7957	5.6991	8.2267	10.0275	14.0135
880	5.6030	8.0899	9.8627	13.7899	5.6953	8.2215	10.0212	14.0052
900	5.6005	8.0864	9.8585	13.7843	5.6918	8.2164	10.0152	13.9972
920	5.5981	8.0830	9.8544	13.7788	5.6883	8.2116	10.0094	13.9894
940	5.5957	8.0797	9.8504	13.7736	5.6849	8.2068	10.0038	13.9819
960	5.5934	8.0765	9.8466	13.7684	5.6817	8.2023	9.9983	13.9747
980	5.5913	8.0731	9.8429	13.7635	5.6785	8.1978	9.9930	13.9676
1000	5.5891	8.0704	9.8393	13.7587	5.6755	8.1935	9.9879	13.9608
∞	5.3853	7.7794	9.4877	13.2767	5.3853	7.7794	9.4877	13.2767

TABLE 4.1a

Approximate values of the confidence level γ that the β-expectation tolerance region (1.3), $k = 2$, *has coverage that exceeds* β

	β			
n	.75	.90	.95	.99
100	.5122	.5258	.5341	.5471
120	.5111	.5237	.5317	.5449
140	.5103	.5221	.5297	.5429
160	.5097	.5208	.5280	.5411
180	.5091	.5197	.5266	.5394
200	.5087	.5187	.5254	.5379
250	.5077	.5169	.5230	.5348
300	.5071	.5154	.5211	.5323
400	.5061	.5134	.5184	.5286
500	.5055	.5120	.5166	.5259
600	.5050	.5110	.5152	.5238
700	.5046	.5102	.5141	.5222
800	.5043	.5096	.5132	.5208
900	.5041	.5090	.5125	.5197
1,000	.5039	.5086	.5118	.5187

TABLE 4.1b

Approximate values of the confidence level γ that the β-expectation tolerance region (1.3), $k = 3$, *has coverage that exceeds* β

	β			
n	.75	.90	.95	.99
100	.5120	.5240	.5306	.5389
120	.5110	.5223	.5287	.5380
140	.5103	.5209	.5272	.5368
160	.5096	.5197	.5258	.5356
180	.5091	.5187	.5246	.5345
200	.5086	.5179	.5236	.5334
250	.5077	.5161	.5214	.5310
300	.5071	.5148	.5198	.5290
400	.5061	.5129	.5173	.5258
500	.5055	.5116	.5156	.5235
600	.5050	.5106	.5143	.5217
700	.5047	.5099	.5133	.5203
800	.5044	.5092	.5125	.5191
900	.5041	.5087	.5118	.5181
1,000	.5039	.5083	.5112	.5172

TABLE 4.1c

Apprximate values of the confidence level γ that the β-expectation tolerance region (1.3), $k = 4$, has coverage that exceeds β

	β			
n	.75	.90	.95	.99
100	.5118	.5225	.5277	.5327
120	.5109	.5210	.5263	.5327
140	.5101	.5198	.5251	.5322
160	.5095	.5188	.5240	.5315
180	.5090	.5179	.5230	.5308
200	.5085	.5171	.5221	.5300
250	.5077	.5155	.5202	.5282
300	.5070	.5143	.5188	.5266
400	.5061	.5125	.5165	.5240
500	.5055	.5113	.5150	.5219
600	.5050	.5103	.5138	.5203
700	.5046	.5096	.5128	.5190
800	.5043	.5090	.5120	.5180
900	.5041	.5085	.5114	.5170
1,000	.5039	.5081	.5108	.5162

TABLE 5.1

Values of the limiting relative efficiency (5.5) of the δ-expectation distribution-free tolerance regions with respect to the δ-expectation regions (1.3)

	δ			
k	.75	.90	.95	.99
1	.5981	.6395	.5525	.2803
2	.6406	.5891	.4723	.2142
3	.6449	.5552	.4304	.1851
4	.6425	.5319	.4040	.1681

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APPENDIX

In this Appendix, we prove the results (2.15), viz.,

$$\text{(A.1)}\qquad \text{(i)}\ \frac{\partial C}{\partial V_\alpha^*}\bigg|_{(*)} = \exp\{-\tfrac{1}{2}K^{(k)}\}[K^{(k)}]^{\frac{1}{2}k}[2^{\frac{1}{2}k-1}\Gamma(\tfrac{1}{2}k)k]^{-1}$$

$$\text{(ii)}\ \frac{\partial C}{\partial V_{\alpha\beta}^*}\bigg|_{(*)} = 0, \qquad \alpha \neq \beta; \qquad \text{(iii)}\ \frac{\partial C}{\partial \bar{X}_\alpha^*}\bigg|_{(*)} = 0$$

where $V_\alpha^* = +V_{\alpha\alpha'}^{*\frac{1}{2}}$; $\alpha, \beta = 1, \cdots, k$, and (*) denotes the point

$$\text{(A.2)}\quad (\bar{X}_1^*, \cdots, \bar{X}_k^*, V_1^*, \cdots, V_k^*, V_{12}^*, \cdots, V_{k-1,k}^*) = (0, \cdots, 0, 1, \cdots, 1, 0, \cdots, 0).$$

To do this we need the following lemma

LEMMA A.1. *If* $V^* = (V_{\alpha\beta}^*)$ *is a* $k \times k$ symmetric *positive definite matrix, with* $(V^*)^{-1} = (C_{\alpha\beta})$, *and if we define*

$$\text{(A.3)}\qquad c_{\alpha\beta}^{(r)} = c_{\alpha\beta}^{(r-1)} - c_{k-r+1,\alpha}^{(r-1)} c_{k-r+1,\beta}^{(r-1)} / c_{k-r+1,k-r+1}^{(r-1)}$$

for $r = 1, \cdots, k-1$, and $\alpha, \beta = 1, \cdots, k-r$ *with* $c_{\alpha\beta}^{(0)} = c_{\alpha\beta}$, *and if* (**) *denotes the point*

$$\text{(A.4)}\qquad (V_{11}^*, \cdots, V_{kk}^*, V_{12}, \cdots, V_{k-1,k}) = (1, \cdots, 1, 0, \cdots, 0), \qquad \textit{then}$$

$$\text{(A.5)}\qquad \text{(i)}\ c_{\alpha\beta}^{(r)}(**) = 1 \quad \text{if} \quad \alpha = \beta, \qquad = 0 \quad \text{if} \quad \alpha \neq \beta,$$

for $r = 0, 1, \cdots, k-1$ *and* $\alpha, \beta = 1, \cdots, k-r$;

$$\text{(A.6)}\qquad \text{(ii)}\ \frac{\partial c_{\alpha\beta}^{(r)}}{\partial V_{\alpha'\beta'}^*}(**) = -1 \quad \textit{if} \quad \alpha = \alpha'; \quad \beta = \beta', \qquad = 0 \quad \textit{otherwise};$$

$r = 0, 1, \cdots, k-1$ *and* $\alpha, \beta = 1, \cdots, k-r$.

PROOF. If $A_{\alpha\beta}$ is the co-factor of $V_{\alpha\beta}^*$ in V^*, then it is easy to see that $A_{\alpha\beta}(**) = 1$ if $\alpha = \beta$, and 0 otherwise. But $c_{\alpha\beta} = A_{\beta\alpha}/|V^*| = A_{\alpha\beta}/|V^*|$, so that

$$\text{(A.7)}\qquad c_{\alpha\beta}(**) = 1 \quad \text{if} \quad \alpha = \beta, \qquad = 0 \quad \text{otherwise},$$

since $|V^*|(**)$ is the determinant of the identity matrix of order k. Now we note that if $r = 1$, we have

$$\text{(A.8)}\qquad c_{\alpha\beta}^{(1)} = c_{\alpha\beta} - c_{k\alpha} c_{k\beta} / c_{kk}, \qquad \alpha, \beta = 1, \cdots, k-1.$$

Because α and β in (A.8) do not equal k, the last term of (A.8) vanishes when $c_{\alpha\beta}^{(1)}$ is evaluated at (**). Thus

$$\text{(A.9)} \qquad \begin{aligned} c_{\alpha\beta}^{(1)}(**) = c_{\alpha\beta}(**) &= 1 \quad \text{if} \quad \alpha = \beta, \\ &= 0 \quad \text{if} \quad \alpha \neq \beta, \end{aligned}$$

if α, $\beta = 1, \cdots, k-1$. Now suppose (A.5) is true for $r = t-1$, i.e.,

$$\text{(A.10)} \qquad \begin{aligned} c_{\alpha\beta}^{(t-1)} &= 1 \quad \text{if} \quad \alpha = \beta, \\ &= 0 \quad \text{if} \quad \alpha \neq \beta; \qquad \alpha, \beta = 1, \cdots, k-t+1. \end{aligned}$$

Using the definition of $c_{\alpha\beta}^{(t)}$ given by (A.3), we see that

$$\text{(A.11)} \qquad c_{\alpha\beta}^{(t)}(**) = c_{\sigma\beta}^{(t-1)}(**)$$

that is, if (A.5) is true for $r = 0$, it is true for $r = 1$, and if true for $r = t-1$, then true for $r = t$, that is, (A.5) true for $r = 0, 1, \cdots, k-1$.

Now to prove (ii) of this lemma, we first note that of a seemingly possible sixteen cases, just 9 cases are possible, viz.,

(i) $\alpha = \beta,\ \alpha' = \beta',\ \alpha = \alpha',\ \beta = \beta'$
(ii) $\alpha = \beta,\ \alpha' = \beta',\ \alpha \neq \alpha',\ \beta \neq \beta'$
(iii) $\alpha = \beta,\ \alpha' \neq \beta',\ \alpha \neq \alpha',\ \beta = \beta'$
(iv) $\alpha \neq \beta,\ \alpha' = \beta',\ \alpha \neq \alpha',\ \beta = \beta'$
(v) $\alpha \neq \beta,\ \alpha' \neq \beta',\ \alpha = \alpha',\ \beta = \beta'$
(vi) $\alpha = \beta,\ \alpha' \neq \beta',\ \alpha \neq \alpha',\ \beta \neq \beta'$
(vii) $\alpha \neq \beta,\ \alpha' = \beta',\ \alpha \neq \alpha',\ \beta \neq \beta'$
(viii) $\alpha \neq \beta,\ \alpha' \neq \beta',\ \alpha \neq \alpha',\ \beta = \beta'$
(ix) $\alpha \neq \beta,\ \alpha' \neq \beta',\ \alpha \neq \alpha',\ \beta \neq \beta'$.

We first consider the above for $r = 0$. Using the method of a proof given by Goldberger (1964, page 43) for non-symmetric matrices, it is easily proved for V^*, symmetric, and $(V^*)^{-1} = (c_{\alpha\beta})$, that

$$\text{(A.12)} \qquad \begin{aligned} \frac{\partial c_{\alpha\beta}}{\partial V_{\alpha'\beta'}^*} &= -[c_{\alpha\alpha'} c_{\beta'\beta} + c_{\alpha\beta'} c_{\alpha'\beta}] \quad \text{if} \quad \alpha' \neq \beta', \\ &= -c_{\alpha\alpha'} c_{\alpha'\beta} \quad \text{if} \quad \alpha' = \beta'. \end{aligned}$$

Using (A.5) and (A.12), it is quickly verified that

$$\text{(A.13)} \qquad \begin{aligned} \frac{\partial c_{\alpha\beta}}{\partial V_{\alpha'\beta'}^*}(**) &= -1 \quad \text{if} \quad \alpha = \alpha'; \quad \beta = \beta', \\ &= 0 \quad \text{otherwise}. \end{aligned}$$

We turn, then, to the case $r = 1$. We have from (A.3) that

$$\text{(A.14)} \qquad \frac{\partial c_{\alpha\beta}^{(1)}}{\partial V_{\alpha'\beta'}^*} = \frac{\partial c_{\alpha\beta}}{\partial V_{\alpha'\beta'}^*} - \left\{ c_{kk}\left[c_{k\alpha}\frac{\partial c_{k\beta}}{\partial V_{\alpha'\beta'}^*} + c_{k\beta}\frac{\partial c_{k\alpha}}{\partial V_{\alpha'\beta'}^*}\right] - c_{k\alpha} c_{k\beta}\frac{\partial c_{kk}}{\partial V_{\alpha'\beta'}^*}\right\} \Big/ c_{kk}^2$$

for α, $\beta = 1, \cdots, k-1$. Using (A.5), we see that

$$\text{(A.15)} \qquad \begin{aligned} \frac{\partial c_{\alpha\beta}^{(1)}}{\partial V_{\alpha'\beta'}^*}(**) = \frac{\partial c_{\alpha\beta}}{\partial V_{\alpha'\beta'}} &= -1 \quad \text{if} \quad \alpha = \alpha'; \quad \beta = \beta', \\ &= 0 \quad \text{otherwise}, \end{aligned}$$

since α and $\beta \neq k$, and proceeding by induction as in (i) of this lemma, we easily establish

$$\frac{\partial c_{\alpha\beta}^{(r)}}{\partial V_{\alpha'\beta'}^{*}}(**) = -1 \quad \text{if} \quad \alpha = \alpha'; \quad \beta = \beta',$$
$$= 0 \quad \text{otherwise}$$

for $\alpha, \beta = 1, \cdots, k-r$, and $r = 0, 1, \cdots, k-1$, and the lemma is proved.

Derivation of (A.1). We proceed then to the derivation of (A.1). From (2.9), we see that we may write C as

(A.16) $$C = \int_{l_{11}}^{l_{21}} g(Y_1^*)\, dY_1^*$$ where

(A.16a)

$$g(Y_1^*) = \phi(Y_1^*) \int_{l_{12}}^{l_{22}} \int_{l_{13}}^{l_{23}} \cdots \int_{l_{1,k-1}}^{l_{2,k-1}} \int_{l_{1k}}^{l_{2k}} \{\textstyle\prod_{j=2}^{k} \phi(Y_j^*)\}\, dY_k^*\, dY_{k-1}^* \cdots dY_3^*\, dY_2^*,$$

with the l_{ir} defined as in (2.10) and (2.11).

Using a well-known identity, we have

(A.17) $$\frac{\partial C}{\partial V_{\alpha\beta}^*} = \int_{l_{11}}^{l_{21}} \frac{\partial}{\partial V_{\alpha\beta}^*} g(Y_1^*)\, dY_1^* - g(Y_1^* = l_{11}) \frac{\partial l_{11}}{\partial V_{\alpha\beta}^*} + g(Y_1^* = l_{21}) \frac{\partial l_{21}}{\partial V_{\alpha\beta}^*}$$

and this is to be evaluated at (*), the point defined by (2.13). But from (2.11), we may write the limits l_{i2} as

(A.18) $$l_{i2} = l_{i2}(Y_1^*) = \bar{X}_2 - c_{21}^{(k-2)}(Y_1^* - \bar{X}_1^*)/c_{22}^{(k-2)}$$
$$\pm (c_{22}^{(k-2)})^{-\frac{1}{2}}\{K^{(k)} - c_{11}^{(k-1)}(Y_1^* - \bar{X}_1^*)^2\}^{\frac{1}{2}}$$

so that

(A.19) $$l_{i2}(Y_1^* = l_{i1}) = \bar{X}_2 - c_{21}^{(k-2)}(l_{i1} - \bar{X}_1^*)/c_{22}^{(k-2)}$$
$$\pm (c_{22}^{(k-2)})^{-\frac{1}{2}}\{K^{(k)} - c_{11}^{(k-1)}(l_{i1} - \bar{X}_1^*)^2\}^{\frac{1}{2}}.$$

Now l_{i1}, defined by (2.11), is such that $l_{i1}(*) = \pm\{K^{(k)}\}^{\frac{1}{2}}$, with the result that

(A.20) $$l_{i2}(Y_1^* = l_{i1})\big|_{(*)} = \pm\{K^{(k)} - \{(\pm K^{(k)})^{\frac{1}{2}}\}^2\}^{\frac{1}{2}} = 0,$$

using (i) of Lemma A.1. Hence, the last two terms of (A.17) will vanish when $\partial C/\partial V_{\alpha\beta}^*$ is evaluated at (*), since the limits of integration of Y_2^* in (A.16a) are both equal to zero. In fact, using definitions (2.10) and (2.11), it can easily be verified that

(A.21) $$l_{i,s+1}(Y_s^* = l_{is})\big|_{(*)} = 0$$

so that successive applications of the "differentiation under the integral sign" identity in (A.17) yield

$$\text{(A.22)}\quad \frac{\partial C}{\partial V_{\alpha\beta}^*}=\int_{l_{11}}^{l_{21}}\cdots\int_{l_{1,k-1}}^{l_{2,k-1}}\left\{\prod_{j=1}^{k-1}\phi(Y_j^*)\right\}\left[\phi(l_{2k})\frac{\partial l_{2k}}{\partial V_{\alpha\beta}^*}-\phi(l_{1k})\frac{\partial l_{1k}}{\partial V_{\alpha\beta}^*}\right]dY_{k-1}^*\cdots dY_1^*$$

$$+\text{terms that vanish at the point }(*).$$

We need, then, the form of the derivatives $\partial l_{ik}/\partial V_{\alpha\beta}^*$, $i=1,2$. From (2.10) we have that

$$\text{(A.23)}\quad \frac{\partial l_{ik}}{\partial V_{\alpha\beta}^*}=-\sum_{j=1}^{k-1}(Y_j^*-\bar{X}_j^*)\left[c_{kk}\frac{\partial c_{kj}}{\partial V_{\alpha\beta}^*}-c_{kj}\frac{\partial c_{kk}}{\partial V_{\alpha\beta}^*}\right]\div c_{kk}^2 \pm\left[\frac{1}{2D_k^{\frac{1}{2}}c_{kk}^{\frac{1}{2}}}\frac{\partial D_k}{\partial V_{\alpha\beta}^*}-\frac{D_k^{\frac{1}{2}}}{2c_{kk}^{\frac{3}{2}}}\frac{\partial c_{kk}}{\partial V_{\alpha\beta}^*}\right]$$

where

$$\text{(A.23a)}\quad D_k=K^{(k)}-\textstyle\sum_{t=2}^{k-1}c_{tt}^{(k-t)}\left[Y_t^*-\bar{X}_t^*+\sum_{j=1}^{t-1}(Y_j^*-\bar{X}_j^*)c_{tj}^{(k-t)}/c_{tt}^{(k-t)}\right]^2 -c_{11}^{(k-1)}(Y_1^*-\bar{X}_1^*)^2$$

so that

$$\text{(A.23b)}\quad \frac{\partial D_k}{\partial V_{\alpha\beta}^*}=-\sum_{t=2}^{k-1}\frac{\partial c_{tt}^{(k-t)}}{\partial V_{\alpha\beta}^*}\left[Y_t^*-\bar{X}_t^*+\sum_{j=1}^{t-1}(Y_j^*-\bar{X}_j^*)c_{tj}^{(k-t)}/c_{tt}^{(k-t)}\right]^2$$
$$-(Y_1^*-\bar{X}_1^*)^2\frac{\partial c_{11}^{(k-1)}}{\partial V_{\alpha\beta}^*}$$
$$-2\sum_{t=2}^{k-1}(c_{tt}^{(k-t)})^{-1}\left[Y_t^*-\bar{X}_t^*+\sum_{j=1}^{t-1}(Y_j^*-\bar{X}_j^*)c_{tj}^{(k-t)}/c_{tt}^{(k-t)}\right]$$
$$\cdot\sum_{j=1}^{t-1}(Y_j^*-\bar{X}_j^*)\left[c_{tt}^{(k-t)}\frac{\partial c_{tj}^{(k-t)}}{\partial V_{\alpha\beta}^*}-c_{tj}^{(k-t)}\frac{\partial c_{tt}^{(k-t)}}{\partial V_{\alpha\beta}^*}\right].$$

We first evaluate $\partial C/\partial V_{\alpha\alpha}^*$, $\alpha=1,\cdots,k-1,k$. For the case $\alpha=k$, and using Lemma A.1, we find

$$\text{(A.24)}\quad D_k(*)=K^{(k)}-\textstyle\sum_1^{k-1}Y_t^{*2}$$

$$\text{(A.24a)}\quad \left.\frac{\partial D_k}{\partial V_{kk}^*}\right|_{(*)}=0,\qquad \frac{\partial l_{ik}}{\partial V_{kk}^*}=\pm\frac{1}{2}\left[K^{(k)}-\sum_1^{k-1}Y_t^{*2}\right]^{\frac{1}{2}}$$

and from (2.10) and (2.11) we also note that

$$\text{(A.25)}\quad l_{is}(*)=\pm\left[K^{(k)}-\textstyle\sum_1^{s-1}Y_t^{*2}\right]^{\frac{1}{2}}\qquad s=k,\cdots,2,\text{ and}$$
$$l_{i1}(*)=\pm\{K^{(k)}\}^{\frac{1}{2}}.$$

From (A.22), then, we have, using the above results, that

$$\left.\frac{\partial C}{\partial V_{kk}^*}\right|_{(*)} = \int_{-[K^{(k)}]^{1/2}}^{[K^{(k)}]^{1/2}} \int_{-[K^{(k)}-Y_1^{*2}]^{1/2}}^{[K^{(k)}-Y_1^{*2}]^{1/2}} \cdots \int_{-[K^{(k)}-\sum_1^{k-2}Y_t^{*2}]^{1/2}}^{[K^{(k)}-\sum_1^{k-2}Y_t^{*2}]^{1/2}} \left\{\prod_1^{k-1} \phi(Y_j^{*2})\right\}$$

$$\times 2\left\{\phi\left[\left\{K^{(k)}-\sum_1^{k-1} Y_t^{*2}\right\}^{\frac{1}{2}}\right]\right\}\frac{1}{2}\left[K^{(k)}-\sum_1^{k-1} Y_t^{*2}\right]^{\frac{1}{2}} dY_{k-1}^* \cdots dY_2^*\,dY_1^*$$

$$= \frac{1}{2}\int_{-[K^{(k)}]^{1/2}}^{[K^{(k)}]^{1/2}} \int_{-[K^{(k)}-Y_1^{*2}]^{1/2}}^{[K^{(k)}-Y_1^{*2}]^{1/2}} \cdots \int_{-[K^{(k)}-\sum_1^{k-2}Y_t^{*2}]^{1/2}}^{[K^{(k)}-\sum_1^{k-2}Y_t^{*2}]^{1/2}}$$

$$\times \left\{\frac{1}{(2\pi)^{\frac{1}{2}(k-1)}}\exp\left\{-\sum_1^{k-1} Y_t^{*2}\right\}\right\}\frac{1}{(2\pi)^{\frac{1}{2}}}\exp\left\{-\frac{1}{2}\left[K^{(k)}-\sum_1^{k-1} Y_t^{*2}\right]\right.$$

$$\times 2\left(K^{(k)}-\sum_1^{k-1} Y_t^{*2}\right)^{\frac{1}{2}} dY_{k-1}^* \cdots dY_2^*\,dY_1^*$$

$$= \frac{1}{2}\frac{\exp\{-\frac{1}{2}K^{(k)}\}}{(2\pi)^{\frac{1}{2}k}}$$

$$\times \int_{-[K^{(k)}]^{1/2}}^{[K^{(k)}]^{1/2}} \int_{-[K^{(k)}-Y_1^{*2}]^{1/2}}^{[K^{(k)}-Y_1^{*2}]^{1/2}} \cdots \int_{-[K^{(k)}-\sum_1^{k-2}Y_t^{*2}]^{1/2}}^{[K^{(k)}-\sum_1^{k-1}Y_t^{*2}]^{1/2}} \int_{-[K^{(k)}-\sum_1^{k-1}Y_t^{*2}]^{1/2}}^{[K^{(k)}-\sum_1^{k-1}Y_t^{*2}]^{1/2}}$$

$$\times dY_k^*\,dY_{k-1}^* \cdots dY_2^*\,dY_1^*$$

$$= \frac{1}{2}\frac{\exp(-\frac{1}{2}K^{(k)})}{(2\pi)^{\frac{1}{2}k}} \times \{\text{Volume of sphere, radius } (K^{(k)})^{\frac{1}{2}}, \text{ in } k \text{ dimensions}\}.$$

The second factor of the above is given by Coxeter (1948); he showed that the volume of such a sphere has value

$$\text{(A.26)} \qquad \frac{2\pi^{\frac{1}{2}k}}{\Gamma(\frac{1}{2}k)k}(K^{(k)})^{\frac{1}{2}k}.$$

Hence we have that

$$\text{(A.26a)} \qquad \left.\frac{\partial C}{\partial V_{kk}^*}\right|_{(*)} = \frac{1}{2}\frac{\exp\{-\frac{1}{2}K^{(k)}\}}{(2\pi)^{\frac{1}{2}k}}\left\{\frac{2\pi^{\frac{1}{2}k}}{k\Gamma(\frac{1}{2}k)}(K^{(k)})^{\frac{1}{2}k}\right\}$$

$$= \frac{1}{2}\frac{(K^{(k)})^{\frac{1}{2}k}\exp\{-\frac{1}{2}K^{(k)}\}}{k\Gamma(\frac{1}{2}k)2^{\frac{1}{2}k-1}}.$$

Now, $\dfrac{\partial C}{\partial V_k^*} = \dfrac{\partial C}{\partial V_{kk}^*}\dfrac{\partial V_{kk}^*}{\partial V_k^*} = 2V_k^*\dfrac{\partial C}{\partial V_{kk}^*}$, so that we have

$$\text{(A.26b)} \qquad \left.\frac{\partial C}{\partial V_k^*}\right|_{(*)} = \frac{(K^{(k)})^{\frac{1}{2}k}\exp\{-\frac{1}{2}K^{(k)}\}}{k\Gamma(\frac{1}{2}k)2^{\frac{1}{2}k-1}}.$$

We now wish to evaluate $\partial C/\partial V_{\alpha\alpha}^*$, $1 \leqq \alpha \leqq k-1$, at the point (*). From (A.23b) and Lemma A.1, we have, for such an α, that

$$\left.\frac{\partial D_k}{\partial V_{\alpha\alpha}^*}\right|_{(*)} = Y_\alpha{}^{*2}. \tag{A.27}$$

Hence, using (A.24), and (A.27) and Lemma A.1, we find from (A.33) that

$$\left.\frac{\partial l_{ik}}{\partial V_{\alpha\alpha}^*}\right|_{(*)} = \pm\left[\frac{1}{2}\left(K^{(k)} - \sum_1^{k-1} Y_t{}^{*2}\right)^{-\frac{1}{2}} Y_\alpha{}^{*2}\right] \tag{A.28}$$

for $1 \leqq \alpha \leqq k-1$. Returning to (A.22), we find, with the help of (A.25), that

$$\begin{aligned}\left.\frac{\partial C}{\partial V_{\alpha\alpha}^*}\right|_{(*)} &= \frac{1}{2}\frac{\exp\{-\frac{1}{2}K^{(k)}\}}{(2\pi)^{\frac{1}{2}k}} \\ &\times \int_{-[K^{(k)}]^{1/2}}^{[K^{(k)}]^{1/2}} \int_{-[K^{(k)}-Y_1{}^{*2}]}^{[K^{(k)}-Y_1{}^{*2}]^{1/2}} \cdots \int_{-[K^{(k)}-\sum_1^{k-2}Y_t{}^{*2}]^{1/2}}^{[K^{(k)}-\sum_1^{k-2}Y_t{}^{*2}]^{1/2}} \\ &\times 2Y_\alpha{}^* \Big/ \left(K^{(k)} - \sum_1^{k-1} Y_t{}^{*2}\right)^{\frac{1}{2}} dY_{k-1}^* \cdots dY_2{}^* \, dY_1{}^*.\end{aligned} \tag{A.28a}$$

But by inverting the order of integration and making a dummy change of variable (A.28a) may then be expressed as

$$\begin{aligned}\left.\frac{\partial C}{\partial V_{\alpha\alpha}^*}\right|_{(*)} &= \frac{1}{2}\frac{\exp\{-\frac{1}{2}K^{(k)}\}}{(2\pi)^{\frac{1}{2}k-1}} \int_{-[K^{(k)}]^{1/2}}^{[K^{(k)}]^{1/2}} y_1{}^2 \\ &\times \left\{\int_{-[K^{(k)}-y_1{}^2]^{1/2}}^{[K^{(k)}-y_1{}^2]^{1/2}} \cdots \int_{-[K^{(k)}-\sum_1^{k-3}y_t{}^2]^{1/2}}^{[K^{(k)}-\sum_1^{k-3}y_t{}^2]^{1/2}} dy_{k-2} \cdots dy_2\right\} dy_1.\end{aligned} \tag{A.28b}$$

Now the inner integral of (A.28b) is the volume of a sphere in $k-3$ dimensions, radius $[K^{(k)}-y_1{}^2]^{\frac{1}{2}}$, that is, has value

$$\frac{2\pi^{\frac{1}{2}(k-3)}(K^{(k)}-y_1{}^2)^{\frac{1}{2}(k-3)}}{\Gamma(\frac{1}{2}(k-3))(k-3)} = \frac{\pi^{\frac{1}{2}(k-3)}(K^{(k)}-y_1{}^2)^{\frac{1}{2}(k-3)}}{\Gamma(\frac{1}{2}(k-1))}.$$

Hence

$$\left.\frac{\partial C}{\partial V_{\alpha\alpha}^*}\right|_{(*)} = \frac{1}{2}\frac{(\exp\{-\frac{1}{2}K^{(k)}\})\pi^{\frac{1}{2}(k-3)}}{(2\pi)^{\frac{1}{2}k-1}\Gamma((k-1)/2)} \int_{-[K^{(k)}]^{1/2}}^{[K^{(k)}]^{1/2}} y_1{}^2(K^{(k)}-y_1{}^2)^{\frac{1}{2}(k-3)} \, dy_1 \tag{A.28c}$$

and on making the transformation $y_1 = [K^{(k)}]^{\frac{1}{2}} \sin\theta$, we arrive at

$$\frac{\partial C}{\partial V_{\alpha\alpha}^*} = \frac{1}{2}\frac{(\exp\{-\frac{1}{2}K^{(k)}\})[K^{(k)}]^{\frac{1}{2}k}}{2^{\frac{1}{2}k-1}k\Gamma(\frac{1}{2}k)} \tag{A.29}$$

for $\alpha = 1, \cdots, k-1$. Hence

$$\left.\frac{\partial C}{\partial V_\alpha{}^*}\right|_{(*)} = \left.\frac{\partial C}{\partial V_k{}^*}\right|_{(*)} = \frac{[K^{(k)}]^{\frac{1}{2}k}\exp\{-\frac{1}{2}K^{(k)}\}}{2^{\frac{1}{2}k-1}k\Gamma(\frac{1}{2}k)},$$

proving part (i) of A.1.

To prove part (ii) of A.1, we first consider the cases $\alpha = k$, $\beta < k$. We then have, from (A.23b) and using Lemma A.1, that

$$\left.\frac{\partial D_k}{\partial V^*_{k\beta}}\right|_{(*)} = 0 \tag{A.30}$$

so that (A.23) takes the value, at (*), given by

$$\left.\frac{\partial l_{ik}}{\partial V^*_{k\beta}}\right|_{(*)} = Y_\beta{}^*, \qquad \beta = 1, \cdots, k-1. \tag{A.30a}$$

From (A.22), then, we see that $\partial C/\partial V^*_{k\beta}|_{(*)}$ has as integrand

$$\{\textstyle\prod_{j=1}^{k-1} \phi(Y_j{}^*)\}\{\phi[(K^{(k)} - \sum_1^{k-1} Y_t{}^{*2})^{\frac{1}{2}}]\}\{Y_\beta{}^* - Y_\beta{}^*\} = 0, \quad \text{i.e.} \tag{A.31}$$

$$\left.\frac{\partial C}{\partial V^*_{k\beta}}\right|_{(*)} = 0, \qquad \beta = 1, \cdots, k-1.$$

We consider now the cases $\alpha, \beta = 1, \cdots, k-1$, $\alpha \neq \beta$. Again, using Lemma A.1, we have from (A.23b) that

$$\left.\frac{\partial D_k}{\partial V^*_{\alpha\beta}}\right|_{(*)} = 2Y_\alpha{}^* Y_\beta{}^*. \tag{A.32}$$

This in turn implies that

$$\left.\frac{\partial l_{ik}}{\partial V^*_{\alpha\beta}}\right|_{(*)} = \pm\left[\frac{1}{2(K^{(k)} - \sum_1^{k-1} Y_t{}^{*2})^{\frac{1}{2}}} 2Y_\alpha{}^* Y_\beta{}^*\right], \qquad 1 \leqq \alpha, \beta \leqq k-1, \alpha \neq \beta. \tag{A.32a}$$

Using the above and consulting (A.22), and making a dummy change of variable, we may write

$$\begin{aligned}\left.\frac{\partial C}{\partial V^*_{\alpha\beta}}\right|_{(*)} &= 2\frac{\exp\{-\frac{1}{2}K^{(k)}\}}{(2\pi)^{\frac{1}{2}k}} \int_{-[K^{(k)}]^{1/2}}^{[K^{(k)}]^{1/2}} y_1 \int_{-[K^{(k)}-y_1{}^2]^{1/2}}^{[K^{(k)}-y_1{}^2]^{1/2}} \\ &\times \left\{\left(\int_{-[K^{(k)}-\sum_1{}^2 y_t{}^2]^{1/2}}^{[K^{(k)}-\sum_1{}^2 y_t{}^2]^{1/2}} \cdots \int_{-[K^{(k)}-\sum_1{}^{k-2} y_t{}^2]^{1/2}}^{[K^{(k)}-\sum_1{}^{k-2} y_t{}^2]^{1/2}}\right.\right. \\ &\times \left.\left(K^{(k)} - \sum_1^{k-1} y_t{}^2\right)^{-\frac{1}{2}} dy_{k-1} \cdots dy_3\right\} dy_2\, dy_1.\end{aligned} \tag{A.33}$$

But the inner integral of (A.33) is

$$\begin{aligned}&\pi \times \text{ Volume of a sphere in } k-4 \text{ dimensions, radius } [K^{(k)} - \textstyle\sum_1^2 y_t{}^2]^{\frac{1}{2}} \\ &= \pi \frac{2\pi^{\frac{1}{2}(k-4)}}{\Gamma(\frac{1}{2}(k-4))(k-4)} \left(K^{(k)} - \sum_1^2 y_t{}^2\right)^{\frac{1}{2}(k-4)}.\end{aligned} \tag{A.34}$$

Inserting (A.34) in (A.33) yields

$$\text{(A.35)}\quad \left.\frac{\partial C}{\partial V_{\alpha\beta}^*}\right|_{(*)} = \frac{\exp\{-\frac{1}{2}K^{(k)}\}}{\Gamma(\frac{1}{2}(k-2))2^{\frac{1}{2}k-1}\pi}\int_{-[K^{(k)}]^{1/2}}^{[K^{(k)}]^{1/2}} y_1 \times\left\{\int_{-[K^{(k)}-y_1^2]^{1/2}}^{[K^{(k)}-y_1^2]^{1/2}} y_2\left(K^{(k)}-\sum_1^2 y_t^2\right)^{\frac{1}{2}(k-4)} dy_2\right\}dy_1$$

$= 0,$ since the inner integral obviously vanishes.

The results (A.30) and (A.35) establish part (ii) of (A.1).

It remains to prove part (iii) of (A.1). From the definitions (2.10) and (2.4), we see that $l_{i1}, \cdots, l_{i,\alpha-1}$ do not involve $\overline{X}_\alpha^*$. Hence we may write (see (2.9))

$$\text{(A.36)}\quad \frac{\partial C}{\partial \overline{X}_\alpha^*} = \int_{l_{11}}^{l_{21}}\cdots\int_{l_{1,\alpha-1}}^{l_{2,\alpha-1}}\left\{\frac{\partial}{\partial \overline{X}_\alpha^*}\int_{l_{1\alpha}}^{l_{2\alpha}}\cdots\int_{l_{1k}}^{l_{2k}}\left\{\prod_1^k \phi(Y_j^*)\right\}dY_k^*\cdots dY_\alpha^*\right\} \times dY_{\alpha-1}^*\cdots dY_1^*.$$

Using arguments similar to these that led to (A.22), and with the help of (A.21), we see that we may write (A.36) as

$$\text{(A.37)}\quad \frac{\partial C}{\partial \overline{X}_\alpha^*} = \int_{l_{11}}^{l_{21}}\cdots\int_{l_{1,k-1}}^{l_{2,k-1}}\left\{\prod_{j=1}^{k-1}\phi(Y_j^*)\right\}\left\{\phi(l_{2k})\frac{\partial l_{2k}}{\partial \overline{X}_\alpha^*}-\phi(l_{1k})\frac{\partial l_{1k}}{\partial \overline{X}_\alpha^*}\right\} \times dY_{k-1}^*\cdots dY_1^*$$

$+$ terms that vanish when evaluated at (*)

for $\alpha = 1, \cdots, k$. There are four cases to be distinguished, viz (i) $\alpha = k$; (ii) $\alpha = k-1$; (iii) $2 \leqq \alpha \leqq k-2$, and (iv) $\alpha = 1$.

We begin with the case $\alpha = k$. From (2.10) we see that

$$\text{(A.38)}\quad \frac{\partial l_{ik}}{\partial \overline{X}_k^*} = 1, \quad \text{so that} \quad \left.\frac{\partial l_{ik}}{\partial \overline{X}_k^*}\right|_{(*)} = 1.$$

Hence, when we evaluate $\partial C/\partial \overline{X}_k^*$, as given by (A.37) with $\alpha = k$, at (*), the resulting integral has integrand

$$\text{(A.39)}\quad \{\textstyle\prod_1^{k-1}\phi(Y_j^*)\}\,\phi[(K^{(k)}-\sum_1^{k-1}Y_j^{*2})^{\frac{1}{2}}](1-1) = 0, \qquad \text{i.e., we have}$$

$$\text{(A3.9a)}\quad \left.\frac{\partial C}{\partial \overline{X}_k^*}\right|_{(*)} = 0.$$

For the second case, $\alpha = k-1$. Consulting (2.10), we have that

$$\text{(A.40)}\quad l_{ik} = \overline{X}_k^* - \textstyle\sum_{j=1}^{k-1} C_{kj}(Y_j^* - \overline{X}_j^*)/C_{kk} \pm C_{kk}^{-\frac{1}{2}}D_k^{\frac{1}{2}}$$

where D_k is given by (A.23a). Thus

$$\text{(A.41)}\quad \frac{\partial l_{ik}}{\partial \overline{X}_{k-1}^*} = C_{k,k-1}/C_{kk} \pm \frac{1}{C_{kk}^{\frac{1}{2}}D_k^{\frac{1}{2}}}\frac{\partial D_k}{\partial \overline{X}_{k-1}^*}.$$

But from (A.23a), we have that

$$\text{(A.42)}\quad \frac{\partial D_k}{\partial \overline{X}^*_{k-1}} = 2c^{(1)}_{k-1,k-1}\left[Y^*_{k-1} - \overline{X}^*_{k-1} + \sum_{j=1}^{k-2} (Y_j^* - \overline{X}_j^*)c^{(1)}_{k-1,j}/c^{(1)}_{k-1,k-1}\right].$$

From (A.24), (A.41) and (A.42), with Lemma A.1, we obtain

$$\text{(A.43)}\qquad \left.\frac{\partial l_{ik}}{\partial \overline{X}_{k-1}}\right|_{(*)} = \left\{\pm\left(K^{(k)} - \sum_{1}^{k-1} Y_t^{*2}\right)^{-\frac{1}{2}} Y^*_{k-1}\right\} + c_{k,k-1}/c_{kk}$$

so that (A.37) gives us

$$\left.\frac{\partial C}{\partial \overline{X}^*_{k-1}}\right|_{(*)} = 2\,\frac{\exp\{-\frac{1}{2}K^{(k)}\}}{(2\pi)^{\frac{1}{2}k}} \int_{-[K^{(k)}]^{1/2}}^{[K^{(k)}]^{1/2}} \cdots$$

$$\times\left\{\int_{-[K^{(k)}-\sum_1^{k-2} Y_t^{*2}]^{1/2}}^{[K^{(k)}-\sum_1^{k-2} Y_t^{*2}]^{1/2}} Y^*_{k-1}\left(K^{(k)} - \sum_{1}^{k-1} Y_t^{*2}\right)^{-\frac{1}{2}} dY^*_{k-1}\right\}\cdots dY_1^*$$

and the inner integral is clearly zero, so that

$$\text{(A.43a)}\qquad \left.\frac{\partial C}{\partial \overline{X}^*_{k-1}}\right|_{(*)} = 0.$$

We turn now to the cases $2 \leqq \alpha \leqq k-2$. Consulting (A.23a) we have

$$\text{(A.44)}\quad \frac{\partial D_k}{\partial \overline{X}_\alpha^*} = -2c^{(k-\alpha-1)}_{\alpha+1,\alpha+1}\left[Y^*_{\alpha+1} - \overline{X}^*_{\alpha+1} + (Y_\alpha^* - \overline{X}_\alpha^*)\frac{c^{(k-\alpha-1)}_{\alpha+1,\alpha}}{c^{(k-\alpha-1)}_{\alpha+1,\alpha+1}} + \sum_{j=1}^{\alpha-1}(Y_j^* - \overline{X}_j^*) \times \frac{c^{(k-\alpha-1)}_{\alpha+1,j}}{c^{(k-\alpha-1)}_{\alpha+1,\alpha+1}}\right]\frac{(-c^{(k-\alpha-1)}_{\alpha+1,\alpha})}{(c^{(k-\alpha-1)}_{\alpha+1,\alpha+1})}$$

$$-2c^{(k-\alpha)}_{\alpha\alpha}\left[Y_\alpha^* - \overline{X}_\alpha^* + \sum_{j=1}^{\alpha-1}(Y_j^* - \overline{X}_j^*)\frac{c^{(k-\alpha)}_{\alpha j}}{c^{(k-\alpha)}_{\alpha\alpha}}\right](-1).$$

From (A.40), for $2 \leqq \alpha \leqq k-2$, we see that

$$\text{(A.45)}\qquad \frac{\partial l_{ik}}{\partial \overline{X}_\alpha^*} = \frac{c_{k\alpha}}{c_{kk}} \pm \frac{1}{2c_{kk}^{\frac{1}{2}} D_k^{\frac{1}{2}}}\frac{\partial D_k}{\partial \overline{X}_\alpha^*}.$$

From (A.24), (A.44), (A.45) and Lemma A.1, we have, if $2 \leqq \alpha \leqq k-2$,

$$\text{(A.46)}\qquad \left.\frac{\partial l_{ik}}{\partial \overline{X}_\alpha^*}\right|_{(*)} = \left\{\pm\frac{1}{2(K^{(k)} - \sum_1^{k-1} Y_t^{*2})^{\frac{1}{2}}}\,2Y_\alpha^*\right\} + \frac{c_{k\alpha}}{c_{kk}}.$$

Using (A.37) and (A.46), it may be seen that $\partial C/\partial \bar{X}_\alpha^*$, $\alpha = 2, \cdots, k-2$ may be written as

$$\text{(A.46a)}\quad \left.\frac{\partial C}{\partial \bar{X}_\alpha^*}\right|_{(*)} = 2\frac{\exp\{-\frac{1}{2}K^{(k)}\}}{(2\pi)^{\frac{1}{2}k}} \int_{-[K^{(k)}]^{1/2}}^{[K^{(k)}]^{1/2}} \cdots \int_{-[K^{(k)}-\sum_1^{k-3}y_t^2]^{1/2}}^{[K^{(k)}-\sum_1^{k-3}y_t^2]^{1/2}} \times \left\{\int_{-[K^{(k)}-\sum_1^{k-2}y_t^2]^{1/2}}^{[K^{(k)}-\sum_1^{k-2}y_t^2]^{1/2}} y_{k-1}\left(K^{(k)} - \sum_1^{k-1} y_t^2\right)^{-\frac{1}{2}} dy_{k-1}\right\} \times dy_{k-2}\cdots dy_1 = 0$$

for $2 \leqq \alpha \leqq k-2$.

Finally, we turn to the case $\alpha = 1$. From (A.23a), we have that

$$\text{(A.47)}\quad \frac{\partial D_k}{\partial \bar{X}_1^*} = -2c_{22}^{(k-2)}\left[Y_2^* - \bar{X}_2^* + (Y_1^* - \bar{X}_1^*)\frac{c_{21}^{(k-2)}}{c_{22}^{(k-2)}}\right]\left(\frac{-c_{21}^{(k-2)}}{c_{22}^{(k-2)}}\right) - 2c_{11}^{(k-1)}(Y_1^* - \bar{X}_1^*)(-1).$$

Also, from (A.40), we have

$$\text{(A.48)}\quad \frac{\partial l_{ik}}{\partial \bar{X}_1^*} = \frac{c_{k1}}{c_{kk}} \pm \frac{1}{2c_{kk}^{\frac{1}{2}} D_k^{\frac{1}{2}}} \frac{\partial D_k}{\partial \bar{X}_1^*}.$$

From (A.24), (A.47), (A.48) and Lemma A.1, we have

$$\text{(A.49)}\quad \left.\frac{\partial l_{ik}}{\partial \bar{X}_1^*}\right|_{(*)} = \left\{\pm\frac{1}{2(K^{(k)} - \sum_1^{k-1} Y_t^{*2})^{\frac{1}{2}}} Y_1^*\right\} + \frac{c_{k1}}{c_{kk}}$$

so that, on using (A.37) and similar arguments as above, we have

$$\text{(A.49a)}\quad \left.\frac{\partial C}{\partial \bar{X}_1^*}\right|_{(*)} = 2\frac{\exp\{-\frac{1}{2}K^{(k)}\}}{(2\pi)^{\frac{1}{2}k}} \int_{-[K^{(k)}]^{1/2}}^{[K^{(k)}]^{1/2}} \cdots \int_{-[K^{(k)}-\sum_1^{k-3}y_t^2]^{1/2}}^{[K^{(k)}-\sum_1^{k-3}y_t^2]^{1/2}} \times \left\{\int_{-[K^{(k)}-\sum_1^{k-2}y_t^2]^{1/2}}^{[K^{(k)}-\sum_1^{k-2}y_t^2]^{1/2}} \frac{y_{k-1}\, dy_{k-1}}{(K^{(k)} - \sum_1^{k-1} y_t^2)^{\frac{1}{2}}}\right\} dy_{k-2}\cdots dy_1. = 0.$$

The results (A.39a), (A.43a), (A.46a) and (A.49a) have proved (iii) of (A.1), that is, we have established the results (2.15).

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