Construction of fractal interpolation surfaces on rectangular grids

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January, 2008

Abstract

We present a general method of generating continuous fractal interpolation surfaces by iterated function systems on an arbitrary data set over rectangular grids and estimate their Box-counting dimension.

Keywords: Iterated function system(IFS); Fractal interpolation function(FIF); Box-counting dimension

1 Introduction

A fractal interpolation function(FIF) is an interpolation function whose graph is a fractal. In 1986, by M. Barnsley[2], FIFs were introduced and after that they have widely been used in many scientific applications like approximation theory(to approximate discrete sequences of data), image compression, computer graphics, modelling of the natural surfaces(terrains, metals, planets, water) and so on. On the basis of the construction of fractal interpolation functions, fractal surfaces are usually generated as graphs of bivariate fractal interpolation functions(BFIF) and they are called fractal interpolation surfaces(FIS).

In many papers (see $[4-7,\ 11-16]$) constructions of self-affine FISs are considered, which are attractors of some iterated function systems(IFSs) associated with a given data set. Massopust [13] presented the construction of self-affine FISs on triangular data sets, at which the interpolation points on the boundary data are coplanar. By Geronimo and Hardin [11] this construction was generalized to allow more general boundary data, and Zhao [16] improved that.

Papers [14, 15] introduced the construction of an attractor that contains the interpolation points of a rectangular data set, but generally is not a graph of a continuous function, and in [8] this problem was solved for the special case where the interpolation points on the boundary data are collinear. This method was generalized by Malysz [12]. In that article, the free vertical contractivity factor is constant and the IFS consists of linear horizontal(domain) contraction

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transformations and vertical contraction mappings which are quadratic polynomials. This type of an IFS was also used by Bouboulis and others [4, 5]. In [6] a general construction of a FIF in $\mathbf{R}^N$ was introduced, but that still constrains the domain contraction transformations and the vertical contractivity factors. To solve these problems, Bouboulis and Dalla [7] introduced a construction of non self-affine FISs on the basis of fractal curves.

In this paper we introduce a new construction of FISs using an even more general IFS with a vertical contraction factor function on an arbitrary data set which can generate self-affine and non self-affine fractal surfaces, and give lower and upper bounds for the (fractal)Box-counting dimension of the constructed surface. Finally, we consider the generalization of our construction for a data set over a grid in $\mathbf{R}^N$.

2 Construction of the BFIF

Let the data set over the rectangular grid be

$$\mathrm{P} = \left\{ (x_i, y_j, z_{ij}) \in \mathbf{R}^3;\ i = 0, 1, \ldots, m,\ j = 0, 1, \ldots, n \right\},$$

such that $x_0 < x_1 < \ldots < x_m,\ y_0 < y_1 < \ldots < y_n$. Let denote

$$\begin{aligned}
&\mathrm{N}_{mn} = \{1, \ldots, m\} \times \{1, \ldots, n\},\ \mathrm{I}_x = [x_0, x_m],\ \mathrm{I}_y = [y_0, y_n],\\
&\mathrm{I}_{x_i} = [x_{i-1}, x_i],\ \mathrm{I}_{y_j} = [y_{j-1}, y_j],\ \mathrm{E}_{ij} = \mathrm{I}_{x_i} \times \mathrm{I}_{y_j},\ \mathrm{E} = \mathrm{I}_x \times \mathrm{I}_y, \quad \text{for } (i,j) \in \mathrm{N}_{mn},\\
&\mathrm{P}_{x_\alpha} = \{(x_\alpha, y_l, z_{\alpha l}) \in \mathrm{P};\ l = 0, 1, \ldots, n\}, \qquad \text{for } \alpha \in \{0, \ldots, m\},\\
&\mathrm{P}_{y_\beta} = \{(x_k, y_\beta, z_{k\beta}) \in \mathrm{P};\ k = 0, 1, \ldots, m\}, \quad \text{for } \beta \in \{0, \ldots, n\},
\end{aligned}$$

In $\mathbf{R}^2$, we use the metric $\rho_0\left((x,y), (x',y')\right) = |x - x'| + |y - y'|$, for $(x,y), (x',y') \in \mathbf{R}^2$.

We construct an IFS$\left\{\mathbf{R}^3; W_{ij} = (L_{ij}, F_{ij}),\ i = 1, \ldots, m,\ j = 1, \ldots, n\right\}$, whose attractor is the graph of some bivariate interpolation function of the data set P. We define the domain contraction transformations $L_{ij} : \mathrm{E} \to \mathrm{E}_{ij}$, for $(i,j) \in \mathrm{N}_{mn}$ by

$$L_{ij}(x,y) = \left(L_{x_i}(x), L_{y_j}(y)\right),$$

where $L_{x_i} : \mathrm{I}_x \to \mathrm{I}_{x_i},\ L_{y_j} : \mathrm{I}_y \to \mathrm{I}_{y_j}$ are contractive homeomorphisms with contractivity factors $a_{x_i},\ a_{y_j}$ obeying

(i) $L_{x_i} : \{x_0, x_m\} \to \{x_{i-1}, x_i\}, \qquad L_{y_j} : \{y_0, y_n\} \to \{y_{j-1}, y_j\}$,

(ii) For any $i \in \{1, \ldots, m-1\},\ j \in \{1, \ldots, n-1\}$, there exist $x_k \in \{x_0, x_m\},\ y_l \in \{y_0, y_n\}$ such that

$$L_{x_{i+1}}(x_k) = L_{x_i}(x_k) = x_i,\ L_{y_{j+1}}(y_l) = L_{y_j}(y_l) = y_j. \tag{1}$$

Denote $a_{ij} = \mathrm{Max}\left\{a_{x_i}, a_{y_j}\right\}$, for $(i,j) \in \mathrm{N}_{mn}$. Then a_{ij} are contractivity factors of the mappings L_{ij}.

Let $F_{ij} : \mathrm{E} \times \mathbf{R} \to \mathbf{R}$, for $(i,j) \in \mathrm{N}_{mn}$ be defined by

$$F_{ij}(x,y,z) = d\left(L_{ij}(x,y)\right)\left(z - g(x,y)\right) + h\left(L_{ij}(x,y)\right), \tag{2}$$

where $d(x,y)$ is a vertical continuous contraction such that $|d(x,y)| < 1$ on E, $h(x,y),\ g(x,y)$ are continuous Lipschitz mappings on E with the Lipschitz constants $\mathrm{L}_h,\ \mathrm{L}_g$ each satisfying

$$\begin{aligned}
&g(x_\alpha, y_\beta) = z_{\alpha,\beta}, &&\text{for } (\alpha, \beta) \in \{0, m\} \times \{0, n\},\\
&h(x_i, y_j) = z_{ij}, &&\text{for } (i,j) \in \{0, 1, \ldots, m\} \times \{0, 1, \ldots, n\}.
\end{aligned}$$

Then, the F_{ij} satisfy 'join up' conditions

$$F_{ij}\left(x_\alpha, y_\beta, z_{\alpha\beta}\right) = z_{\sigma\left(L_{ij}\left(x_\alpha, y_\beta\right)\right)}, \qquad \text{for } \alpha \in \{0, m\}, \ \beta \in \{0, n\},$$

where $\sigma\left(L_{ij}\left(x_\alpha, y_\beta\right)\right) = \sigma\left(x_k, y_l\right) = (k, l)$, for $(k, l) \in \{i-1, i\} \times \{j-1, j\}$. By (1), (2) we have on the common borders $\{x_i\} \times [y_{j-1}, y_j]$, $[x_{i-1}, x_i] \times \{y_j\}$

$$\begin{array}{ll} F_{i+1\ j}\left(x_k, y, z\right) = F_{ij}\left(x_k, y, z\right), & \text{for } i \in \{1, \ldots, m-1\}, \ j \in \{1, \ldots, n\}, \\ F_{i\ j+1}\left(x, y_l, z\right) = F_{ij}\left(x, y_l, z\right), & \text{for } i \in \{1, \ldots, m\}, \ j \in \{1, \ldots, n-1\}, \end{array}$$

where $x_k, \ x_i, \ y_l, \ y_j$ obey (1).

Hence, for $(i, j) \in \mathrm{N}_{mn}$ the transformations W_{ij}, coincide on common borders. The following theorem shows that the above IFS has an attractor.

Theorem 1 *There exists some metric ρ that is equivalent to the Euclidean metric on $\mathbf{R}^3$ such that the W_{ij} are contractions for all $(i, j) \in \mathrm{N}_{mn}$ with respect to ρ.*

Proof. Let denote $d_{max} = \mathrm{Max}_{\mathrm{E}} \left|d(x, y)\right|$, $d_{min} = \mathrm{Min}_{\mathrm{E}} \left|d(x, y)\right|$. We define a metric ρ on $\mathbf{R}^3$ for $(x, y, z), (x', y', z') \in \mathbf{R}^3$ by

$$\rho\left((x, y, z), (x', y', z')\right) = |x - x'| + |y - y'| + \theta |z - z'|,$$

where θ is a positive real number which is specified below. It is obvious that this metric is equivalent to the Euclidean metric on $\mathbf{R}^3$.

The distance between two points $W_{ij}(x, y, z)$ and $W_{ij}(x', y', z')$, for $(i, j) \in \mathrm{N}_{mn}$ is as follows:

$$\begin{aligned} &\rho\left(W_{ij}(x, y, z), W_{ij}(x', y', z')\right) \\ &= \left|L_{x_i}(x) - L_{x_i}(x')\right| + \left|L_{y_j}(y) - L_{y_j}(y')\right| + \theta \left|F_{ij}(x, y, z) - F_{ij}(x', y', z')\right| \\ &\leq \left|a_{x_i}\right| |x - x'| + \left|a_{y_j}\right| |y - y'| + \\ &\quad \theta \left(d_{max} |z - z'| + d_{max} \mathrm{L}_g \left(|x - x'| + |y - y'|\right) + \mathrm{L}_h \left(\left|a_{x_i}\right| |x - x'| + \left|a_{y_j}\right| |y - y'|\right)\right) \\ &= \left(\left|a_{x_i}\right| + \theta \left(d_{max} \mathrm{L}_g + \mathrm{L}_h \left|a_{x_i}\right|\right)\right) |x - x'| + \left(\left|a_{y_j}\right| + \theta \left(d_{max} \mathrm{L}_g + \mathrm{L}_h \left|a_{y_j}\right|\right)\right) |y - y'| + \\ &\quad \theta d_{max} |z - z'|. \end{aligned} \tag{3}$$

We choose

$$\theta = \frac{1 - \mathrm{Max}\left\{\left|a_{x_i}\right|, \left|a_{y_j}\right|; \ i = 1, \ldots, m, \ j = 1, \ldots, n\right\}}{2\mathrm{Max}\left\{d_{max} \mathrm{L}_g + \mathrm{L}_h a_{x_i}, \ d_{max} \mathrm{L}_g + \mathrm{L}_h a_{y_j}; \ i = 1, \ldots, m, \ j = 1, \ldots, n\right\}}.$$

Then, by(3) it follows that

$$\begin{aligned} \rho\left(W_{ij}(x, y, z), W_{ij}(x', y', z')\right) &\leq a |x - x'| + a |y - y'| + \theta d_{max} |z - z'| \\ &\leq \mathrm{Max}\left\{a, d_{max}\right\} \rho\left((x, y, z), (x', y', z')\right), \end{aligned}$$

where

$$a = \frac{1 + \mathrm{Max}\left\{\left|a_{x_i}\right|, \left|a_{y_j}\right|; \ i = 1, \ldots, m, \ j = 1, \ldots, n\right\}}{2} < 1.$$

This complets the proof. □

Let denote

$$\mathbf{C}(\mathrm{E}) = \left\{\varphi \in \mathbf{C}^0(\mathrm{E}) : \ \varphi(x_i, y_j) = z_{ij}, \ i = 0, \ldots, m, \ j = 0, 1 \ldots, n\right\},$$
$$\mathbf{F}(\mathrm{E}) = \{f | f : \mathrm{E} \to \mathbf{R}\}.$$

Defining an operator $T : \mathbf{C}(\mathrm{E}) \to \mathbf{F}(\mathrm{E})$ by

$$(T\varphi)(x, y) = F_{ij}\left(L_{x_i}^{-1}(x), \ L_{y_j}^{-1}(y), \ \varphi\left(L_{x_i}^{-1}(x), L_{y_j}^{-1}(y)\right)\right), \qquad \text{for } (x, y) \in \mathrm{E}_{ij},$$

it can be easily proved that the operator T has the following properties:

(i) $T\varphi \in \mathbf{C}(\mathrm{E})$

(ii) T is contractive in the sup-norm $\|\cdot\|_\infty$ with contractivity factor d_{max}.

Thus, according to the fixed point theorem in the complete metric space $\mathbf{C}(\mathrm{E})$, the operator T has a unique fixed point $f \in \mathbf{C}(\mathrm{E})$ with

$$f(L_{ij}(x, y)) = F_{ij}(x, y, f(x, y)), \qquad \text{for } (x, y) \in \mathrm{E}_{ij}, \ \ (i, j) \in \mathrm{N}_{mn}.$$

This means that

$$Gr(f) = \bigcup_{(i,j) \in N_{mn}} W_{ij}(Gr(f)),$$

where $Gr(f)$ is the graph of the function f.

Therefore, $Gr(f)$ is the attractor of the IFS$\left\{\mathbf{R}^3; W_{ij}, \ i = 1, \ldots, m, \ j = 1, \ldots, n\right\}$ defined above. This gives the following theorem:

Theorem 2 *There exists a bivariate fractal interpolation function* $(BFIF)$ *of the data set* P *whose graph is the attractor of the IFS defined above.*

Remark 1 To be simple, let the endpoints of interval $[x_0, \ x_m]$ be 0 *and* 1. Then, the above L_{x_i} can take two ways of corresponding endpoints of intervals as follows:

$$L_{x_i}^{(1)}(\alpha) = \begin{cases} x_{i-1+\alpha} & i : \text{odd} \\ x_{i-1+(1-\alpha)} & i : \text{even} \end{cases}$$

or

$$L_{x_i}^{(2)}(\alpha) = \begin{cases} x_{i-1+(1-\alpha)} & i : \text{odd} \\ x_{i-1+\alpha} & i : \text{even} \end{cases}$$

for $\alpha \in \{0, 1\}$. $L_{y_j}^{(1)}, \ L_{y_j}^{(2)}$ are the same forms. Thus, for L_{ij} there are 4 cases(Figure 1):

$$\left(L_{x_i}^{(1)}, \ L_{y_j}^{(1)}\right), \ \left(L_{x_i}^{(1)}, \ L_{y_j}^{(2)}\right), \ \left(L_{x_i}^{(2)}, \ L_{y_j}^{(1)}\right), \ \left(L_{x_i}^{(2)}, \ L_{y_j}^{(2)}\right).$$

In the paper[12], L_{ij} has the form $\left(L_{x_i}^{(1)}, \ L_{y_j}^{(1)}\right)$, where $L_{x_i}^{(1)}, \ L_{y_j}^{(1)}$ are linear mappings.

Remark 2 $d(L_{ij}(x, y))$ is the vertical contraction factor function on subdomain E_{ij}. In (2), $d(L_{ij}(x, y))$ can be replaced by $d\left(L_{x_i}^u(x), \ L_{y_j}^v(y)\right)$, where

$$L_{x_i}^u(x) = \begin{cases} L_{x_{i_1}}^{\theta_{i_1}} \circ \cdots \circ L_{x_{i_u}}^{\theta_{i_u}}(x) & u \in \mathbf{Z}^+ \\ x & u = 0 \end{cases} \tag{4}$$

and $\theta_{i_k} \in \{-1, \ 1\}, \ i_k \in \{1, \ldots, m\}, \ k = 1, \ldots, u$. $L_{y_j}^v(y)$ is of the same form. This can improve more flexibility of the IFS, but normally it's attractor is not the graph of some continuous fractal interpolation function, excepting the case where in (2) d is given by $d\left(L_{x_i}(x), \ L_{y_j}(y)\right)$ and $d(x, y)$ from (4).

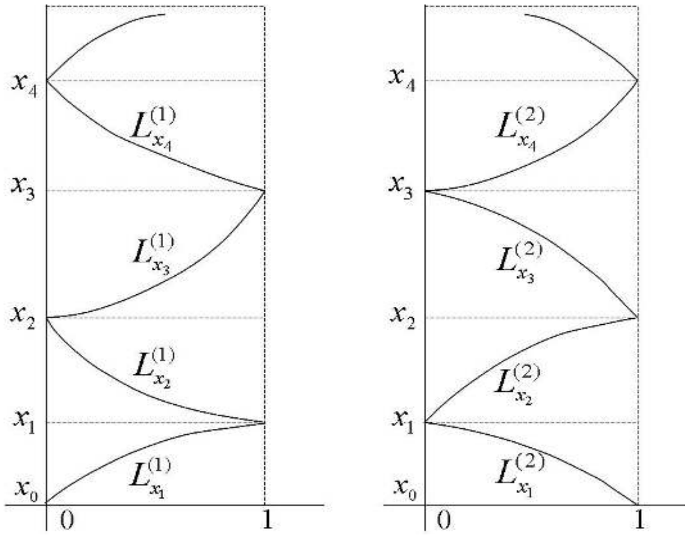

Figure 1: Shaps of $L^{(1)}_{x_i}$, $L^{(2)}_{x_i}$

3 Box-counting dimension of the graph of BFIF

In this section, we get lower and upper bounds for the Box-counting dimension of the graph of the fixed point f of T in the case where a data set is

$$\mathrm{P} = \left\{ \left(x_0 + \frac{x_n - x_0}{n} i,\ y_0 + \frac{y_n - y_0}{n} j,\ z_{ij} \right) \in \mathbf{R}^3;\ i,\ j = 0, 1 \ldots, n \right\}.$$

The calculation of the fractal dimension is similar to that one in [9, 12]. Since there exists a bi-Lipschitz mapping which maps a rectangle $[0,1] \times [0,1]$ to some rectangle $\mathrm{E} \subset \mathbf{R}^2$, we can assume that $\mathrm{E} = [0,1] \times [0,1]$. Then $\mathrm{P} = \left\{ \left(\frac{i}{n},\ \frac{j}{n},\ z_{ij} \right) \in \mathbf{R}^3;\ i,\ j = 0, 1 \ldots, n \right\}$. We denote

$$F_{ij}(x, y, z) = d(x, y)\, z + \varphi(x, y), \qquad \text{for } (x, y, z) \in \mathrm{E} \times \mathbf{R}, \tag{5}$$

where $\varphi(x, y) = h(L_{ij}(x, y)) - d(L_{ij}(x, y))\, g(x, y)$, which is a Lipschitz mapping with Lipschitz constant $b = \mathrm{L}_h \frac{1}{n} + d_{max} \mathrm{L}_g$.

Let the maximum range of a function f be denoted by

$$R_f[\mathrm{M}] = \mathbf{sup}_{u, v \in \mathrm{M}} |f(u) - f(v)| \qquad \text{for } \mathrm{M} \subset \mathbf{R}^2,$$

and denote $\mathrm{D} = \left\{ \left(\frac{i}{n}, \frac{j}{n} \right) \in \mathrm{E};\ \left(\frac{i}{n}, \frac{j}{n}, z_{ij} \right) \in \mathrm{P},\ i,\ j = 0, 1, \ldots, n \right\}$,

$$L(\mathrm{E}) = \bigcup_{(k,l) \in \mathrm{N}_{nn}} L_{kl}(\mathrm{E}), \qquad L^k(\mathrm{E}) = \underbrace{L \circ \cdots \circ L}_{k-times}(\mathrm{E}).$$

Lemma 1 *Let f be the BFIF in Theorem 2. If $d_{max} < 1$, then*

$$R_f[\mathrm{E}] \leq 2n^2 \frac{d_{max} \Delta z + 2(d_{max} \mathrm{L}_g + \mathrm{L}_h)}{1 - d_{max}},$$

where $\Delta z = \mathrm{Max}\{|z_{kl} - z_{00}|;\ k,\ l = 0, 1, \ldots, n\}$.

Proof Let

$$\Lambda_k = \mathrm{Max}\left\{\left|f(u) - z_{00}\right|; u \in L^{k-1}(\mathrm{D})\right\},$$
$$\lambda_k = \mathrm{Max}_{\mathrm{N}_{nn}}\left\{\left|f(u) - z_{\sigma(L_{ij}(0,0))}\right|; u \in L^{k-1}(\mathrm{D}) \cap \mathrm{E}_{ij}\right\},$$

where $L^0(\mathrm{D}) = \mathrm{D}$.

Then, for $u \in L^k(\mathrm{D}) \cap \mathrm{E}_{ij}$,

$$f(u) = F_{ij}(u', f(u')) = d(u)\left[f(u') - g(u')\right] + h(u),$$

where $u' = L_{ij}^{-1}(u) \in L^{k-1}(\mathrm{D}) \cap \mathrm{E}_{ij}$, and

$$f(u) - z_{\sigma(L_{ij}(0,0))} = d(u)\left[(f(u') - z_{00}) + (g(0,0) - g(u'))\right] + h(u) - h(L_{ij}(0,0)),$$

where $g(0,0) = z_{00},\ h(L_{ij}(0,0)) = z_{\sigma(L_{ij}(0,0))}$. Therefore, we obtain

$$\lambda_{k+1} \leq d_{max}\Lambda_k + 2\left(d_{max}\mathrm{L}_g + \mathrm{L}_h\right).$$

Since $\Lambda_k \leq \Lambda_1 + \lambda_k$, by induction it follows that

$$\begin{aligned}\lambda_{k+1} &\leq \left(d_{max}\Lambda_1 + 2\left(d_{max}\mathrm{L}_g + \mathrm{L}_h\right)\right)\left(\sum_{\alpha=0}^{k-1} d_{max}^{\alpha}\right) \\ &\leq \frac{d_{max}\Lambda_1 + 2\left(d_{max}\mathrm{L}_g + \mathrm{L}_h\right)}{1 - d_{max}}.\end{aligned}$$

Hence, we get

$$\mathbf{sup}_{\mathrm{E_{ij}}}\left|f(u) - z_{\sigma(L_{ij}(0,0))}\right| \leq \frac{d_{max}\Lambda_1 + 2\left(d_{max}\mathrm{L}_g + \mathrm{L}_h\right)}{1 - d_{max}}.$$

That gives the result. $\square$

Let $\mathrm{E}_{\mathrm{i}_k,\mathrm{i}_{k-1},\ldots,\mathrm{i}_1} = L_{\mathrm{i}_k} \circ L_{\mathrm{i}_{k-1}} \circ \cdots \circ L_{\mathrm{i}_1}(\mathrm{E})$ for $\mathrm{i}_1, \ldots, \mathrm{i}_k \in \mathrm{N}_{nn}$ for any non-negative integer k and $\mathrm{E}_{\mathrm{i}_0} = \mathrm{E}$.

Lemma 2 *If $d_{max} > \frac{1}{n}$, then there exists $H \in \mathbf{R}$ such that*

$$R_f\left[\mathrm{E}_{\mathrm{i}_k,\mathrm{i}_{k-1},\ldots,\mathrm{i}_1}\right] \leq H d_{max}^k.$$

Otherwise

$$R_f\left[\mathrm{E}_{\mathrm{i}_k,\mathrm{i}_{k-1},\ldots,\mathrm{i}_1}\right] \leq \frac{1}{n^k} R_f[\mathrm{E}] + bk\frac{1}{n^{k-1}}.$$

Proof. By (5), we obtain

$$R_f\left[\mathrm{E}_{\mathrm{i}_k,\mathrm{i}_{k-1},\ldots,\mathrm{i}_1}\right] \quad \leq \quad d_{max} R_f\left[\mathrm{E}_{\mathrm{i}_{k-1},\ldots,\mathrm{i}_1}\right] + b\left(\frac{1}{n}\right)^{k-1}$$

Therefore, by induction

$$R_f\left[\mathrm{E}_{\mathrm{i}_k,\mathrm{i}_{k-1},\ldots,\mathrm{i}_1}\right] \leq d_{max}^k R_f[\mathrm{E}] + b\sum_{\alpha=0}^{k-1} d_{max}^{\alpha}\left(\frac{1}{n}\right)^{k-1-\alpha}. \tag{6}$$

If $d_{max} > \frac{1}{n}$, then by (6), we have

$$\begin{aligned} R_f \left[\mathrm{E}_{\mathrm{i}_k, \mathrm{i}_{k-1}, \ldots, \mathrm{i}_1} \right] &\leq d_{max}^k \left(R_f \left[\mathrm{E} \right] + \frac{b}{d_{max}} \sum_{\alpha=0}^{k-1} \left(\frac{1}{d_{max} n} \right)^{k-1-\alpha} \right) \\ &\leq d_{max}^k \left(R_f \left[\mathrm{E} \right] + \frac{b}{d_{max} \left(1 - \frac{1}{d_{max} n} \right)} \right) = H d_{max}^k , \end{aligned}$$

where

$$H = 2n^2 \frac{2 \left(d_{max} \mathrm{L}_g + \mathrm{L}_h \right) + d_{max} \Delta z}{1 - d_{max}} + \frac{b}{d_{max} \left(1 - \frac{1}{d_{max} n} \right)}.$$

Otherwise, replacing d_{max} by $\frac{1}{n}$ in (6) gives the inequality

$$R_f \left[\mathrm{E}_{\mathrm{i}_k, \mathrm{i}_{k-1}, \ldots, \mathrm{i}_1} \right] \leq \frac{1}{n^k} R_f \left[\mathrm{E} \right] + bk \frac{1}{n^{k-1}}. \quad \square$$

Theorem 3 *Let f be the BFIF of the data set* P *defined above.*

(i) *If there exists $\beta \in \{0, \ldots, n\}$* (or $\alpha \in \{0, \ldots, n\}$) *such that the points of* P_{y_β} (or P_{x_α}) *are non-collinear and $d_{min} > \frac{1}{n}$, then*

$$3 + \log_n^{d_{min}} \leq \dim_{\mathrm{B}} Gr \left(f \right) \leq 3 + \log_n^{d_{max}} . \tag{7}$$

(ii) *If $d_{max} \leq \frac{1}{n}$, then* $\dim_{\mathrm{B}} Gr \left(f \right) = 2$.

Proof. As usual, $\dim_{\mathrm{B}} Gr \left(f \right)$ denotes the Box-counting dimension of graph(f) and is defined by

$$\dim_{\mathrm{B}} Gr \left(f \right) = \lim_{\delta \to 0} \frac{\log N_\delta \left(Gr \left(f \right) \right)}{- \log \delta}$$

(if this limit exists), where $N_\delta \left(Gr \left(f \right) \right)$ is any of the following [9]:

(i) the smallest number of closed balls of radius δ that cover $Gr \left(f \right)$;

(ii) the smallest number of cubes of side δ that cover $Gr \left(f \right)$;

(iii) the number of δ-mesh cubes that intersect $Gr \left(f \right)$;

(iv) the smallest number of sets of diameter at most δ that cover $Gr \left(f \right)$;

(v) the largest number of disjoint balls of radius δ with centres in $Gr \left(f \right)$.

(i) By the assumption, we may assume without loss of generality that $q_1 = \left(\frac{k_1}{n}, \frac{j}{n}, z_{k_1 j} \right)$, $q_2 = \left(\frac{k_2}{n}, \frac{j}{n}, z_{k_2 j} \right)$, $q_3 = \left(\frac{k_3}{n}, \frac{j}{n}, z_{k_3 j} \right)$ are non-collinear. Let $W_{\mathrm{i}_k, \ldots, \mathrm{i}_1} = W_{\mathrm{i}_k} \circ \cdots \circ W_{\mathrm{i}_1}$, for $\mathrm{i}_1, \ldots, \mathrm{i}_k \in \mathrm{N}_{mn}$. Then, the points $W_{\mathrm{i}_k, \ldots, \mathrm{i}_1} \left(q_i \right) \left(i = 1, 2, 3 \right)$ are contained in $W_{\mathrm{i}_k, \ldots, \mathrm{i}_1} \left(Gr \left(f \right) \right)$. The height of the triangle with these vertices is at least $h d_{min}^k$, where h is the distance between the point q_2 and an intersection point at which the vertical line to the rectangle E through the point q_2 intersects the segment $\left[q_1, q_3 \right]$. Thus, by Lemma 2 the range of the function f over $\mathrm{E}_{\mathrm{i}_k, \ldots, \mathrm{i}_1}$ satisfies

$$h d_{min}^k \leq R_f \left[\mathrm{E}_{\mathrm{i}_k, \ldots, \mathrm{i}_1} \right] \leq H d_{max}^k.$$

Let $\varepsilon_k = \left(\frac{1}{n}\right)^k$ and $N(\varepsilon_k)$ denote the smallest number of cubes of the side length ε_k which cover $Gr(f)$. Then,

$$n^{2k}\left(hd_{min}^k n^k\right) \leq N(\varepsilon_k) \leq n^{2k}\left(2 + Hd_{max}^k n^k\right).$$

Taking logarithms and using the definition of the Box-counting dimension gives (7).

(ii) If $d_{max} \leq \frac{1}{n}$, then by Lemma 2, we get

$$\begin{aligned} N(\varepsilon_k) &\leq n^{2k}\left(2 + \left(\frac{1}{n^k}R_f[E] + bk\frac{1}{n^{k-1}}\right)n^k\right) \\ &= n^{2k}\left(2 + R_f[E] + bkn\right). \end{aligned}$$

Therefore, $\dim_{\mathrm{B}} Gr(f) \leq 2$. Since the Box-counting dimension of any surface is at least 2, $\dim_{\mathrm{B}} Gr(f) = 2$. □

Remark 3 If for all $(i,j) \in \mathrm{N}_{mn}$, L_{x_i}, L_{y_j} are similitudes, for all $(\alpha,\beta) \in \{0,\ldots,m\} \times \{0,\ldots,n\}$, the points of $P_{x_\alpha}, P_{y_\beta}$ are collinear and $d(x,y) = d_0$, then $\dim_{\mathrm{B}} Gr(f) = 2$.

4 Examples

In this section, we consider the special case where the L_{ij} are quadratic transformations, F_{ij} are power functions and $\mathrm{E} = [0,1] \times [0,1]$.

$L_{x_i}^{(1)}$, $L_{x_i}^{(2)}$ are given by

$$\begin{aligned} L_{x_i}^{(1)}(x) &= \left[(-1)^{i+1}(x_i - x_{i-1}) + (-1)^i b\right]x^2 + (-1)^{i+1}bx + x_{i-\gamma(i)}, \\ L_{x_i}^{(2)}(x) &= \left[(-1)^i(x_i - x_{i-1}) + (-1)^{i+1} b\right]x^2 + (-1)^i bx + x_{i-1+\gamma(i)}, \end{aligned}$$

where b obeys $0 \leq b \leq 2(x_i - x_{i-1})\,(b \neq x_i - x_{i-1})$ and $\gamma(i) = i \bmod 2$. $L_{y_j}^{(1)}$, $L_{y_j}^{(2)}$ are of the same form. In the case where $b = 2(x_i - x_{i-1})$

$$L_{x_i}^{(1)}(x) = (-1)^i(x_i - x_{i-1})x^2 + (-1)^{i+1}2(x_i - x_{i-1})x + x_{i-\gamma(i)} \tag{8}$$

and

$$L_{x_i}^{(2)}(x) = (-1)^{i+1}(x_i - x_{i-1})x^2 + (-1)^i 2(x_i - x_{i-1})x + x_{i-1+\gamma(i)}. \tag{9}$$

If $b = (x_i - x_{i-1})$, then

$$\begin{aligned} L_{x_i}^{(1)}(x) &= (-1)^{i+1}(x_i - x_{i-1})x + x_{i-\gamma(i)}, \\ L_{x_i}^{(2)}(x) &= (-1)^i(x_i - x_{i-1})x + x_{i-1+\gamma(i)}, \end{aligned}$$

and this is exactly the example [12]:

$$L_{x_i}^{(1)}(x) = \frac{(-1)^{\gamma(i+1)}}{n}x + \frac{i - \gamma(i)}{n}, \tag{10}$$

$$L_{y_j}^{(1)}(y) = \frac{(-1)^{\gamma(j+1)}}{n}y + \frac{j - \gamma(j)}{n}. \tag{11}$$

For $(i,j) \in \mathrm{N}_{mn}$ F_{ij} is as follows:

$$F_{ij}(x,y,z) = d\left(L_{x_i}(x), L_{y_j}(y)\right)(z - g(x,y)) + h\left(L_{x_i}(x), L_{y_j}(y)\right),$$

where

$$
\begin{aligned}
g(x,y) &= z_{00}(1-x)^{s_{11}}(1-y)^{t_{11}} + z_{0n}(1-x)^{s_{12}}y^{t_{12}} \\
&\quad + z_{m0}x^{s_{13}}(1-y)^{t_{13}} + z_{mn}x^{s_{14}}y^{t_{14}}, \\
h(x,y) &= z_{\sigma(L_{ij}(0,0))}\left(1-L_{x_i}^{-1}(x)\right)^{s_{21}}\left(1-L_{y_j}^{-1}(y)\right)^{t_{21}} \\
&\quad + z_{\sigma(L_{ij}(0,1))}\left(1-L_{x_i}^{-1}(x)\right)^{s_{22}}\left(L_{y_j}^{-1}(y)\right)^{t_{22}} \\
&\quad + z_{\sigma(L_{ij}(1,0))}\left(L_{x_i}^{-1}(x)\right)^{s_{23}}\left(1-L_{y_j}^{-1}(y)\right)^{t_{23}} \\
&\quad + z_{\sigma(L_{ij}(1,1))}\left(L_{x_i}^{-1}(x)\right)^{s_{24}}\left(L_{y_j}^{-1}(y)\right)^{t_{24}}
\end{aligned}
$$

and $s_{ij},\ t_{ij} \in \mathbf{R}^+,\ \ i,j \in \{1,2,3,4\}$, and $d(x,y)$ is any continuous function obeying $|d(x,y)| < 1$. In [12] F_{ij} is the case where $s_{ij} = t_{ij} = 1,$ for $i,\ j \in \{1,2,3,4\}$ and $d(x,y) = d_0$. Figure 2 shows FISs constructed according to the above conditions on the data set [12], i.e.
$\mathrm{E} = \left\{(0,0,0), \left(0,\frac{1}{2},0\right), (0,1,0), \left(\frac{1}{2},0,0\right), \left(\frac{1}{2},\frac{1}{2},1\right), \left(\frac{1}{2},1,0\right), (1,0,0), \left(1,\frac{1}{2},0\right), (1,1,0)\right\}$.

5 Construction in the N-dimensional space $\mathbf{R}^N$

Since the principle of the construction of a FIF in $\mathbf{R}^N$ is similar to that in $\mathbf{R}^3$, we introduce only the results.

Let now the data set be denoted by

$$
\mathrm{P} = \left\{(x_{1,i_1}, x_{2,i_2}, \ldots, x_{M,i_M}, z_{i_1,i_2,\ldots,i_M}) \in \mathbf{R}^{M+1};\ i_k = 0,1,\ldots,m_k,\ m_k \in \mathbf{N},\ k = 1,\ldots,M\right\},
$$

where $x_{k,0} < x_{k,1} < \ldots < x_{k,m_k}$ for $k \in \{1,\ldots,M\}$, $M,\ m_k \in \mathbf{N}$, and denote $\mathrm{P}_{k,i_k} = \left\{(x_{1,i_1},\ldots,x_{k,i_k},\ldots,x_{M,i_M}, z_{i_1,\ldots,i_k,\ldots,i_M}) \in \mathrm{P};\ i_l = 0,\ldots,m_l,\ l = 1,\ldots,k-1,k+1,\ldots,M\right\}$, for $k \in \{1,\ldots,M\}$, $i_k \in \{0,\ldots,m_k\}$. We use the following notations;

$$
\begin{aligned}
&\mathrm{I}_k = [x_{k,0}, x_{k,m_k}], \quad \mathrm{I}_{k,l} = [x_{k,l-1}, x_{k,l}], \\
&\mathrm{E_i} = \mathrm{I}_{1,i_1} \times \ldots \times \mathrm{I}_{M,i_M}, \\
&\Omega = \{(1,i_1),\ldots,(M,i_M);\ i_k = 1,\ldots,m_k,\ k = 1,\ldots,M\},
\end{aligned}
\tag{12}
$$

where $\mathrm{i} \in \Omega,\ \ i_k \in \{1,\ldots,m_k\},\ k \in \{1,\ldots,M\}$. Then $\mathrm{E} = \mathrm{I}_1 \times \ldots \times \mathrm{I}_M = \bigcup_{\mathrm{i}} \mathrm{E_i},\ \mathrm{I}_k = \bigcup_{l=1}^{m_k} \mathrm{I}_{k,l}$.

We construct an IFS$\left\{\mathbf{R}^{M+1};\ W_{\mathrm{i}} = (L_{\mathrm{i}}, F_{\mathrm{i}});\mathrm{i} \in \Omega\right\}$. The domain contraction transformations $L_{\mathrm{i}} : \mathrm{E} \to \mathrm{E_i}$ with contractivity factors a_{i} are defined by

$$
L_{\mathrm{i}}(x_1,\ldots,x_M) = \left(L_{1,i_1}(x_1),\ldots,L_{M,i_M}(x_M)\right),
$$

where $L_{k,i_k} : \mathrm{I}_k \to \mathrm{I}_{k,i_k}$, for $i_k \in \{1,\ldots,m_k\},\ k \in \{1,\ldots,M\}$ are contractive homeomorphisms with the contractivity factors a_{k,i_k} satisfying

(i) $L_{k,i_k} : \{x_{k,0}, x_{k,m_k}\} \to \{x_{k,i_k-1}, x_{k,i_k}\},\ i_k \in \{1,\ldots,m_k\}$,

(ii) For any $x_{k,i_k} \in \{x_{k,1},\ldots,x_{k,m_k-1}\}$, there exist $x_{k,l} \in \{x_{k,0}, x_{k,m_k}\}$ such that

$$
L_{k,i_k+1}(x_{k,l}) = L_{k,i_k}(x_{k,l}) = x_{k,i_k},
\tag{13}
$$

and $a_{\mathrm{i}} = \mathrm{Max}\{a_{1,i_1},\ldots,a_{M,i_M}\}$.

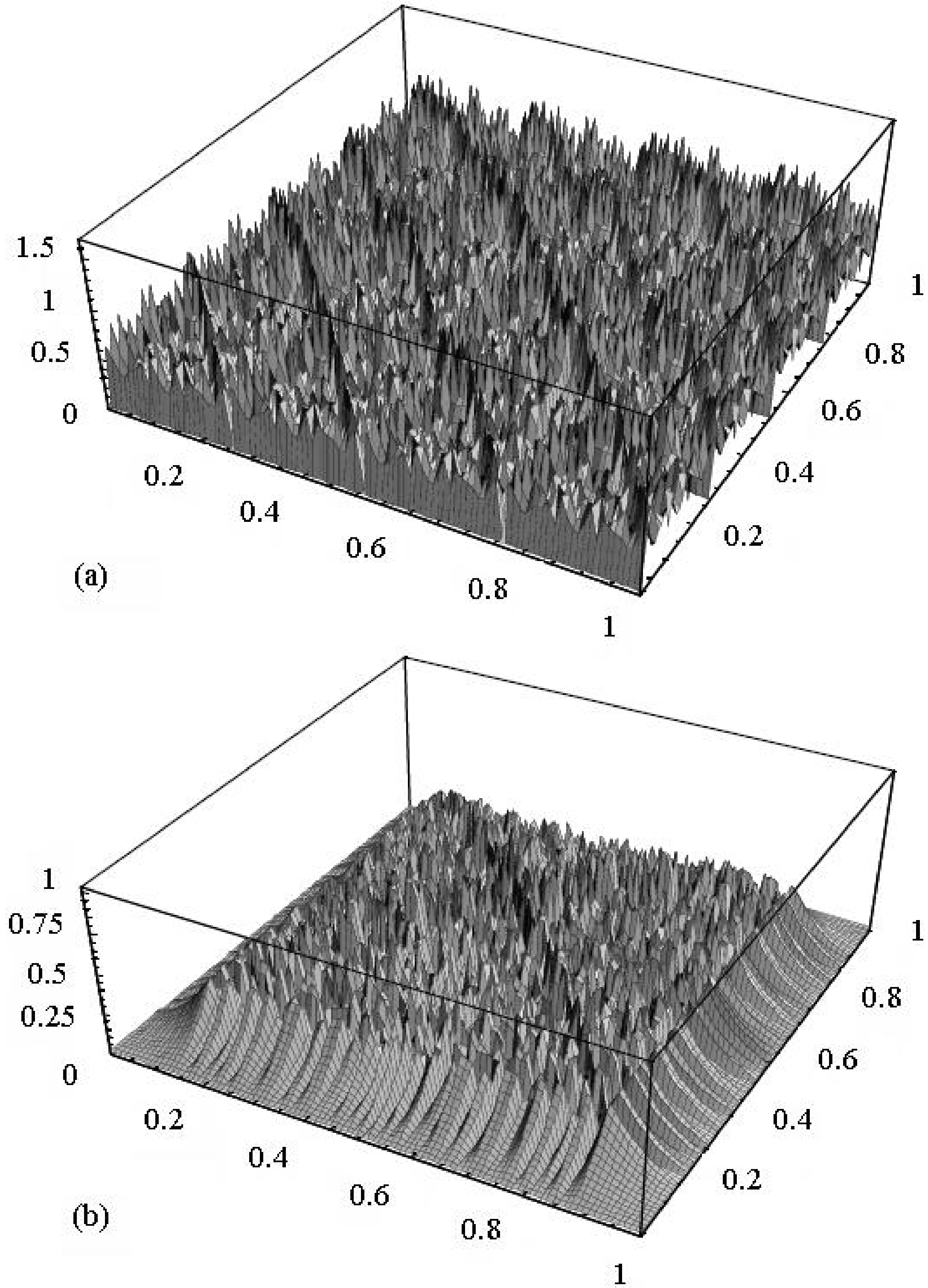

Figure 2: FISs: In (a), $L^{(1)}_{x_i},\ L^{(1)}_{y_j}$ are defined by (10), (11). In (b), L_{ij} are given by (8), (9) : (b) : $\left(L^{(1)}_{x_i},\ L^{(1)}_{y_j}\right)$. Here $s_{ij} = t_{ij} = 2.5,\ d(x,y) = 0.9$. $\dim_{\mathrm{B}} Gr(f) \approx 2.848$.

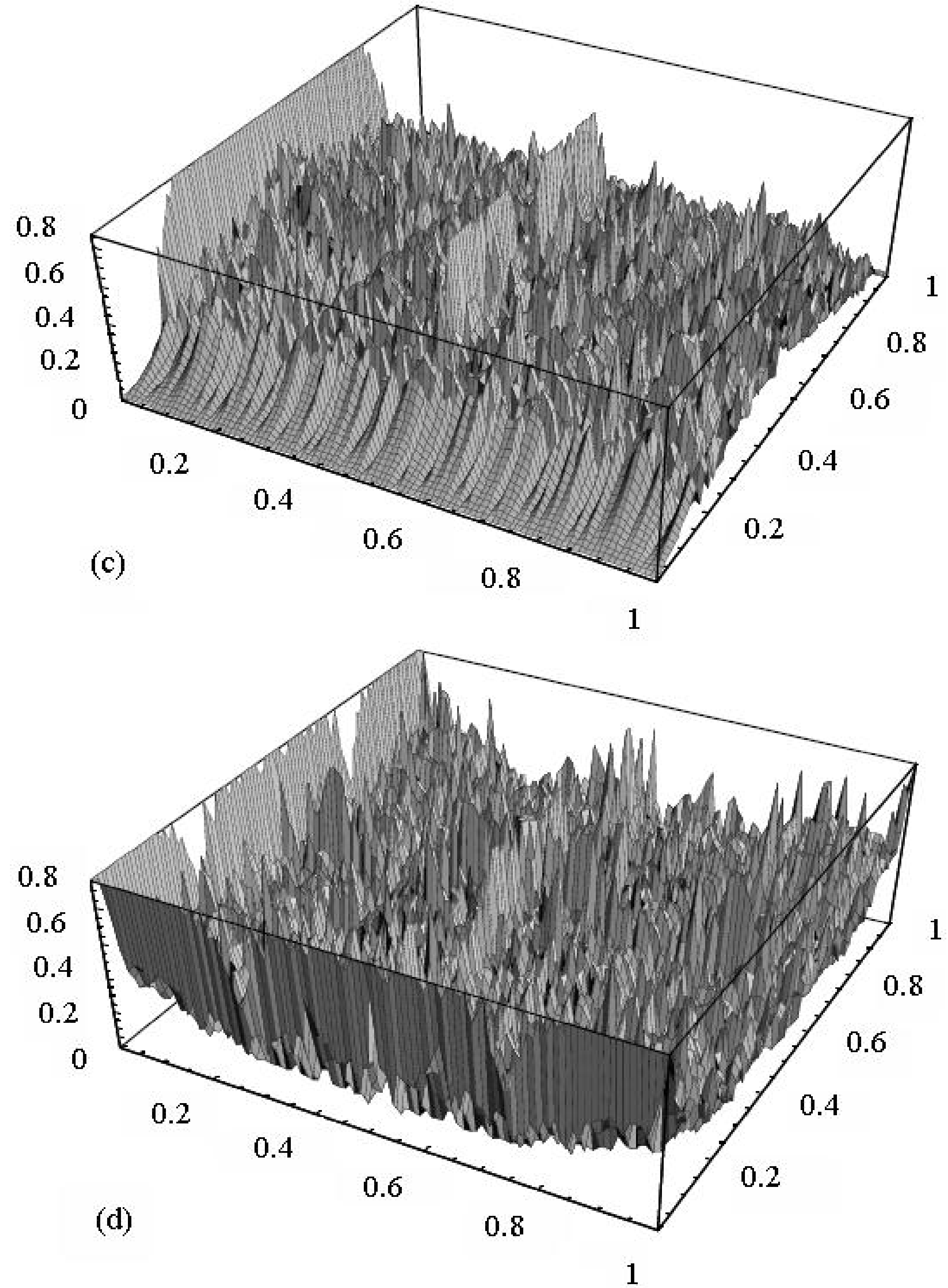

Figure 3: FISs: In (c), (d), L_{ij} are given by (8), (9): (c) : $\left(L_{x_i}^{(1)},\ L_{y_j}^{(2)}\right)$, (d) : $\left(L_{x_i}^{(2)},\ L_{y_j}^{(2)}\right)$. Case $\left(L_{x_i}^{(2)},\ L_{y_j}^{(1)}\right)$ is symmetric to (c). Here $s_{ij} = t_{ij} = 2.5,\ d\,(x, y) = 0.9$. $\dim_{\mathrm{B}} Gr\,(f) \approx 2.848$.

We define the vertical contraction functions $F_{\mathrm{i}} : \mathrm{E} \times \mathbf{R} \to \mathbf{R}$ by

$$F_{\mathrm{i}}(\mathbf{x}, z) = d(L_{\mathrm{i}}(\mathbf{x}))(z - g(\mathbf{x})) + h(L_{\mathrm{i}}(\mathbf{x})), \qquad \text{for } (\mathbf{x}, z) \in \mathrm{E} \times \mathbf{R}, \tag{14}$$

where $d(\mathbf{x})$ obeys $|d(\mathbf{x})| < 1$ and g, h are continuous Lipschitz mappings on E with Lipschitz constants $\mathrm{L}_h, \mathrm{L}_g$ satisfying

$$g(x_{1,e_1}, \ldots, x_{M,e_M}) = z_{e_1,\ldots,e_M} \text{ for } (e_1, \ldots, e_M) \in \{0, m_1\} \times \ldots \times \{0, m_M\}, \text{ and}$$
$$h(x_{1,i_1}, \ldots, x_{M,i_M}) = z_{i_1,\ldots,i_M} \text{ for } (i_1, \ldots, i_M) \in \{1, \ldots, m_1\} \times \ldots \times \{0, \ldots, m_M\}.$$

Then, the F_{i} satisfy 'join-up' conditions

$$F_{\mathrm{i}}(x_{1,e_1}, \ldots, x_{M,e_M}, z_{e_1,\ldots,e_M}) = z_{\sigma\left(L_{\mathrm{i}}\left(x_{1,e_1},\ldots,x_{M,e_M}\right)\right)},$$

where $\sigma(L_{\mathrm{i}}(x_{1,e_1}, \ldots, x_{M,e_M})) = \sigma(x_{1,k_1}, \ldots, x_{k,k_M}) = (k_1, \ldots, k_M), (e_1, \ldots, e_M) \in \{0, m_1\} \times \ldots \times \{0, m_M\}$, $(k_1, \ldots, k_M) \in \{i_1 - 1,\ i_1\} \times \ldots \times \{i_M - 1,\ i_M\}$.

Consequently, W_{i} are contractive transformations for all $\mathrm{i} \in \Omega$ with respect to some metric which is equivalent to Euclidean metric on $\mathbf{R}^{M+1}$. Furthermore, there exists an interpolation function $f : \mathrm{E} \to \mathbf{R}$ of a data set P whose graph is the attractor of the above IFS. The following theorem gives the Box-counting dimension of the graph of this function f in the case where the data set

$$\mathrm{P} = \left\{ \left(\frac{i_1}{n}, \ldots, \frac{i_M}{n}, z_{i_1,\ldots,i_M} \right) \in \mathrm{R}^{M+1};\ i_k = 0, 1, \ldots, n,\ k = 1, \ldots, M \right\}$$

and $\mathrm{E} = [0, 1]^M$.

Theorem 4 *Let f be the above FIF of the data set* P.

i) *If there exists $k \in \{1, \ldots, n\}$ such that the points of the data set P_{k,i_k} are not in $M - 1$ dimension space and $d_{min} > \frac{1}{n}$, then*

$$M + 1 + \log_n^{d_{min}} \leq \dim_{\mathrm{B}} Gr(f) \leq M + 1 + \log_n^{d_{max}}.$$

ii) *If $d_{max} \leq \frac{1}{n}$, then $\dim_{\mathrm{B}} Gr(f) = M$, where $d_{max} = \mathrm{Max}_{\mathrm{E}} |d(\mathbf{x})|$, $d_{min} = \mathrm{Min}_{\mathrm{E}} |d(\mathbf{x})|$.*

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