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CONSTRUCTION OF INVARIANTS

By

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1. Introduction.

Let G be a connected reductive group defined over the complex number field $\boldsymbol{C}$, V a finite dimensional vector space and $\rho: G\rightarrow GL(V)$ a rational representation of G. Such a triplet (G, ρ, V) is called a *prehomogeneous vector space* if V has an open G-orbit, and called *irreducible* if ρ is an irreducible representation. A complete list of irreducible prehomogeneous vector spaces is given by M. Sato and T. Kimura [12]. The purpose of this paper is to construct explicitly an irreducible relative invariant for every irreducible prehomogeneous vector space. If (G, ρ, V) and (G', ρ', V') are in the same castling class, then an irreducible relative invariant of (G, ρ, V) can be constructed from that of (G', ρ', V'). (See proposition 18 in [12, section 4].) Hence it is enough to consider irreducible reduced prehomogeneous vector spaces. (See [12, section 2] for the generalities concerning the castling transformations.) In the tables I and II of [12, section 7], irreducible relative invariants are given except for the following six cases;

(6) $(GL(7), \Lambda_3, V(35))$,

(7) $(GL(8), \Lambda_3, V(56))$,

(10) $(SL(5)\times GL(3), \Lambda_2\otimes\Lambda_1, V(10)\otimes V(3))$,

(20) $(\mathrm{Spin}\,(10)\times GL(2), (\text{half spin})\otimes\Lambda_1, V(16)\otimes V(2))$,

(21) $(\mathrm{Spin}\,(10)\times GL(3), (\text{half spin})\otimes\Lambda_1, V(16)\otimes V(3))$,

(24) $(GL(1)\times \mathrm{Spin}\,(14), (\text{half spin}), V(64))$.

Irreducible relative invariants of (6) and (7) are constructed by T. Kimura [8], and that of (20) is constructed by H. Kawahara [7]. (Concerning a construction of an invariant of (7), see the last section of the present paper.) Hence our task is to construct irreducible relative invariants of (10), (21) and (24).

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2. Invariants of $SL(5)\times GL(3)$.

Let $\Lambda^2 C^5$ be the Grassmann tensor product of C^5 of the second order. If $\{e_1, \cdots, e_5\}$ is a basis of C^5, a general element x of $\Lambda^2 C^5$ is uniquely expressed as

$$x=\sum_{1\leq i<j\leq 5} x_{ij} e_i \wedge e_j .$$

In this section, we reserve the letters x, y, z, w and u for such elements. Their coordinates are written as x_{ij}, y_{ij} etc. and we put $x_{ji}=-x_{ij}$ etc. A general element of the representation space $V=(\Lambda^2 C^5)\otimes C^3$ can be regarded as a triplet (x, y, z) and the action ρ of $G=SL(5)\times GL(3)$ on V is given by

$$\rho(g_1, g_2)(x, y, z)=(g_1 x, g_1 y, g_1 z)\cdot {}^t g_2$$

for $(g_1, g_2)\in G$, where $g_1 x$ etc. are the natural action of $SL(5)$ on $\Lambda^2 C^5$. Consider the following polynomials;

$$f_1(x)=x_{23}x_{45}-x_{24}x_{35}+x_{25}x_{34} ,$$

$$f_2(x)=x_{34}x_{51}-x_{35}x_{41}+x_{31}x_{45} ,$$

$$f_3(x)=x_{45}x_{12}-x_{41}x_{52}+x_{42}x_{51} ,$$

$$f_4(x)=x_{51}x_{23}-x_{52}x_{13}+x_{53}x_{12} ,$$

$$f_5(x)=x_{12}x_{34}-x_{13}x_{24}+x_{14}x_{23} .$$

REMARK 1. We introduced these polynomials by a representation theoretic consideration as in [8], so that the property (3) below is satisfied.

Let $D_{y,x}$ be the polarization which transforms a letter x to y [13], In our case

$$D_{y,x}=\sum_{1\leq i<j\leq 5} y_{ij}\frac{\partial}{\partial x_{ij}} .$$

Let

$$g_i(x, y)=D_{y,x} f_i(x) ,$$

and

$$P(x, y, z, w, u)=\sum_{i,j=1}^{5} g_i(x, y) g_j(z, w) u_{i,j} .$$

By the definition of P,

(1) $$P(x, y, z, w, u)=P(y, x, z, w, u)=-P(z, w, x, y, u) .$$

Hence

(2) $$P(x, y, x, y, z)=0 .$$

LEMMA. *The polynomial P is a relative invariant with respect to $GL(5)$. More precisely,*

(3) $$P(gx, gy, gz, gw, gu)=(\det g)^2 P(x, y, z, w, u)$$

for $g\in GL(5)$.

PROOF. Invariance with respect to the scalar action of $GL(1)$ is obvious. By the symmetry, it is enough to show the invariance with respect to the matrix unit E_{12}. Note that $-E_{12}$ acts as the polarization which transforms 1 to 2. Hence $-E_{12}f_2=-f_1$ and $E_{12}f_i=0$ for $i\neq 2$. Hence $-E_{12}g_2=-g_1$ and $E_{12}g_i=0$ for $i\neq 2$. Using this fact, we can easily show that $E_{12}P(x, y, z, w, u)=0$. □

(4) If at most two kinds of letters appear among $\{x, y, z, w, u\}$, then $P(x, y, z, w, u)=0$, e.g., $P(x, x, x, y, y)=0$ etc.

PROOF. In such a case, P gives a relative invariant of $(GL(5), \Lambda_2\oplus\Lambda_2, V(10)\oplus V(10))$ which is a prehomogeneous vector space without relative invariant other than constants [12; p 94]. This fact can also be shown by a representation theoretic consideration as in [8]. □

By (4), $P(z, z, y, y, y)=0$. By the polarization $D_{x,y}$, we get

(5) $$2P(z, z, x, y, y)+P(z, z, y, y, x)=0.$$

Hence by (1),

(6) $$2P(z, z, x, y, y)=P(y, y, z, z, x).$$

By (4), $P(y, y, y, x, x)=0$. By the polarization $D_{z,x}$, we get

(7) $$P(y, y, y, x, z)+P(y, y, y, z, x)=0.$$

By (1) and (7),

(8) $$P(y, y, y, z, x)=-P(y, y, y, x, z)=P(x, y, y, y, z).$$

By multiplying the both sides of (6) and (8),

(9) $$2P(y, y, y, z, x)P(z, z, x, y, y)=P(x, y, y, y, z)P(y, y, z, z, x).$$

THEOREM 1. *Put*

$$\begin{aligned}F(x, y, z)=&P(x, x, x, y, z)P(y, y, z, z, x)^2+P(y, y, y, z, x)P(z, z, x, x, y)^2\\&+P(z, z, z, x, y)P(x, x, y, y, z)^2\\&-P(x, x, y, y, z)P(y, y, z, z, x)P(z, z, x, x, y)\\&-4P(x, x, x, y, z)P(y, y, y, z, x)P(z, z, z, x, y).\end{aligned}$$

Then F is an irreducible relative invariant of $(SL(5)\times GL(3), \Lambda_2\otimes\Lambda_1, V(10)\otimes V(3))$ *which corresponds to the character*

$$(g_1, g_2) \longrightarrow (\det g_2)^5, \qquad (g_1, g_2)\in SL(5)\times GL(3).$$

PROOF. Since the degree of an irreducible relative invariant is known to be 15 [12, section 7, Table I (10)], it is enough to prove the relative invariance of F. The invariance with respect to $SL(5)\times GL(1)$ is obvious, where $GL(1)$ is the set of scalar matrices in $GL(3)$. Hence it is enough to see the invariance with respect to the actions of the matrix units $E_{ij}\in\mathrm{Lie}(GL(3))$ for $i\neq j$. Since x, y and z appears symmetrically in F, it is enough to consider only one of them. The action of $\{E_{ij}|i\neq j\}$ are nothing but the polarizations $D_{y,x}$ etc. Hence it is enough to show that $D_{y,x}F(x, y, z)=0$. By (2) and (4), we have

$$\begin{aligned} D_{y,x}F(x,y,z) &= P(x,x,y,y,z)P(y,y,z,z,x)\{P(y,y,z,z,x)-2P(z,z,x,y,y)\} \\ &\quad +2P(z,z,x,x,y)\{2P(y,y,y,z,x)P(z,z,x,y,y)-P(x,y,y,y,z)P(y,y,z,z,x)\} \\ &\quad +4P(x,x,y,y,z)P(z,z,z,x,y)\{P(x,y,y,y,z)-P(y,y,y,z,x)\} \end{aligned}$$

By (6), (9) and (8), the right hand side equals zero. □

REMARK 2. Let G be any reductive group and $\rho: G\to GL(V)$ any rational representation. Let $[v_0]\in V/G$ be a generic point, v_0 a point in the closed G-orbit lying above $[v_0]$, G_{v_0} the isotropy subgroup of G at v_0, T a maximal torus of G_{v_0}, N the normalizer of T in G, $V^T=\{v\in V\,|\,tv=v,\ t\in T\}$, $\boldsymbol{C}[V]$ the set of polynomial functions on V, ϕ a rational character of G and

$$\boldsymbol{C}[V]^{G,\phi}=\{f\in\boldsymbol{C}[V]\,|\,f(gv)=\phi(g)f(v),\ g\in G\}.$$

Define $\boldsymbol{C}[V^T]^{N,\phi}$ in the same way. Then we have an isomorphism of Chevalley type

$$\boldsymbol{C}[V]^{G,\phi}\cong\boldsymbol{C}[V^T]^{N,\phi}.$$

which is given by the restriction. (See [11; Appendix 2].) For many prehomogeneous vector spaces (G, ρ, V), it is quite easy to give a non-zero element of $\boldsymbol{C}[V^T]^{N,\phi}$. Thus we can describe the restriction of an irreducible relative invariant in $\boldsymbol{C}[V]^{G,\phi}$ to V^T. In our case, this description gave us enough information to determine the explicit form of F in our theorem.

REMARK 3. In our case (G, ρ, V) has a unique split $\boldsymbol{Z}$-form [3]. For this $\boldsymbol{Z}$-form, $V(\boldsymbol{Z})$ may be identified with the lattice of $V(\boldsymbol{C})$ generated by

$$(e_i\wedge e_j, 0, 0),\ (0, e_i\wedge e_j, 0),\ (0, 0, e_i\wedge e_j),$$

where $1 \leqq i < j \leqq 5$. Then $\pm 2^{-5}F(x, y, z)$ are the irreducible relative invariants in $\boldsymbol{Z}[V]$.

In fact, since $g_i(x, x) = 2f_i(x)$, we can show that $2^{-2}P(x, x, y, y, z)$, $2^{-1}P(y, y, y, z, x)$ etc. belong to $\boldsymbol{Z}[V]$. If we take

$$(e_1 \wedge e_2 + e_3 \wedge e_4,\ e_2 \wedge e_3 + e_4 \wedge e_5,\ e_1 \wedge e_3 + e_2 \wedge e_5)$$

as v_0 in remark 2, then we can take

$$\{\mathrm{diag}(1, t, t^{-1}, t^2, t^{-2}) \times \mathrm{diag}(t^{-1}, 1, t) \mid t \in \boldsymbol{C} - \{0\}\}$$

as T. Then $C = V^T$ is the linear span of the following elements;

$$(e_1 \wedge e_2, 0, 0),\ (e_3 \wedge e_4, 0, 0),$$

$$(0, e_2 \wedge e_3, 0),\ (0, e_4 \wedge e_5, 0),$$

$$(0, 0, e_1 \wedge e_3),\ (0, 0, e_2 \wedge e_5).$$

An easy calculation shows that

$$2^{-5}F(x, y, z)|_C = -x_{12}^3 x_{34}^2 y_{23} y_{45}^4 z_{13}^3 z_{25}^2 .$$

Hence $2^{-5}F(x, y, z)$ is irreducible in $\boldsymbol{Z}[V]$. Note that we have also shown that

$$\boldsymbol{Z}[V]^{G, \phi} \cong \boldsymbol{Z}[V^T]^{N, \phi} .$$

in our case.

3. Invariant of $Spin(10) \times GL(3)$.

The purpose of this section is to construct an irreducible relative invariant of $(Spin(10) \times GL(3)$, (half spin)$\otimes \Lambda_1$, $V(16) \otimes V(3))$. In this section, we need the theory of spinors. See [12; pp. 110-114] and [1] for the generalities concerning the spinor groups and spinor representations. Here we use the same notations as in [12].

A general element x of the representation space $V(16)$ of the even half spin representation of $Spin(10)$ can be written uniquely as

$$x = x_0 + \sum_{1 \leq i < j \leq 5} x_{ij} e_i e_j + \sum_{1 \leq i < j < k < l \leq 5} x_{ijkl} e_i e_j e_k e_l .$$

In this section, we reserve the letters x, y, z and w for such elements. Their coordinates are written as x_0, y_{ij} etc., and we put $x_{ji} = -x_{ij}$ and

$$x_{p(i), p(j), p(k), p(l)} = \mathrm{sign}(p) x_{ijkl}$$

for any permutation p of $i < j < k < l$. A general element of the representation space $V(16) \otimes V(3)$ can be regarded as a triplet (x, y, z) and the action $\rho = \rho_1 \otimes \rho_2$

of $G=Spin(10)\times GL(3)$ on V is given by

$$\rho(g_1, g_2)(x, y, z)=(\rho_1(g_1)x, \rho_1(g_1)y, \rho_1(g_1)z)\cdot {}^t\rho_2(g_2)$$

for $(g_1, g_2)\in G$, where ρ_1 is the even half spin representation of $Spin(10)$ on $V(16)$ and ρ_2 is the natural representation of $GL(3)$ on $V(3)$.

Consider the following polynomials;

$$f_1(x)=-x_{12}x_{1345}+x_{13}x_{1245}-x_{14}x_{1235}+x_{15}x_{1234},$$

$$f_2(x)=-x_{23}x_{2451}+x_{24}x_{2351}-x_{25}x_{2341}+x_{21}x_{2345},$$

$$f_3(x)=-x_{34}x_{3512}+x_{35}x_{3412}-x_{31}x_{3452}+x_{32}x_{3451},$$

$$f_4(x)=-x_{45}x_{4123}+x_{41}x_{4523}-x_{42}x_{4513}+x_{43}x_{4512},$$

$$f_5(x)=-x_{51}x_{5234}+x_{52}x_{5134}-x_{53}x_{5124}+x_{54}x_{5123},$$

$$f_6(x)=x_0x_{2345}-x_{23}x_{45}+x_{24}x_{35}-x_{25}x_{34},$$

$$f_7(x)=x_0x_{3451}-x_{34}x_{51}+x_{35}x_{41}-x_{31}x_{45},$$

$$f_8(x)=x_0x_{4512}-x_{45}x_{12}+x_{41}x_{52}-x_{42}x_{51},$$

$$f_9(x)=x_0x_{5123}-x_{51}x_{23}+x_{52}x_{13}-x_{53}x_{12},$$

$$f_{10}(x)=x_0x_{1234}-x_{12}x_{34}+x_{13}x_{24}-x_{14}x_{23},$$

$$g_i(x, y)=D_{xy}f_i(x),$$

$$P(x, y, z, w)=\sum_{i=1}^{5}(g_i(x, y)g_{i+5}(z, w)+g_{i+5}(x, y)g_i(z, w)),$$

Then by the definition of P,

(1) $$P(x, y, z, w)=P(y, x, z, w)=P(z, w, x, y).$$

The polynomials f_i are known as spinor invariants [1]. Concerning the properties of the spinor invariants, what is necessary for our purpose is the following fact;

$$f_i(\rho_1(g)v)=\sum_{j=1}^{10}\chi(g)_{ij}f_j(v)$$

for $g\in Spin(10)$ and $1\leqq j\leqq 10$. Here χ denotes the vector representation of $Spin(10)$ ([12]), and $\chi(g)_{ij}$ denote the matrix components. Since the image of χ is the special orthogonal group which preserves the symmetric bilinear form

$$\sum_{i=1}^{5}(\xi_i\eta_{i+5}+\xi_{i+5}\eta_i),$$

the polynomial P is a $Spin(10)$-invariant, i.e.,

(3) $$P(gx, gy, gz, gw)=P(x, y, z, w)$$

for $g\in Spin(10)$. Here we wrote gx etc. for $\rho_1(g)x$ etc. Of course, (3) can also be shown by a direct calculation as in section 2. Since $P(x, x, x, x)$ is an (absolute) invariant of the non-regular prehomogeneous vector space ($Spin(10)$, half spin, $V(16)$) without relative invariants other than constants [12; section 7, Table III (6′)],

(4) $$P(x, x, x, x)=0\,.$$

Polarizing (4) by D_{yx}, we get

(5) $$P(x, x, x, y)=0\,.$$

(Here we used (1).) Polarizing (5) again by D_{yx}, we get

(6) $$P(x, x, y, y)+2P(x, y, x, y)=0\,.$$

Polarizing (6) by D_{zy}, we get

(7) $$P(x, x, y, z)+2P(x, y, x, z)=0\,.$$

THEOREM 2 (H. Kawahara [7]). *An irreducible relative invariant of* ($Spin(10)\times GL(2)$, (half spin)$\otimes\Lambda_1$, $V(16)\otimes V(2)$) *is given by* $F_2(x, y)=P(x, y, x, y)$.

PROOF. It is easy to see that $F_2(x, y)\neq 0$. (See remark 4 below.) By (3), the invariance with respect to $Spin(10)\times GL(1)$ is obvious, where $GL(1)$ is the set of scalar matrices in $GL(2)$. By (1) and (5), we have

$$D_{xy}F_2(x, y)=P(x, x, x, y)+P(x, y, x, x)=0\,.$$

Since $F_2(x, y)=F_2(y, x)$, $F_2(x, y)$ is a relative invariant with respect to $Spin(10)\times GL(2)$. Since the degree of an irreducible relative invariant is known to be 4 [12; section 7, Table I (20)], F_2 is irreducible. □

REMARK 4. In the case treated in theorem 2, (G, ρ, V) has a unique split $\boldsymbol{Z}$-form [3]. For this $\boldsymbol{Z}$-form, $V(\boldsymbol{Z})$ may be identified with the lattice of $V(\boldsymbol{C})$ generated by the elements

$$(1, 0), (0, 1),$$

$$(e_ie_j, 0), (0, e_ie_j), \quad (1\leqq i<j\leqq 5),$$

$$(e_ie_je_ke_l, 0), (0, e_ie_je_ke_l), \quad (1\leqq i<j<k<l\leqq 5)\,.$$

Then $\pm F_2(x, y)$ are the irreducible relative invariants in $\boldsymbol{Z}[V]$. In order to prove this, take

$$(1+e_1e_2e_3e_4,\ e_1e_5+e_2e_3e_4e_5)$$

as v_0 in remark 2. Then we can take as T the inverse image by ($\chi\times$identity) of the set of

$$\mathrm{diag}\,(1, t_2, t_3, t_4, t_5^2; 1, t_2^{-1}, t_3^{-1}, t_4^{-1}, t_5^{-2})\times\mathrm{diag}\,(t_5, t_5^{-1})$$

where $t_2, t_3, t_4, t_5\in \boldsymbol{C}-\{0\}$ and $t_2t_3t_4=1$. Then $C=V^T$ is the linear span of the following 4 elements;

$$(1, 0), (e_1e_2e_3e_4, 0), (0, e_1e_5), (0, e_2e_3e_4e_5).$$

An easy calculation shows that

$$F_2(x, y)|_C=x_2x_{1334}y_{15}y_{2345}.$$

Hence F_2 is irreducible in $\boldsymbol{Z}[V]$. We have also shown that

$$\boldsymbol{Z}[V]^{G,\phi}\cong\boldsymbol{Z}[V^T]^{N,\phi}$$

in our case.

THEOREM 3. *An irreducible relative invariant of* $((Spin(10)\times GL(3)$, (half spin)$\otimes\Lambda_1$, $V(16)\otimes V(3))$ *is given by*

$$\begin{aligned}F_3(x, y, z)=&P(x, x, y, y)P(x, y, z, z)^2+P(y, y, z, z)P(y, z, x, x)^2\\&+P(z, z, x, x)P(z, x, y, y)^2-P(x, x, y, y)P(y, y, z, z)P(z, z, x, x)\\&+2P(x, x, y, z)P(y, y, z, x)P(z, z, x, y).\end{aligned}$$

PROOF. It is easy to see that $F_3(x, y, z)\neq 0$. (See remark 5 below.) By (3), the invariance with respect to $Spin(10)\times GL(1)$ is obvious, where $GL(1)$ is the set of scalar matrices of $GL(3)$. Since the degree of an irreducible relative invariant is known to be 12 [12; section 7, Table I (21)], it is enough to show that $D_{xy}F_3(x, y, z)=0$. By (1) and (5), we have

$$\begin{aligned}D_{xy}F_3(x, y, z)=&2P(x, y, z, z)P(x, x, y, z)\{P(x, x, y, z)+2P(x, y, x, z)\}\\&+2P(x, x, z, z)P(x, z, y, y)\{P(x, x, y, z)+2P(x, y, x, z)\}.\end{aligned}$$

Hence by (7), $D_{xy}F_3(x, y, z)=0$. □

REMARK 5. In the case treated in theorem 3, (G, ρ, V) has a unique split $\boldsymbol{Z}$-form [3]. For this $\boldsymbol{Z}$-form, $V(\boldsymbol{Z})$ may be identified with the lattice of $V(\boldsymbol{C})$ generated by

$$(1, 0, 0), (0, 1, 0), (0, 0, 1),$$

$$(e_ie_j, 0, 0), (0, e_ie_j, 0), (0, 0, e_ie_j), \quad 1\leqq i<j\leqq 5,$$

$$(e_ie_je_ke_l, 0, 0), (0, e_ie_je_ke_l, 0), (0, 0, e_ie_je_ke_l), \quad 1\leqq i<j<k<l\leqq 5.$$

Then $\pm 2^{-4}F_3(x, y, z)$ are the irreducible relative invariants in $\boldsymbol{Z}[V]$. In fact, since $g_i(x, x)=2f_i(x)$, we can show that $2^{-2}P(x, x, y, y)$, $2^{-1}P(x, y, z, z)$ etc. belong to $\boldsymbol{Z}[V]$. Hence $2^{-4}F_3(x, y, z)\in\boldsymbol{Z}[V]$. If we take

$$(1+e_1e_2e_3e_4,\ e_1e_5+e_2e_3e_4e_5,\ e_1e_2+e_1e_3e_4e_5)$$

as v_0 in remark 2, then we can take as T the inverse image by $(\chi\times\text{identity})$ of the set of

$$\operatorname{diag}(1, (t_1t_2)^{-1}, t_1, t_2, (t_1t_2)^{-2}\,;\, 1, t_1t_2, t_1^{-1}, t_2^{-1}, (t_1t_2)^2)\times\operatorname{diag}((t_1t_2)^{-1}, t_1t_2, 1),$$

where $t_1, t_2\in\boldsymbol{C}-\{0\}$. Then $C=V^T$ is the linear span of the following 6 elements;

$$(1, 0, 0),\ (e_1e_2e_3e_4, 0, 0),$$
$$(0, e_1e_5, 0),\ (0, e_2e_3e_4e_5, 0),$$
$$(0, 0, e_1e_2),\ (0, 0, e_1e_3e_4e_5).$$

An easy calculation shows that

$$2^{-4}F_3(x, y, z)|_C=-x_0^3x_{1234}y_{15}y_{2345}^3z_{12}^2z_{1345}^2.$$

Hence $2^{-4}F_3(x, y, z)$ is irreducible in $\boldsymbol{Z}[V]$. We have also shown that

$$\boldsymbol{Z}[V]^{G,\phi}\cong\boldsymbol{Z}[V^T]^{N,\phi}$$

in our case.

4. Invariants of $(GL(1)\times GL(7),\ \Lambda_3\oplus\Lambda_1,\ V(35)\oplus V(7))$.

The purpose of this and next sections are to construct an irreducible relative invariant of $(GL(1)\times Spin(14)$, (odd half spin), $V(64))$, where $GL(1)$ acts on $V(64)$ as scalars. First, we need to construct irreducible relative invariants of $(GL(1)\times GL(7),\ \Lambda_3\oplus\Lambda_1,\ V(35)\oplus V(7))$, where $GL(1)$ acts on $V(7)$ as scalars. A construction of the irreducible relative invariants of this prehomogeneous vector space is given by T. Kimura. See [8; p. 96, Table A (14)]. Here we give another construction.

Let $\{e_1, \cdots, e_7\}$ be a basis of $V(7)$. Then $\{e_i\wedge e_j\wedge e_k\,|\,1\leqq i<j<k\leqq 7\}$ is a basis of $V(35)$. We write e_{ijk} for $e_i\wedge e_j\wedge e_k$. A general element of $V=V(35)\oplus V(7)$ can be uniqely expressed as

$$x=\sum_{1\leqq i<j<k\leqq 7} x_{ijk}e_{ijk}\oplus\sum_{i=1}^{7} x_ie_i.$$

Put $x_{jik}=-x_{ijk}$ etc. If we take

$$(e_{123}+{}_{567}+e_{145}+e_{246}+e_{347})\oplus e_4$$

as v_0 in remark 2, we can take

$$\{\operatorname{diag}(t_1, t_2, t_3, 1, t_1^{-1}, t_2^{-1}, t_3^{-1}) \mid t_1 t_2 t_3 = 1\}$$

as the maximal torus T of G_{v_0}, where $G = GL(1) \times GL(7)$. (See remark 2 for the notations.) Then $C = V^T$ is the linear span of the following 6 elements;

$$e_{123}, e_{567}, e_{145}, e_{246}, e_{347}, e_4 .$$

The relative invariants of (N, V^T) are products of

(4.1) $$x_{123}^2 x_{567}^2 x_4^2 ,$$

(4.2) $$x_{123}^2 x_{567}^2 x_{145} x_{246} x_{347} ,$$

and scalars. Let J_6 and J_7 be the relative invariants of (G, V) whose restrictions are (4.1) and (4.2) respectively.

THEOREM **4.** (1) *We have*

$$\begin{aligned} J_6 = & \textstyle\sum' x_{123}^2 x_{456}^2 x_7^2 \\ & -2\textstyle\sum' x_{123}^2 x_{456} x_{457} x_6 x_7 \\ & -2\textstyle\sum' x_{123} x_{124} x_{356} x_{456} x_7^2 \\ & +2\textstyle\sum' x_{123} x_{124} x_{356} x_{457} x_6 x_7 \\ & +2\textstyle\sum' x_{123} x_{124} x_{356} x_{567} x_4 x_7 \\ & -4\textstyle\sum' x_{123} x_{156} x_{246} x_{345} x_7^2 \\ & -4\textstyle\sum' x_{123} x_{145} x_{246} x_{357} x_6 x_7 , \end{aligned}$$

where $\sum' x_{123}^2 x_{456}^2 x_7^2$ *etc. means the sum of distinct terms among*

$$\{x_{p(1),p(2),p(3)}^2 x_{p(4),p(5),p(6)}^2 x_{p(7)}^2 \mid p \in \mathfrak{S}_7\}.$$

The relative invariant J_6 *corresponds to the character*

$$(g_1, g_2) \longrightarrow g_1^2 (\det g_2)^2 , \qquad (g_1, g_2) \in GL(1) \times GL(7) .$$

(2) *We have*

$$\begin{aligned} J_7 = & \textstyle\sum' \pm x_{123} x_{124} x_{135} x_{246} x_{357} x_{467} x_{567} \\ & -\textstyle\sum' \pm x_{123}^2 x_{145} x_{246} x_{357} x_{467} x_{567} \\ & +\textstyle\sum' \pm x_{123}^2 x_{145} x_{246} x_{347} x_{567}^2 \\ & +\textstyle\sum' \pm x_{123} x_{124} x_{135} x_{256} x_{347} x_{467} x_{567} \\ & +\textstyle\sum' \pm x_{123} x_{124} x_{135} x_{256} x_{367} x_{457} x_{467} \end{aligned}$$

$$+\sum{}'\pm x_{123}x_{124}x_{156}x_{257}x_{346}x_{357}x_{467}$$

$$-2\sum{}'\pm x_{123}x_{124}x_{156}x_{257}x_{345}x_{367}x_{467}$$

$$-4\sum{}'\pm x_{123}x_{246}x_{356}x_{257}x_{145}x_{167}x_{347}\,.$$

where $\sum{}'\pm x_{123}x_{124\dots}$ *etc. means the sum of distinct terms among*

$$\{\mathrm{sign}\,(p)x_{p(1),\,p(2),\,p(3)}x_{p(4),\,p(5),\,p(6)\dots}\,|\,p\in\mathfrak{S}_7\}\,.$$

The relative invariant J_7 *corresponds to the character*

$$(g_1,\,g_2)\longrightarrow(\det g_2)^3\,,\qquad (g_1,\,g_2)\in GL(1)\times GL(7)\,.$$

REMARK 6. The above formula for J_7 is already obtained by J. Igusa [5]. A different formula for J_7 is given in [2]. (See also [8].)

PROOF OF (1). We write

$$(abc,\ def,\ \cdots,\ i,\ j,\ \cdots)$$

for the monomial

$$x_{abc}x_{def}\cdots x_i x_j\cdots\,,$$

and

$$p(abc,\ \cdots)$$

for

$$(p(a)p(b)p(c),\ \cdots)\,,$$

where p is a permutation. Put

$$m_1=(123,123,456,456,7,7)\,,$$

$$m_2=(123,123,456,457,6,7)\,,$$

$$m_3=(123,124,345,567,6,7)\,,$$

$$m_4=(123,124,356,456,7,7)\,,$$

$$m_5=(123,124,356,457,6,7)\,,$$

$$m_6=(123,124,356,567,4,7)\,,$$

$$m_7=(123,156,246,345,7,7)\,,$$

$$m_8=(123,145,246,357,6,7)\,.$$

By considering the invariance with respect to the maximal torus of $GL(1)\times GL(7)$ and the permutation matrices in $GL(7)$, we can show that J_6 is of the form

$$\sum_{k=1}^{8} a_k(\Sigma' m_k),$$

with $a_1=1$. Since $(34)m_3=-m_3$, $\Sigma' m_3=0$. So we may suppose that $a_3=0$. Let us consider the derivation D_{ij} $(i\neq j)$ such that

$$D_{ij}x_{klm}=\delta_{jk}x_{ilm}+\delta_{jl}x_{kim}+\delta_{jm}x_{kli} \qquad (1\leqq k<l<m\leqq 7),$$

and

$$D_{ij}x_k=\delta_{jk}x_i \qquad (1\leqq k\leqq 7).$$

Since $-D_{ij}$ is nothing but the action of the matrix unit E_{ji}, it is enough to determine a_k's so that

$$D_{ij}\sum_k a_k(\Sigma' m_k)=0.$$

If $(ij)=(76)$

$$(123, 123, 456, 457, 7, 7)$$

appears only in

$$D_{76}(123, 123, 456, 456, 7, 7)=D_{76}m_1,$$

$$D_{76}(123, 123, 456, 457, 6, 7)=D_{76}m_2,$$

Hence $2a_1+a_2=0$, $a_2=-2$. If $(ij)=(34)$,

$$(123, 123, 356, 456, 7, 7)$$

appears only in

$$D_{34}(123, 124, 356, 456, 7, 7)=D_{34}m_4,$$

$$D_{34}(123, 123, 456, 456, 7, 7)=D_{34}m_1.$$

Hence $2a_1+a_4=0$, $a_4=-2$. If $(ij)=(34)$,

$$(123, 123, 356, 457, 6, 7)$$

appears only in

$$D_{34}(123, 124, 356, 457, 6, 7)=D_{34}m_5.$$

$$D_{34}(123, 123, 456, 457, 6, 7)=D_{34}m_2.$$

Hence $a_5=-a_2=2$. If $(ij)=(34)$,

$$(123, 123, 356, 567, 4, 7)$$

appears only in

$$D_{34}(123, 124, 356, 567, 4, 7)=D_{34}m_6,$$

$$D_{34}(123, 123, 456, 567, 4, 7)=D_{34}(46)m_2.$$

Hence $a_6=-a_2=2$. If $(ij)=(25)$,

$$(123, 126, 246, 345, 7, 7)$$

appears only in

$$D_{25}(153, 126, 246, 345, 7, 7) = -D_{25}(13)(25)m_4 ,$$

$$D_{25}(123, 156, 246, 345, 7, 7) = D_{25}m_7 ,$$

$$D_{25}(123, 126, 546, 345, 7, 7) = -D_{25}(465)m_4 .$$

Hence $a_7 = 2a_4 = -4$. If $(ij) = (34)$.

$$(123, 135, 246, 357, 6, 7)$$

appears only in

$$D_{34}(124, 135, 246, 357, 6, 7) = D_{34}(124653)m_6 ,$$

$$D_{34}(123, 145, 246, 357, 6, 7) = D_{34}m_8 ,$$

$$D_{34}(123, 135, 246, 457, 6, 7) = D_{34}(23)(45)m_5 .$$

Hence $a_5 + a_6 + a_8 = 0$, $a_8 = -4$. Thus we have completed the proof of (1). □

REMARK 7. Let $P_i = \{p \in \mathfrak{S}_7 \mid pm_i = m_i\}$. Then

$$P_1 = (\mathfrak{S}(123)\mathfrak{S}(456)) \rtimes \langle (14)(25)(36) \rangle ,$$

$$P_2 = \mathfrak{S}(123) \times \langle (45), (67) \rangle ,$$

$$P_4 = \langle (12), (56), (15)(26) \rangle \times \langle (34) \rangle ,$$

$$P_5 = \langle (12), (34)(67) \rangle ,$$

$$P_6 = (\mathfrak{S}(12) \times \mathfrak{S}(56)) \rtimes \langle (15)(26)(47) \rangle ,$$

$$P_7 = \langle (26)(35), (12)(45), (23)(56) \rangle \cong \mathfrak{S}_4 ,$$

$$P_8 = \langle (24)(35), (23)(45)(67) \rangle ,$$

where an isomorphism $\mathfrak{S}_4 \to P_7$ is given by

$$(12) \to (26)(35), (23) \to (12)(45), (34) \to (23)(56) .$$

Hence the number of terms apearing in $\sum' m_i$ $(i = 1, 2, 4, 5, 6, 7, 8)$ are 70, 210, 315, 1260, 630, 210 and 1260 respectively. Let $f^\vee = f = J_6$. Then $f^\vee(\mathrm{grad}) f^{s+1} = b(s) f^s$ with a polynomial

$$b(s) = b_0(s+1)\left(s + \frac{5}{2}\right)\left(s + \frac{7}{2}\right)^2 (s+4)(s+5)$$

[6]. Since $b(0) = f^\vee(\mathrm{grad}) f = 2^5 5^2 7^2$, $b_0 = 2^6$.

PROOF OF (2). We keep the conventions above. Put

$$m_1 = (123, 124, 135, 246, 357, 467, 567) ,$$

$$m_2=(123, 124, 134, 256, 357, 467, 567),$$
$$m_3=(123, 123, 145, 246, 357, 467, 567),$$
$$m_4=(123, 123, 145, 246, 347, 567, 567),$$
$$m_5=(123, 124, 135, 256, 347, 467, 567),$$
$$m_6=(123, 124, 135, 256, 367, 457, 467),$$
$$m_7=(123, 124, 135, 267, 367, 456, 457),$$
$$m_8=(123, 124, 156, 257, 346, 357, 467),$$
$$m_9=(123, 124, 156, 257, 345, 367, 467),$$
$$m_{10}=(123, 246, 356, 257, 145, 167, 347),$$
$$m_{11}=(123, 123, 123, 456, 457, 467, 567),$$
$$m_{12}=(123, 123, 124, 345, 467, 567, 567),$$
$$m_{13}=(123, 123, 124, 356, 457, 467, 567),$$
$$m_{14}=(123, 123, 145, 245, 367, 467, 567),$$
$$m_{15}=(123, 124, 125, 345, 367, 467, 567),$$
$$m_{16}=(123, 124, 125, 346, 357, 467, 567),$$

By considering the invariance with respect to the maximal torus of $GL(1)\times GL(7)$ and the permutation matrices of $GL(7)$, we can show that J_7 is of the form

$$\sum_{k=1}^{16} a_k(\Sigma' \pm m_k),$$

with $a_4=1$. Since $(23)(67)m_2=-m_2$, $(45)m_{11}=m_{11}$, $(56)m_{13}=m_{13}$ and $(12)m_{14}=m_{14}$, we have

$$\Sigma' \pm m_2=\Sigma' \pm m_{11}=\Sigma' \pm m_{13}=\Sigma' \pm m_{14}=0.$$

So we may suppose that $a_2=a_{11}=a_{13}=a_{14}=0$. As in the proof of (1), let us determine the coefficients a_k so that

$$D_{ij} \sum_k a_k(\Sigma' \pm m_k)=0.$$

If $(ij)=(34)$,

$$(123, 123, 123, 345, 467, 567, 567),$$

appears only in

$$D_{34}(123, 123, 124, 345, 467, 567, 567)=D_{34}m_{12}$$

Hence $a_{12}=0$. If $(ij)=(34)$,

$$(123, 123, 125, 345, 367, 467, 567)$$

appears only in

$$D_{34}(123, 124, 125, 345, 367, 467, 567) = D_{34}m_{15}\,,$$

$$D_{34}(123, 123, 125, 345, 467, 467, 567) = -D_{34}(45)m_{12}\,.$$

Hence $a_{15} = -2a_{12} = 0$. If $(ij) = (34)$,

$$(123, 123, 125, 346, 357, 467, 567)$$

appears only in

$$D_{34}(123, 124, 125, 346, 357, 467, 567) = D_{34}m_{16}\,,$$

$$D_{34}(123, 123, 125, 346, 457, 467, 567) = -D_{34}(45)m_{13}\,.$$

Hence $a_{16} = -a_{13} = 0$. If $(ij) = (54)$,

$$(123, 123, 145, 246, 357, 567, 567)$$

appears only in

$$D_{54}(123, 123, 145, 246, 347, 567, 567) = D_{54}m_4\,,$$

$$D_{54}(123, 123, 145, 246, 357, 467, 567) = D_{54}m_3\,,$$

Hence $a_3 = -a_4 = -1$. If $(ij) = (34)$,

$$(123, 123, 135, 246, 357, 467, 567)$$

appears only in

$$D_{34}(123, 124, 135, 246, 357, 467, 567) = D_{34}m_1\,,$$

$$D_{34}(123, 123, 145, 246, 357, 467, 567) = D_{34}m_3\,,$$

$$D_{34}(123, 123, 135, 246, 457, 467, 567) = -D_{34}(23)(45)m_{13}\,.$$

Hence $a_1 + a_3 - a_{13} = 0$, $a_1 = 1$. If $(ij) = (34)$,

$$(123, 123, 135, 256, 347, 467, 567)$$

appears only in

$$D_{34}(123, 124, 135, 256, 347, 467, 567) = D_{34}m_5\,,$$

$$D_{34}(123, 123, 145, 256, 347, 467, 567) = -D_{34}(45)m_3\,,$$

Hence $a_3 + a_5 = 0$, $a_5 = 1$. If $(ij) = (34)$,

$$(123, 123, 135, 256, 367, 457, 467)$$

appears only in

$$D_{34}(123, 124, 135, 256, 367, 457, 467) = D_{34}m_6\,,$$

$$D_{34}(123, 123, 145, 256, 367, 457, 467) = -D_{34}(12)(456)m_3\,,$$

$$D_{34}(123, 123, 135, 256, 467, 457, 467) = D_{34}(23)(4567)m_{12}\,,$$

Hence $a_6+a_3+2a_{12}=0$, $a_6=1$. If $(ij)=(34)$,

$$(123, 123, 135, 267, 367, 456, 457)$$

appears only in

$$D_{34}(123, 124, 135, 267, 367, 456, 457)=D_{34}m_7\,,$$

$$D_{34}(123, 123, 145, 267, 367, 456, 457)=D_{34}(123)(46)(57)m_{14}\,,$$

$$D_{34}(123, 123, 135, 267, 467, 456, 457)=D_{34}(23)(457)m_{13}\,.$$

Hence $a_7+a_{14}-a_{13}=0$, $a_7=0$. If $(ij)=(34)$,

$$(123, 123, 156, 257, 346, 357, 467)$$

appears only in

$$D_{34}(123, 124, 156, 257, 346, 357, 467)=D_{34}m_8\,,$$

$$D_{34}(123, 123, 156, 257, 346, 457, 467)=D_{34}(23)(46)m_3\,.$$

Hence $a_8+a_3=0$, $a_8=1$. If $(ij)=(34)$,

$$(123, 123, 156, 257, 345, 367, 467)$$

appears only in

$$D_{34}(123, 124, 156, 257, 345, 367, 467)=D_{34}m_9\,,$$

$$D_{34}(123, 123, 156, 257, 345, 467, 467)=-D_{34}(4567)m_4\,.$$

Hence $a_9+2a_4=0$, $a_9=-2$. If $(ij)=(34)$,

$$(123, 236, 356, 257, 145, 167, 347)$$

appears only in

$$D_{34}(124, 236, 356, 257, 145, 167, 347)=-D_{34}(24573)m_9\,.$$

$$D_{34}(123, 246, 356, 257, 145, 167, 347)=D_{34}m_{10}\,,$$

$$D_{34}(123, 236, 456, 257, 145, 167, 347)=D_{34}(123)(4657)m_9\,.$$

Hence $-a_9+a_{10}-a_9=0$, $a_{10}=-4$. Thus we have completed the proof of (2). □

REMARK 8. Let $P_i=\{p\in\mathfrak{S}_7 \mid pm_i=\operatorname{sign}(p)m_i\}$. Then

$$P_1=\langle(1357642), (17)(26)(35)\rangle\cong \boldsymbol{Z}_2\ltimes \boldsymbol{Z}_7\,,$$

$$P_3=\langle(23)(45)(67)\rangle\cong \boldsymbol{Z}_2\,,$$

$$P_4=\langle(12)(56), (23)(67)\rangle\rtimes\langle(17)(26)(35)\rangle\cong\mathfrak{S}_3\rtimes \boldsymbol{Z}_2\,,$$

$$P_5=\langle(23)(45)(67)\rangle\cong \boldsymbol{Z}_2\,,$$

$$P_6=\langle(17)(26)(34)\rangle\cong \boldsymbol{Z}_2\,,$$

$$P_8=\langle(156)(274)\rangle\cong \boldsymbol{Z}_3,$$

$$P_9=\langle(12)(67),(16)(27)\rangle\rtimes\langle(34)\rangle\cong \boldsymbol{Z}_2^2\rtimes \boldsymbol{Z}_2,$$

$$P_{10}\cong SL_3(\boldsymbol{Z}_2).$$

(Note that $SL_3(\boldsymbol{Z}_2)$ is of order 168 and is the automorphism group of the finite projective plane over $\boldsymbol{Z}_2$.) Hence the numbers of terms appearing in $\sum'\pm m_i$ ($i=1, 3, 4, 5, 6, 8, 9, 10$) are 360, 2520, 420, 2520, 2520, 1680, 630 and 30 respectively. Let $f^\vee=f=J_7$. Then $f^\vee(\mathrm{grad})f^{s+1}=b(s)f^s$ with a polynomial

$$b(s)=b_0(s+1)(s+2)\left(s+\frac{5}{2}\right)(s+3)\left(s+\frac{7}{2}\right)(s+4)(s+5)$$

[9]. Since $b(0)=f^\vee(\mathrm{grad})f=2^6 3 5^2 7$, $b_0=2^4$.

5. Invariant of $GL(1)\times Spin(14)$.

Our purpose here is to construct an irreducible relative invariant J_8 of the odd half spin representation ($Sipn(14)$, ρ, $V(64)$). Our method of construction is similar to that of J. Igusa [4]. In this section, we use the same notations as in [12].

A general element x of $V(64)$ can be uniquely expressed as

$$x=\sum_i x_i e_i+\sum_{i<j<k} x_{ijk}e_{ijk}+\sum_{i<j} x^*_{ij}e^*_{ij}+x_L e_L,$$

where

$$e_{ijk}=e_ie_je_k \qquad \text{etc.},$$

$$e_L=e_{1234567},$$

$$e_{ij}e^*_{ij}=e_L.$$

Put $f_{ij}=f_if_j$ etc. $x_{jik}=-x_{ijk}$ etc. and $x_{1\cdots\hat{i}\cdots\hat{j}\cdots 7}=(-1)^{i+j-1}x^*_{ij}$.

LEMMA. *In general, put*

$$\Big(\prod_{r<s}(1+y_{rs}f_{rs})\Big)\Big(\sum_{i_0} x_{i_0}e_{i_0}+\sum_{i_0<i_1<i_2} x_{i_0i_1i_2}e_{i_0i_1i_2}+\cdots\Big)$$
$$=\sum_{i_0} z_{i_0}e_{i_0}+\sum_{i_0<i_1<i_2} z_{i_0i_1i_2}e_{i_0i_1i_2}+\cdots$$

Then

$$z_{i_0i_1\cdots i_{2q}}=x_{i_0i_1\cdots i_{2q}}+\sum_{p>q}(-1)^{p-q}\sum_{i_{2q+1}<\cdots<i_{2p}}\mathrm{Pf}(y_{i_ri_s})_{2q<r,s\le 2p}\cdot x_{i_0\cdots i_{2p}},$$

where Pf *denotes the Pfaffian.*

In fact,

$$z_{i_0 i_1 \cdots i_{2q}} e_{i_0 i_1 \cdots i_{2q}}$$

$$=\sum_{p\geq q} \frac{1}{2^{p-q}(p-q)!} \sum_{i_{2q+1} \cdots i_{2p}} (y_{i_{2q+1} i_{2q+2}} \cdots y_{i_{2p-1} i_{2p}})(f_{i_{2q+1} i_{2q+2}} \cdots f_{i_{2p-1} i_{2p}})$$

$$\cdot x_{i_0 \cdots i_{2p}} e_{i_0 \cdots i_{2p}}$$

$$=\sum_{p\geq q} \frac{1}{2^{p-q}(p-q)!} \sum_{i} (y_{i_{2q+1} i_{2q+2}} \cdots y_{i_{2p-1} i_{2p}}) x_{i_0 \cdots i_{2p}} (-1)^{p-q} e_{i_0 \cdots i_{2q}}$$

$$\sum_{p\geq q} \sum_{j_{2q+1}<\cdots<j_{2p}} \frac{1}{2^{p-q}(p-q)!} \sum_{(i_{2q+1} \cdots i_{2p})=(j_{2q+1} \cdots j_{2q})} (y_{i_{2q+1} i_{2q+2}} \cdots y_{i_{2p-1} i_{2p}})$$

$$\cdot \operatorname{sign} \begin{pmatrix} j_{2q+1} & \cdots & j_{2p} \\ i_{2q+1} & \cdots & i_{2p} \end{pmatrix} x_{i_0 \cdots i_{2q} j_{2q+1} \cdots j_{2p}} (-1)^{p-q} e_{i_0 \cdots i_{2q}}$$

$$=\sum_{p\geq q} \sum_{i_{2q+1}<\cdots<i_{2p}} \operatorname{Pf}(y_{i_r i_s})_{2q<r,s\leq 2p} \cdot x_{i_0 \cdots i_{2p}} (-1)^{p-q} e_{i_0 \cdots i_{2q}}.$$

By the above lemma, we have

$$(5.1) \qquad (\prod_{r<s} (1+x_L^{-1} x_{rs}^* f_{rs}))x = \sum_{i_0} z_{i_0} e_{i_0} + \sum_{i_0<i_1<i_2} z_{i_0 i_1 i_2} e_{i_0 i_1 i_2} + x_L e_L,$$

where

$$(5.2) \qquad \begin{aligned} z_{i_0} = x_L^{-2} \{ & (-1)^{i_0-1} \sum_{i_1<\cdots<i_6} \operatorname{Pf}(x_{i_r i_s}^*)_{1\leq r,s\leq 6} \\ & + \sum_{i_1<\cdots<i_4} \operatorname{Pf}(x_{i_r i_s}^*)_{1\leq r,s\leq 4} \cdot x_{i_0 i_1 \cdots i_4} \\ & + x_L \sum_{i_1<i_2} x_{i_1 i_2}^* x_{i_0 i_1 i_2} \} + x_{i_0}, \end{aligned}$$

and

$$(5.3) \qquad \begin{aligned} z_{i_0 i_1 i_2} = x_L^{-1} \{ & (-1)^{i_0+i_1+i_2} \sum_{i_3<\cdots<i_6} \operatorname{Pf}(x_{i_r i_s}^*)_{3\leq r,s\leq 6} \\ & - \sum_{i_3<i_4} x_{i_3 i_4}^* x_{i_0 i_1 \cdots i_4} \} + x_{i_0 i_1 i_2}. \end{aligned}$$

As is easily seen, every generic element of $\boldsymbol{C}^7 \oplus \Lambda^3 \boldsymbol{C}^7 / SL_7(\boldsymbol{C})$ has a representative of the form

$$w' e_7 + w(e_{123} + e_{456}) + w^{-1}(e_{147} + e_{257} + e_{367}),$$

(cf. [9; Prop. 2.14]). Hence if we put

$$z = \sum z_{i_0} e_{i_0} + \sum z_{i_0 i_1 i_2} e_{i_0 i_1 i_2} + x_L e_L,$$

then

$$(5.4) \qquad \rho(g)z = w' e_7 + w(e_{123} + e_{456}) + w^{-1}(e_{147} + e_{257} + e_{367}) + x_L e_L$$

with some w, w' and $g \in SL(7)(\subset Spin(14))$. By theorem 4,

$$J_6(z) = J_6(\rho(g)z) = w^4 w'^2,$$

and

$$J_7(z)=J_7(\rho(g)z)=-w\,.$$

Here we regard J_6 and J_7 as polynomial functions on $V(64)$ via the natural projection $V(64)\to \boldsymbol{C}^7\oplus(\wedge^3\boldsymbol{C}^7)$. Hence

(5.5) $$w=-J_7(z), \qquad w'=(J_6(z)J_7(z)^{-4})^{1/2}\,.$$

Let U be the linear span of

$$\{e_7,\ e_{123},\ e_{456},\ e_{147},\ e_{257},\ e_{367},\ e_L\}\,.$$

Since J_8 is invariant with respect to the action of $\{\prod_{i=1}^7(t_ie_if_i+t_i^{-1}f_ie_i)\,|\,t_i\in \boldsymbol{C}-\{0\}\}$ and $\deg J_8=8$ [12; section 7, Table I(24)], we can see that $J_8|_U$ is of the form

$$ax_7^2x_{123}^2x_{456}^2x_L^2+bx_{123}^2x_{456}^2x_{147}x_{257}x_{367}x_L\,.$$

Since

$$(1+2^{-1}f_{14})(1+2^{-1}f_{25})(1+2^{-1}f_{36})(1+e_{14})(1+e_{25})(1+e_{36})$$
$$\cdot(e_7+e_{123}+e_{456}+e_{1425367})$$
$$=e_{123}+e_{456}+2^{-1}(e_{147}+e_{257}+e_{367})+2e_{1425367}\,,$$

we have

$$J_8(e_7+e_{123}+e_{456}-e_L)$$
$$=J_8(e_{123}+e_{456}+2^{-1}(e_{147}+e_{257}+e_{367})-2e_L)\,.$$

Hence $a=-b/4$, and

(5.6) $$J_8|_U=x_7^2x_{123}^2x_{456}^2x_L^2-4x_{123}^2x_{456}^2x_{147}x_{257}x_{367}x_L$$

up to non-zero scalar multiple. Thus

$$J_8(x)=J_8(z), \qquad \textit{by } (5.1)$$
$$=J_8(w'e_7+w(e_{123}+e_{456})+w^{-1}(e_{147}+e_{257}+e_{367})+x_Le_L), \qquad \textit{by } (5.4)$$
$$=w'^2w^4x_L^2-4wx_L\,, \qquad \textit{by } (5.6)$$
$$=J_6(z)x_L^2+4J_7(z)x_L\,, \qquad \textit{by } (5.5)$$

THEOREM 5. *An irreducible relative invariant* J_8 *of* $(GL(1)\times Spin(14)$, (odd half spin), $V(64))$ *is given by*

$$J_8(x)=J_6(z)x_L^2+4J_7(z)x_L$$

with

$$z=\sum z_{i_0}e_{i_0}+\sum_{i_0<i_1<i_2} z_{i_0i_1i_2}e_{i_0i_1i_2}+x_Le_L\,,$$

where z_{i_0} *and* $z_{i_0i_1i_2}$ *are given by* (5.2) *and* (5.3).

REMARK 9. In the case treated in theorem 5, (G, ρ, V) has a unique split $\boldsymbol{Z}$-form [3]. For this $\boldsymbol{Z}$-form, $V(\boldsymbol{Z})$ may be identified with the lattice of $V(\boldsymbol{C})$ generated by

$$e_{i_0}e_{i_1}\cdots e_{i_{2k}}, \qquad 0\leqq k\leqq 3,\ 1\leqq i_0<\cdots<i_{2k}\leqq 7.$$

Then $\pm J_8(x)$ are the irreducible relative invariants in $\boldsymbol{Z}[V]$. In fact, as is seen from theorem 4, (5.2), (5.3) and theorem 5, $J_8(x)\in\boldsymbol{Z}[V, x_L^{-1}]\cap\boldsymbol{C}[V]=\boldsymbol{Z}[V]$. As is seen from (5.6), J_8 is irreducible in $\boldsymbol{Z}[V]$. If we take

$$e_7+e_{123}+e_{456}+e_L$$

as v_0 in remark 2, then we can take as T the inverse image by χ: $Spin(14)\to SO(14)$ of the set of

$$\mathrm{diag}(t_1, t_2, t_3, t_4, t_5, t_6, 1\,;\ t_1^{-1}, t_2^{-1}, t_3^{-1}, t_4^{-1}, t_5^{-1}, t_6^{-1}, 1),$$

where $t_1t_2t_3=t_4t_5t_6=1$. Then $C=V^T$ is the linear span of the following 4 elements;

$$e_7,\ e_{123},\ e_{456},\ e_L.$$

As is seen from (5.6),

$$\boldsymbol{Z}[V]^{G,\phi}\cong\boldsymbol{Z}[V^T]^{N,\phi}$$

in our case.

By a direct calculation, we can show that

$$(\mathrm{grad}\log J_8)(v_0)=2v_0.$$

As is seen from (5.6), $J_8(v_0)=1$. Hence $J_8((\mathrm{grad}\log J_8)(v_0))J_8(v_0)=2^8$, and $J_8^{\vee}(\mathrm{grad})J_8^{s+1}=b(s)J_8^s$ with the polynomial

$$b(s)=2^8(s+1)\Big(s+\frac{5}{2}\Big)\Big(s+\frac{7}{2}\Big)(s+4)(s+5)\Big(s+\frac{11}{2}\Big)\Big(s+\frac{13}{2}\Big)(s+8),$$

(cf. [11]).

6. Invariant of $GL(8)$.

In [8; Remark 4.6], a construction of an irreducible relative invariant of $(GL(8), \Lambda_3, V(56))$ is given. In order to write down this relative invariant explicitly, we need to know the explicit form of polynomials

$$F_{i_1^1\cdots i_{m-2}^q}(x) \qquad (q=3,\ m=3)$$

appeared in [8; Example (II)]. It would be worth noting that, although the explicit form of these polynomials are not given in [8], they can be constructed immediately as follows: Let D_{8i} be the polarization $i\to 8$, i.e., $D_{8,i}x_{\alpha\beta i}=x_{\alpha\beta 8}$

(α, $\beta \neq 8, 1 \leqq i \leqq 8$). Then

$$F_{i_1^1, i_1^2, i_1^3} = D_{8, i_1^1} D_{8, i_1^2} D_{8, i_1^3} f ,$$

where f is an irreducible relative invariant of $(GL(7), \Lambda_3, V(35))$. In order to see that these polynomials satisfy (4.7) and (4.8) of [8], it is enough to notice that D_{8i} is nothing but the action of the matrix unit $-E_{i8}$.

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