CONSTRUCTION OF ISOLATED LEFT ORDERINGS VIA PARTIALLY CENTRAL CYCLIC AMALGAMATION

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Abstract. We give a new method to construct isolated left orderings of groups whose positive cones are finitely generated. Our construction uses an amalgamated free product of two groups having an isolated ordering. We construct a lot of new examples of isolated orderings, and give an example of isolated left orderings with various properties which previously known isolated orderings do not have.

1. Introduction. A total ordering $<_G$ on a group G is a *left ordering* if $g <_G g'$ implies $hg <_G hg'$ for all $g, g', h \in G$. The *positive cone* of a left ordering $<_G$ is a sub-semigroup $P(<_G)$ of G consisting of $<_G$-positive elements.

The set of all left orderings of G is denoted by $\mathrm{LO}(G)$. For $g \in G$, let U_g be a subset of $\mathrm{LO}(G)$ defined by

$$U_g = \{<_G \in \mathrm{LO}(G) \mid 1 <_G g\}.$$

The set $\mathrm{LO}(G)$ can be equipped with a topology so that $\{U_g\}_{g \in G}$ is an open sub-basis of the topology. This topology is understood as follows. For a left ordering $<_G$ of G, G is decomposed as a disjoint union $G = P(<_G) \sqcup \{1\} \sqcup P(<_G)^{-1}$. Conversely, a sub-semigroup P of G having this property is a positive cone of a left ordering of G: An ordering $<_P$ defined by $g <_P g'$ if $g^{-1} g' \in P$ is a left-ordering whose positive cone is P. Thus $\mathrm{LO}(G)$ is identified with a subset of the powerset $2^{G-\{1\}}$. The topology of $\mathrm{LO}(G)$ defined as above coincides with the relative topology as the subspace of $2^{G-\{1\}}$, equipped with the topology as the product of copies of the discrete space $2 = \{+, -\}$.

In this paper, we always consider countable groups, so we simply refer a countable group as a group unless otherwise specified. Then $\mathrm{LO}(G)$ is a compact, metrizable, and totally disconnected [12]. Moreover, $\mathrm{LO}(G)$ is either uncountable or finite [7]. Thus as a topological space, $\mathrm{LO}(G)$ is rather similar to the Cantor set. The main difference is that the space $\mathrm{LO}(G)$ might be non-perfect, that is, $\mathrm{LO}(G)$ might have isolated points. Indeed, if $\mathrm{LO}(G)$ has no isolated points and is not a finite set, then $\mathrm{LO}(G)$ is homeomorphic to the Cantor set. We call a left ordering which is an isolated point of $\mathrm{LO}(G)$ an *isolated ordering*.

It is known that a left ordering $<_G$ whose positive cone is a finitely generated semigroup is isolated. In this paper we will concentrate our attention to study such an isolated ordering. We say a finite set of non-trivial elements of G, $\mathcal{G} = \{g_1, \ldots, g_r\}$ *defines* an isolated left ordering $<_G$ of G if the positive cone of $<_G$ is generated by $\mathcal{G}$ as a semigroup. For an

isolated left ordering $<_G$ of a group G, the *rank* of $<_G$ is the minimal number of generators of the positive cone and denoted by $r(<_G)$. (If $P(<_G)$ is not finitely generated semigroup we define $r(<_G) = \infty$).

We say an isolated ordering $<_G$ of G is *genuine* if $LO(G)$ is not a finite set. Then $LO(G)$ contains (uncountably many) non-isolated points [7]. The classification of groups having non-genuine isolated orderings, namely, the classification of groups having finitely many left-orderings is given by Tararin (see [6]).

On the other hand, it is difficult to construct genuine isolated left orderings, and few examples are known. At this moment to the best of the author's knowledge, there are only two families of genuine isolated left orderings. (After the first version of this paper appeared in a preprint form in July 2011, Dehornoy gave other construction of isolated orderings by using word-reversing method [1]. It is an interesting problem to understand the relationship between our construction and Dehornoy's one.)

(A) *Dubrovina-Dubrovin ordering* [2], [3].

Let $\sigma_1, \ldots, \sigma_{n-1}$ be the standard generator of the n-strand braid group B_n. The *Dubrovina-Dubrovin ordering* $<_{DD}$ is an isolated left ordering of B_n whose positive cone is generated by $\{a_1, \ldots, a_{n-1}\}$, where a_i is given by

$$a_i = (\sigma_{n-i}\sigma_{n-i+1}\cdots\sigma_{n-1})^{(-1)^i}.$$

The rank of the Dubrovina-Dubrovin ordering $<_{DD}$ is $n - 1$. See [2], [3] for details.

(B) *Isolated orderings of $\mathbb{Z} *_{\mathbb{Z}} \mathbb{Z}$* [5], [10].

Let $G = \mathbb{Z} *_{\mathbb{Z}} \mathbb{Z}$ be the group obtained as an amalgamated free product of two infinite cyclic groups over $\mathbb{Z}$. Thus, G is presented as

$$G = \langle x, y \mid x^m = y^n \rangle$$

by using some positive integers m and n. Then the generating set $\{xy^{1-n}, y\}$ defines an isolated left ordering $<_A$ of G, which is genuine if $(m, n) \neq (2, 2)$. The rank of $<_A$ is 2. This example was found by Navas [10] for the case $m = 2$, and by the author [5] for general cases. We remark here that if $(m, n) = (2, 3)$ then $G_{m,n}$ is the 3-braid group B_3, and the isolated ordering $<_A$ is the same as the Dubrovina-Dubrovin ordering $<_{DD}$.

Thus, it is desirable to find more examples or general constructions of isolated left orderings. In particular, Dubrobvina-Dubrobin orderings is unique known example of genuine isolated ordering of rank greater than 2.

In author's previous paper [5], we gave one general method to construct isolated orderings by using rather combinatorial approach. Following [10], we introduced the notion

of *Dehornoy-like ordering*. This is a left-ordering whose positive cone consists of certain kind of words over a special generating set $\mathcal{S}$ of G, which we called $\sigma(\mathcal{S})$-*positive words*. A Dehornoy-like ordering is a generalization of the Dehornoy ordering of the braid groups, one of the most interesting left orderings: The Dehornoy ordering has various stimulating features and a lot of interesting interpretations that relate many aspects of braid groups and orderings. See [2] for the theory of the Dehornoy ordering. One fascinating property of the Dehornoy ordering is that one get the Dubrovina-Dubrovin ordering by modifying the Dehornoy ordering.

We showed that, under some condition which we called the Property F, Dehornoy-like orderings and the Dehornoy ordering share various properties. In particular, we have shown that a Dehornoy-like ordering produces an isolated ordering and vice versa. Indeed, it is shown that the above two families of known isolated orderings are derived from Dehornoy-like orderings.

However it seems to be more difficult to find an example of a Dehornoy-like ordering than to find an example of an isolated ordering directly, since the definition of Dehornoy-like orderings includes complicated combinatorics.

The aim of this paper is to give a new construction of isolated left orderings by means of the *partially central cyclic amalgamation*. From two groups having (not necessarily genuine) isolated orderings, we construct a new group having an isolated left ordering by using amalgamated free product over $\mathbb{Z}$.

In almost all cases, the obtained isolated orderings are genuine. Our construction can be seen as an extension of (B) of known examples, but it is completely different from the Dehornoy-like orderings construction. In fact, we will see that some the orderings constructed in this paper cannot be obtained from Dehornoy-like orderings.

The following is a summary of the main results of this paper. Recall that for $g \in G$ and a left ordering $<_G$ of G, $<_G$ is called a g-*right invariant ordering* if the ordering $<_G$ is preserved by the right multiplication by g, that is, $a <_G b$ implies $ag <_G bg$ for all $a, b \in G$.

THEOREM 1.1 (Construction of isolated left ordering via partially central cyclic amalgamation). *Let G and H be finitely generated groups. Let z_G be a non-trivial central element of G, and z_H be a non-trivial element of H.*

Let $\mathcal{G} = \{g_1, \ldots, g_m\}$ be a finite generating set of G which defines an isolated left ordering $<_G$ of G. We take a numbering of elements of $\mathcal{G}$ so that $1 <_G g_1 <_G \cdots <_G g_m$ holds. Similarly, let $\mathcal{H} = \{h_1, \ldots, h_n\}$ be a finite generating set of H which defines an isolated left ordering $<_H$ of H such that the inequalities $1 <_H h_1 <_H \cdots <_H h_n$ hold.

We assume the cofinality assumptions [**CF(G)**], [**CF(H)**], *and the invariance assumption* [**INV(H)**].

$$[\textbf{CF(G)}] \quad g_i <_G z_G \text{ holds for all } i.$$
$$[\textbf{CF(H)}] \quad h_i <_H z_H \text{ holds for all } i.$$
$$[\textbf{INV(H)}] \quad <_H \text{ is a } z_H\text{-right invariant ordering}.$$

Let $X = G *_{\mathbb{Z}} H = G *_{\langle z_G = z_H \rangle} H$ be an amalgamated free product of G and H over $\mathbb{Z}$. For $i = 1, \ldots, m$, let $x_i = g_i z_H^{-1} h_1$. Then we have the following results:

(i) *The generating set $\{x_1, \ldots, x_m, h_1, \ldots, h_n\}$ of X defines an isolated left ordering $<_X$ of X.*

(ii) *The isolated ordering $<_X$ does not depend on the choice of the generating sets $\mathcal{G}$ and $\mathcal{H}$. Thus, $<_X$ only depends on the isolated orderings $<_G$, $<_H$ and the elements z_G, z_H.*

(iii) *The natural inclusions $\iota_G : G \to X$ and $\iota_H : H \to X$ are order-preserving homomorphisms.*

(iv) *$1 <_X x_1 <_X \cdots <_X x_m <_X h_1 <_X \cdots <_X h_n <_X z_H = z_G$. Moreover, $z = z_G = z_H$ is $<_X$-positive cofinal and the isolated ordering $<_X$ is a z-right invariant ordering.*

(v) *$r(<_X) \leq r(<_G) + r(<_H)$.*

(vi) *Let Y be a non-trivial proper subgroup of X. If Y is $<_X$-convex, then $Y = \langle x_1 \rangle$, the infinite cyclic group generated by x_1.*

We call the construction of isolated ordering described in Theorem 1.1 the *partially central cyclic amalgamation construction*.

As we will see in Lemma 2.3 in Section 2.1, the cofinality assumption **[CF(G)]** (resp. **[CF(H)]**) is understood as an assumption on z_G and $<_G$ (resp. z_H and $<_H$). Thus Theorem 1.1 (ii) shows that the choice of the generating sets $\mathcal{G}$ and $\mathcal{H}$ is not important though it is useful to describe and understand the isolated ordering $<_X$. The generating sets $\mathcal{G}$ and $\mathcal{H}$ are not essential and play rather auxiliary roles as for the partially central cyclic amalgamation construction. This makes a sharp contrast with the construction using Dehornoy-like orderings, since in the Dehornoy-like ordering construction we need to use a special generating set derived from Dehornoy-like ordering having the nice property which we called the Property F.

On the other hand, it should be emphasized that the proof of Theorem 1.1 (i) is constructive, and will actually provide an algorithm to determine the isolated ordering $<_X$. In particular, the isolated ordering $<_X$ can be determined algorithmically if we have algorithms to compute the isolated orderings $<_G$ and $<_H$, as we will see in Section 2.7. This is why we added auxiliary information of generators of $P(<_G)$ and $P(<_H)$ – They provide a useful, explicit, and more combinatorial description of the resulting isolated ordering $<_X$. By utilizing a combinatorial expression of certain isolated orderings from Theorem 1.1, in [4] we construct chain domains with exceptional prime ideal. This is highly non-trivial object in a theory of non-commutative valuation rings, and illustrates a usefulness of partially central cyclic amalgamation construction.

Theorem 1.1 (iii) shows that the partially central cyclic amalgamation construction can be seen as a mixing of two isolated orderings $<_G$ and $<_H$. We remark that Theorem 1.1 (iv) ensures that we can iterate the partially central cyclic amalgamation construction: The resulting group X, its isolated ordering $<_X$ and z can always be used in the role of H, $<_H$

and z_H in Theorem 1.1. Thus, we can actually produce many isolated orderings by using the partially central cyclic amalgamation constructions from known (not necessarily genuine) isolated orderings, like the standard ordering of $\mathbb{Z}$.

It is interesting to compare our amalgamation with other natural operations on groups. Unlike the partial central cyclic amalgamation (the amalgamated free product over $\mathbb{Z}$, used in Theorem 1.1), the usual free product does not preserve the property that the group has an isolated left ordering. Here is the simplest counter-example: the free group of rank two $F_2 = \mathbb{Z} * \mathbb{Z}$ has no isolated orderings [9], whereas the infinite cyclic group $\mathbb{Z}$ has (non-genuine) isolated orderings, since it admits only two left orderings. Indeed, recently Rivas [11] proved that free products of groups do not have any isolated left orderings. Similarly, the direct products of groups also do not preserve the property that the group has an isolated left ordering: the free abelian group of rank two $\mathbb{Z} \times \mathbb{Z}$ has no isolated orderings [12].

The plan of this paper is as follows: In Section 2 we prove Theorem 1.1. The main technical tool of the proof is a *reduced standard factorization*, which serves as some kind of normal form of elements in X, adapted to the generating set $\{x_1, \ldots, x_m, h_1, \ldots, h_n\}$. In Section 3 we give some examples of isolated orderings obtained by applying Theorem 1.1. We observe that our examples have various interesting properties, which do not occur in the previously known examples.

2. Construction of isolated left orderings.

Let $\mathcal{S} = \{s_1, \ldots, s_n\}$ be a finite generating set of G and let $\mathcal{S}^{-1} = \{s_1^{-1}, \ldots, s_n^{-1}\}$. We denote by $\mathcal{S}^*$ the free semigroup generated by $\mathcal{S}$. That is, $\mathcal{S}^*$ is the set of non-empty words over $\mathcal{S}$. We say an element of $\mathcal{S}^*$ (resp. $(\mathcal{S}^{-1})^*$) is an $\mathcal{S}$-*positive word* (resp. an $\mathcal{S}$-*negative word*). We will often use a symbol $P(\mathcal{S})$ (resp. $N(\mathcal{S})$) to represent some $\mathcal{S}$-positive (resp. $\mathcal{S}$-negative) words, whose actual form may depend on the context.

2.1. Cofinality and Invariance assumptions.

First of all we review the assumptions in the statement of Theorem 1.1 again, and deduce their direct consequences. This clarifies the role of each hypothesis in Theorem 1.1.

Let G and H be finitely generated groups having an isolated left ordering $<_G$ and $<_H$ respectively. Let $z_G \in G$ be a non-trivial central element of G, and let z_H be a non-trivial element of H, which might be noncentral. We consider the group X obtained as an amalgamated free product over $\mathbb{Z}$,

$$X = G *_{\mathbb{Z}} H = G *_{\langle z_G = z_H \rangle} H .$$

Let $\mathcal{G} = \{g_1, \ldots, g_m\}$ be a generating set of G which defines an isolated left ordering $<_G$ of G. We take a numbering of elements of $\mathcal{G}$ so that $1 <_G g_1 <_G \cdots <_G g_m$ holds. Similarly, let $\mathcal{H} = \{h_1, \ldots, h_n\}$ be a generating set of H which defines an isolated left ordering $<_H$ of H, and we assume that the inequalities $1 <_H h_1 <_H \cdots <_H h_n$ hold.

Recall that an element $g \in G$ is called the $<_G$-*minimal positive element* if g is the $<_G$-minimal element in the positive cone $P(<_G)$. In other words, the inequality $1 <_G g' \leq_G g$ implies $g = g'$. A left ordering $<_G$ is called *discrete* if $<_G$ has a $<_G$-minimal positive element. Otherwise, $<_G$ is called *dense*.

As the next lemma shows, the choice of the numbering of $\mathcal{G}$ (resp. $\mathcal{H}$) implies that g_1 (resp. h_1) is the $<_G$-minimal (resp. $<_H$-minimal) positive element. In particular, g_1 (resp. h_1) is independent of the choice of the generating set $\mathcal{G}$ (resp. $\mathcal{H}$).

LEMMA 2.1. *Let $\mathcal{G} = \{g_1, \ldots, g_m\}$ be a generating set of a group G which defines an isolated left ordering $<_G$ of G. Assume that g_1 is the $<_G$-minimal element in the set $\mathcal{G}$. Then g_1 is the $<_G$-minimal positive element. In particular, $<_G$ is discrete. Moreover, $<_G$ is a g_1-right invariant ordering.*

PROOF. Assume $g \in G$ satisfies the inequalities $1 <_G g \leq_G g_1$.
$1 <_G g$ means that g is written as a $\mathcal{G}$-positive word $g = g_{i_1} \cdots g_{i_l}$. Then

$$g_1^{-1} g = (g_1^{-1} g_{i_1}) g_{i_2} \cdots g_{i_l} \leq_G 1 .$$

This inequality holds only if $i_1 = 1$ and $l = 1$, that is, $g = g_1$.

The g_1-right invariance of the ordering $<_G$ now follows from the fact that g_1 is the $<_G$-minimal positive element: If $a <_G b$, then $1 <_G a^{-1}b <_G a^{-1}bg_1$. Thus, $g_1 <_G a^{-1}bg_1$ so $ag_1 <_G bg_1$. □

To obtain an isolated ordering of X from $<_G$ and $<_H$, we impose the following assumptions, which we call the *cofinality assumption* for G and H, and the *invariance assumption*.

$$[\textbf{CF(G)}] \quad g_i <_G z_G \text{ holds for all } i .$$
$$[\textbf{CF(H)}] \quad h_i <_H z_H \text{ holds for all } i .$$
$$[\textbf{INV(H)}] \quad <_H \text{ is a } z_H\text{-right invariant ordering} .$$

Here we remark that the invariance assumption for $<_G$ is automatically satisfied: that is, $<_G$ is a z_G-right invariant ordering since we have chosen z_G so that it is a central element.

First we observe the following simple lemma.

LEMMA 2.2. *Let $<_H$ be a discrete left ordering of a group H, and let h_1 be the $<_H$-minimal positive element. If $<_H$ is an h-right invariant ordering for $h \in H$, then h commutes with h_1.*

PROOF. $<_H$ is an h-right invariant ordering, so $hh_1h^{-1} >_H 1$ and $h^{-1}h_1h >_H 1$. h_1 is the $<_H$-minimal positive element, so $hh_1h^{-1} \geq_H h_1$ and $h^{-1}h_1h \geq_H h_1$. Thus, we get $hh_1 \geq_H h_1h$ and $h_1h \geq_H hh_1$, hence $hh_1 = h_1h$. □

By Lemma 2.1 and Lemma 2.2, the invariance assumption [**INV(H)**] implies that z_H commutes with h_1.

For a left-ordering $<_G$ of G, an element $g \in G$ is called $<_G$-*cofinal* if for all $g' \in G$, there exist integers m and M such that $g^m <_G g' <_G g^M$ holds. Although the cofinality assumptions [**CF(G)**] and [**CF(G)**] involve the generating sets $\mathcal{G}$ and $\mathcal{H}$, if we assume the invariance assumption [**INV(H)**] then these assumptions should be regarded as assumptions on z_G, z_H and the isolated orderings $<_G$, $<_H$ as the next lemma shows.

LEMMA 2.3. *Assume the invariance assumption* **[INV(H)]** *is satisfied. A generating set $\mathcal{H}$ satisfying the cofinality assumption* **[CF(H)]** *exists if and only if z_H is $<_H$-positive cofinal and $H \neq \langle z_H \rangle$. Here $\langle z_H \rangle$ represents the subgroup of H generated by z_H. Moreover, in such case we may choose a generating set $\mathcal{H}$ so that the cardinal of $\mathcal{H}$ is equal to the rank of the isolated ordering $<_H$.*

PROOF. In the following, we assume the invariance assumption **[INV(H)]**.

Assume that a generating set $\mathcal{H}$ satisfies the cofinality assumption **[CF(H)]**. Then by the invariance assumption **[INV(H)]**, z_H is $<_H$-positive cofinal and $H \neq \langle z_H \rangle$.

We show the converse: if z_H is $<_H$-positive cofinal and $H \neq \langle z_H \rangle$, then we can choose a generating set $\mathcal{H} = \{h_1, \dots, h_k\}$ so that $\mathcal{H}$ defines the isolated ordering $<_H$, and that $\mathcal{H}$ satisfies **[CF(H)]**. Moreover, we will show that we can choose k, the cardinal of $\mathcal{H}$, so that k is equal to $r(<_H)$.

Let us take a generating set $\mathcal{H}' = \{h'_1, \dots, h'_k\}$ of H which defines the isolated ordering $<_H$. By definition of rank, we may choose $\mathcal{H}'$ so that $k = r(<_H)$ holds. With no loss of generality, we may assume that

$$h'_1 <_H \cdots <_H h'_s \leq_H z_H <_H h'_{s+1} <_H \cdots <_H h'_k.$$

Since z_H is $<_H$-cofinal, for each i there is a non-negative integer N_i such that $1 <_H z_H^{-N_i} h'_i \leq z_H$. Let us put $h_i = z_H^{-N_i} h'_i$. By assumption, $h'_i = h_i$ if $i \leq s$.

By the hypothesis $H \neq \langle z_H \rangle$, we have a strict inequality $z_H >_H h'_1 = h_1$. Thus if necessary, by replacing h_i with $h_1^{-1} h_i$, we may assume that $h_i \neq z_H$ for all i.

We show that z_H is written as an $\{h'_1, \dots, h'_s\}$-positive word. Assume that $z_H = V h'_i W$, where $i > s$ and V, W are $\mathcal{H}'$-positive or non-empty words. Then $z_H W^{-1} = V h'_i >_H V z_H$, hence we get $1 \geq_H W^{-1} >_H z_H^{-1} V z_H$. However, $<_H$ is a z_H-right invariant ordering, hence $z_H^{-1} V z_H \geq_H 1$. This is a contradiction.

Therefore, the generating set $\mathcal{H} = \{h_1, \dots, h_k\}$ also defines the isolated ordering $<_H$. By construction, $\mathcal{H}$ is a generating set which satisfies the cofinality assumption **[CF(H)]** with cardinal $k = r(<_G)$. $\square$

Thus, under the invariance assumption **[INV(H)]**, we can always find a generating set $\mathcal{H}$ which defines $<_H$ and satisfies the cofinality assumption **[CF(H)]**, if the conditions on $<_H$ and z_H in Lemma 2.3 are satisfied. Moreover, if necessary we may choose $\mathcal{H}$ so that the cardinal of $\mathcal{H}$ is equal to the rank of $<_H$.

Since for z_G and $<_G$, the invariance assumption is automatically satisfied, we can always find a generating set $\mathcal{G}$ which defines $<_G$ and satisfies the cofinality assumption **[CF(G)]** if z_G is $<_G$-positive cofinal and $G \neq \langle z_G \rangle$.

Now we put $\Delta_H = z_H h_1^{-1}$. Since z_H and h_1 do not depend on the choice of the generating set $\mathcal{H}$, the same holds for Δ_H. As an element of H, Δ_H is characterized by the following property.

LEMMA 2.4. *Δ_H is the $<_H$-maximal element which is strictly smaller than z_H.*

PROOF. Assume that $z_H h_1^{-1} = \Delta_H \leq_H h <_H z_H$ holds for some $h \in H$. Then $h_1^{-1} \leq_H z_H^{-1} h <_H 1$. By Lemma 2.1, h_1^{-1} is the $<_H$-maximal element which is strictly smaller than 1, so $z_H^{-1} h = h_1^{-1}$. Hence $h = z_H h_1^{-1}$. $\square$

Finally, we put $x_i = g_i \Delta_H^{-1} = g_i z_H^{-1} h_1$ and let $\mathcal{X} = \{x_1, \ldots, x_m\}$. Then $\{\mathcal{X}, \mathcal{H}\}$ generates the group X. The following lemma is rather obvious, but plays an important role in the proof of Theorem 1.1.

LEMMA 2.5. $z_H = z_G$ commutes with all x_i.

PROOF. By Lemma 2.2, z_H commutes with $\Delta_H = z_H h_1^{-1}$. Since $z_H = z_G$ commutes with all g_i, we conclude that z_H commutes with all $x_i = g_i \Delta_H^{-1}$. $\square$

2.2. Property A and Property C criteria. To prove that $\{\mathcal{X}, \mathcal{H}\}$ defines an isolated left ordering $<_X$ of X, we use the following criterion which was used in the theory of the Dehornoy ordering of the braid groups [2] and Dehornoy-like orderings [5, 10]. Here we give the most general form of this kind of arguments.

DEFINITION 2.6. Let $\mathcal{S} = \{s_1, \ldots, s_m\}$ be a generating set of a group G and let W be a sub-semigroup of $(\mathcal{S} \cup \mathcal{S}^{-1})^*$.

  (1) We say W has the *Property A (Acyclic Property)* if no word in W represents the trivial element of G.
  (2) We say W has the *Property C (Comparison Property)* if for each non-trivial element $g \in G$, either g or g^{-1} is represented by a word $w \in W$.

PROPOSITION 2.7. *Let W be a sub-semigroup of $(\mathcal{S} \cup \mathcal{S}^{-1})^*$. Let $P = \pi(W)$, where $\pi : (\mathcal{S} \cup \mathcal{S}^{-1})^* \to G$ is the natural projection. Then P is equal to a positive cone of a left ordering of G if and only if W has Properties A and C.*

PROOF. If W is a positive cone of a left ordering, then it is obvious that W has Properties A and C. We show the converse. Since W is a sub-semigroup, P is a sub-semigroup of G. By Property C, $G = P \cup \{1\} \cup P^{-1}$. Property A implies that $1 \notin P$, hence G is decomposed as a disjoint union $G = P \sqcup \{1\} \sqcup P^{-1}$. This shows that P is a positive cone of a left ordering. $\square$

DEFINITION 2.8. The set of words W in Proposition 2.7 is called the *language defining the corresponding left-ordering*.

It is an interesting problem to ask if one can choose a language defining an arbitrary left-ordering $<_G$ so that it is a regular language over a finite alphabet: This is related to order-decision problems which we will consider in Section 2.7, but in this paper we will not treat this problem.

As a special case, we get a criterion for a finite generating set to define an isolated ordering, which will be used to show $\{\mathcal{X}, \mathcal{H}\}$ indeed defines an isolated ordering.

COROLLARY 2.9. *A finite generating set $\mathcal{G} = \{g_1, \ldots, g_m\}$ of a group G defines an isolated ordering of G if and only if the following conditions* **[Property A]** *and* **[Property C]** *hold:*

Property A: *If $g \in G$ is represented by a $\mathcal{G}$-positive word, then $g \neq 1$.*

Property C: *If $g \neq 1$, then g is represented by either a $\mathcal{G}$-positive or a $\mathcal{G}$-negative word.*

2.3. Reduced standard factorization. Now we start to show that $\{\mathcal{X}, \mathcal{H}\}$ indeed defines an isolated left ordering of X. From now on, we take $G, H, X, <_G, <_H, \mathcal{G}, \mathcal{H}, \mathcal{X}$ as in assumptions in Theorem 1.1, and we always assume the cofinality assumptions **[CF(G)]**, **[CF(H)]**, and the invariance assumption **[INV(H)]**.

As the first step of the proof, we introduce a notion of reduced standard factorization, which serves as a certain kind of normal form of X adapted to the generating set $\{\mathcal{X}, \mathcal{H}\}$.

Let $P\mathcal{X}$ be the sub-semigroup of X generated by $\mathcal{X} = \{x_1, \ldots, x_m\}$. A *standard factorization* of $x \in X$ is a factorization of $x \in X$ of the form

$$\mathcal{F}(x) = r p_1 q_1 \cdots p_l q_l$$

where $r, q_1, \ldots, q_l \in H$, $p_1, \ldots, p_l \in P\mathcal{X}$ satisfy the conditions

(1) $q_i >_H 1$ $(i \neq l)$, and $q_l \geq_H 1$, and
(2) $q_i \neq z_H^N$ for all $N > 0$.

It is not hard to see that every x admits a standard factorization. Actually, we will show that every x admits a standard factorization which is the simplest in certain sense.

The *complexity* of a standard factorization $\mathcal{F}(x) = r p_1 q_1 \cdots p_l q_l$ is defined to be l, and denoted by $c(\mathcal{F})$.

A *distinguished subfactorization* of a standard factorization $\mathcal{F}(x)$ is, roughly saying, a part of the standard factorization $\mathcal{F}(x)$ which can be regarded as a $\mathcal{G}$-positive word, defined as follows.

We say a subfactorization

$$(2.1) \qquad\qquad w = (q_i p_{i+1} q_{i+1} \cdots p_{i+r} q_{i+r})$$

in a standard factorization $\mathcal{F}(x)$ is a *distinguished subfactorization* if it satisfies the following two conditions:

(1) $q_j = \Delta_H$ for all $j = i, i+1, \ldots, i+r$.
(2) $p_j \in \mathcal{X}$ for all $j = i+1, \ldots, i+r$.

That is, a distinguished subfactorization is a part of standard factorization which is written as

$$(2.2) \qquad\qquad w = \Delta_H x_{j_{i+1}} \Delta_H x_{j_{i+2}} \cdots x_{j_{i+r}} \Delta_H.$$

We will express the distinguished subfactorization w (2.2) by using a $\mathcal{G}$-positive word g_w as follows: Let us take $x_a \in \mathcal{X}$ so that $p_i' = p_i x_a^{-1} \in P\mathcal{X} \cup \{1\}$ (such a choice of x_a might be not unique), and write a standard factorization $\mathcal{F}(x)$ as

$$\mathcal{F}(x) = r p_1 q_1 \cdots p_l q_l$$
$$= r p_1 q_1 \cdots p_{i-1} q_{i-1} (p_i x_a^{-1})(x_a q_i \cdots p_{i+r} q_{i+r}) p_{i+r+1} q_{i+r+1} \cdots p_l q_l$$

$$= r p_1 q_1 \cdots p_{i-1} q_{i-1} p_i' (x_a \Delta_H x_{j_{i+1}} \Delta_H x_{j_{i+2}} \cdots x_{j_{i+r}} \Delta_H) p_{i+r+1} q_{i+r+1} \cdots p_l q_l.$$

Let us put

$$g_w = g_{j_{i+1}} \cdots g_{j+i+r}.$$

We call g_w the *corresponding $\mathcal{G}$-positive word (element)* of the distinguished subfactorization w. Since $g_i = x_i \Delta_H$,

$$x_a \Delta_H x_{j_{i+1}} \Delta_H x_{j_{i+2}} \cdots x_{j_{i+r}} \Delta_H = g_a g_{j_{i+1}} \cdots g_{j_{i+r}} = g_a g_w.$$

Thus, if w is a distinguished subfactorization in $\mathcal{F}(x)$, by choosing x_a we may express x as

$$x = r p_1 q_1 \cdots p_{i-1} q_{i-1} p_i' [g_a g_w] p_{i+r+1} q_{i+r+1} \cdots p_l q_l$$

by using the corresponding $\mathcal{G}$-positive word g_w.

Next we introduce a notion of *reducible distinguished subfactorization*. Let w be a distinguished subfactorization of $\mathcal{F}(x)$ as taken in (2.1). Let us take $x_u \in \mathcal{X}$ so that $p_{i+r+1}' = x_u^{-1} p_{i+r+1} \in P\mathcal{X} \cup \{1\}$. As for the choice of x_a above, such x_u may not unique. If such x_u does not exist, that is, $p_{i+r+1} = 1$, we take $x_u = 1$. We say a distinguished subfactorization w is *reducible* if for *any* choice of such x_a and x_u, we have the inequality $g_a g_w g_u \geq_G z_G$. Otherwise, that is, if one can choose x_a and x_u so that $g_a g_w g_u <_G z_G$ holds, then we say w is *irreducible*.

Now we define the notion of a *reduced standard factorization*, which plays an important role in the proof of both Property A and Property C.

DEFINITION 2.10 (Reduced standard factorization). Let $\mathcal{F}(x) = r p_1 q_1 \cdots p_l q_l$ be a standard factorization. We say $\mathcal{F}$ is *reduced* if $q_i <_H z_H$ for all i and $\mathcal{F}$ contains no reducible distinguished subfactorization.

We say a distinguished subfactorization w of a standard factorization $\mathcal{F}$ is *maximal* if there is no other distinguished subfactorization w' of $\mathcal{F}$ whose corresponding $\mathcal{G}$-positive word $g_{w'}$ contains g_w as its subword. For any $<_G$-positive elements g, g', g'', since z_G is central, if $g \geq_G z_G$ then $g' g g'' \geq_G z_G$. Thus, to see whether a standard factorization is reducible or not, it is sufficient to check that all maximal distinguished subfactorization are irreducible.

EXAMPLE 2.11. A distinguished subfactorization and related notions are slightly complex, so here we give an example. Let us consider the case $\mathcal{X} = \{x_1, x_2\}$ and take a standard factorization of the form

(2.3) $$\mathcal{F}(x) = (x_1 x_2) \Delta_H x_1 \Delta_H x_2 \Delta_H (x_1^3 x_2) h_1,$$

for example.

In the standard factorization (2.3) $w = \Delta_H x_1 \Delta_H$ is a distinguished subfactorization. The corresponding $\mathcal{G}$-positive word is $g_w = g_1$. In this case, we may choose $x_a = x_2$ since $(x_1 x_2) x_2^{-1} = x_1 \in P\mathcal{X}$. So we are able to write x as

$$x = x_1 [g_2 g_1] x_2 \Delta_H (x_1^3 x_2) h_1.$$

The distinguished sub-factorization w is not maximal: it is included in another distinguished subfactorization $w' = \Delta_H x_1 \Delta_H x_2 \Delta_H$, and we may write

$$x = x_1[g_2(g_1 g_2)](x_1^3 x_2)h_1 .$$

The distinguished subfactorization w' is maximal.

Is w' reducible ? To see this, first we need to determine all possibilities of x_a and x_u in the definition of reducible distinguished subfactorization. Assume that $(x_1 x_2)x_1^{-1} \in P\mathcal{X} \cup \{1\}$, but $x_2^{-1}(x_1^3 x_2) \notin P\mathcal{X} \cup \{1\}$. Then we may choose $x_a = x_1$ or x_2, and $x_u = x_1$. Hence our definition says, w' is reducible if and only if

$$g_1(g_1 g_2)g_1 \geq_G z_G, \text{ and } g_2(g_1 g_2)g_1 \geq_G z_G$$

hold.

First we show the existence of the reduced standard subfactorization. The proof of the next lemma utilizes the standard form of amalgamated free products, and mainly works in the generating set $\{\mathcal{G}, \mathcal{H}\}$. This explains how the notion of reduced standard factorization appears—a reduced standard factorization corresponds to a standard form in amalgamated free products, taken so that each G-factor g satisfy $1 <_G g <_G z_G$.

LEMMA 2.12. *Every element $x \in X$ admits a reduced standard subfactorization.*

PROOF. Since X is an amalgamated free product of G and H, every $x \in X$ is written as

$$x = q_0 f_1 q_1 f_2 q_2 \cdots f_l q_l$$

where $q_i \in H$, $f_i \in G$, and $q_i \neq z_H^N$ and $f_i \neq z_G^N$ for any $N \in \mathbb{Z}$ and $i > 0$.

Since z_G is $<_G$-cofinal, for each $i > 0$ there exists $N_i \in \mathbb{Z}$ which satisfies

$$z_G^{N_i} <_G f_i <_G z_G^{N_i+1} .$$

We put $f_i^* = z_G^{-N_i} f_i$. Then f_i^* satisfies the inequality

$$1 <_G f_i^* <_G z_G .$$

Similarly, since z_H is $<_H$-cofinal, for each $i > 0$ there exists M_i which satisfies the inequality

$$z_H^{M_i} \leq_H \Delta_H q_i <_H z_H^{M_i+1} .$$

Let $L_i = \sum_{j>i}(N_j + M_j)$, and put $q_i^* = z_H^{-L_i}(z_H^{-M_i} \Delta_H q_i)z_H^{L_i}$. Since $<_H$ is a z_H-right invariant ordering, $1 \leq_H q_i^* <_H z_H$ holds. We have assumed that $q_i \neq z_H^N$, so we have $q_i^* \neq \Delta_H$. Thus, $1 \leq_H q_i^* <_H \Delta_H$.

Then we get a reduced standard factorization of x as follows. First we modify the first expression of x as

$$\begin{aligned}
x &= q_0 f_1 q_1 \cdots f_l q_l \\
&= q_0(z_G^{N_1} f_1^*)q_1(z_G^{N_2} f_2^*) \cdots (z_G^{N_{l-1}} f_{l-1}^*)q_{l-1}(z_G^{N_l} f_l^*)q_l \\
&= (q_0 z_H^{N_1})f_1^*(q_1 z_H^{N_2}) \cdots f_{l-1}^*(q_{l-1} z_H^{N_l})f_l^* q_l \\
&= (q_0 z_H^{N_1})f_1^*(q_1 z_H^{N_2}) \cdots f_{l-1}^*(q_{l-1} z_H^{N_l})f_l^* \Delta_H^{-1} z_H^{M_l}(z_H^{-M_l} \Delta_H q_l)
\end{aligned}$$

$$
\begin{aligned}
&= (q_0 z_H^{N_1}) f_1^* (q_1 z_H^{N_2}) \cdots f_{l-1}^* (q_{l-1} z_H^{N_l} z_H^{M_l}) (f_l^* \Delta_H^{-1}) q_l^* \\
&= (q_0 z_H^{N_1}) f_1^* (q_1 z_H^{N_2}) \cdots f_{l-1}^* \Delta_H^{-1} z_H^{N_l+M_l} (z_H^{-N_l-M_l} \Delta_H q_{l-1} z_H^{N_l+M_l}) (f_l^* \Delta_H^{-1}) q_l^* \\
&= (q_0 z_H^{N_1}) f_1^* (q_1 z_H^{N_2}) \cdots z_H^{N_l+M_l} (f_{l-1}^* \Delta_H^{-1}) q_{l-1}^* (f_l^* \Delta_H^{-1}) q_l^* \\
&= \cdots \\
&= (q_0 z_H^{L_0}) (f_1^* \Delta_H^{-1}) q_1^* \cdots (f_{l-1}^* \Delta_H^{-1}) q_{l-1}^* (f_l^* \Delta_H^{-1}) q_l^* .
\end{aligned}
$$

Now let us write $f_i^* = P_i(\mathcal{G}) g_{k_i}$, where $P_i(\mathcal{G})$ is a $\mathcal{G}$-positive or an empty word. Since $g_i = x_i \Delta_H$, we may express $P_i(\mathcal{G})$ as an $\{\mathcal{X}, \mathcal{H}\}$-positive (or empty) word. Hence by rewriting each $P_i(\mathcal{G})$ as an $\{\mathcal{X}, \mathcal{H}\}$-positive (or empty) word, we get a standard factorization

$$
(2.4) \qquad \mathcal{F}(x) = (q_0 z_H^{L_0})[P_1(\mathcal{G})] x_{k_1} q_1^* \cdots [P_{l-1}(\mathcal{G})] x_{k_{l-1}} q_{l-1}^* [P_l(\mathcal{G})] x_{k_l} q_l^* .
$$

Since $q_i^* \neq \Delta_H$ for all i, every distinguished subfactorization of (2.4) comes from $[P_i(\mathcal{G})]$. (For example, if $P_1(\mathcal{G}) = g_1 g_2 g_3 = x_1 \Delta_H x_2 \Delta_H x_3 \Delta_H$, then it yields a (maximal) distinguished subfactorization $\Delta_H x_2 \Delta_H x_3 \Delta_H x_{k_1}$). Therefore for a maximal distinguished subfactorization w in $\mathcal{F}(x)$, we may choose g_a and g_u so that $g_a g_w g_u = P_i(\mathcal{G}) g_{k_i}$ holds for some i. Since $P_i(\mathcal{G}) g_{k_i} = f_i^* <_G z_G$, this implies that all distinguished sub-factorizations are irreducible. Hence $\mathcal{F}(x)$ is a reduced standard factorization. $\qquad\square$

2.4. Reducing operation and the proof of Property A. In the proof of Lemma 2.12 given in previous section, we mainly used the generating set $\{\mathcal{G}, \mathcal{H}\}$. In this section we give an alternative way to get a reduced standard factorization, which gives a proof of Property A. This method has an advantage since we work on words over $\{\mathcal{X}, \mathcal{H}\}$.

We say a standard factorization $\mathcal{F}(x) = r p_1 q_1 \cdots p_l q_l$ is *pre-reduced* if $1 <_H q_i <_H z_H$ holds for all i. It is rather easy to see pre-reduced standard factorization exists.

LEMMA 2.13 (Existence of pre-reduced standard factorization). *Every element $x \in X$ admits a pre-reduced standard factorization.*

PROOF. Let $\mathcal{F}(x) = r p_1 q_1 \cdots p_l q_l$ be a standard factorization. For each i, take $M_i \geq 0$ so that $z_H^{M_i} <_H q_i <_H z_H^{M_i+1}$. Let $L_i = \sum_{j \geq i} M_j$ and $q_i^* = z_H^{-L_i} q_i z_H^{L_{i+1}} = z_H^{-L_{i+1}} (z_H^{-M_i} q_i) z_H^{L_{i+1}}$. Since $<_H$ is z_H-right invariant, $1 <_H q_i^* <_H z_H$. Therefore, we get a pre-reduced standard factorization

$$
\begin{aligned}
x &= r p_1 q_1 \cdots p_l q_l \\
&= r p_1 q_1 \cdots p_{l-1} q_{l-1} p_l (z_H^{M_l} q_l^*) \\
&= r p_1 q_1 \cdots p_{l-1} (q_{l-1} z_H^{M_l}) p_l q_l^* \\
&= r p_1 q_1 \cdots p_{l-1} z_H^{-M_l-M_{l-1}} (z_H^{-M_l-M_{l-1}} q_{l-1} z_H^{M_l}) p_l q_l^* \\
&= r p_1 q_1 \cdots p_{l-1} z_H^{-L_{l-1}} q_{l-1}^* p_l q_l^* \\
&= \cdots \\
&= (r z_H^{-L_0}) p_1 q_1^* \cdots p_l q_l^* .
\end{aligned}
$$

$\qquad\square$

To show we are actually able to get reduced standard factorization, we observe that we are able to eliminate all reducible distinguished subfactorizations. Let $d(\mathcal{F})$ be the number of maximal reducible distinguished subfactorizations. The next lemma gives alternative proof that a reduced standard factorization exists. It says that by induction on $(d(\mathcal{F}), c(\mathcal{F}))$ for pre-reduced factorization $\mathcal{F}$, we are able to get reduced standard factorization.

LEMMA 2.14 (Reducing operation). *Let* $\mathcal{F}(x) = r p_1 q_1 \cdots p_l q_l$ *be a pre-reduced standard factorization of* $x \in X$. *If* $\mathcal{F}(x)$ *contains a reducible distinguished subfactorization, then we can find another pre-reduced standard factorization* $\mathcal{F}'(x) = r' p'_1 q'_1 \cdots$ *which satisfies* $d(\mathcal{F}') < d(\mathcal{F})$ *or,* $d(\mathcal{F}') = d(\mathcal{F})$ *and* $c(\mathcal{F}') < c(\mathcal{F})$. *Moreover, if* $r >_H 1$ *then* $r' >_H 1$.

PROOF. Let $w = q_i p_{i+1} \cdots p_{s-1} q_{s-1}$ be a reducible maximal distinguished subfactorization in $\mathcal{F}(x)$. Thus, we may assume that the pre-reduced standard factorization $\mathcal{F}(x)$ is written as

$$\mathcal{F}(x) = r p_1 q_1 \cdots p_{i-1} q_{i-1} p'_i [g_a g_w] x_u p'_s q_s \cdots p_l q_l$$

where

(1) $p'_i = p_i x_a^{-1}$ and $p'_s = x_u^{-1} p_s$,
(2) $p'_i, p'_s \in P\mathcal{X} \cup \{1\}$,
(3) $g_a g_w g_u \geq_G z_G$.

Now take $N > 0$ so that $z_G^N <_G g_a g_w g_u \leq_G z_G^{N+1}$, and for $j < i$ let $q_j^* = z_H^{-N} q_j z_H^N$. Then we may write x as

$$\begin{aligned}
x &= r p_1 q_1 \cdots p_{i-1} q_{i-1} p'_i [g_a g_w] x_u p'_s q_s \cdots p_l q_l \\
&= r p_1 q_1 \cdots p_{i-1} q_{i-1} p'_i z_G^N (z_G^{-N} g_a g_w g_u) \Delta_H^{-1} p'_s q_s \cdots p_l q_l \\
&= (r z_H^N) p_1 q_1^* \cdots p_{i-1} q_{i-1}^* p'_i (z_G^{-N} g_a g_w g_u) \Delta_H^{-1} p'_s q_s \cdots p_l q_l.
\end{aligned}$$

First assume that $(z_G^{-N} g_a g_w g_u) = z_G = z_H$. Then we write x as

$$\begin{aligned}
x &= (r z_H^N) p_1 q_1^* \cdots p_{i-1} q_{i-1}^* p'_i (z_H \Delta_H^{-1}) p'_s q_s \cdots p_l q_l \\
&= (r z_H^N) p_1 q_1^* \cdots p_{i-1} q_{i-1}^* p'_i h_1 p'_s q_s \cdots p_l q_l.
\end{aligned}$$

If $p'_i \neq 1$ and $p'_s \neq 1$, then we get a pre-reduced standard factorization

(2.5) $$\mathcal{F}'(x) = (r z_H^N) p_1 q_1^* \cdots p_{i-1} q_{i-1}^* p'_i h_1 p'_s q_s \cdots p_l q_l.$$

In $\mathcal{F}'(x)$, we removed the reducible distinguished subfactorization w and no distinguished subfactorization is created, so $d(\mathcal{F}') < d(\mathcal{F})$.

If $p'_i = 1$ or $p'_s = 1$, then the standard factorization (2.5) might fail to be pre-reduced. We construct a pre-reduced standard factorization $\mathcal{F}''$ from the standard factorization (2.5) by using the argument of proof of Lemma 2.13. In such case, we might produce one new reducible maximal distinguished subfactorization, so in general $d(\mathcal{F}'') \leq d(\mathcal{F})$ although we have removed w from $\mathcal{F}(x)$. In this case we have $c(\mathcal{F}'') < c(\mathcal{F})$.

(Here is a simple example where $d(\mathcal{F}'')$ does not decrease: assume that $p'_i = 1$, $p'_s \in \mathcal{X}$, and $q_{i-1}^* h_1 = \Delta_H$, and that $w = q_s \cdots$ is a distinguished subfactorization: then we get a new

maximal distinguished subfactorization $w = \cdots (q_{i-1}^* h_1) p_s' q_s \cdots$ in $\mathcal{F}'(x)$. This maximal distinguished subfactorization might be reducible, so $d(\mathcal{F}'') = d(\mathcal{F})$ may occur.)

Next assume that $(z_G^{-N} g_a g_w g_u) \neq z_G$. Let us put $g' = (z_G^{-N} g_a g_w g_u) g_1^{-1}$ and write x as

$$\begin{aligned}
\mathcal{F}(x) &= (r z_H^N) p_1 q_1^* \cdots p_{i-1} q_{i-1}^* p_i'(z_G^{-N} g_a g_w g_u) \Delta_H^{-1} p_s' q_s \cdots p_l q_l \\
&= (r z_H^N) p_1 q_1^* \cdots p_{i-1} q_{i-1}^* p_i' g' g_1 \Delta_H^{-1} p_s' q_s \cdots p_l q_l \\
&= (r z_H^N) p_1 q_1^* \cdots p_{i-1} q_{i-1}^* p_i' g'(x_1 p_s') q_s \cdots p_l q_l \, .
\end{aligned}$$

If $g' = 1$, then we get a pre-reduced standard factorization

$$\mathcal{F}'(x) = (r z_H^N) p_1 q_1^* \cdots p_{i-1} q_{i-1}^* (p_i' x_1 p_s') q_s \cdots p_l q_l$$

such that $d(\mathcal{F}') < d(\mathcal{F})$.

If $g' >_G 1$, then let us write $g' = g_{a'} P(\mathcal{G})$ where $P(\mathcal{G})$ is a $\mathcal{G}$-positive, or empty word. By rewriting $P(\mathcal{G})$ as an $\{\mathcal{X}, \mathcal{H}\}$-positive word, we get a new pre-reduced standard factorization

$$\mathcal{F}'(x) = (r z_H^N) p_1 q_1^* \cdots p_{i-1} q_{i-1}^* p_i' [g_{a'} P(\mathcal{G})](x_1 p_s') q_s \cdots p_l q_l.$$

Observe that $P(\mathcal{G})$ gives rise to a maximal distinguished subfactorization w' in $\mathcal{F}'(x)$ such that $g_{w'} = P(\mathcal{G})$. By Lemma 2.1, $<_G$ is a g_1-right invariant ordering, so $1 \leq_G g' <_G z_G g_1^{-1}$, so $g_{a'} g_{w'} g_1 <_G z_G$. Hence the maximal distinguished subfactorization w' in $\mathcal{F}'(x)$ is irreducible. By construction, all other maximal reducible distinguished subfactorizations in $\mathcal{F}'(x)$ are derived from the pre-reduced factorization $\mathcal{F}(x)$. Since we have removed the maximal reducible distinguished subfactorization w in $\mathcal{F}(x)$, $d(\mathcal{F}') < d(\mathcal{F})$.

Moreover, by construction we have always $r \leq_H r'$. In particular, $1 <_H r'$ if $1 <_H r$, $\square$

Now we are ready to prove Property A.

PROPOSITION 2.15 (Property A). *If x is expressed as an $\{\mathcal{X}, \mathcal{H}\}$-positive word, then $x \neq 1$.*

PROOF. Assume that x is expressed by an $\{\mathcal{X}, \mathcal{H}\}$-positive word. Such a word expression can be modified to a standard factorization which is also an $\{\mathcal{X}, \mathcal{H}\}$-positive word: By the proof of Lemma 2.13, we can modify such a standard factorization so that it is pre-reduced, preserving the property that it is also an $\{\mathcal{X}, \mathcal{H}\}$-positive word. By Lemma 2.14, we may modify the $\{\mathcal{X}, \mathcal{H}\}$-positive pre-reduced standard expression $\mathcal{F}(x)$ so that it is an $\{\mathcal{X}, \mathcal{H}\}$-positive reduced standard factorization.

Now let us rewrite $\mathcal{F}(x)$ as a word over $\{\mathcal{G}, \mathcal{H}\}$ as follows. Let w be a maximal distinguished subfactorization in $\mathcal{F}(x)$ so we may write $\mathcal{F}(x)$ as

$$\mathcal{F}(x) = r p_1 q_1 \cdots p_{i-1} q_{i-1} p_i' [g_a g_w] p_s q_s \cdots p_l q_l \, .$$

Since w is irreducible, we may choose g_a and $x_u \in \mathcal{X}$ so that $p_s' = x_u^{-1} p_s$, $p_i' \in P\mathcal{X} \cup \{1\}$ and that $g_a g_w g_u <_G z_G$. Then we write x as

$$\mathcal{F}(x) = r p_1 q_1 \cdots p_{i-1} q_{i-1} p_i'(g_a g_w g_u) \Delta_H^{-1} p_s' q_s \cdots p_l q_l \, ,$$

and regard $(g_a g_w g_u)$ as a $\mathcal{G}$-positive word.

Iterating this rewriting procedure for each maximal distinguished subword, and rewriting the rest of x_i in $\mathcal{F}(x)$ as a word over $\{\mathcal{G}, \mathcal{H}\}$ by using the relation $x_i = g_i \Delta_H^{-1}$, we finally write x as

$$(2.6) \qquad\qquad x = W_0 V_1 W_1 \cdots V_n W_n$$

where W_i is a word over $\mathcal{H}^{\pm 1}$ and V_i is a word over $\mathcal{G}^{\pm 1}$. By construction, $V_i \in \mathcal{G}$ or $V_i = g_a g_w g_u$ where g_w is a maximal distinguished subfactorization in $\mathcal{F}(x)$. Since we have chosen $g_a g_w g_u <_G z_G$, this implies that, $V_i \notin \langle z_G \rangle$ for all i. Similarly, the assumption that $\mathcal{F}(x)$ is reduced implies that we may choose $W_i \notin \langle z_H \rangle$ for $i > 0$. This implies that the expression (2.6) is a normal form of an amalgamated free product $X = G *_{\langle z_G = z_H \rangle} H$ so $x \neq 1$.

$\square$

2.5. Proof of Property C. Next we give a proof of Property C. To begin with, we observe a simple, but useful observation.

LEMMA 2.16.
$$h_j^{-1} x_i = N(\mathcal{X}, \mathcal{H}) \Delta_H^{-1}$$
where $N(\mathcal{X}, \mathcal{H})$ represents an $\{\mathcal{X}, \mathcal{H}\}$-negative word.

PROOF. Since $z_H = z_G$ and $x_i = g_i \Delta_H^{-1}$, we have

$$z_H = g_i g_i^{-1} z_G g_1^{-1} g_1 = x_i \Delta_H (g_i^{-1} z_G g_1^{-1}) x_1 \Delta_H .$$

Therefore

$$h_j^{-1} x_i = (h_j^{-1} z_H \Delta_H^{-1}) x_1^{-1} (z_G^{-1} g_1 g_i) \Delta_H^{-1} = (h_j^{-1} h_1) x_1^{-1} (z_G^{-1} g_1 g_i) \Delta_H^{-1} .$$

Since $z_G^{-1} g_i <_G 1$ and g_1 is the $<_G$-minimal positive element, $z_G^{-1} g_i \leq_G g_1^{-1}$. Hence $z_G^{-1} g_1 g_i \leq_G 1$. Thus, $(h_j^{-1} h_1) x_1^{-1} (z_G^{-1} g_1 g_i)$ is written as an $\{\mathcal{X}, \mathcal{H}\}$-negative word.

$\square$

Now we are ready to prove Property C.

PROPOSITION 2.17 (Property C). *Each non-trivial element $x \in X$ is expressed by an $\{\mathcal{X}, \mathcal{H}\}$-positive word or an $\{\mathcal{X}, \mathcal{H}\}$-negative word.*

PROOF. Let x be a non-trivial element of X and take a reduced standard factorization of x,

$$\mathcal{F}(x) = r p_1 q_1 \cdots p_l q_l .$$

If $r \geq_H 1$, r can be written as an $\mathcal{H}$-positive or empty word, hence we may express x as an $\{\mathcal{X}, \mathcal{H}\}$-positive word.

By induction on $l = c(\mathcal{F})$, we prove that x is expressed by an $\{\mathcal{X}, \mathcal{H}\}$-negative word under the assumption that $r <_H 1$.

First assume that $q_1 \neq \Delta_H$. Since $r <_H 1$, we can express r as $r = N(\mathcal{H}) h_1^{-1}$, where $N(\mathcal{H})$ is an $\mathcal{H}$-negative word or an empty word. Take an $\mathcal{X}$-positive word expression of

$p_1 = x_{i_1} x_{i_2} \cdots x_{i_p}$. Then by Lemma 2.16,

$$
\begin{aligned}
r p_1 p_2 q_2 \cdots &= (N(\mathcal{H}) h_1^{-1})(x_{i_1} x_{i_2} \cdots x_{i_p}) q_1 p_2 q_2 \cdots \\
&= N(\mathcal{H})(h_1^{-1} x_{i_1}) x_{i_2} \cdots x_{i_p} q_1 p_2 q_2 \cdots \\
&= N(\mathcal{X}, \mathcal{H}) \Delta_H^{-1} x_{i_2} \cdots x_{i_p} q_1 p_2 q_2 \cdots \\
&= N(\mathcal{X}, \mathcal{H})(h_1^{-1} x_{i_2}) \cdots x_{i_p} q_1 p_2 q_2 \cdots \\
&= \cdots \\
&= N(\mathcal{X}, \mathcal{H}) \Delta_H^{-1} q_1 p_2 q_2 \cdots .
\end{aligned}
$$

$N(\mathcal{X}, \mathcal{H})$ represents an $\{\mathcal{X}, \mathcal{H}\}$-negative word.

Since $\mathcal{F}(x)$ is a reduced standard factorization, $q_1 <_H z_H$. By Lemma 2.4 Δ_H is the $<_H$-maximal element of H which is strictly smaller than z_H, so $q_1 \leq_H \Delta_H$. We have assumed that $q_1 \neq \Delta_H$ so $(\Delta_H^{-1} q_1) <_H 1$. Thus the subword $(\Delta_H^{-1} q_1) p_2 q_2 \cdots p_l q_l$ is a reduced standard factorization with complexity $(l-1)$. By induction, $(\Delta_H^{-1} q_1) p_2 q_2 \cdots p_l q_l$ is written as an $\{\mathcal{X}, \mathcal{H}\}$-negative word, hence we conclude that x is written as an $\{\mathcal{X}, \mathcal{H}\}$-negative word.

Next assume that $q_1 = \Delta_H$. Let $w = q_1 p_2 q_2 \cdots p_{s-1} q_{s-1}$ be a maximal distinguished subfactorization of $\mathcal{F}(x)$ which contains q_1. Thus, the reduced standard factorization S is written as

$$
\mathcal{F}(x) = r p_1'[g_a g_w] x_u p_s' q_s p_{s+1} \cdots p_l q_l
$$

where $p_1' = p_1 x_a^{-1}$, $p_s' = x_u^{-1} p_s \in P\mathcal{X} \cup \{1\}$.

Then by Lemma 2.16,

$$
\begin{aligned}
x &= r p_1'[g_a g_w] x_u p_s' q_s p_{s+1} \cdots p_l q_l \\
&= N(\mathcal{X}, \mathcal{H}) h_1^{-1} [g_a g_w] x_u \Delta_H \Delta_H^{-1} p_s' q_s \cdots p_l q_l \\
&= N(\mathcal{X}, \mathcal{H}) h_1^{-1} [g_a g_w g_u] \Delta_H^{-1} p_s' q_s \cdots p_l q_l \\
&= N(\mathcal{X}, \mathcal{H}) \Delta_H (z_G^{-1} g_a g_w g_u) \Delta_H^{-1} p_s' q_s \cdots p_l q_l .
\end{aligned}
$$

The distinguished subfactorization w is irreducible so we may choose x_u and x_a so that $z_G^{-1} g_a g_w g_u <_G 1$ holds. This implies that $z_G^{-1} g_a g_w g_u$ is written as a $\mathcal{G}$-negative word. By expressing a $\mathcal{G}$-negative word expression of $z_G^{-1} g_a g_w g_u$ as an $\{\mathcal{X}, \mathcal{H}\}$-negative word, we conclude that $z_G^{-1} g_a g_w g_u$ is written as a word of the form $\Delta_H^{-1} N(\mathcal{X}, \mathcal{H})$. Hence

$$
\begin{aligned}
x &= N(\mathcal{X}, \mathcal{H}) \Delta_H \Delta_H^{-1} N(\mathcal{X}, \mathcal{H})(\Delta_H^{-1} p_s' q_s \cdots p_l q_l) \\
&= N(\mathcal{X}, \mathcal{H})(\Delta_H^{-1} p_s' q_s \cdots p_l q_l) .
\end{aligned}
$$

If $p_s' \neq 1$, then $(\Delta_H^{-1} p_s' q_s \cdots p_l q_l)$ is a reduced standard factorization having the complexity less than l. Hence by induction, $(\Delta_H^{-1} p_s' q_s \cdots p_l q_l)$ is expressed by an $\{\mathcal{X}, \mathcal{H}\}$-negative word.

If $p'_s = 1$, then $q_s \neq \Delta_H$ since w was a maximal distinguished subfactorization. Hence $q_s <_H \Delta_H$, and $(\Delta_H^{-1} q_s) p_{s+1} \cdots p_l q_l$ is a reduced standard factorization with complexity less than l. By induction, $(\Delta_H^{-1} q_s) p_{s+1} \cdots p_l q_l$ is expressed by an $\{\mathcal{X}, \mathcal{H}\}$-negative word.

Thus in either case, we conclude x is expressed by an $\{\mathcal{X}, \mathcal{H}\}$-negative word. $\qquad\square$

2.6.   Proof of Theorem 1.1.   Now we are ready to prove our main theorem.

Proof of Theorem 1.1.   (i): In Proposition 2.15 and Proposition 2.17, we have already confirmed the Properties A and C for the generating set $\{\mathcal{X}, \mathcal{H}\}$. By Corollary 2.9 the generating set $\{\mathcal{X}, \mathcal{H}\}$ indeed defines an isolated left ordering $<_X$ of X.

(ii): Let $\mathcal{G}' = \{g'_1, \dots\}$ and $\mathcal{H}' = \{h'_1, \dots\}$ be other generating sets of G and H satisfying **[CF(G)]** and **[CF(H)]**. Recall that $\Delta_H = z_H h_1^{-1}$ does not depend on the choice of a generating set $\mathcal{H}$. Let $x_i = g_i \Delta_H^{-1}$, $x'_i = g'_i \Delta_H^{-1}$, $\mathcal{X} = \{x_1, \dots, \}$, and $\mathcal{X}' = \{x'_1, \dots, \}$.

Since $\mathcal{H}$ and $\mathcal{H}'$ are generators of the same semigroup, we may write h_i as an $\mathcal{H}'$-positive word. Similarly, since $\mathcal{G}$ and $\mathcal{G}'$ are generators of the same semigroup, we may write g_i as a $\mathcal{G}'$-positive word $g_i = g'_{i_1} g'_{i_2} \cdots g'_{i_l}$. Thus,

$$x_i = g_i \Delta_H^{-1} = g'_{i_1} g'_{i_2} \cdots g'_{i_l} \Delta_H^{-1} = x'_{i_1} \Delta_H x'_{i_2} \Delta_H \cdots x'_{i_{l-1}} \Delta_H x'_{i_l}$$

so x_i is written as an $\{\mathcal{X}', \mathcal{H}'\}$-positive word. Thus, if $x \in X$ is expressed by an $\{\mathcal{X}, \mathcal{H}\}$-positive word, then x is also represented by an $\{\mathcal{X}', \mathcal{H}'\}$-positive word. By interchanging the roles of $\{\mathcal{G}, \mathcal{H}\}$ and $\{\mathcal{G}', \mathcal{H}'\}$, we conclude that $\{\mathcal{X}, \mathcal{H}\}$ and $\{\mathcal{X}', \mathcal{H}'\}$ generate the same sub-semigroup of X so they define the same isolated ordering of X.

(iii): This is obvious from the definition of $<_X$.

(iv): The inequality $h_1 <_X h_2 <_X \cdots <_X h_n$ follows from the definition of $<_X$. By Lemma 2.16, $x_i <_X h_1$ for all i. Now we show $x_i <_X x_j$ if $i < j$. Since $g_i <_G g_j$ if $i < j$, $g_i^{-1} g_j$ is written as a $\mathcal{G}$-positive word. Now by definition $g_i = x_i \Delta_H$, so we may express a $\mathcal{G}$-positive word expression of $g_i^{-1} g_j$ as an $\{\mathcal{X}, \mathcal{H}\}$-positive word expression of the form $P_{i,j}(\mathcal{X}, \mathcal{H}) \Delta_H$, where $P_{i,j}(\mathcal{X}, \mathcal{H})$ represents an $\{\mathcal{X}, \mathcal{H}\}$-positive word. Therefore $x_i^{-1} x_j = \Delta_H g_i^{-1} g_j \Delta_H^{-1} = \Delta_H P_{i,j}(\mathcal{X}, \mathcal{H})$, so $x_i <_X x_j$. The assertion that $z = z_G = z_H$ is $<_X$-positive cofinal is obvious. To see that $<_X$ is a z-right invariant ordering, we observe that $z^{-1} x_i z = x_i >_X 1$ and $z^{-1} h_j z >_X 1$. Now for $x, x' \in X$, assume $x <_X x'$, so $x^{-1} x'$ is written as $\{\mathcal{X}, \mathcal{H}\}$-positive word $w = s_1 \cdots s_m$, where s_i denotes x_j or h_j. Then $z^{-1}(x^{-1} x')z = (z^{-1} s_1 z) \cdots (z^{-1} s_m z) >_X 1$, hence $xz <_X x'z$.

(v): Recall that by Lemma 2.3, we may choose the generating sets $\mathcal{G}$ and $\mathcal{H}$ so that the cardinal of $\mathcal{G}, \mathcal{H}$ are equal to $r(<_G), r(<_H)$ respectively. Thus, $r(<_X) \leq r(<_G) + r(<_H)$.

(vi): We prove that $\langle x_1 \rangle$ is the unique $<_X$-convex non-trivial proper subgroup of X. Recall by (2), (4) and Lemma 2.1, x_1 is the minimal $<_X$-positive element of X, hence x_1 does not depend on a choice of $\mathcal{G}$ and $\mathcal{H}$. In particular, $\langle x_1 \rangle$ is a non-trivial $<_X$-convex subgroup.

Let C be a $<_X$-convex subgroup of X. Assume that $C \supset \langle x_1 \rangle$. Let $y \in C - \langle x_1 \rangle$ be an $<_X$-positive element. Then y is written as $y = x_1^m x_j P(\mathcal{X}, \mathcal{H})$ or $y = x_1^m h_l P(\mathcal{X}, \mathcal{H})$ where $m \geq 0, l > 0, j > 1$ and $P(\mathcal{X}, \mathcal{H})$ is an $\{\mathcal{X}, \mathcal{H}\}$-positive word. Since $x_1 \in C$, we may choose y so that $m = 0$ by considering $x_1^{-m} y$ instead.

First we consider the case $\mathcal{X} \not\subset \langle x_1 \rangle$. Then we may choose y so that $1 < x_2 \leq_X y$ holds, so the convexity assumption implies $x_2 \in C$. Now observe that $x_1^{-1} x_2 = \Delta_H g_1^{-1} g_2 \Delta_H^{-1} = \Delta_H P_{1,2}(\mathcal{X}, \mathcal{H})$, hence

$$1 <_X h_p \leq z_H h_1^{-1} = \Delta_H <_X \Delta_H P_{1,2}(\mathcal{X}, \mathcal{H}) = x_1^{-1} x_2.$$

Since $x_1^{-1} x_2 \in C$, this implies $\mathcal{X} \cup \mathcal{H} = \{x_1, \ldots, x_m, h_1, \ldots, h_n\} \subset C$. Therefore we conclude $C = X$.

Next we consider the case $\mathcal{X} \subset \langle x_1 \rangle$. This happens only when $G = \mathbb{Z} = \langle g_1 \rangle$ and $z_G = g_1^N$. Then we may choose y so that $1 < h_1 \leq_X y$ holds, so $h_1 \in C$. Then $x_1^{-1} h_1 = \Delta_H g_1^{-1} h_1 = h_1^{-1} z_G g_1^{-1} h_1$ so $z_G g_1^{-1} = g_1^{N-1} \in C$. This implies $z_G = z_H \in C$, so $C = X$.

$\square$

2.7. Computational issues.

In this section we briefly mention the computational issue concerning the isolated ordering $<_X$. Let $G = \langle \mathcal{S} \mid \mathcal{R} \rangle$ be a group presentation and $<_G$ be a left ordering of G. The *order-decision problem* for $<_G$ is the algorithmic problem of deciding for an element $g \in G$ given as a word over $\mathcal{S} \cup \mathcal{S}^{-1}$, whether $1 <_G g$ holds or not. Clearly, the order-decision problem is harder than the word problem, since $1 <_G g$ implies $1 \neq g$. It is interesting to find an example of a left ordering $<_G$ of a group G, such that the order-decision problem for $<_G$ is unsolvable but the word problem for G is solvable.

There is another algorithmic problem which is also related to the order-decision problem of isolated orderings. We say a word over $\mathcal{G} \cup \mathcal{G}^{-1}$ is $\mathcal{G}$-*definite* if w is $\mathcal{G}$-positive or $\mathcal{G}$-negative, or empty. If $\mathcal{G}$ defines an isolated ordering of G, then every $g \in G$ admits a $\mathcal{G}$-definite word expression. The $\mathcal{G}$-*definite search problem* is a problem to find a $\mathcal{G}$-definite word expression of a given element of G.

THEOREM 2.18. *Let us take $G, H, X, <_G, <_H, z_G, z_H, \mathcal{G}, \mathcal{H}, \mathcal{X}$ as in Theorem 1.1.*

(1) *The order-decision problem for $<_X$ is solvable if and only if the order-decision problems for $<_G$ and $<_H$ are solvable.*

(2) *The $\{\mathcal{X}, \mathcal{H}\}$-definite search problem is solvable if and only if the $\mathcal{G}$-definite search problem and the $\mathcal{H}$-search problem are solvable.*

PROOF. Since the restriction of $<_X$ to G and H yields the ordering $<_G$ and $<_H$ respectively, if the order-decision problem for $<_X$ is solvable, then so is for $<_G$ and $<_H$. Similarly, if $\{\mathcal{X}, \mathcal{H}\}$-definite search problem is solvable, then we are able to get $\mathcal{G}$-positive (resp. $\mathcal{H}$-positive) word by transforming $\{\mathcal{X}, \mathcal{H}\}$-positive word representing elements of G (resp. H) by using $g_i = x_i \Delta_H$ so $\mathcal{G}$-definite ($\mathcal{H}$-definite) search problem is also solvable.

The proof of converse is implicit in the proof of Theorem 1.1 (i). Recall that in the proof of Property C (Proposition 2.17), we have shown that for a reduced standard factorization $\mathcal{F}(x) = rp_1q_1 \cdots p_lq_l$, $x >_X 1$ if $r \geq_H 1$ and $x <_X 1$ if $r_H < 1$. Moreover, the proof of Property C (Proposition 2.17) is constructive, hence we can algorithmically compute an $\{\mathcal{X}, \mathcal{H}\}$-negative word expression of x if $r <_H 1$ if the $\mathcal{G}$-definite search problem and the $\mathcal{H}$-search problem is solvable.

Thus, to solve the order-decision problem or the $\{\mathcal{X}, \mathcal{H}\}$-definite search problem, it is sufficient to compute a reduced standard factorization. We have established two different methods to compute a reduced standard factorization, in the proof of Lemma 2.12 and Lemma 2.14. Both proofs are constructive, hence we can algorithmically compute a reduced standard expression. □

It is not difficult to analyze the computational complexity of order-decision problem or the $\{\mathcal{X}, \mathcal{H}\}$-definite search problems based on the algorithm obtained from the proof of Proposition 2.17, Lemma 2.12 and Lemma 2.14. In particular, we observe the following results.

PROPOSITION 2.19. *Let us take* $G, H, X, <_G, <_H, z_G, z_H, \mathcal{G}, \mathcal{H}, \mathcal{X}$ *as in Theorem 1.1.*

(1) *If the order-decision problems for* $<_G$ *and* $<_H$ *are solvable in polynomial time with respect to the length of the input of words, then the order-decision problem for* $<_X$ *is also solvable in polynomial time.*

(2) *If the* $\mathcal{G}$-*definite search problem and the* $\mathcal{H}$-*definite search problem are solvable in polynomial time, then the* $\{\mathcal{X}, \mathcal{H}\}$-*definite search problem is also solvable in polynomial time.*

(3) *Moreover, if one can always find a* $\mathcal{G}$-*definite and an* $\mathcal{H}$-*definite word expression whose length are polynomial with respect to the length of the input word, then one can always find an* $\{\mathcal{X}, \mathcal{H}\}$-*definite word expression whose length is polynomial with respect to the length of the input word.*

3. Examples. In this section we give examples of isolated left orderings produced by Theorem 1.1. All examples in this section are new, and have various properties which previously known isolated orderings do not have. For the sake of simplicity, in the following examples we only use the infinite cyclic group $\mathbb{Z}$, the most fundamental example of group having isolated orderings, as a basic building block.

Other groups with isolated orderings, such as groups having only finitely many left-orderings, or the braid group B_n with the Dubrovina-Dubrovin ordering $<_{DD}$, also can be used to construct new examples of isolated orderings.

3.1. Group having many distinct isolated orderings. Let $a_1, \ldots, a_m$ $(m > 1)$ be positive integers bigger than one and consider the group obtained as a central cyclic amalgamated free product of m infinite cyclic groups $\mathbb{Z}^{(i)} = \langle x_i \rangle$.

$$G = G_{a_1,\ldots,a_m} = *_{\mathbb{Z}} \mathbb{Z}^{(i)}$$

$$= \left\langle x_1, \ldots, x_m \mid x_1^{a_1} = x_2^{a_2} = \cdots = x_m^{a_m} \right\rangle.$$

Recall that an infinite cycle group $\mathbb{Z}$ have exactly two left orderings, the standard one and its opposite. Using the standard left ordering for each factor $\mathbb{Z}^{(i)}$, by Theorem 1.1 we are able to construct an isolated left ordering $<_G$ so that the restriction of $<_G$ to the i-th factor $\mathbb{Z}^{(i)}$ is the standard left ordering.

First we give a detailed exposition of $<_G$ for the case $m = 2$ and $m = 3$.

EXAMPLE 3.1.

(i) First we begin with the case $m = 2$, which was already considered in [5], [10]:

$$G_{a_1, a_2} = \mathbb{Z}^{(1)} *_{\mathbb{Z}} \mathbb{Z}^{(2)} = \langle x_1, x_2 \mid x_1^{a_1} = x_2^{a_2} \rangle.$$

By Theorem 1.1, we get an isolated ordering $<_G$ defined by the generating set $\{x_1 x_2^{1-a_2}, x_2\}$.

(ii) Next we consider the case $m = 3$. There are two different ways to express G as an amalgamated free products of $\mathbb{Z}$.

(a) First we regard $G_{a_1, a_2, a_3} = G_{a_1, a_2} *_{\mathbb{Z}} \mathbb{Z}^{(3)} = (\mathbb{Z}^{(1)} *_{\mathbb{Z}} \mathbb{Z}^{(2)}) *_{\mathbb{Z}} \mathbb{Z}^{(3)}$.

By (1), G_{a_1, a_2} have an isolated ordering defined by $\{x_1 x_2^{1-a_2}, x_2\}$. By applying Theorem 1.1 again, we get the isolated ordering $<_{(\bullet\bullet)\bullet}$ defined by $\{x_1 x_2^{1-a_2} x_3^{1-a_3}, x_2 x_3^{1-a_3}, x_3\}$.

(b) Next we regard $G_{a_1, a_2, a_3} = \mathbb{Z}^{(1)} * G_{a_2, a_3} = \mathbb{Z}^{(1)} * (\mathbb{Z}^{(2)} *_{\mathbb{Z}} \mathbb{Z}^{(3)})$. By applying Theorem 1.1, we get the isolated ordering $<_{\bullet(\bullet\bullet)}$ defined by $\{x_1 x_3^{-a_3} x_2 x_3^{1-a_3} = x_1 x_2^{1-a_2} x_3^{1-a_3}, x_2 x_3^{1-a_3}, x_3\}$.

Thus two orderings $<_{(\bullet\bullet)\bullet}$ and $<_{\bullet(\bullet\bullet)}$ derived from different factorizations are the same ordering.

As Example 3.1 (ii) suggests, the isolated orderings constructed from Theorem 1.1 are independent of the way of factorization as amalgamated free products, that is, the way of putting parenthesis in the expression $\mathbb{Z}^{(1)} *_{\mathbb{Z}} \mathbb{Z}^{(2)} *_{\mathbb{Z}} \cdots *_{\mathbb{Z}} Z^{(m)}$. All factorizations give the same isolated ordering $<_G$ defined by $\{s_1, \ldots, s_m\}$, where s_i is given by

$$s_i = x_i x_{i+1}^{1-a_{i+1}} \cdots x_m^{1-a_m}.$$

This is checked by induction on m. Take a factorization of G as $G = G_1 *_{\mathbb{Z}} G_2 = G_{a_1, \ldots, a_k} *_{\mathbb{Z}} G_{a_{k+1}, \ldots, a_m}$. By induction, the isolated ordering $<_1$ of G_1 is independent of a choice of a factorization of G_1, and is defined by

$$s_i' = x_i x_{i+1}^{1-a_{i+1}} \cdots x_k^{1-a_k} \quad (i = 1, \ldots, k).$$

Similarly, the isolated ordering $<_2$ of G_2 is independent of a choice of a factorization of G_2, and is defined by

$$s_j'' = x_j x_{j+1}^{1-a_{j+1}} \cdots x_m^{1-a_m} \quad (j = k+1, \ldots, m).$$

Thus by Theorem 1.1, we get an isolated ordering $<_G$ of G defined by

$$
s_i = \begin{cases} s_i' x_{k+1}^{-a_{k+1}} x_{k+1} x_{k+2}^{1-a_{j+2}} \cdots x_m^{1-a_m} & (i = 1, \ldots, k), \\ x_i x_{i+1}^{1-a_{i+1}} \cdots x_m^{1-a_m} & (i = k+1, \ldots, m) \end{cases}
$$
$$
= x_i x_{i+1}^{1-a_{i+1}} \cdots x_m^{1-a_m}.
$$

The group G is the simplest example of a group with isolated orderings constructed by Theorem 1.1. Nevertheless the group G and its isolated ordering $<_G$ have various interesting properties which have not appeared in the previous examples:

(1): *The isolated ordering $<_G$ of G is not derived from Dehornoy-like orderings if G is not generated by two elements.*

As we mentioned earlier, the special kind of left-orderings called *Dehornoy-like orderings* produces isolated orderings, and all previously known examples of genuine isolated orderings are derived from Dehornoy-like orderings.

In [5] it is proved that an isolated ordering derived from Dehornoy-like orderings has a lot of convex subgroups: if the isolated orderings $<_H$ of a group H is derived from the Dehornoy-like orderings, then there are at least $r(<_H) - 1$ proper, $<_H$-convex nontrivial subgroups. On the other hand Theorem 1.1 (vi) shows the isolated orderings $<_G$ has only one proper, $<_G$-convex nontrivial subgroup.

If G is not generated by two elements, then $r(<_G) > 2$. This implies that the isolated ordering $<_G$ of G is not derived from a Dehornoy-like ordering. This provides a counter example of somewhat optimistic conjecture: every genuine isolated ordering is derived from Dehornoy-like ordering. (Recall that all previously known examples of genuine isolated orderings are constructed by Dehornoy-like orderings.)

We remark that it is known that the group $G = G_{a_1, \ldots, a_m}$ is a two-generator group if and only if a_i and a_j are not coprime for some $i \neq j$ [8]. Therefore for example, the isolated ordering of $G_{2,3,4}$ in Example 3.1 (ii) is an isolated ordering which is not derived from a Dehornoy-like ordering.

(2): *The natural right G-action on $LO(G)$ has at least $2(m-1)!$ distinct orbits derived from isolated orderings.*

There is a natural, continuous right G-action on $LO(G)$, defined as follows: For a left ordering $<$ of G and $g \in G$, we define the left ordering $< \cdot g$ by $h \, (< \cdot g) \, h'$ if $hg < h'g$. This action sends an isolated ordering to an isolated ordering. Although this action is natural and important, little is known about the quotient $LO(G)/G$.

Recall that G is written as the amalgamated free products of m infinite cyclic groups $\mathbb{Z}^{(i)}$. As we have seen, the way of decomposition of G (the way of putting parenthesis) does not affect the obtained isolated ordering $<_G$.

On the other hand, for a permutation $\sigma \in S_m$, $G_{a_1,\ldots,a_m} = G_{a_{\sigma(1)},\ldots,a_{\sigma(m)}}$. By viewing $G_{a_1,\ldots,a_m} = G_{a_{\sigma(1)},\ldots,a_{\sigma(m)}}$ and applying the construction above, we get an isolated ordering $<_\sigma$ whose minimal positive element is

$$x_{\sigma(1)} x_{\sigma(2)}^{1-a_{\sigma(2)}} \cdots x_{\sigma(m)}^{1-a_{\sigma(m)}} = x_1^{-(m-1)a_1} x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(m)}.$$

Thus, for two permutations σ and τ, if $x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(m)}$ and $x_{\tau(1)} x_{\tau(2)} \cdots x_{\tau(m)}$ are not conjugate, then two isolated orderings $<_\sigma$ and $<_\tau$ belong to distinct G-orbits. Hence, we are able to construct $(m-1)!$ distinct G-orbits of isolated orderings.

Recall that these orderings are constructed from the standard left orderings of $\mathbb{Z}^{(i)}$. By using the opposite of the standard left-ordering of $\mathbb{Z}$ instead, we get other $(m-1)!$ distinct G-orbits of isolated orderings in a similar way. Thus we have at least $2(m-1)!$ different G-orbits derived from isolated orderings.

(3): *The natural right $Aut(G)$-action on $LO(G)$ has at least $(m-1)!$ distinct orbits derived from isolated orderings if all $a_1, \ldots, a_m$ are distinct.*

As in the group G itself, there is a natural right $Aut(G)$-action on $LO(G)$. For a left ordering $<$ of G and $\theta \in Aut(G)$, we define the left ordering $< \cdot \theta$ by $h <\cdot\theta\ g$ if $h\theta < g\theta$. The right G-action on $LO(G)$ can be regarded as the restriction of the natural $Aut(G)$-action to the subgroup $Inn(G)$.

There is one symmetry which reduces the number of orbits: the involution defined by $x_i \mapsto x_i^{-1}$ $(i = 1, \ldots, m)$. This amounts to taking the opposite ordering. If all $a_1, \ldots, a_m$ are distinct, $\phi(a_i) \neq a_j^{\pm 1}$ for any $\phi \in Aut(G)$. Hence by a similar argument as (2), by looking at the minimal positive elements, we show that there are $(m-1)!$ distinct $Aut(G)$-orbit derived from isolated orderings.

Thus, the properties (2) and (3) show that the group G has quite a lot of essentially different isolated orderings.

3.2. Centerless group with isolated ordering. Next we consider the construction of the case z_H is non-central. First of all, let $G_{m,n} = \langle b, c \mid b^m = c^n \rangle$. By Example 3.1 (i), $G_{m,n}$ has an isolated left ordering $<_{m,n}$ which is defined by $\{bc^{1-n}, c\}$.

Let us consider a non-central element $bc = bc^{1-n} \cdot b^m$. Then it satisfies the inequality $b^m <_{m,n} bc <_{m,n} b^{2m}$. Since b^m is $<_{m,n}$-cofinal central element, this shows that bc is also $<_{m,n}$-positive cofinal.

$<_{m,n}$ is a (bc^{1-n})-right invariant ordering by Lemma 2.1, and $<_{m,n}$ is also a b^m-right invariant ordering since b^m is central. Thus, $<_{m,n}$ is a (bc)-right invariant ordering.

Thus, we can take the non-central element bc as an element z_H in Theorem 1.1 and we are able to apply the partially central cyclic amalgamation construction. Now we consider the group $H = H_{p,q,m,n} = \mathbb{Z} *_{\mathbb{Z}} G_{m,n} = \mathbb{Z} *_{\mathbb{Z}} (\mathbb{Z} *_{\mathbb{Z}} \mathbb{Z})$ defined by

$$\langle a, b, c \mid b^m = c^n, a^p = (bc)^q \rangle.$$

This group has an isolated left ordering $<_H$, defined by $\{a(bc)^{1-q}, bc^{1-n}, c\}$. Let us put $x = a(bc)^{1-q}$, $y = (bc)^{1-n}$, and $z = c$. Then the group $H_{p,q,m,n}$ is presented as

$$H_{p,q,m,n} = \langle x, y, z \mid (yz^{n-1})^m = z^n, (x(yz^n)^{q-1})^p = (yz^n)^q \rangle$$

by using the generator $\{x, y, z\}$.

Clearly, H has trivial center. This gives a first example of centerless group having isolated orderings. In fact, Theorem 1.1 will allow us to construct many examples of centerless group having isolated orderings.

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