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## Construction of Soliton-Cnoidal Wave Interaction Solution for the (2+1)-Dimensional Breaking Soliton Equation\*

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**Abstract** In this paper, the truncated Painlevé analysis and the consistent tanh expansion (CTE) method are developed for the (2+1)-dimensional breaking soliton equation. As a result, the soliton-cnoidal wave interaction solution of the equation is explicitly given, which is difficult to be found by other traditional methods. When the value of the Jacobi elliptic function modulus  $m = 1$ , the soliton-cnoidal wave interaction solution reduces back to the two-soliton solution. The method can also be extended to other types of nonlinear evolution equations in mathematical physics.

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**Key words:** (2+1)-dimensional breaking soliton equation, soliton-cnoidal wave interaction solution, CTE method, truncated Painlevé analysis

### 1 Introduction

The investigation of exact solutions for nonlinear evolution equations (NLEEs) arising from many science fields is of an important significance. The exact solutions of NLEEs can be constructed by many powerful methods, such as the inverse scattering transformation,<sup>[1]</sup> the Darboux transformations (DT),<sup>[2]</sup> the Bäcklund transformation (BT),<sup>[3]</sup> Hirota's bilinear method,<sup>[4]</sup> Lie group method,<sup>[5–6]</sup> Painlevé analysis<sup>[7]</sup> and various function expansion methods,<sup>[8–15]</sup> etc. Many kinds of nonlinear waves such as the solitons, cnoidal periodic waves, Painlevé waves are found by various effective methods. However, except for the interactions among solitons, finding the interactions among these nonlinear waves is very difficult. Recently, according to the result of the symmetry reductions with nonlocal symmetries,<sup>[16–18]</sup> Lou *et al.*<sup>[19–24]</sup> proposed the consistent tanh expansion (CTE) method, which is a simple but effective method to look for interaction solutions between solitons and other types of nonlinear excitations and possible new physical properties.

We consider the (2+1)-dimensional breaking soliton equation<sup>[25]</sup>

$$v_t + v_{xxy} - 4vv_y - 2v_x \partial_x^{-1} v_y = 0, \quad (1)$$

with  $\partial_x^{-1} = \int \cdot dx$ . Setting  $v = u_x$ , Eq. (1) becomes

$$u_{xt} + u_{xxx} - 4u_x u_{xy} - 2u_{xx} u_y = 0, \quad (2)$$

which describes the (2+1)-dimensional interaction of a Riemann wave propagating along the  $y$ -axis with a long wave along the  $x$ -axis.<sup>[26]</sup> For  $y = x$ , Eq. (1) is reduced to the KdV equation. The Painlevé property, the Lax pair, the Hamiltonian structure, and various exact solutions have

been studied.<sup>[27–36]</sup> In this paper, the analytic interaction solution between the soliton and the cnoidal periodic wave for the (2+1)-dimensional breaking soliton equation is shown by means of the truncated Painlevé analysis and CTE method.

The paper is arranged as follows. In Sec. 2, for the (2+1)-dimensional breaking equation, we derive explicit interaction solution between the soliton and the cnoidal periodic wave with the help of the truncated Painlevé analysis and the CTE method. In the last section, some conclusions and discussions are given.

### 2 Soliton-Cnoidal Wave Interaction Solution for the (2+1)-Dimensional Breaking Soliton Equation

It is well known that painlevé test is a systematic method to identify the integrability of NLEEs. Moreover, the painlevé test can be used to solve special solutions for NLEEs. Balancing the nonlinear and dispersive terms in Eq. (2), we have the truncated Painlevé expansion in the form

$$u = \frac{u_0}{\phi} + u_1, \quad (3)$$

where  $u_0$ ,  $u_1$  and  $\phi$  are arbitrary functions with respect to  $x$ ,  $y$ , and  $t$ . Substituting Eq. (3) into Eq. (2) and vanishing coefficients of the different powers  $1/\phi$ , we obtain

$$u_0 = -2\phi_x, \quad u_1 = \frac{1}{2} \frac{\phi_{xx}}{\phi_x} + \frac{1}{4} \int \frac{\phi_{xx}^2}{\phi_x^2} dx, \quad (4)$$

which yields the solution of Eq. (2) as follows:

$$u = -\frac{2\phi_x}{\phi} + \frac{1}{2} \frac{\phi_{xx}}{\phi_x} + \frac{1}{4} \int \frac{\phi_{xx}^2}{\phi_x^2} dx, \quad (5)$$

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with the associated compatibility condition of  $\phi$

$$\begin{aligned} \phi_{xt} + \phi_{xxy} - \frac{3\phi_{xx}\phi_{xy}}{\phi_x} - \frac{\phi_{xx}\phi_t}{\phi_x} \\ - \frac{\phi_{xy}\phi_{xxx}}{\phi_x} + \frac{3\phi_{xy}\phi_{xx}^2}{\phi_x^2} = 0. \end{aligned} \quad (6)$$

Therefore,

$$v = u_x = \frac{2\phi_x^2}{\phi^2} - \frac{2\phi_{xx}}{\phi} + \frac{1}{2} \frac{\phi_{xxx}}{\phi_x} - \frac{1}{4} \frac{\phi_{xx}^2}{\phi_x^2} \quad (7)$$

is a solution of Eq. (1).

Compatibility condition (6) can be written as Schwarzian form of Eq. (2)

$$C_x + K S_x + 2SK_x + K_{xxx} = 0, \quad (8)$$

where

$$C = \frac{\phi_t}{\phi_x}, \quad K = \frac{\phi_y}{\phi_x}, \quad S = \frac{\phi_{xxx}}{\phi_x} - \frac{3}{2} \frac{\phi_{xx}^2}{\phi_x^2}. \quad (9)$$

The above Schwarzian equation (8) is consistent with the result of Ref. [37]. Due to all the quantities  $C$ ,  $K$ , and  $S$  are Möbius transformation

$$\phi \rightarrow \frac{a + b\phi}{c + d\phi}, \quad (ad \neq bc), \quad (10)$$

invariants, the Schwarzian equation (8) is invariant under the Möbius transformation (10).

Using the following straightening transformation

$$\phi = \frac{2}{\tanh(w) - 1}, \quad (11)$$

the solution (7) can be rewritten as

$$\begin{aligned} v = 2w_x^2 \tanh^2(w) - 2w_{xx} \tanh(w) - w_x^2 \\ + \frac{1}{2} \frac{w_{xxx}}{w_x} - \frac{1}{4} \frac{w_{xx}^2}{w_x^2}, \end{aligned} \quad (12)$$

with the equivalent compatibility condition of  $w$

$$C_x + \mathcal{K} S_x + 2S\mathcal{K}_x + \mathcal{K}_{xxx} - 4w_x w_{xy} = 0, \quad (13)$$

where

$$\mathcal{C} = \frac{w_t}{w_x}, \quad \mathcal{K} = \frac{w_y}{w_x}, \quad \mathcal{S} = \frac{w_{xxx}}{w_x} - \frac{3}{2} \frac{w_{xx}^2}{w_x^2}. \quad (14)$$

It is seen that solution (12) can be considered as the generalization of the usual tanh function expansion method. Here we can obtain the solution (12) via the CTE method.<sup>[19–24]</sup> By using the leading order analysis for Eq. (2), we may take the following generalized truncated tanh function expansion

$$u = u_0 + u_1 \tanh(w), \quad (15)$$

where  $u_0$ ,  $u_1$ , and  $w$  are functions of  $(x, y, t)$  to be determined later. Substituting Eq. (15) into Eq. (2) and

setting zero the coefficients of  $\tanh^i(w)$ , we have six over-determined equations for only three undetermined functions. It is fortunate to find that these over-determined equations are consistent and possess the following solution

$$u_1 = -2w_x, \quad u_0 = \frac{1}{2} \frac{w_{xx}}{w_x} + \int w_x^2 dx + \frac{1}{4} \int \frac{w_{xx}^2}{w_x^2} dx, \quad (16)$$

then we deduce the same solution (12) of Eq. (1) with the compatibility condition (13).

Due to the fact that the single soliton solution of the Eq. (1) is only the straightened solution  $w = k_0x + l_0y + \omega_0t$  of Eq. (13), the interaction solutions between solitons and other nonlinear excitations of Eq. (1) can be constructed by solving Eq. (13). In order to obtain the interaction solution of Eq. (1), we consider  $w$  in the form

$$w = k_0x + l_0y + \omega_0t + g, \quad (17)$$

where  $g$  is a function of  $x$ ,  $y$  and  $t$ . In this study, we only discuss the solutions with the form

$$w = k_0x + l_0y + \omega_0t + W(X), \quad X = k_1x + l_1y + \omega_1t. \quad (18)$$

Substituting Eq. (18) into Eq. (13), we can find that  $W_1(X)$  satisfies

$$\begin{aligned} W_1(X)_X^2 = 4W_1(X)^4 + 2a_1W_1(X)^3 \\ + 2a_2W_1(X)^2 + a_3W_1(X) + a_4, \end{aligned} \quad (19)$$

with

$$W(X)_X = W_1(X),$$

$$a_1 = \frac{4k_0 - C_1k_1^3}{k_1},$$

$$a_2 = \frac{2k_0^2 - 3C_1k_0k_1^3 + C_2k_1^3}{k_1^2},$$

$$a_3 = \frac{k_1\omega_0 - k_0\omega_1 - 6C_1k_0^2k_1^3l_1 + 4C_2k_0k_1^3l_1}{k_1^3l_1},$$

$$a_4 = \frac{k_0(k_1\omega_0 - k_0\omega_1 - 2C_1k_0^2k_1^3l_1 + 2C_2k_0k_1^3l_1)}{k_1^4l_1}, \quad (20)$$

and  $C_1$ ,  $C_2$  are arbitrary constants.

It is known that the general solution of Eq. (19) can be written out in terms of Jacobi elliptic functions. To show more clearly of this kind of solution, two special cases are listed.

**Case 1** A special solution of Eq. (19) reads

$$W(X) = cE_\pi(\text{sn}(X, m), n, m), \quad (21)$$

which leads to the soliton-cnoidal wave interaction solution of Eq. (1):

$$\begin{aligned} v = \left[ 2k_0^2 - \frac{4ck_0k_1}{nS^2 - 1} + \frac{2c^2k_1^2}{(nS^2 - 1)^2} \right] \tanh^2(k_0x + l_0y + \omega_0t + cE_\pi(S, n, m)) \\ - \frac{4ck_1^2nSCD}{(nS^2 - 1)^2} \tanh(k_0x + l_0y + \omega_0t + cE_\pi(S, n, m)) - \frac{[ck_1 - k_0(nS^2 - 1)]^2}{(nS^2 - 1)^2} \\ - \frac{ck_1^3nD^2\{k_0[n^2S^4(2C^2 + 1) - 2nS^2 - 2C^2 + 1] + ck_1[-nS^2(C^2 + 1) - 2C^2 + 1]\}}{[ck_1 - k_0(nS^2 - 1)]^2(nS^2 - 1)^2} \\ + \frac{ck_1^3nm^2S^2C^2}{(nS^2 - 1)[ck_1 - k_0(nS^2 - 1)]}, \end{aligned} \quad (22)$$

where  $\{k_1, l_0, l_1, \omega_1, m, n\}$  are arbitrary constants,  $S \equiv \text{sn}(k_1x + l_1y + \omega_1t, m)$ ,  $C \equiv \text{cn}(k_1x + l_1y + \omega_1t, m)$ ,  $D \equiv \text{dn}(k_1x + l_1y + \omega_1t, m)$ , and

$$c^2 = -\frac{(n-1)(m^2-n)}{n}, \quad k_0 = -ck_1, \quad \omega_0 = 4ck_1^2l_1n - c\omega_1. \tag{23}$$

In solution (22),  $E_\pi(\zeta, n, m)$  is the third type of incomplete elliptic integral.

The solution given in Eq. (22) denotes the analytic interaction solution between the soliton and the cnoidal periodic wave. The simulation of soliton-cnoidal wave solution (22) is illustrated in Figs. 1 and 2 at two different choices of the arbitrary parameters. In Fig. 1, we plot the interaction solution between the solitary wave and the cnoidal wave when the value of the Jacobi elliptic function modulus  $m \neq 1$ . We can see that a soliton propagates on a cnoidal wave background instead of on the plane continuous wave background. This kind of solution can be easily applicable to the analysis of physically interesting processes. If setting the modulus  $m = 1$ , the soliton-cnoidal wave interaction solution reduces back to the two-soliton solution, whose interaction behaviors are displayed in Fig. 2.

**Case 2** Another special solution of Eq. (19) is given as

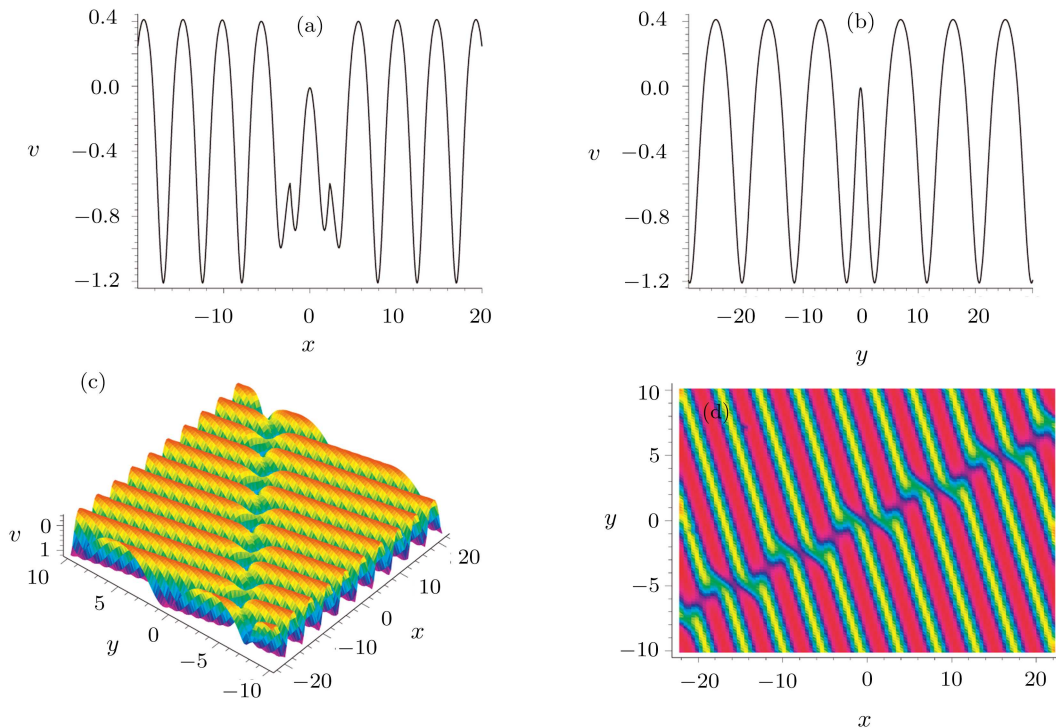
$$W(X) = A \text{arctanh}(\text{sn}(X, m)), \tag{24}$$

which yields the soliton-cnoidal wave interaction solution of Eq. (1):

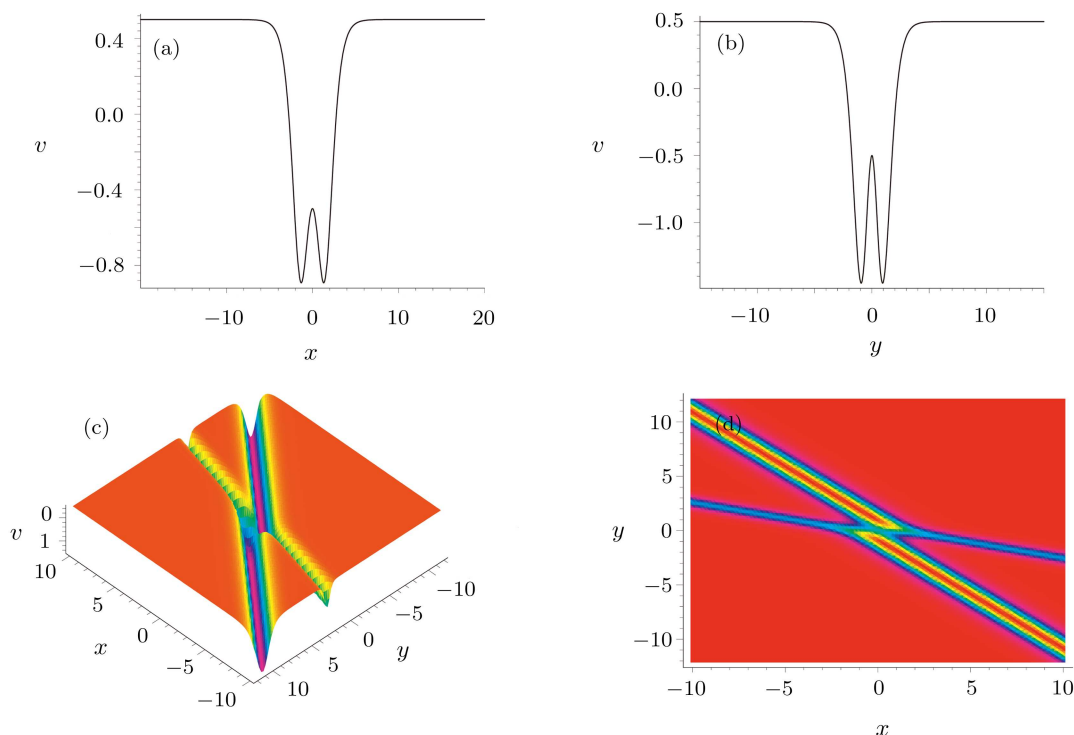
$$\begin{aligned} v = & \frac{k_1^2[S^4(m^4 + 6m^2 + 1) - 4S^2(m^2 + 1)(CD + 2) + 8(CD + 1)]}{2(1 - S^2 + CD)^2} \tanh^2\left(k_0x + l_0y + \omega_0t + \frac{1}{2}\text{arctanh}(S)\right) \\ & + \frac{k_1^2[S^3(1 - m^4) + 2S(m^2 - 1)(CD + 1)]}{(1 - S^2 + CD)^2} \tanh\left(k_0x + l_0y + \omega_0t + \frac{1}{2}\text{arctanh}(S)\right) \\ & - \frac{k_1^2[S^4(1 + 8m^2 - m^4) - S^2(m^2 + 3)(2CD + m^2 + 3) + 2(m^2 + 3)(CD + 1)]}{4(1 - S^2 + CD)^2}, \end{aligned} \tag{25}$$

where  $\{k_1, l_0, l_1, \omega_1, m\}$  are arbitrary constants,  $S \equiv \text{sn}(k_1x + l_1y + \omega_1t, m)$ ,  $C \equiv \text{cn}(k_1x + l_1y + \omega_1t, m)$ ,  $D \equiv \text{dn}(k_1x + l_1y + \omega_1t, m)$ , and

$$A = \frac{1}{2}, \quad k_0 = \frac{k_1}{2}, \quad \omega_0 = k_1^2l_1(m^2 - 1) + \frac{\omega_1}{2}. \tag{26}$$



**Fig. 1** Soliton-cnoidal wave interaction solution for Eq. (1) given by Eq. (22), with the parameters  $k_1 = 1$ ,  $l_0 = 1.2$ ,  $l_1 = 0.5$ ,  $\omega_1 = 0.8$ ,  $m = 0.9$ , and  $n = 0.5$ . (a) One-dimensional image at  $t = 0$  and  $y = 0$ ; (b) One-dimensional image at  $t = 0$  and  $x = 0$ ; (c) The two-dimensional perspective view of the corresponding solution.



**Fig. 2** Two-soliton solution for Eq. (1) given by Eq. (22), with the parameters  $k_1 = l_0 = l_1 = \omega_1 = 1$ ,  $m = 1$ , and  $n = 0.5$ . (a) One-dimensional image at  $t = 0$  and  $y = 0$ ; (b) One-dimensional image at  $t = 0$  and  $x = 0$ ; (c) The two-dimensional perspective view of the corresponding solution.

### 3 Summary and Discussion

In conclusion, the (2+1)-dimensional breaking soliton equation is investigated by using the truncated Painlevé analysis and the CTE method to find the soliton-cnoidal wave solution. This kind of solution can be easily applicable to the analysis of physically interesting processes. Despite the simplicity of the CTE method, it did provide us with the result which is quite nontrivial and difficult to

be obtained by other traditional approaches.

The method presented here could be applied to other kinds of integrable models, especially for supersymmetric models and discrete ones. The details on the CTE method and other methods to solve interaction solutions among different kinds of nonlinear waves will be discussed in our future research work.

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