Archives

Ref. TH.915

CONSTRUCTION OF THE SCATTERING AMPLITUDE FROM THE DIFFERENTIAL CROSS - SECTIONS

André Martin
CERN - Geneva

A B S T R A C T

We establish rigorously a sufficient condition for the existence of a scattering amplitude corresponding to a given angular distribution for scalar particles in the elastic region. The condition is

$$\operatorname{Max}\left[\frac{1}{4\pi}\int |F(13)|\,|F(23)|\,d\Omega_3 \Big/ |F(12)|\right] = \sin\mu < 1$$

We show that if $|\sin\mu| < 0.79$ the amplitude is unique, except for one obvious ambiguity. Further by examining the case of a finite, but arbitrarily large number of partial waves, we make it very likely that the solution is still unique for $0.79 < \sin\mu < 1$. We also discuss the number of solutions in other situations.

68/867/5-TH.915
8 July 1968

1. INTRODUCTION

The problem of constructing the scattering amplitude from the exact knowledge of the differential cross-section at a given energy has been considered by Crichton [1] and very recently by Newton [2]. It is a problem which is generally thought to be trivial for the case of scalar particles in the elastic region, but which is in fact very difficult. Since I have myself investigated this question and got some results which go somewhat beyond those of Ref. 2), I think that it is of some interest to report what I know on the subject. I shall try to make a self-contained presentation of the question. The problem is this : you take the simplest situation in the world, the scattering of two scalar particles at an energy which is below the first inelastic threshold. You measure with "infinite" accuracy the differential cross-section at this energy. The question is whether this fixes the amplitude if you take into account the unitarity condition. In fact there are two questions :

1) is a given angular distribution acceptable, i.e., does there exist a unitary scattering amplitude fitting this angular distribution ?

2) for a given angular distribution is there a unique scattering amplitude ?

About question 2) we know the obvious ambiguity $F(s,\cos\theta) \rightarrow -F^{*}(s,\cos\theta)$ which amounts to replace each phase shift by its opposite in the partial wave expansions. But apart from this ambiguity, Crichton [1] has exhibited a more subtle ambiguity in the relatively simple case of a distribution with S, P, D waves. On the other hand, Newton has obtained a sufficient condition for uniqueness which, as we shall see, is too restrictive. Concerning the question of existence, we get the same sufficient condition as Ref. 2), at the price of a much bigger effort, because, at least to us, the character "completely continuous" of the non-linear operator involved is not obvious.

Outside the rather obvious case where solutions do not exist because the optical theorem is violated, we construct other examples, without solution, where the optical inequality in the forward direction is satisfied.

Finally, we discuss whether the sufficient condition for the existence of some solution also ensures uniqueness. Although no definite conclusions can be drawn, we take as an indication of uniqueness the fact that if the number of partial waves is finite (but arbitrarily large) the solution can be shown to be unique under the sufficient condition for existence.

2. GENERAL CONSIDERATIONS

We normalize the scattering amplitude as

$$F(s, \cos\theta)$$

such that

$$\frac{d\sigma}{d\Omega} = \frac{1}{k^2} \left| F(s, \cos\theta) \right|^2 \qquad (1)$$

or

$$F(s, \cos\theta) = \sum (2\ell+1) e^{i\delta_\ell} \sin\delta_\ell \, P_\ell(\cos\theta) \qquad (2)$$

where the δ_ℓ's are real.

We shall use the notation $F(12)$ to denote $F(s, \cos\theta_{12})$ where θ_{12} is the angle between direction 1 and direction 2.

We shall also write

$$F(12) = |F(12)| \exp i\phi(12) \qquad (3)$$

Under these circumstances, in the elastic region, the unitarity condition can be written

$$\sin\phi(12)\,|F(12)| = \frac{1}{4\pi}\int d\Omega_3\,|F(13)|\,|F(23)| \times \cos\left[\phi(13)-\phi(23)\right] \tag{4}$$

For given $|F|$ this is a non-linear equation for ϕ.

For various reasons, which will appear later, we shall mainly study the case where

$$\underset{(1,2)}{\mathrm{Max}}\ \frac{\frac{1}{4\pi}\int |F(13)|\,|F(23)|\,d\Omega_3}{|F(12)|} = \sin\mu < 1 \tag{5}$$

Under these circumstances, we have $|\sin\phi| < \sin\mu$, i.e., $-\mu < \phi < +\mu$, if we impose $\mathrm{Re}\,F(\theta=0) > 0$.

Let us prove, however, that continuous solutions are necessarily such that under condition (5) ϕ stays between 0 and μ [if $\phi(0)$ is taken to be between $-\pi/2$ and $+\pi/2$].

Indeed, assume $\phi_{min} \leqslant \phi \leqslant \phi_{max}$, then if $(\phi_{max}-\phi_{min}) < \frac{\pi}{2}$, clearly from (4), $\sin\phi$ is positive. If, on the other hand, $\phi_{max}-\phi_{min} > \frac{\pi}{2}$, then

$$\sin\phi_{min} > \sin\mu\,\cos(\phi_{Max}-\phi_{min})$$

i.e.,

$$\mathrm{tg}\,\phi_{min} > \frac{\sin\mu\,\cos\phi_{Max}}{1-\sin\mu\,\sin\phi_{Max}} > 0$$

which is absurd.

3. THE QUESTION OF EXISTENCE

In a problem where you have to solve a non-linear equation like (4), it is useful to consider the non-linear operation

$$\phi' = O(\phi)$$

i.e.,

$$\sin \phi'(12) = \frac{\frac{1}{4\pi}\int |F(13)||F(23)| \cos\left(\phi(13)-\phi(23)\right) d\Omega_3}{|F(12)|} \tag{6}$$

and ϕ belongs to some function space. ϕ' is only well defined if one has a prescription to get it from $\sin\phi'$. This is the case if (5) is fulfilled. Then if ϕ is taken to be a continuous function $0 < \phi(\theta) < \mu$ $\phi'(\theta)$ belongs to the same space.

To show the existence of a solution to the non-linear equation, (4), one wants to apply the Leray-Schauder principle [3]. For this one has to show that the convex closure of $O(\phi)$ or of any iterate $O\,O\,O\,(\phi)$ is <u>compact</u>, and for this the simplest thing to do is to use the Ascoli-Arzela theorem, i.e., to show that the functions in the space are bounded (which is obvious) and equicontinuous (which is not obvious !).

In the present case we proceed as follows : to a given ϕ corresponds $F_\phi = |F| \exp i\,\phi$. Now

$$F_\phi = \sum (2\ell+1)\, e^{i\delta_\phi^\ell} \sin \delta_\phi^\ell \; P_\ell(\cos\theta) \tag{7}$$

and hence

$$\mathrm{Im}\, F_{\phi'} = \sum (2\ell+1) \; \sin^2 \delta_\phi^\ell \; P_\ell(\cos\theta) \tag{8}$$

Of course δ_ϕ^ℓ depends on ϕ, but

$$\sum (2\ell+1)\left(\sin \delta_\phi^\ell\right)^2 = \frac{1}{4\pi}\int d\Omega \, |F|^2 \tag{9}$$

depends only on $|F|$ and not on ϕ.

Now, from the inequality

$$\left|P_\ell(\cos\theta_1) - P_\ell(\cos\theta_2)\right| < C \frac{\sqrt{|\theta_1 - \theta_2|}}{\left[\sin\theta_1 \, \sin\theta_2\right]^{1/4}} \tag{10}$$

established in Appendix I, it follows that

$$\left|\operatorname{Im} F_{\phi'}(\theta_1) - \operatorname{Im} F_{\phi'}(\theta_2)\right| < C \frac{\sqrt{|\theta_1 - \theta_2|}}{\left(\sin\theta_1 \, \sin\theta_2\right)^{1/4}} \int |F|^2 \frac{d\Omega}{4\pi} \tag{11}$$

If we except $\theta = 0$ and $\theta = \pi$, condition (11) is precisely an equi-continuity condition. However, we must get rid of the singularities at the extremities. Assume now that $|F(\theta)|$ also satisfies a Hölder condition

$$\left|\,|F(\theta_1)| - |F(\theta_2)|\,\right| < C \sqrt{|\theta_1 - \theta_2|} \tag{12}$$

Then, under condition (5), one can also show that

$$\left|\phi'(\theta_1) - \phi'(\theta_2)\right| < C \frac{\sqrt{|\theta_1 - \theta_2|}}{\left(\sin\theta_1 \, \sin\theta_2\right)^{1/4}} \tag{13}$$

The space we shall consider now is the space of the ϕ' plus their convex closure, which we shall call $\hat{\phi}'$. Clearly since the initial space ϕ is convex, $\hat{\phi}'$ is a subspace of ϕ and $\phi'' = O(\hat{\phi}')$ is a subspace of $\hat{\phi}'$. Then we have

$$\operatorname{Im} F_{\phi''}(12) - \operatorname{Im} F_{\phi''}(1'2) = \frac{1}{4\pi} \int \left[F_{\hat{\phi}'}(13) - F_{\hat{\phi}'}(1'3) \right] F^{*}(32)\, d\Omega_3$$

and hence

$$\begin{aligned} &\left| \operatorname{Im} F_{\phi''}(12) - \operatorname{Im} F_{\phi''}(1'2) \right| \\ &\leqslant C \int \frac{\sqrt{|\theta_{13} - \theta_{1'3}|}\ \ |F(32)|}{\left[\sin\theta_{13} \sin\theta_{1'3}\right]^{1/4}}\, d\Omega_3 \\ &\leqslant C \sqrt{\theta_{11'}} \int \frac{|F(32)|\, d\Omega_3}{\left[\sin\theta_{13} \sin\theta_{1'3}\right]^{1/4}} \\ &\leqslant C \sqrt{\theta_{11'}} \end{aligned} \qquad (14)$$

if $|F(32)|$ is <u>bounded</u>.

Hence the space of the ϕ'' and $\hat{\phi}''$, its convex closure, is made of equicontinuous functions and is <u>compact</u>.

To apply the Leray-Schauder principle, we must also check that O is continuous. If we take as the norm of ϕ its maximum, this is rather trivial :

$$\left| \operatorname{Im} F_{\phi_1'} - \operatorname{Im} F_{\phi_2'} \right| =$$

$$\left| \frac{1}{4\pi} \int |F(13)|\,|F(23)| \begin{bmatrix} \cos[\phi_1(13)-\phi_1(23)] \\ -\cos[\phi_2(13)-\phi_2(23)] \end{bmatrix} d\Omega_3 \right|$$

$$\leq \frac{1}{\pi} \left(\int |F|\,|F| \right) \times \operatorname{Sup} |\phi_1 - \phi_2| \tag{15}$$

Hence if

$$\operatorname{Sup} \frac{\frac{1}{4\pi}\int |F(13)|\,|F(23)|\, d\Omega_3}{|F(12)|} < 1$$

there exists at least one unitary amplitude with modulus $|F|$.

Though this condition is only a sufficient condition for the existence of a solution, it is clear that the numerical constant, 1, cannot be changed, because in the forward direction condition

$$\frac{\frac{1}{4\pi}\int |F(13)|^2 d\Omega_3}{|F(11)|} < 1 \tag{16}$$

is <u>necessary</u>.

As pointed out in Ref. 2), an angular distribution which violates this condition is unacceptable. Let us now give an example where (16) is satisfied but for which there is <u>no</u> solution. Take $|F(\theta)| = \lambda f(\theta)$ where $f(\pi) = 0$ and $df/d\cos\theta$ is finite and non-zero around $\theta = \pi$. Then for λ small enough there is no acceptable unitary amplitude. We have

$$|\sin\phi(12)| < \lambda \frac{\frac{1}{4\pi}\int f(13) f(23)\, d\Omega_3}{f(12)}$$

For $\cos(12) > -1 + \lambda^{\frac{1}{2}}$ we have

$$f(12) > C\,\lambda^{1/2}$$

Hence, for $\cos(12) > -1 + \lambda^{\frac{1}{2}}$ we have

$$|\sin\phi(12)| < C\,\lambda^{1/2} \quad \text{or} \quad |\phi(12)| < C\,\lambda^{1/2}$$

Thus we have

$$\mathrm{Im}\,F(\theta = \pi) \geq \lambda \int_{-1+\lambda^{1/2}}^{1-\lambda^{1/2}} f(\cos\theta)\, f(-\cos\theta)\, d\cos\theta\, [1-\lambda]$$
$$- 2\lambda \int_{1-\lambda^{1/2}}^{1} f(\cos\theta)\, f(-\cos\theta)\, d\cos\theta$$

For λ small enough, the right-hand side is positive and hence $|F|$ cannot vanish in the backward direction, which is a contradiction.

4. THE UNIQUENESS PROBLEM

Let us state first the best result obtained in this direction which is that, except for the obvious ambiguity $F \to -F^*$, the solution is unique for $\sin\mu < 0.79$ [in Ref. 2) one gets $\sin\mu < 1/\sqrt{5}$]. It is, however, quite clear that this can be improved and we shall give later a heuristic argument to support our belief that the solution is unique as long as $\sin\mu < 1$.

To start, let us establish the uniqueness for $\sin\mu < 1/\sqrt{2}$ which is easy. Let F and G be two solutions. We have

$$\mathrm{Im}\,F(12) - \mathrm{Im}\,G(12)$$

$$= \frac{1}{4\pi}\int\left[\begin{array}{l}(\mathrm{Re}\,F - \mathrm{Re}\,G)_{13}(\mathrm{Re}\,F + \mathrm{Re}\,G)_{23} \\ +(\mathrm{Im}\,F - \mathrm{Im}\,G)_{13}(\mathrm{Im}\,F + \mathrm{Im}\,G)_{23}\end{array}\right] d\Omega_3 \qquad (17)$$

We also have from $|F| = |G|$

$$0 = (\mathrm{Im}\,F + \mathrm{Im}\,G)(\mathrm{Im}\,F - \mathrm{Im}\,G) + (\mathrm{Re}\,F + \mathrm{Re}\,G)(\mathrm{Re}\,F - \mathrm{Re}\,G) \qquad (18)$$

Hence

$$(\mathrm{Im}\,F - \mathrm{Im}\,G)_{12} =$$

$$\frac{1}{4\pi}\int(\mathrm{Im}\,F - \mathrm{Im}\,G)_{13}\left[\left(\frac{\mathrm{Im}\,F + \mathrm{Im}\,G}{\mathrm{Re}\,F + \mathrm{Re}\,G}\right)_{23} - \left(\frac{\mathrm{Im}\,F + \mathrm{Im}\,G}{\mathrm{Re}\,F + \mathrm{Re}\,G}\right)_{13}\right] \times (\mathrm{Re}\,F + \mathrm{Re}\,G)_{23}\, d\Omega_3 \qquad (19)$$

Notice that $\mathrm{Re}\,F + \mathrm{Re}\,G$ is always positive. Hence

$$|\mathrm{Im}\,F - \mathrm{Im}\,G| \leq \mathrm{Max}|\mathrm{Im}\,F - \mathrm{Im}\,G|\ \mathrm{tg}\,\mu\ \frac{1}{4\pi}\int(\mathrm{Re}\,F + \mathrm{Re}\,G)\,d\Omega$$

but $\frac{1}{4\pi}\int \mathrm{Re}\,F\,d\Omega = \mathrm{Re}\,f_0$, is the S wave real part which has to be less than $\frac{1}{2}$ in modulus. Thus

$$|\mathrm{Im}\,F - \mathrm{Im}\,G| < \mathrm{Max}|\mathrm{Im}\,F - \mathrm{Im}\,G|\ \mathrm{tg}\,\mu \qquad (20)$$

So for $\operatorname{tg}\mu < 1$ the solution is unique. This method has the inconvenience that it does not prove that the iteration converges because we already assume that we have a true solution.

Let us now try to go further. But first we shall study the problem in differential form, i.e., we shall ask ourselves whether there can be two solutions very close to one another. In mathematical language we look at the Fréchet derivative of the non-linear operator. Let $\Delta\phi(\theta)$ be the phase difference between the two solutions. We have

$$|F(12)| \cos\phi(12)\,\Delta\phi(12) = \frac{-1}{4\pi}\int |F(13)|\,|F(23)| \sin[\phi(13)-\phi(23)] \times [\Delta\phi(13)-\Delta\phi(23)]\, d\Omega_3 \tag{21}$$

and

$$|F(12)| \sin\phi(12) = \frac{1}{4\pi}\int |F(13)|\,|F(23)| \cos[\phi(13)-\phi(23)]\, d\Omega_3 \tag{22}$$

Now we shall try to exploit $\sin^2\chi + \cos^2\chi = 1$ and for this purpose we replace (21) and (22) by Schwarz inequalities :

$$|F(12)|^2 [\cos\phi(12)]^2 [\Delta\phi(12)]^2 \leq \frac{1}{4\pi}\int |F(13)|\,|F(23)|\,|\Delta\phi(13)-\Delta\phi(23)|^2\, d\Omega_3 \times \frac{1}{4\pi}\int |F(13)|\,|F(23)| \sin^2(\phi(13)-\phi(23))\, d\Omega_3 \tag{23}$$

$$|F(12)|^2 \left[\sin\phi(12)\right]^2$$

$$\leqslant \frac{1}{4\pi}\int |F(13)|\,|F(23)|\,d\Omega_3 \tag{24}$$

$$\times \frac{1}{4\pi}\int |F(13)|\,|F(23)|\cos^2\left[\phi(13)-\phi(23)\right]d\Omega_3$$

and hence

$$|F(12)|^2\cos^2\phi(12)\,\left(\Delta\phi(12)\right)^2$$

$$\leqslant \frac{1}{4\pi}\int |F(13)|\,|F(23)|\,\left|\Delta\phi(13)-\Delta\phi(23)\right|^2 d\Omega_3 \;\times$$

$$\times\left[\frac{1}{4\pi}\int |F(13)|\,|F(23)|\,d\Omega_3 - \frac{|F(12)|^2\sin^2\phi(12)}{\frac{1}{4\pi}\int |F(13)|\,|F(23)|\,d\Omega_3}\right]$$

or

$$|F(12)|\,|\Delta\phi(12)|^2 \leqslant$$

$$\frac{1}{4\pi}\int |F(13)|\,|F(23)|\,\left|\Delta\phi(13)-\Delta\phi(23)\right|^2 d\Omega_3 \;\times$$

$$\times\frac{\left[\frac{1}{4\pi}\int |F(13)|\,|F(23)|\,d\Omega_3\right]^2 - |F(12)|^2\left[\sin\phi(12)\right]^2}{|F(12)|\,\frac{1}{4\pi}\int |F(13)|\,|F(23)|\,d\Omega_3\,\cos^2\phi(12)} \tag{25}$$

We notice that in (25) the last fraction is a homographic function of $\sin^2\phi(12)$. Now we have

$$\cos\mu \, \frac{\frac{1}{4\pi}\int |F(13)|\,|F(23)|\,d\Omega_3}{|F(12)|} < \sin\phi(12) < \frac{\frac{1}{4\pi}\int |F(13)|\,|F(23)|\,d\Omega_3}{|F(12)|} \qquad (26)$$

and it is easy to see that it is the minimum of $\sin\phi$ which makes the right-hand side of (25) maximum. Hence, using also (5)

$$|F(12)|\,|\Delta\phi(12)|^2 \leqslant \frac{1}{4\pi}\int |F(13)|\,|F(23)|\,|\Delta\phi(13) - \Delta\phi(23)|^2 d\Omega_3 \times \qquad (27)$$

$$\times \frac{(\sin\mu)^3}{1 - (\cos\mu\,\sin\mu)^2}$$

Now integrate the right-hand side and left-hand side of (27) with $d\Omega_1/4\pi$ and $d\Omega_2/4\pi$, we get :

$$\frac{1}{4\pi}\int |F(12)|\,|\Delta\phi(12)|^2 d\Omega_1 \leqslant \left[\frac{1}{4\pi}\int |F(13)|\,|\Delta\phi(13)|^2 d\Omega_3 \, \frac{1}{4\pi}\int |F(23)|\,d\Omega_2 - \left(\frac{1}{4\pi}\int F(13)\,\Delta\phi(13)\,d\Omega_1\right)^2 \right] \qquad (28)$$

$$\times \frac{2\,(\sin\mu)^3}{1 - \cos^2\mu\,\sin^2\mu}$$

Therefore we get

$$1 \leqslant 2 \frac{(\sin\mu)^3}{1-\sin^2\mu\cos^2\mu}\left[\frac{1}{4\pi}\int |F|\, d\Omega\right]$$

Now it is easy to see that since

$$4\pi |F(12)| > \sin\mu \int |F(13)||F(23)|\, d\Omega_3$$

we have

$$\frac{1}{4\pi}\int |F|\, d\Omega < \sin\mu$$

Hence, a condition to have isolated solutions is

$$\frac{2(\sin\mu)^4}{1-\sin^2\mu\cos^2\mu} < 1 \tag{29}$$

which corresponds to

$$\sin\mu < 0{,}79 \tag{30}$$

What matters in this improvement with respect to (20) is that we go beyond the threshold $\mathrm{Im}\, F/\mathrm{Re}\, F \leqslant 1$. Then the imaginary part of the S wave is not any more uniquely determined by its real part and we might have expected ambiguities.

We should also notice that $1/4\pi \int |F|\, d\Omega$ is strictly <u>less</u> than $\sin\mu$ except if we have a pure S wave scattering. So the condition to have isolated solutions

$$\frac{2(\sin\mu)^3 \dfrac{\int |F|\, d\Omega}{4\pi}}{1-\cos^2\mu\sin^2\mu} < 1 \tag{31}$$

is slightly better than (29). In fact, conditions (29) or (31) can be shown to be conditions for the uniqueness of the solution. This is somewhat more painful to do because of notations : if $|F|e^{i\phi}$ and $|F|e^{i\psi}$ are solutions, we have

$$|F(12)| \sin\frac{\phi(12)+\psi(12)}{2}\cos\frac{\phi(12)-\psi(12)}{2}$$

$$= \frac{1}{4\pi}\int |F(13)|\,|F(23)| \cos\left[\frac{\phi(13)-\phi(32)+\psi(13)-\psi(23)}{2}\right] \tag{32}$$

$$\times \cos\left[\frac{\phi(13)-\phi(23)-\psi(13)+\psi(23)}{2}\right] d\Omega_3$$

and

$$|F(12)| \cos\frac{\phi(12)+\psi(12)}{2}\sin\frac{\phi(12)-\psi(12)}{2}$$

$$= \frac{-1}{4\pi}\int |F(13)|\,|F(23)| \sin\frac{\phi(13)-\phi(23)-\psi(13)+\psi(23)}{2} \times$$

$$\times \sin\frac{\phi(13)-\phi(23)+\psi(13)-\psi(23)}{2}\, d\Omega_3$$

Using the Schwarz inequality as before, we get

$$|F(12)|^2 \cos^2\left[\frac{\phi(12)+\psi(12)}{2}\right] \sin^2\left[\frac{\phi(12)-\psi(12)}{2}\right]$$

$$\leq \frac{1}{4\pi}\int |F(13)||F(23)| \sin^2\left[\frac{\phi(13)-\psi(13)-\phi(23)+\psi(23)}{2}\right] d\Omega_3$$

$$\times \left[\frac{\frac{1}{4\pi}\int |F(13)||F(23)| d\Omega_3 - |F(12)|^2 \sin^2\left[\frac{\phi(12)+\psi(12)}{2}\right] \cos^2\left[\frac{\phi(12)-\psi(12)}{2}\right]}{\frac{1}{4\pi}\int |F(12)||F(23)| \cos^2\left(\frac{\phi(13)-\psi(13)-\phi(23)+\psi(23)}{2}\right) d\Omega_3}\right]$$

It can be shown (rather painfully) that if $\sin \mu < (\sqrt{3}/2)$ then the maximum of

$$\frac{\left[\frac{1}{4\pi}\int |F(13)||F(23)| d\Omega_3\right]^2 - |F(12)|^2 \sin^2\left(\frac{\phi(12)+\psi(12)}{2}\right) \cos^2\left(\frac{\phi(12)-\psi(12)}{2}\right)}{|F(12)| \cos^2\left(\frac{\phi(12)+\psi(12)}{2}\right)}$$

is obtained by giving to ϕ and ψ their minimum value. This is the case under condition (29). Then one has

$$|F(12)| \sin^2 \frac{\phi(12)-\psi(12)}{2}$$

$$\leq \frac{1}{4\pi}\int |F(13)||F(23)| \sin^2 \frac{\phi(13)-\psi(13)-\phi(23)+\psi(23)}{2} d\Omega_3 \qquad (33)$$

$$\times \frac{(\sin\mu)^3}{1-\sin^2\mu \cos^2\mu}$$

and it is then easy to get condition (29) for uniqueness. Condition (31), on the other hand, applies only for $\sin\mu < (\sqrt{3}/2)$.

The question now remains : what happens for $0.79 < \sin\mu < 1$. Let us first notice that if two solutions are such that they never intersect (except for $\theta = 0$), i.e., $\Delta\phi > 0$. Then from (33) we get

$$\sin^2 \frac{\Delta\phi_{Max}}{2} \leq \frac{\sin^4\mu \, \sin^2 \frac{\Delta\phi_{Max}}{2}}{1 - \sin^2\mu \cos^2\mu}$$

so that $\Delta\phi = 0$ is the only solution. Also, if we follow the chain from (23) to (28), we see that a series of majorizations has been made, some of them being extremely drastic (such as applying the Schwarz inequality to $\int AB$ where A and B are very different). So my own conviction [as opposed to the hypothesis made in Ref. 2] is that as long as $\sin\mu < 1$ the solution is unique. This view will be supported by examining the case where the number of partial waves is finite.

On the other hand, if

$$\mathrm{Max} \, \frac{\frac{1}{4\pi}\int |F(13)|\,|F(23)|\,d\Omega_3}{|F(12)|}$$

is larger than unity, then there is no reason to have a unique solution and in fact an explicit counter-example has been given by Crichton [1], with only S, P and D waves. We shall study the question of these ambiguities in the following Section and restrict ourselves to the case of a finite number of partial waves.

Before coming to this, we wish to comment a bit more on the case $\sin\mu < 1$. First of all, it is obvious that in this case one has $\mathrm{Re}\,F > 0$, $\mathrm{Im}\,F > 0$ at all angles [if the convention $\mathrm{Re}\,F(\theta = 0) > 0$ is made]. But in addition to that, we have

$$\mathrm{Re}\, f_0 \pm \mathrm{Re}\, f_\ell = \frac{1}{2}\int_{-1}^{+1} \mathrm{Re}\, F(\cos\theta)\left[1 \pm P_\ell(\cos\theta)\right] d\cos\theta$$

Hence $\mathrm{Re}\, f_0 > |\mathrm{Re}\, f_\ell|$ for $\ell > 0$. Similarly

$$\mathrm{Im}\, f_0 \pm \mathrm{Im}\, f_\ell = \frac{1}{2}\int_{-1}^{+1} \mathrm{Im}\, F(\cos\theta)\left[1 \pm P_\ell(\cos\theta)\right] d\cos\theta$$

and hence $\mathrm{Im}\, f_0 > \mathrm{Im}\, f_\ell \geq 0$ for $\ell > 0$. It is easy to see on the Argand diagram of f_ℓ (the circle $|f_\ell - \frac{i}{2}| = \frac{1}{2}$) that this imposes $\mathrm{Im}\, f_\ell < \frac{1}{2}$, $|\mathrm{Re}\, f_\ell| \leq \frac{1}{2}$ or

$$|\delta_\ell| < \frac{\pi}{4} \tag{34}$$

for $\ell > 0$. Therefore only the S wave is allowed to resonate. So condition (5) prevents any resonance.

Notice that (34) can be improved for <u>even</u> waves. Then $1 + 2P_\ell(\cos\theta) > 0$ and hence for <u>ℓ even</u>, $\mathrm{Re}\, f_\ell > -\frac{1}{4}$, so

$$-\frac{\pi}{6} < \delta_\ell < \frac{\pi}{4} \tag{35}$$

However, <u>this does not prevent the total cross-section from being large</u>. One can build examples with very large total cross-sections satisfying $\mathrm{Re}\, F > 0$, $\mathrm{Im}\, F > 0$ at all angles, for instance

$$\mathrm{Re}\, F = \lambda \sum (2\ell+1) x^\ell P_\ell(\cos\theta)$$

with λ very small and x very close to one.

4. THE CASE OF A FINITE NUMBER OF PARTIAL WAVES

The situation in which the scattering amplitude is built of a finite number of partial waves is of course exceptional, more precisely of zero measure among all scattering amplitudes, and also, from a physical point of view inconsistent with all famous principles, in particular crossing symmetry. Notice also that even if the differential cross-section is given by a polynomial in $\cos\theta$, it is most of the time impossible to fit it with a finite number of partial waves, because if $2L$ is the degree of the polynomial in $\cos\theta$ we would have L partial waves, i.e., L unknown for $2L$ algebraic equations.

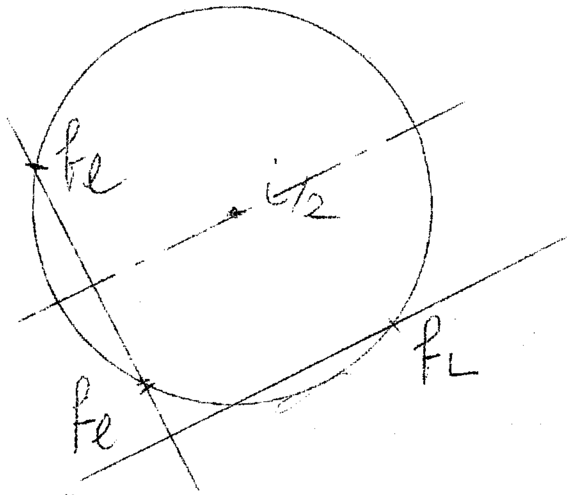
However, in spite of all this, we shall study this case because, if you put no restriction on the maximum angular momentum, the set of scattering amplitudes with a finite number of partial waves is dense in the set of all scattering amplitudes. Any reasonable scattering amplitude can be approximated arbitrarily well by a finite number of partial wave amplitudes.

Now when the number of partial waves is finite, the construction procedure the most simple conceptually (if not practically) is to start from the highest wave : we have

$$k^2 \frac{d\sigma}{d\Omega} = C_0 + C_1 P_1 + \dots + C_L P_L + \dots\dots + C_{2L} P_{2L}$$

$$F = f_0 + 3 f_1 P_1 + \dots\dots\dots + (2L+1) f_L P_L$$

and obviously c_{2L} is proportional to $|f_L|^2$. One can then for instance choose $\mathrm{Re}\, f_L > 0$. The next coefficient c_{2L-1} is proportional to $\mathrm{Re}\, f_L^* f_{L-1}$; so it gives us the projection of f_{L-1} on the direction of f_L. We have therefore an ambiguity (see Figure). In fact, once a choice



is made, one gets $\mathrm{Re}\, f_L^* f_{L-2}$ and so on. So, knowing $c_{L+1}, c_{L+2}, \ldots, c_{2L}$, we have a priori 2^L solutions. Then $c_0, c_1, \ldots, c_L$ give additional constraints which in most cases cannot be satisfied and in the remaining cases hopefully pick a unique solution among the 2^L candidates, a fact which is not always true as shown by Crichton.

What we want to show is that if condition (5) is fulfilled and if there is a maximum L, however large it may be, the solution is unique. In fact, the only conditions we shall use are $\mathrm{Im}\, f_{\ell \neq 0} < \frac{1}{2}$ and $\mathrm{Re}\, F(\cos\theta) > 0$ for $-1 < \cos\theta < +1$. Then what we shall prove is that for any $\ell > 0$

$$|\delta_\ell| < \frac{1}{2}\left(\frac{\pi}{2} - |\delta_L|\right) \tag{36}$$

where δ_L is the phase shift corresponding to the highest wave. Then the ambiguity described in the Figure cannot occur and $\mathrm{Re}(f_L^* f_\ell)$ fixes entirely f_ℓ.

We start from

$$\mathrm{Re}\, f_0 \pm \mathrm{Re}\, f_\ell = \frac{1}{2}\int_{-1}^{+1} \left[1 \pm P_\ell(\cos\theta)\right] \mathrm{Re}\, F(\cos\theta)\, d\cos\theta \tag{37}$$

$\mathrm{Re}\, F(\cos\theta)$ is by assumption a polynomial of degree L which does not vanish in -1 $+1$. Notice in particular that

$$\mathrm{Re}\, F(\cos\theta) = K(L)(2L+1)\, \mathrm{Re} f_L (\cos\theta)^L + \ldots\ldots \tag{38}$$

where $K(L)$ is the coefficient of $(\cos\theta)^L$ in $P_L(\cos\theta)$:

$$K(L) = \frac{(2L)!}{2^L (L!)^2} \tag{39}$$

Now let us notice that for $\ell \geqslant 1$

$$1 \pm P_\ell(\cos\theta) \geqslant \frac{1-\cos^2\theta}{2} \tag{40}$$

(see Appendix II).

Then, taking into account (38), we are reduced to the problem of finding the minimum of $\mathrm{Re}\, f_0 \pm \mathrm{Re}\, f_\ell$ knowing that

$$\mathrm{Re} f_0 \pm \mathrm{Re} f_\ell > K(L) \left| \mathrm{Re} f_L \right| (2L+1) \frac{1}{4} \int_{-1}^{+1} (1-x^2)\, \Pi_L(x)\, dx$$

where $\Pi_L(x)$ is a polynomial of degree L, positive for $-1 < x < +1$ and such that $|\Pi_L(x)| \sim x^L$ for $x \to \infty$.

The answer to this question is easy to find. It is shown in Appendix III that the minimum is obtained - for L even - by taking

$$\Pi_L = C \left[P^{1,1}_{\frac{L}{2}}(x) \right]^2 \equiv C' \left[P'_{\frac{L}{2}+1}(x) \right]^2$$

or

$$C''(1-x^2) \left[P^{2,2}_{\frac{L}{2}-1}(x) \right]^2 \equiv C'''(1-x^2) \left[P''_{\frac{L}{2}+1}(x) \right]^2$$

for L odd

$$\Pi_L = C(1+x)\left[P^{1,2}_{\frac{L-1}{2}}(x)\right]^2 \quad \text{or} \quad C(1-x)\left[P^{2,1}_{\frac{L-1}{2}}(x)\right]^2$$

where the $P_n^{\alpha,\beta}$ are Jacobi polynomials. One finds in this way

$$\mathrm{Re}\, f_0 \pm \mathrm{Re}\, f_\ell > c(L)\left|\mathrm{Re}\, f_L\right| \tag{41}$$

where $c(L)$ is given in Appendix III, and we only need to know that

$$c(L) \geqslant C(2) = \frac{1}{2} \quad \text{for} \quad L \geqslant 2$$

and hence

$$\frac{1}{2} \geqslant \mathrm{Re}\, f_0 \geqslant \left|\mathrm{Re}\, f_\ell\right| + \frac{1}{2}\left|\mathrm{Re}\, f_L\right| \tag{42}$$

Now condition (36), which ensures the uniqueness of the solution once the sign of $\mathrm{Re}\, f_L$ is fixed, is equivalent to

$$\left|\sin 2\delta_\ell\right| < \cos\delta_L \tag{43}$$

While (42) gives

$$\left|\sin 2\delta_\ell\right| < 1 - \left|\sin\delta_L\right|\left|\cos\delta_L\right|$$

this quantity can be shown to be less than $\cos\delta_L$ at least as long as $|\delta_L| < \frac{\pi}{4}$, which as we know is necessarily the case if $\mathrm{Re}\, F > 0$ and $\mathrm{Im}\, F > 0$ in all the range of integration.

The conclusion is : for $L \geqslant 2$, $\ell \geqslant 1$, once a choice is made for the sign of $\mathrm{Re}\, f_L$, f_ℓ is completely fixed, without ambiguity. As for f_0, it is clear that $\mathrm{Im}\, f_0$ is given by $\mathrm{Im}\, F(\theta=0) - \sum_1^L (2\ell+1)\, \mathrm{Im}\, f_\ell$. Now the knowledge of $\mathrm{Re}(f_0^* f_L)$ gives two values of f_0 which do not have opposite real part, except if $\delta_L = \pi/2$, which is a priori excluded. Only one value of $\mathrm{Re}\, f_0$ will therefore fit with the $\mathrm{Im}\, f_0$ obtained by subtraction.

We believe that this result is a strong indication that if condition (5) is fulfilled (actually $\mathrm{Re}\, F > 0$ and $\mathrm{Im}\, F > 0$ are sufficient conditions), the solution is unique. Indeed, the maximum number of partial waves disappears from our result and we can approximate a scattering amplitude arbitrarily well with a finite number of partial waves. As it is stated, this result is not rigorous but it could be that somebody with sufficient mathematical knowledge can make it rigorous.

To get ambiguities, $\mathrm{Re}\, F$ or $\mathrm{Im}\, F$ or both must vanish somewhere in the physical region, and this is precisely what happens with the examples given by Crichton in Ref. 1) : the two sets

$$\delta_0 = -23^\circ 20' \qquad \delta_1 = -43^\circ 27' \qquad \delta_2 = 20^\circ$$

$$\delta_0' = 98^\circ 50 \qquad \delta_1 = -26^\circ 33' \qquad \delta_2 = 20^\circ$$

give the same angular distribution. One can check that both amplitudes have the structures :

$$\frac{2}{15} e^{i\delta_2} \sin\delta_2 \left[\cos\theta - z_1\right]\left[\cos\theta - \frac{4}{5} + i\,\frac{\cot\delta_2}{5}\right]$$

$$\frac{2}{15} e^{i\delta_2} \sin\delta_2 \left[\cos\theta - z_1^*\right]\left[\cos\theta - \frac{4}{5} + i\,\frac{\cot\delta_2}{5}\right]$$

where $\mathrm{Re}\, z_1$ and $|z_1|^2$ are given as functions of δ_2. This is the most general form of the ambiguity for $L = 2$.

One might want to have an upper limit for the number of solutions which would not depend on L. We shall not try to get the best possible condition but just show that simple conditions exist. For instance, suppose that the integral

$$I = \frac{1}{2}\int_{-1}^{+1} \frac{|F(\cos\theta)|}{\sqrt{\sin\theta}}\, d\cos\theta \tag{44}$$

exists, which is obvious if F remains finite. Then we have

$$|f_\ell| < \sqrt{\frac{2}{\pi(\ell+\frac{1}{2})}}\; I \tag{45}$$

$$\left(\qquad |P_\ell(\cos\theta)| < \sqrt{\frac{2}{\pi(\ell+\frac{1}{2})\sin\theta}} \right)$$

and for

$$\ell > L_0 = \frac{8}{\pi} I^2 - \frac{1}{2}$$

$$|f_\ell| < \frac{1}{2} \qquad |\delta_\ell| < \frac{\pi}{6}$$

So two things can happen : either $L < L_0$ and then we have 2^L solutions, or $L > L_0$ and then $|\delta_L| < \frac{\pi}{6}$ and as long as $\delta_\ell < \frac{\pi}{6}$, $\mathrm{Re}\, f_\ell f_L^*$ fixes entirely f_ℓ. So we have at most

$$2^{\frac{8}{\pi} I^2 - \frac{1}{2}}$$

solutions, which is of course a possibly very large number, but which is independent of L. Again, I have a tendency to believe that this result holds even if the number of partial waves is infinite.

5. CONCLUSIONS

We have seen that neither the uniqueness nor the existence of the solution is easy. We have made it very likely that the sufficient condition (5) for the existence of a solution is at the same time a condition for uniqueness. The problem would be a nice, well-defined, mathematical problem for a good applied mathematician. Unfortunately this race is becoming more and more rare in the Western hemisphere. From the point of view of physics, one can wonder whether we are not losing our time since : continuity with energy, Coulomb interference, and inaccuracy of experimental data change considerably the problem. Let me point out, however, that in a case of uniqueness like that of condition (29), one can control the uncertainties, i.e., give explicit upper bounds for the uncertainty on ϕ corresponding to the uncertainty on $|F|$.

6. ACKNOWLEDGEMENTS

The author is indebted to Mr. Léon Motchane and Professor Louis Michel for the hospitality extended to him in 1965 at the "Institut des hautes Etudes scientifiques", when this investigation was started. Thanks are due to K. Chadan, F. Lurçat, M. Good, D. Ruelle and V. Glaser for discussions. The author thanks Professor R.G. Newton for sending him a preprint of his work, and Professor Calogero for pointing out to him the work of Crichton.

APPENDIX I

The question is to find a bound on $|P_\ell(\cos\theta_1)-P_\ell(\cos\theta_2)|$ which vanishes when $\theta_1-\theta_2\to 0$ and is at the same time independent of ℓ.

1) We know that [4]

$$|P_\ell(\cos\theta)| < \sqrt{\frac{2}{\pi(\ell+\frac{1}{2})}}\,\frac{1}{\sqrt{\sin\theta}}$$

Hence

$$|P_\ell(\cos\theta_1)-P_\ell(\cos\theta_2)| < \sqrt{\frac{2}{\pi(\ell+\frac{1}{2})}}\left[\frac{1}{\sqrt{\sin\theta_1}}+\frac{1}{\sqrt{\sin\theta_2}}\right] \quad \text{(AI.1)}$$

2) We also have

$$P_\ell(\cos\theta_1)-P_\ell(\cos\theta_2)=\int_{\cos\theta_1}^{\cos\theta_2} P_\ell'(\cos\theta)\,d\cos\theta$$

and since, according to Szegö [5]

$$|P_\ell'(\cos\theta)| < C\,\frac{\sqrt{\ell+\frac{1}{2}}}{(\sin\theta)^{3/2}}$$

we get for $0<\theta_1,\ \theta_2<\frac{2\pi}{3}$, using $0.7\,\theta<\sin\theta<\theta$

$$|P_\ell(\cos\theta_1)-P_\ell(\cos\theta_2)| < C'\sqrt{\ell+\frac{1}{2}}\,\left|\theta_1^{1/2}-\theta_2^{1/2}\right| \quad \text{(AI.2)}$$

Hence multiplying (AI.1) by (AI.2), we get for

$$\left| P_\ell(\cos\theta_1) - P_\ell(\cos\theta_2) \right| < \text{Const} \frac{\sqrt{|\theta_1 - \theta_2|}}{\theta_1^{1/4}\ \theta_2^{1/4}} \qquad \text{(AI.3)}$$

and it is quite clear that if we replace (AI.3) by

$$\left| P_\ell(\cos\theta_1) - P_\ell(\cos\theta_2) \right| < \text{Const} \frac{\sqrt{|\theta_1 - \theta_2|}}{\left[\sin\theta_1\ \sin\theta_2\right]^{1/4}} \qquad \text{(AI.4)}$$

Eq. (AI.4) holds for $0 < \theta_1$ and $\theta_2 < \frac{2\pi}{3}$ and $\frac{\pi}{3} < \theta_1$ and $\theta_2 < \pi$. The only case not covered by this is

$$\theta_{1,2} < \pi/3 \qquad \theta_{2,1} > \frac{2\pi}{3}$$

but then

$$\left| P_\ell(\cos\theta_1) - P_\ell(\cos\theta_2) \right| < 2$$

and (AI.4) is still valid.

APPENDIX II

Let us prove that

$$1 \pm P_\ell(\cos\theta)$$

for $\ell \geqslant 1$. The simplest is to use a bound which has been obtained by the author many years ago [6]

$$\left|P_\ell(\cos\theta)\right| < \frac{1}{\left[1+\ell(\ell+1)\sin^2\theta\right]^{1/4}}$$

So for $\ell \geqslant 4$

$$\left|P_\ell(\cos\theta)\right| < \frac{1}{\left[1+20\sin^2\theta\right]^{1/4}}$$

and it is easy to show that

$$1 - \left[1+20\sin^2\theta\right]^{-1/4} > \frac{\sin^2\theta}{2}$$

The individual cases are easy :

$$\ell = 1 \qquad 1 \pm \cos\theta > \frac{1-\cos^2\theta}{2}$$

$$\ell = 2 \qquad \begin{cases} \frac{3}{2}\left(1-\cos^2\theta\right) > \frac{1-\cos^2\theta}{2} \\ \frac{3\cos^2\theta+1}{2} > \frac{1-\cos^2\theta}{2} \end{cases}$$

$$\ell = 3 \qquad 1 - P_3 = \frac{1-\cos\theta}{2}\left[5\cos^2\theta + 5\cos\theta + 2\right]$$

— N even :

we have either

$$\alpha)\quad \Pi_N = \left[\Pi_{\frac{N}{2}}\right]^2$$

or

$$\beta)\quad \Pi_N = (1-x^2)\left[\Pi_{\frac{N}{2}-1}(x)\right]^2$$

where $\Pi_{N/2}$ and $\Pi_{N/2-1}$ are polynomials of degree $N/2$ and $N/2-1$.

In case α) we can write

$$\Pi_{\frac{N}{2}} = \sum_0^{N/2} c_p\, P_p^w(x)$$

where the $P_p^w(x)$ are orthogonal polynomials on $-1\ +1$ with weight w. Then

$$\int_{-1}^{+1} w(x)\left(\Pi_{\frac{N}{2}}\right)^2 dx = \sum_0^{N/2} (c_p)^2\, N_p^w \qquad \text{(AIII.4)}$$

where $$N_p^w = \int_{-1}^{+1}\left[P_p^w(x)\right]^2 dx$$

It is clear that only $c_{N/2}$ is fixed by the condition $|\Pi_N| \sim x^N$ for $x \to \infty$. (AIII.4) will therefore be minimized by taking

$$\Pi_N = \left[P^{w}_{\frac{N}{2}}(x)\right]^2 \times \text{const} \qquad \text{(AIII.5)}$$

If we consider case β), we shall have to take

$$\Pi_N = (1-x^2)\left[P^{\tilde{w}}_{\frac{N}{2}-1}(x)\right]^2 \times \text{const} \qquad \text{(AIII.6)}$$

where $\tilde{w}(x) = w(x)\left[1-x^2\right]$. What remains to be done is to find what is the best of the two possibilities. This will depend on $w(x)$.

-- N odd :

Then clearly, repeating the same arguments, we find that the candidates to minimize (AIII.1) are

$$\text{const}\,(1-x)\left[P^{\approx}_{\frac{N-1}{2}}(x)\right]^2 \qquad \text{(AIII.7)}$$

where $P^{\approx w}_{N-1/2}$ is an orthogonal polynomial with weight $\approx w = (1-x)w$, or

$$\text{const}\,(1+x)\left[P^{\approx}_{\frac{N-1}{2}}(x)\right]^2 \qquad \text{(AIII.8)}$$

where $\approx w = (1+x)w$.

In the present case $w = (1-x^2)$. The P^w are $P^{1,1}$ Jacobi polynomials, i.e., derivatives of Legendre polynomials. The $P^{\tilde{w}}$ are $P^{2,2}$ Jacobi polynomials, i.e., second derivatives of Legendre polynomials. The $P^{\tilde{\tilde{w}}}$ are $P^{2,1}$ Jacobi's. It is easy to decide between possibilities α and β for n even. One has :

$$\Pi_N^{\alpha} = \left[\frac{P'_{\frac{N}{2}+1}}{\left(\frac{N}{2}+1\right) K\left(\frac{N}{2}+1\right)}\right]^2$$

$$\Pi_N^{\beta} = (1-x^2)\left[\frac{P''_{\frac{N}{2}+1}}{\left(\frac{N}{2}+1\right)\frac{N}{2}\;K\left(\frac{N}{2}+1\right)}\right]^2$$

with

$$K(\nu) = \frac{(2\nu)!}{2^{\nu}(\nu!)^2}$$

and hence [see Magnus and Oberettinger [7)]]

$$\frac{\int(1-x^2)\,\Pi_N^{\beta}\,dx}{\int(1-x^2)\,\Pi_N^{\alpha}\,dx} = \frac{N+6}{N}$$

The minimum is therefore given by Π_N^{α}. Hence for L <u>even</u> we have

$$\mathrm{Re} f_o \pm \mathrm{Re} f_e > |\mathrm{Re} f_L| \frac{(L+1/2)(L+4)}{(L+3)(L+2)} \frac{K(L)}{\left[K\left(\frac{L}{2}+1\right)\right]^2} \qquad \text{(AIII.9)}$$

For large L the coefficient of $|\mathrm{Re}\, f_L|$ is $\sim \frac{1}{8}\sqrt{\pi L}$. What matters is that it increases with L which can be checked by comparing L with $L+2$. In particular for L even $\geqslant 2$ we have

$$\mathrm{Re}f_0 \pm \mathrm{Re}f_e > |\mathrm{Re}f_L| \, \frac{1}{2} \qquad \text{(AIII.10)}$$

for L odd we have to take

$$\Pi_L = \mathrm{Const} \times \left[P^{2,1}_{\frac{L-1}{2}}(x) \right]^2 (1-x)$$

Again, we shall get inequalities with a coefficient increasing with L. Let us therefore look only at $L = 3$. Then

$$P^{2,1}_1(x) = \mathrm{Const} \left(x + \frac{1}{5} \right)$$

in this way one gets, for $L = 3$

$$\mathrm{Re}f_0 \pm \mathrm{Re}f_e \geqslant |\mathrm{Re}f_{L=3}| \, \frac{7}{10} \qquad \text{(AIII.11)}$$

The conclusion is that one can take (AIII.10) as valid in all cases, for $L \geqslant 2$.

R E F E R E N C E S

1) J.H. Crichton, Nuovo Cimento 45, 256 (1966).

2) R.G. Newton, preprint.

3) See, for instance, N.A. Krasnosel'skii, "Topological methods in in the theory of non-linear integral equations", MacMillan Company, New York (1964).

4) Except for the replacement $\ell \to \ell+\frac{1}{2}$ this is established in G. Szegö, "Orthogonal Polynomials", American Mathematical Society Colloquium publication, Vol.XXIII, New York (1959), p.164. The replacement $\ell \to \ell+\frac{1}{2}$ can be justified easily.

5) G. Szegö, Ref. 4), p.170. Notice that Szegö is the only author giving the optimum angular dependence of the bound. Even standard textbooks like Magnus and Oberettinger, the Bateman Project, Gradshteyn and Ryzhik, give bounds of the form

$$\left|\left(\frac{d}{d\cos\theta}\right)^n P_\ell(\cos\theta)\right| < C\left(\ell+\tfrac{1}{2}\right)^{n-\frac{1}{2}}(\sin\theta)^{-2n-\frac{1}{2}}$$

while the best bounds are of the form

$$\left|\left(\frac{d}{d\cos\theta}\right)^n P_\ell(\cos\theta)\right| < C\left(\ell+\tfrac{1}{2}\right)^{n-\frac{1}{2}}(\sin\theta)^{-n-\frac{1}{2}}$$

6) A. Martin, Phys.Rev. 129, 1432 (1963).

7) F. Magnus and G. Oberettinger, "Formulas and Theorems for Theoretical Physics", Chelsea Publishing Company, New York (1954), p.55.