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CONSTRUCTION OF UNITS IN INTEGRAL GROUP RINGS OF FINITE NILPOTENT GROUPS

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It is a basic question, in the theory of group rings, to describe constructively the group of units of the integral group ring **Z**G of a finite group G. Let

$$U(\mathbf{Z}G) = U = \{u \in \mathbf{Z}G \mid \text{ there exists } v \in \mathbf{Z}G \text{ with } uv = 1\}$$

be the unit group of **Z**G. Bass and Milnor **[1]** have given generators for a subgroup of finite index in U if G is abelian. We shall do the same for nilpotent groups with a few exceptions arising from the Sylow 2-group of G.

We begin by describing two key classes of units of **Z**G.

(a) THE BASS CYCLIC UNITS. Let $a \in G$ be an element of order d. Let $|G| = n$, $\varphi(n) = m$, where φ is the Euler function. For a natural number i, less than d and relatively prime to d, the element

$$u = (1 + a + \cdots + a^{i-1})^m + ((1 - i^m)/d)\hat{a}, \qquad \hat{a} = 1 + a + \cdots + a^{d-1}$$

belongs to **Z**G as $i^m \equiv 1 \pmod d$ since $\varphi(d) \mid \varphi(n)$ when $d \mid n$. Moreover, u is a unit as seen by applying various characters of $\langle a \rangle$. These units correspond to the cyclotomic units and Bass **[1]** has proved that if G is cyclic then a subset of units of the above type obtained for all $\langle a \rangle$ gives a linear independent set of generators of a subgroup of finite index in U. We call the units obtained by varying $\langle a \rangle$ and i, the Bass cyclic units of **Z**G.

(b) THE BICYCLIC UNITS. Let $a, b \in G$. Then

$$u_{a,b} = 1 + (a - 1)b\hat{a}$$

is a unit whose inverse is $1 - (a - 1)b\hat{a}$. It is easily seen that $u_{a,b} \neq 1$ if and only if b does not normalize $\langle a \rangle$. These units are called the bicyclic

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units of $\mathbf{Z}G$. They together with the Bass cyclic units turn out to essentially play the same role for nilpotent groups as the Bass cyclics do for abelian groups.

In order to state our first main theorem we denote by $\mathbf{H}_k$, $k \geq 3$, the Hamiltonian quaternion algebra over the real field $\mathbf{Q}(\zeta_{2^{k-1}} + \zeta_{2^{k-1}}^{-1})$, where $\zeta_{2^{k-1}}$ is a primitive 2^{k-1}th root of unity.

THEOREM 1. *Let G be a nilpotent group such that the rational group algebra $\mathbf{Q}G$ has no simple Wedderburn components which are 2×2 matrices over $\mathbf{Q}$ or $\mathbf{Q}(i)$ or $2^r \times 2^r$ matrices, $r \geq 0$ over $\mathbf{H}_k$, $k \geq 3$. Then the Bass cyclic units and the bicyclic units of $\mathbf{Z}G$ generate a subgroup of finite index in $U(\mathbf{Z}G)$.*

All nilpotent groups of odd order clearly satisfy the hypothesis of Theorem 1. If G is a 2-group some restrictions are necessary for the conclusion to hold. It is possible to construct groups G whose rational group algebra has, 2×2 rational matrices, as a simple component and for which the theorem is not true. For instance, if G is the group of order 16, given by

$$G = \langle a, b \mid a^4 = 1 = b^4, a^b = a^{-1} \rangle$$

then $\mathbf{Q}G = 4\mathbf{Q} \oplus 2\mathbf{Q}(i) \oplus \mathbf{Q}_{2\times 2} \oplus \mathbf{Q}_{2\times 2}$ and G has an epimorphic image D_8, the dihedral group of order 8. Corresponding to this a representation of G is given by

$$T(a) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad T(b) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then

$$T(u_{ab,a}) = A = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}, \qquad T(u_{a^3b,a}) = \begin{bmatrix} -1 & -2 \\ 2 & 3 \end{bmatrix} = B.$$

It is easily checked that under this representation the Bass cyclics are all mapped onto I and the bicyclics generate the group

$$\left\langle \begin{bmatrix} 1 & 8 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 8 & 1 \end{bmatrix}, A^2, B^2 \right\rangle,$$

which turns out to be of infinite index in $SL(2, \mathbf{Z})$. This means that in such cases we have to look for some more generators.

Now, a few words to the proof of Theorem 1. We use the congruence theorems of Bass-Milnor-Serre [2] and Serre [4] and an extension thereof which is due to Vaserstein [5] and [6]. To state those let O be the ring of integers in an algebraic number field K, which is finite dimensional over $\mathbf{Q}$. For an ideal Q of O, we denote by $E(Q)$ the subgroup of $SL(n, O)$ generated by all Q-elementary matrices $I + qe_{lm}$, $q \in Q$, $l \neq m$, e_{lm} a matrix unit and by $\tilde{E}(Q)$ its normal closure in $SL(n, O)$. If $n \geq 3$ then the normal subgroup of $E(O)$ generated by $E(Q)$ is normal in $SL(n, O)$ and hence coincides with $\tilde{E}(Q)$.

LEMMA. *Assume that $n \geq 3$ or $n = 2$ and K is neither rational nor imaginary quadratic. Then*

(1) $(SL(n, O) \colon \tilde{E}(Q)) < \infty$ *for any nonzero ideal Q of O.*

(2) *Every noncentral subgroup of* $SL(n, O)$ *normalized by a subgroup of finite index contains* $\tilde{E}(Q)$ *for some nonzero ideal* Q *of* O.
(3) *If* $n \geq 3$, *then* $\tilde{E}(Q^2) \leq E(Q)$, *in particular,* $(SL(n, O): E(Q)) < \infty$.
(4) *If* $n = 2$, *then* $(SL(2, O): E(Q)) < \infty$.

Let B_1, B_2 be the subgroups of U generated by the Bass cyclics and the bicyclics respectively. Let $B = \langle B_1, B_2 \rangle$. We first observe that the simple components of $\mathbf{Q}G$ are matrices over commutative fields. We then use a theorem of Bass **[1]** to deduce that it is enough to show that B_2 contains a subgroup of finite index in $SL(n, O)$ for every component $K_{n\times n}$, $n > 1$. In order to prove this, we have the following main propositions.

PROPOSITION 1. $(SL(n, O): \Pi(B_2)) < \infty$ *for all projections* $\Pi: \mathbf{Q}G \to K_{n\times n}$, $n > 1$.

PROPOSITION 2. *For two different projections* Π_1 *and* Π_2, *there is an element* $b \in B_2$ *such that* $\Pi_1(b) = 1$, $\Pi_2(b) = 1 + xe_{lm}$, $x \neq 0$, $l \neq m$.

These results depend on a new and canonical way of writing an absolutely irreducible representation T of a p-group G as an induced representation. Namely, depending on T we find a maximal subgroup M of G together with an irreducible representation V of M inducing T such that among other properties T and V have the same character field. To state this, let us denote by $|Y|_1$ the number of eigenvalues equal to one of a square matrix Y. We define three groups

$$Q_{2^k} = \langle a, b: a^{2^{k-1}} = 1, a^{2^{k-2}} = b^2, a^b = a^{-1}\rangle, \text{ the generalized quaternion group of order } 2^k,\ k \geq 3$$

$$D_{2^k} = \langle a, b: a^{2^{k-1}} = 1 = b^2, a^b = a^{-1}\rangle, \text{ the dihedral group of order } 2^k,\ k \geq 3$$

$$D^-_{2^{k+1}} = \langle a, b \mid a^{2^k} = 1 = b^2, a^b = a^{2^{k-1}-1}\rangle \text{ of order } 2^{k+1}, \qquad k \geq 3.$$

PROPOSITION 3. *Assume that* G *is a* p*-group different from* Q_{2^k}, D_{2^k}, $D^-_{2^{k+1}}$. *Let* T *be an absolutely irreducible faithful representation of* G *of degree* p^n, $n \geq 1$. *Then there exists a maximal abelian normal subgroup* A *in* G, *an element* $a \in A$ *of order* p, *a maximal subgroup* M *of* G *containing* A, *and an irreducible representation* V *of* M *such that* $T = \operatorname{ind}_M^G V$, $V(a) = 1$, $|V(a^b)|_1 = 0$ $(\forall b \notin M)$.

This proposition has another consequence, which settles a conjecture that has been around a while but which, to our knowledge, has not been affirmed in the literature yet. The result for p-groups, p a regular prime, goes back to Schur [**3**, p. 518].

THEOREM 2. *If* G *is a nilpotent group having Schur index one for every representation then each representation of* G *is realizable over the algebraic integers in its character field.*

The proofs will appear elsewhere.

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